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On the Construction of a Geometric Invariant Measuring the Deviation from Kerr Data

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Abstract. This article contains a detailed and rigorous proof of the construction of a geometric invariant for initial data sets for the Einstein vacuum field equations. This geometric invariant vanishes if and only if the initial data set corresponds to data for the Kerr spacetime, and thus, it characterises this type of data. The construction presented is valid for boosted and non-boosted initial data sets which are, in a sense, asymptotically Schwarzschildean. As a preliminary step to the construction of the geometric invariant, an analysis of a characterisation of the Kerr spacetime in terms of Killing spinors is carried out. A space spinor split of the (spacetime) Killing spinor equation is performed to obtain a set of three conditions ensuring the existence of a Killing spinor of the development of the initial data set. In order to construct the geometric invariant, we introduce the notion of approximate Killing spinors. These spinors are symmetric valence 2 spinors intrinsic to the initial hypersurface and satisfy a certain second order elliptic equation—the approximate Killing spinor equation. This equation arises as the Euler-Lagrange equation of a non-negative integral functional. This functional constitutes part of our geometric invariant—however, the whole functional does not come from a variational principle. The asymptotic behaviour of solutions to the approximate Killing spinor equation is studied and an existence theorem is presented.

1. Introduction

The Kerr spacetime is, undoubtedly, one of the most important exact solutions to the vacuum Einstein field equations [32]. It describes a rotating stationary asymptotically flat black hole parametrized by its mass m and its specific angular momentum a. One of the outstanding challenges of contemporary General Relativity is to obtain a full understanding of the properties and the structure of the Kerr spacetime and of its standing in the space of solutions to the Einstein field equations.

There are a number of difficult conjectures and partial results concerning the Kerr spacetime. In particular, it is widely expected to be the only rotating stationary asymptotically flat black hole. This conjecture has been proved if the spacetime is assumed to be analytic (C^ω)—see, e.g. [14] and references within. Recently, there has been progress in the case where the spacetime is assumed to be only smooth (C^∞)—see [29]. Moreover, it has been shown that a regular, non-extremal stationary black hole solution of the Einstein vacuum equations which is suitably close to a Kerr solution must be that Kerr solution—i.e. *perturbative stability* among the class of stationary solutions [1].

Another of the conjectures concerning the Kerr spacetime is that it describes, in some sense, the late time behaviour of a spacetime with dynamical (that is, non-stationary) black holes—this is sometimes known as the *establishment point of view of black holes*, cfr. [40]. A step in this direction is to obtain a proof of the *non-linear stability* of the Kerr spacetime—this conjecture roughly states that the Cauchy problem for the vacuum Einstein field equations with initial data for a black hole which is suitably close to initial data for the Kerr spacetime gives rise to a spacetime with the same global structure as Kerr and with suitable pointwise decay. Numerical simulations support the conjectures described in this paragraph.

A common feature in the problems mentioned in the previous paragraphs is the need of having a precise formulation of what it means that a certain spacetime is *close* to the Kerr solution. Due to the coordinate freedom in General Relativity, it is, in general, difficult to measure how much two spacetimes differ from each other. Statements made in a particular choice of coordinates can be deceiving. In the spirit of the geometrical nature of General Relativity, one would like to make statements which are coordinate and gauge independent. Invariant characterisations of spacetimes provide a way of bridging this difficulty.

Most analytical and numerical studies of the Einstein field equations make use of a 3+1 decomposition of the equations and the unknowns. In this context, the question of whether a given initial data set for the Einstein field equations corresponds to data for the Kerr spacetime arises naturally—an initial data set will be said to be data for the Kerr spacetime if its development is isometric to a portion (or all) of the Kerr spacetime. A related issue arises when discussing the (either analytical or numerical) 3+1 evolution of a spacetime: do the leaves of the foliation approach, as a result of the evolution, hypersurfaces of the Kerr spacetime? In order to address these issues it is important to have a geometric characterisation of the Kerr solution which is amenable to a 3+1 splitting.

A number of invariant characterisations are known in the literature, each with their own advantages and disadvantages. For completeness we discuss some which bear connection to the analysis presented in this article:

The Simon and Mars-Simon tensors. A convenient way of studying stationary solutions to the Einstein field equations is through the quotient manifold of the orbits of the stationary Killing vector. The Schwarzschild spacetime is characterised among all stationary solutions by the vanishing of the Cotton tensor of the metric of this quotient manifold—see, e.g. [20]. In [43] a suitable generalisation of the Cotton tensor of the quotient manifold was introduced—the *Simon tensor*. The vanishing of the Simon tensor together with asymptotic flatness and non-vanishing of the mass characterises the Kerr solution in the class of stationary solutions. In [36,37] a spacetime version of the Simon tensor was introduced—the so-called *Mars-Simon tensor*. The construction of this tensor requires the *a priori* existence of a Killing vector in the spacetime. Accordingly, it is tailored for the problem of the uniqueness of stationary black holes. The vanishing of the Mars-Simon tensor together with some global conditions (asymptotic flatness, non-zero mass, stationarity of the Killing vector) characterises the Kerr spacetime.

Characterisations using concomitants of the Weyl tensor. A concomitant of the Weyl tensor is an object constructed from tensorial operations on the Weyl tensor and its covariant derivatives. An invariant characterisation of the Kerr spacetime in terms of concomitants of the Weyl tensor has been obtained in [19]. This result generalises a similar result for the Schwarzschild spacetime given in [17]. These characterisations consist of a set of conditions on concomitants of the Weyl tensor, which if satisfied, characterise locally the Kerr/Schwarzschild spacetime. An interesting feature of the characterisation is that it provides expressions for the stationary and axial Killing vectors of the spacetime in terms of concomitants of the Weyl tensor. Unfortunately, the concomitants used in the characterisation are complicated, and thus, produce very involved expressions when performing a 3+1 split.

Characterisations by means of generalised symmetries. Generalised symmetries (sometimes also known as hidden symmetries) are generalisations of the Killing vector equation—like the Killing tensors and conformal Killing-Yano tensors. These tensors arise naturally in the discussion of the so-called Carter constant of motion and in the separability of various types of linear equations on the Kerr spacetime—see, e.g. [11,31,42]. In particular, the existence of a conformal Killing-Yano tensor is equivalent to the existence of a valence-2 symmetric spinor satisfying the Killing spinor equation. An important property of the Schwarzschild and Kerr spacetimes is that they admit a Killing spinor. This Killing spinor generates, in a certain sense, the Killing vectors and Killing-Yano tensors of the exact solutions in question [27]. Moreover, as it will be discussed in the main part of this article, for a spacetime which is neither conformally flat nor of Petrov type N, the existence of a Killing spinor associated with a Killing-Yano tensor together with the requirement of asymptotic flatness renders a characterisation of the Kerr spacetime. To the best of our knowledge, this property has only been discussed in the literature—without proof—in [18].

Although at first sight independent, the characterisations of the Schwarzschild and Kerr spacetimes described in the previous paragraphs are interconnected—sometimes in very subtle manners. This is not too surprising as all these characterisations make use in a direct or indirect manner of the fact that the Kerr spacetime is a vacuum spacetime of Petrov type D—see, e.g. [45] for a discussion of the Petrov classification. The art in producing a useful characterisation of the Kerr spacetime lies in finding further conditions on type D spacetimes which are natural and simple to use.

A Characterisation of Kerr Data

Characterisations of initial data sets for the Schwarzschild and Kerr spacetimes have been discussed in [21,22,47]. These characterisations make use of a number of local and global ingredients. For example, in [22] it is necessary to assume the existence of a Killing vector on the development of the spacetime.

In this article we present a rigorous and detailed discussion of a geometric invariant characterising initial data for the Kerr spacetime. A restricted version of this construction has been presented in [2].

The starting point of our construction is the observation that the existence of a Killing spinor in the Kerr spacetime is a key property. It allows to relate the Killing vectors of the spacetime with its curvature in a neat way. The reason for its importance can be explained in the following way: from a specific Killing spinor it is possible to obtain a Killing vector which in general will be complex. It turns out that for the Kerr spacetime this Killing vector is in fact real and coincides with the stationary Killing vector. It can be shown that the Kerr solution is the only asymptotically flat vacuum spacetime with these properties, if one assumes that there are no points where the Petrov type is either N or O.

Given the aforementioned spacetime characterisation of the Kerr solution, the question now is how to make use of it to produce a characterisation in terms of initial data sets. For this, one has to encode the existence of a Killing spinor at the level of the data. The way of doing this was first discussed in [23] and follows the spirit of the well-known discussion of how to encode Killing vectors on initial data—see, e.g. [5].

The conditions on the initial data that ensure the existence of a Killing spinor in its development are called the *Killing spinor initial data equations* and are, like the Killing initial data equations (KID equations), overdetermined. In [15], a procedure was given on how to construct equations which generalise the KID equations for time symmetric data. These generalised equations have the property that for a particular behaviour at infinity they always admit a solution. If the spacetime admits Killing vectors, then the solutions to the generalised KID equations with the same asymptotic behaviour as the Killing vectors are, in fact, Killing vectors. Therefore, one calls the solutions to the generalised KID equations *approximate symmetries*. The total number of approximate symmetries is equal to the maximal number of possible Killing vectors on the spacetime. A peculiarity of this procedure is that if the spacetime is not stationary, the approximate Killing vector associated with a time translation does not have the same asymptotic behaviour as a time translation.[1]

[1] Here, and in what follows, for a time translation it is understood a Killing vector which in some asymptotically Cartesian coordinate system has a leading term of the form ∂_t.

The Killing spinor initial data equations consist of three conditions: one of them differential (the *spatial Killing spinor equation*)[2] and two *algebraic conditions*. Following the spirit of [15] we construct a generalisation of the spatial Killing spinor equation—*the approximate Killing spinor equation*. This equation is elliptic and of second order. This equation is the Euler-Lagrange equation of an integral functional—the L^2-norm of the exact spatial Killing spinor equation. For this equation it is possible to prove the following theorem:

Theorem. *For initial data sets to the Einstein field equations with suitable asymptotic behaviour, there exists a solution to the approximate Killing spinor equation with the same asymptotic behaviour as the Killing spinor of the Kerr spacetime.*

A precise formulation will be given in the main text. In particular, it will be seen that the conditions on the asymptotic behaviour of the initial data are rather mild and amount to requiring the data to be, in a sense, asymptotically Kerr data. Contrasted with the results in [15], this result is notable because, arguably, the most important approximate symmetry of [15] does not share the same asymptotic behaviour as the exact symmetry. The precise version of this theorem generalises the one discussed in [2] in that it allows for boosted data. This generalisation is only possible after a detailed analysis of the asymptotic solutions of the exact Killing spinor equation.

The approximate Killing spinor discussed in the previous paragraphs can be used to construct a geometric invariant for the initial data. This invariant is global and involves the L^2 norms of the Killing spinor initial data equations evaluated at the approximate Killing vector. It should be observed that only part of the invariant satisfies a variational principle—this is a further difference with respect to the construction of [15]. As the initial data set is assumed to be asymptotically Euclidean, one expects its development to be asymptotically flat. This renders the desired characterisation of Kerr data and our main result.

Theorem. *Consider an initial data set for the vacuum Einstein field equations whose development in a small slab is neither of Petrov type N nor O at any point, and such that the L^2 norm of the Killing spinor initial data equations evaluated at the solution (with the same asymptotic behaviour as the Killing spinor of the Kerr spacetime) to the approximate Killing spinor equation vanishes. Then the initial data set is locally data for the Kerr spacetime. Furthermore, if the* 3*-manifold has the same topology as that of hypersurfaces of the Kerr spacetime, then the initial data set is data for the Kerr spacetime.*

There are several advantages of this characterisation over previous ones given in the literature. Most notably, it allows to condense the non-Kerrness of an initial data set in a single number. That this invariant constitutes a

[2] The idea of using the spatial part of spinorial equations to characterise slices of particular spacetimes is not new. In [46] the spatial twistor equation has been used to characterise slices of conformally flat spacetimes. See also [7].

good distance in the space of initial data sets will be discussed elsewhere. Furthermore, the way the invariant is constructed is fully amenable to a numerical implementation—the elliptic solvers that one would need to compute the solution to the approximate Killing spinor equation are, nowadays, standard technology.

Detailed Outline of the Article

The outline of the article is as follows: in Sect. 2 we study Killing spinors, and their influence on the algebraic type of the spacetime. We relate the Killing spinors to Killing vectors and Killing-Yano tensors. Using these results together with a characterisation of the Kerr spacetime by Mars [37], we conclude that the Kerr spacetime can be characterised in terms of existence of a Killing spinor related to a real Killing vector. This has previously been overlooked in the literature, but it is a key element in our analysis.

Section 3 follows with an exposition of space spinors, which will be the main computational tool for the remainder of the paper. Following that, in Sect. 4 we study a 3+1 splitting of the Killing spinor equation. A similar analysis was carried out in [23], but here we manage to condense the result into three simple equations, the *spatial Killing spinor equation* and two algebraic equations. We also present general equations for the spatial derivatives of a general valence 2 spinor, which is not necessarily a Killing spinor. These equations are also used in later parts of the paper.

In Sect. 5 we introduce the new concept of *approximate Killing spinors*. These are introduced as solutions to an elliptic equation formed by composing the spatial Killing spinor operator with its formal adjoint. That this composed operator is indeed elliptic and formally self adjoint is proved. We also see that the approximate Killing spinor equation can be derived from a variational principle.

To get unique solutions to the approximate Killing spinor equation, we need to specify the asymptotic behaviour. For a rigorous treatment of this, we use weighted Sobolev spaces; these are described in Sect. 6. Here, we also study the asymptotics of a Killing spinor on a boosted slice of the Schwarzschild spacetime. In general, we study slices of an arbitrary spacetime with asymptotics similar to those of the Schwarzschild spacetime. Using these assumptions, we can then in Sect. 7 prove existence of spinors with the same asymptotics as the Killing spinor in the Schwarzschild spacetime. We later use these spinors as seeds for solutions to the approximate Killing spinor equation. In this way we get the desired asymptotic behaviour of our approximate Killing spinors.

In Sect. 8 we study the approximate Killing spinor equation in our asymptotically Euclidean manifolds to gain existence and uniqueness of solutions with the desired asymptotics. This is done by means of the Fredholm alternative on weighted Sobolev spaces, transforming the existence problem into a study of the kernel of the Killing spinor operator. In this process we get the first part of the geometric invariant—the L^2 norm of the approximate Killing spinor. This norm is proved to be finite. The geometric invariant is constructed in Sect. 9, by adding the L^2 norms of the algebraic conditions. There follows our main theorem: the invariant vanishes if and only if the spacetime is the Kerr spacetime. The invariant is as a consequence of the construction proved to be finite and well defined.

We also include two appendices. The first describes an alternative proof of finiteness of a particular boundary integral in Sect. 8. The other contains tensor versions of the invariant—this can be useful in applications.

General Notation and Conventions

All throughout, $(\mathcal{M}, g_{\mu\nu})$ will be an orientable and time orientable globally hyperbolic vacuum spacetime. It follows that the spacetime admits a spin structure—see [24,25]. Here, and in what follows, $\mu, \nu, \cdots$ denote abstract 4-dimensional tensor indices. The metric $g_{\mu\nu}$ will be taken to have signature $(+,-,-,-)$. Let ∇_μ denote the Levi-Civita connection of $g_{\mu\nu}$. The sign of the Riemann tensor will be given by the equation

$$\nabla_\mu \nabla_\nu \xi_\zeta - \nabla_\nu \nabla_\mu \xi_\zeta = R_{\nu\mu\zeta}{}^{\eta} \xi_\eta.$$

The triple $(\mathcal{S}, h_{ab}, K_{ab})$ will denote initial data on a hypersurface of the spacetime $(\mathcal{M}, g_{\mu\nu})$. The symmetric tensors h_{ab}, K_{ab} will denote, respectively, the 3-metric and the extrinsic curvature of the 3-manifold $\mathcal{S}$. The metric h_{ab} will be taken to be negative definite—that is, of signature $(-,-,-)$. The indices a, $b, \ldots$ will denote abstract 3-dimensional tensor indices, while i, $j, \ldots$ will denote 3-dimensional tensor coordinate indices. Let D_a denote the Levi-Civita covariant derivative of h_{ab}.

Spinors will be used systematically. We follow the conventions of [41]. In particular, $A, B, \ldots$ will denote abstract spinorial indices, while $\mathbf{A}, \mathbf{B}, \ldots$ will be indices with respect to a specific frame. Tensors and their spinorial counterparts are related by means of the solder form $\sigma_\mu{}^{AA'}$ satisfying $g_{\mu\nu} = \sigma_\mu^{AA'} \sigma_\nu^{BB'} \epsilon_{AB} \epsilon_{A'B'}$, where ϵ_{AB} is the antisymmetric spinor and $\epsilon_{A'B'}$ its complex conjugate copy. One has, for example, that $\xi_\mu = \sigma_\mu{}^{AA'} \xi_{AA'}$. Let $\nabla_{AA'}$ denote the spinorial counterpart of the spacetime connection ∇_μ. Besides the connection $\nabla_{AA'}$, two other spinorial connections will be used: D_{AB}, the spinorial counterpart of the Levi-Civita connection D_a and ∇_{AB}, the Sen connection of $(\mathcal{M}, g_{\mu\nu})$—full details will be given in Sect. 3.

The Kerr spacetime. For the Kerr spacetime the maximal analytic extension of the Kerr metric as described by Boyer and Lindquist [8] and Carter [10] will be understood. When regarding the Kerr spacetime as the development of Cauchy initial data, we will only consider its maximal globally hyperbolic development.

2. Killing Spinors: General Theory

As mentioned in the introduction, our point of departure will be a characterisation of the Kerr spacetime based on the existence in the spacetime of a valence-2 symmetric spinor satisfying the Killing spinor equation. To the best of our knowledge, this characterisation of the Kerr spacetime has not explicitly been discussed in the literature, save for a side remark in [18]. In this section we provide a summary of this characterisation and fill in some technical details.

2.1. Killing Spinors and Petrov Type D Spacetimes

A valence-2 Killing spinor is a symmetric spinor $\kappa_{AB} = \kappa_{(AB)}$ satisfying the equation

$$\nabla_{A'(A}\kappa_{BC)} = 0. \tag{1}$$

Killing spinors offer a way of relating properties of the curvature to properties of the symmetries of the spacetime. Taking a further derivative of equation (1), antisymmetrising and commuting the covariant derivatives one finds the integrability condition

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0, \tag{2}$$

where Ψ_{ABCD} denotes the self-dual Weyl spinor. The above integrability imposes strong restrictions on the algebraic type of the Weyl spinor. More precisely, it follows that if $\Psi_{ABCD} \neq 0$ and $\kappa_{AB} \neq 0$, then

$$\Psi_{ABCD} = \psi\kappa_{(AB}\kappa_{CD)}, \tag{3}$$

where ψ is a scalar. Thus, Ψ_{ABCD} must be of Petrov type D or N—see, e.g. [23,30]. The converse is also true [28,42,48]. Summarising:

Theorem 1 (Walker and Penrose [48]). *A vacuum spacetime admits a valence-2 Killing spinor if and only if it is of Petrov type D, N or O.*

From (3) it can also be seen that Ψ_{ABCD} is of Petrov type N if and only if κ_{AB} is algebraically special. That is, there exists a spinor α_A such that $\kappa_{AB} = \alpha_A\alpha_B$. Thus, an algebraically general Killing spinor $\kappa_{AB} = \alpha_{(A}\beta_{B)}$ is always associated with a vacuum spacetime of Petrov type D.

2.2. The Killing Vector Associated with a Killing Spinor and the Generalised Kerr-NUT Metrics

Given a Killing spinor κ_{AB}, the concomitant

$$\xi_{AA'} = \nabla^{B}{}_{A'}\kappa_{AB}, \tag{4}$$

is a complex Killing vector of the spacetime: its real and imaginary parts are themselves Killing vectors of the spacetime [27]. In relation to this it should be pointed out that all vacuum Petrov type D spacetimes are known [33]. It follows from the analysis in the latter reference that all vacuum, Petrov type D spacetimes have a commuting pair of Killing vectors. A key property of the Kerr spacetime is the following (cfr. [27,42]):

Proposition 2. *Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum Petrov type D spacetime. The Killing vector $\xi_{AA'}$ given by (4) is real in the case of the Kerr spacetime.*

Remark 1. In what follows, the class of Petrov type D spacetimes for which $\xi_{AA'}$ is real will be called the *generalised Kerr-NUT class*—cfr. [18]. This class can be alternatively characterised—see, e.g. [31]—by the existence of a Killing-Yano tensor

$$Y_{\mu\nu} = Y_{[\mu\nu]}, \quad \nabla_{(\mu}Y_{\nu)\lambda} = 0.$$

The correspondence between the Killing spinor κ_{AB} and the spinorial counterpart $Y_{AA'BB'}$ of the Killing-Yano tensor, $Y_{\mu\nu}$, is given by

$$Y_{AA'BB'} \equiv \mathrm{i}\,(\kappa_{AB}\epsilon_{A'B'} - \epsilon_{AB}\bar{\kappa}_{A'B'}),$$

where the overbar denotes the complex conjugate.

Remark 2. In terms of the Kinnersley list of type D metrics, the class of generalised Kerr-NUT metrics contains, in addition to the proper Kerr-NUT metrics (II.C), also the metrics II.E—see, [16].

An important property of the generalised Kerr-NUT metrics involves the Killing form, $F_{AA'BB'} = -F_{BB'AA'}$, of a real Killing vector $\xi_{AA'}$ defined by

$$F_{AA'BB'} \equiv \frac{1}{2}\,(\nabla_{AA'}\xi_{BB'} - \nabla_{BB'}\xi_{AA'}). \tag{5}$$

Let

$$\mathcal{F}_{AA'BB'} \equiv \frac{1}{2}\,(F_{AA'BB'} + \mathrm{i}F^{*}_{AA'BB'}) \tag{6}$$

denote the corresponding *self-dual Killing form*, with $F^{*}_{AA'BB'}$ the Hodge dual of $F_{AA'BB'}$. Due to the symmetries of the Killing form one can write

$$\mathcal{F}_{AA'BB'} = \mathcal{F}_{AB}\epsilon_{A'B'}, \tag{7}$$

with

$$\mathcal{F}_{AB} \equiv \frac{1}{2}F_{AQ'B}{}^{Q'} = \mathcal{F}_{BA}. \tag{8}$$

One has the following result:

Lemma 3. *For generalised Kerr-NUT spacetimes one has that*

$$\mathcal{F}_{AB} = \varkappa\kappa_{AB},$$

where $\varkappa$ is a non-vanishing scalar function, so that the principal spinors of $\mathcal{F}_{AB}$ and Ψ_{ABCD} are parallel. Equivalently, one has that

$$\Psi_{ABPQ}\mathcal{F}^{PQ} = \varphi\mathcal{F}_{AB},$$

with φ a non-vanishing scalar.

Proof. One proceeds by a direct computation. One notes that the expressions (5), (6) and (8) assume that the Killing vector $\xi_{AA'}$ is real. Using Eqs. (4) and (8) and the vacuum commutators for $\nabla_{AA'}$ one finds that

$$\mathcal{F}_{AB} = \frac{3}{4}\Psi_{ABPQ}\kappa^{PQ}.$$

As the spacetime is assumed to be of Petrov type D one has that $\kappa_{AB} = \alpha_{(A}\beta_{B)}$ with $\alpha_A\beta^A = \varsigma$, where ς is a non-vanishing scalar. From Eq. (3) one finds then that $\Psi_{ABCD} = \psi\alpha_{(A}\alpha_B\beta_C\beta_{D)}$, so that

$$\Psi_{ABPQ}\kappa^{PQ} = -\frac{1}{3}\psi\varsigma^2\kappa_{AB},$$

and finally that

$$\mathcal{F}_{AB} = -\frac{1}{4}\psi\varsigma^2\kappa_{AB},$$

from where the desired result follows. □

The property that allows us to single out the Kerr spacetime out of the generalised Kerr-NUT class is given by the following result proved by Mars [36,37]:

Theorem 4 (Mars [36,37]). *Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth vacuum spacetime with the following properties*:

(i) *$(\mathcal{M}, g_{\mu\nu})$ admits a Killing vector $\xi_{AA'}$ such that, $\mathcal{F}_{AB}$, the spinorial counterpart of the Killing form of $\xi_{AA'}$ satisfies*

$$\Psi_{ABPQ}\mathcal{F}^{PQ} = \varphi\mathcal{F}_{AB},$$

with φ a scalar;

(ii) *$(\mathcal{M}, g_{\mu\nu})$ contains a stationary asymptotically flat 4-end, and $\xi_{AA'}$ tends to a time translation at infinity, and the Komar mass of the asymptotic end is non-zero.*

Then $(\mathcal{M}, g_{\mu\nu})$ is locally isometric to the Kerr spacetime.

Remark. A stationary asymptotically flat 4-end is an open submanifold $\mathcal{M}_\infty \subset \mathcal{M}$ diffeomorphic to $I \times (\mathbb{R}^3 \setminus \mathcal{B}_R)$, where $I \subset \mathbb{R}$ is an open interval and $\mathcal{B}_R$ a closed ball of radius R such that in the local coordinates (t, x^i) defined by the diffeomorphism, the metric $g_{\mu\nu}$ satisfies

$$|g_{\mu\nu} - \eta_{\mu\nu}| + |r\partial_i g_{\mu\nu}| \leq Cr^{-\alpha},$$
$$\partial_t g_{\mu\nu} = 0,$$

with C, α constants, $\eta_{\mu\nu}$ is the Minkowski metric and $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. In particular $\alpha \geq 1$. The definition of the Komar mass is given in [34]. In this context it coincides with the ADM mass of the spacetime.

2.3. Non-Degeneracy of the Petrov Type of the Kerr Spacetime

Finally, we note the following result about the non-degeneracy of the Petrov type of the Kerr spacetime [37].

Proposition 5 (Mars [37]). *The Petrov type of the Kerr spacetime is always D—there are no points where it degenerates to type N or O.*

2.4. A Characterisation of the Kerr Spacetime Using Killing Spinors

As a consequence of Theorem 1 and Propositions 2, 5 one obtains the following invariant characterisation of the Kerr spacetimes. From this characterisation we will extract, in the sequel, a characterisation of asymptotically Euclidean Kerr data.

Theorem 6. *Let $(\mathcal{M}, g_{\mu\nu})$ be a smooth vacuum spacetime such that*

$$\Psi_{ABCD} \neq 0, \quad \Psi_{ABCD}\Psi^{ABCD} \neq 0$$

on $\mathcal{M}$. Then $(\mathcal{M}, g_{\mu\nu})$ is locally isometric to the Kerr spacetime if and only if the following conditions are satisfied:

(i) *there exists a Killing spinor, κ_{AB}, such that the associated Killing vector, $\xi_{AA'}$, is real;*

(ii) *the spacetime $(\mathcal{M}, g_{\mu\nu})$ has a stationary asymptotically flat 4-end with non-vanishing mass in which $\xi_{AA'}$ tends to a time translation.*

Proof. Clearly, the conditions (i) and (ii) are necessary to obtain the Kerr spacetime. For the sufficiency, assume that (i) holds, that is, the spacetime has a Killing spinor κ_{AB} such that the associated Killing vector $\xi_{AA'}$ is real. Accordingly, the spacetime must be of type D, N or O. As $\Psi_{ABCD} \neq 0$ and $\Psi_{ABCD}\Psi^{ABCD} \neq 0$ by hypothesis, the spacetime cannot be of types N or O. By the reality of $\xi_{AA'}$ it must be a generalised Kerr-NUT spacetime and the conclusion of Lemma 3 follows. Now, if (ii) holds then by Theorem 4, the spacetime has to be locally the Kerr spacetime. □

Remark. It is of interest to see whether the conditions $\Psi_{ABCD} \neq 0$ and $\Psi_{ABCD}\Psi^{ABCD} \neq 0$ can be removed. An analysis along what is done in the proof of Theorem 4 may allow to do this. This will be discussed elsewhere.

3. Space Spinors: General Theory

As mentioned in the introduction, in this article we will make use of a space spinor formalism to project the longitudinal and transversal parts of the Killing spinor equation (1) with respect to the timelike vector field τ^μ. The space spinor formalism was originally introduced in [44]. Here, we follow conventions and notations similar to those in [23]. For completeness, we introduce all the relevant notation here.

3.1. Basic Definitions

Let τ^μ be a timelike vector field on $(\mathcal{M}, g_{\mu\nu})$ with normalisation $\tau_\mu\tau^\mu = 2$. Define the projector

$$h_{\mu\nu} \equiv g_{\mu\nu} - \frac{1}{2}\tau_\mu\tau_\nu.$$

We also define the following tensors:

$$K_{\mu\nu} = -h_\mu{}^\lambda h_\nu{}^\rho \nabla_\lambda \tau_\rho,$$
$$K^\mu = -\frac{1}{2}\tau^\nu\nabla_\nu\tau^\mu.$$

Note that it is not being assumed that τ^μ is hypersurface orthogonal. Thus, the tensor $K_{\mu\nu}$ as defined above is not necessarily the second fundamental form of a foliation of the spacetime $(\mathcal{M}, g_{\mu\nu})$.

Let $\tau^{AA'}$ denote the spinorial counterpart of τ^μ. One has that $\tau^{AA'} \equiv \sigma_\mu{}^{AA'}\tau^\mu$ so that

$$\tau_{AA'}\tau^{AA'} = 2, \quad \tau^{A}{}_{A'}\tau^{BA'} = \epsilon^{AB}.$$

The spinor $\tau^{AA'}$ allows to introduce the *spatial solder forms*

$$\sigma_\mu{}^{AB} \equiv \sigma_\mu{}^{(A}{}_{A'}\tau^{B)A'}, \quad \sigma^\mu{}_{AB} \equiv \tau_{(B}{}^{A'}\sigma^\mu{}_{A)A'},$$

so that one has

$$\sigma^\mu{}_{AB}\sigma_\nu{}^{AB} = h^\mu{}_\nu, \quad g_{\mu\nu}\sigma^\mu{}_{AB}\sigma^\nu_{CD} = h_{\mu\nu}\sigma^\mu_{AB}\sigma^\nu_{CD} = \frac{1}{2}(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}),$$

$$\tau_\mu\sigma^\mu{}_{AB} = 0, \quad \epsilon_{AB}\epsilon_{A'B'} = \frac{1}{2}\tau_{AA'}\tau_{BB'} + h_{\mu\nu}\sigma^\mu{}_{AE}\sigma^\nu{}_{BF}\tau^{E}{}_{A'}\tau^{F}{}_{B'}.$$

If τ^μ is hypersurface orthogonal, then h_{ab}, K_{ab}, K^a, $\sigma_a{}^{\mathbf{AB}}$, $\sigma^a{}_{\mathbf{AB}}$ denote, respectively, the pull-backs to the hypersurfaces orthogonal to τ^μ of $h_{\mu\nu}$, $K_{\mu\nu}$, K^μ, $\sigma_\mu{}^{\mathbf{AB}}$, $\sigma^\mu{}_{\mathbf{AB}}$—note that these objects are spatial, in the sense that their contraction with τ^μ vanishes, and thus, their pull-backs are well defined. The relevant properties of these tensors apply to their pull-backs. Often we will begin with a space-like hypersurface $\mathcal{S}$, and define τ^μ as the normal to this hypersurface; we then automatically get the desired properties.

3.2. Space Spinor Splittings

The spinor $\tau^{AA'}$ can be used to construct a formalism consisting of unprimed indices. For example, given a spacetime spinor $\zeta_{AA'}$ one can write

$$\zeta_{AA'} = \frac{1}{2}\tau_{AA'}\zeta - \tau_{A'}{}^{P}\zeta_{PA}, \tag{9}$$

with

$$\zeta \equiv \tau^{PP'}\zeta_{PP'}, \quad \zeta_{AB} \equiv \tau_{(A}{}^{P'}\zeta_{B)P'}.$$

This decomposition can be extended in a direct manner to higher valence spinors. Any spatial tensor has a space-spinor counterpart. For example, if $T_\mu{}^\nu$ is a spatial tensor (i.e. $\tau^\mu T_\mu{}^\nu = 0$ and $\tau_\nu T_\mu{}^\nu = 0$), then its space spinor counterpart is given by $T_{AB}{}^{CD} = \sigma^\mu{}_{AB}\sigma_\nu{}^{CD}T_\mu{}^\nu$.

3.3. Spinorial Covariant Derivatives

Applying formally the space spinor split given by (9) to the spacetime spinorial covariant derivative $\nabla_{AA'}$ one obtains

$$\nabla_{AA'} = \frac{1}{2}\tau_{AA'}\nabla - \tau_{A'}{}^{B}\nabla_{AB},$$

where we have introduced the differential operators

$$\nabla \equiv \tau^{AA'}\nabla_{AA'},$$
$$\nabla_{AB} \equiv \tau^{A'}{}_{(A}\nabla_{B)A'} = \sigma^\mu{}_{AB}\nabla_\mu.$$

The latter is referred to as the *Sen connection*. Let K_{ABCD} denote the space spinor counterpart of the tensor $K_{\mu\nu}$. One has that

$$K_{ABCD} = \tau_D{}^{C'}\nabla_{AB}\tau_{CC'}, \quad K_{ABCD} = K_{(AB)(CD)}.$$

In the sequel, it will be convenient to write K_{ABCD} in terms of its irreducible components. For this, define

$$\Omega_{ABCD} \equiv K_{(ABCD)}, \quad \Omega_{AB} \equiv K_{(A}{}^{C}{}_{B)C}, \quad K \equiv K^{AB}{}_{AB},$$

so that one can write

$$K_{ABCD} = \Omega_{ABCD} - \frac{1}{2}\epsilon_{A(C}\Omega_{D)B} - \frac{1}{2}\epsilon_{B(C}\Omega_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}K, \qquad (10)$$

If τ^μ is hypersurface orthogonal, then $\Omega_{AB} = 0$, and thus $K_{\mu\nu}$ can be regarded as the extrinsic curvature of the leaves of a foliation of the spacetime $(\mathcal{M}, g_{\mu\nu})$. Let K_{AB} denote the spinorial counterpart of the acceleration K_μ. It has the symmetry $K_{AB} = K_{(AB)}$ and satisfies

$$K_{AB} = \tau_B{}^{A'}\nabla\tau_{AA'}.$$

If τ^μ is hypersurface orthogonal then the pull-back, D_a, of $D_\mu \equiv h^\nu{}_\mu\nabla_\nu$ corresponds to the Levi-Civita connection of the intrinsic metric of the leaves of the foliation of hypersurfaces orthogonal to τ^μ. Its spinorial counterpart is given by $D_{AB} = D_{(AB)} = \sigma^a{}_{AB}D_a$. The Sen connection, ∇_{AB}, and the Levi-Civita connection, D_{AB}, are related to each other through the spinor K_{ABCD}. For example, for a valence 1 spinor π_C one has that

$$\nabla_{AB}\pi_C = D_{AB}\pi_C + \frac{1}{2}K_{ABC}{}^{D}\pi_D,$$

with the obvious generalisations for higher valence spinors.

3.4. Hermitian Conjugation

Given a spinor π_A, we define its *Hermitian conjugate* via

$$\hat{\pi}_A \equiv \tau_A{}^{E'}\bar{\pi}_{E'}.$$

The Hermitian conjugate can be extended to higher valence symmetric spinors in the obvious way. The spinors ν_{AB} and ξ_{ABCD} are said to be real if

$$\hat{\nu}_{AB} = -\nu_{AB}, \quad \hat{\xi}_{ABCD} = \xi_{ABCD}.$$

It can be verified that $\nu_{AB}\hat{\nu}^{AB}, \xi_{ABCD}\hat{\xi}^{ABCD} \geq 0$. If the spinors are real, then there exist real spatial tensors ν_a, ξ_{ab} such that ν_{AB} and ξ_{ABCD} are their spinorial counterparts.

Notice that the differential operator D_{AB} is real in the sense that

$$\widehat{D_{AB}\pi_C} = -D_{AB}\hat{\pi}_C.$$

Crucially, however, one has that

$$\widehat{\nabla_{AB}\pi_C} = -\nabla_{AB}\hat{\pi}_C + \frac{1}{2}K_{ABC}{}^{D}\hat{\pi}_D.$$

3.5. Commutators

The analysis in the sequel will require intensive use of the commutators of the covariant derivative operators ∇ and ∇_{AB}. These can be derived from a space spinor splitting of the commutator of $\nabla_{AA'}$.

Define

$$\square_{AB} \equiv \nabla_{C'(A}\nabla_{B)}{}^{C'}, \quad \widehat{\square}_{AB} \equiv \tau_A{}^{A'}\tau_B{}^{B'}\square_{A'B'} = \tau_A{}^{A'}\tau_B{}^{B'}\nabla_{C(A'}\nabla_{B')}{}^{C}.$$

The action of these operators on a spinor π_A is given by

$$\square_{AB}\pi_C = \Psi_{ABCQ}\pi^Q + \frac{1}{2}\Lambda\epsilon_{C(A}\pi_{B)}, \quad \widehat{\square}_{AB}\pi_C = \tau_A{}^{A'}\tau_B{}^{B'}\Phi_{FCA'B'}\pi^F,$$

where $\Phi_{ABA'B'}$ and Λ denote, respectively, the spinor counterparts of the tracefree part of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R of the spacetime metric $g_{\mu\nu}$. Clearly, the above expressions simplify in the case of a vacuum spacetime, where we have $\Phi_{ABA'B'} = 0$, $\Lambda = 0$.

In terms of $\square_{AB}$ and $\widehat{\square}_{AB}$, the commutators of ∇ and ∇_{AB} read

$$[\nabla, \nabla_{AB}] = \widehat{\square}_{AB} - \square_{AB} - \frac{1}{2}K_{AB}\nabla + K^D{}_{(A}\nabla_{B)D} - K_{ABCD}\nabla^{CD}, \tag{11a}$$

$$\begin{aligned}[\nabla_{AB}, \nabla_{CD}] = {}& \frac{1}{2}\left(\epsilon_{A(C}\square_{D)B} + \epsilon_{B(C}\square_{D)A}\right) + \frac{1}{2}\left(\epsilon_{A(C}\widehat{\square}_{D)B} + \epsilon_{B(C}\widehat{\square}_{D)A}\right) \\ & + \frac{1}{2}(K_{CDAB}\nabla - K_{ABCD}\nabla) + K_{CDQ(A}\nabla_{B)}{}^{Q} - K_{ABQ(C}\nabla_{D)}{}^{Q}.\end{aligned} \tag{11b}$$

3.6. Decomposition of the Weyl Spinor

The Hermitian conjugation can be used to decompose the Weyl spinor Ψ_{ABCD} in terms of its electric and magnetic parts via

$$E_{ABCD} \equiv \frac{1}{2}\left(\Psi_{ABCD} + \hat{\Psi}_{ABCD}\right), \quad B_{ABCD} \equiv \frac{\mathrm{i}}{2}\left(\hat{\Psi}_{ABCD} - \Psi_{ABCD}\right),$$

so that

$$\Psi_{ABCD} = E_{ABCD} + \mathrm{i}B_{ABCD}.$$

The spinorial Bianchi identity $\nabla^{AA'}\Psi_{ABCD} = 0$ can be split using the space spinor formalism to render

$$\nabla\Psi_{ABCD} = 2\nabla^E{}_A\Psi_{BCDE}, \tag{12a}$$

$$\nabla^{AB}\Psi_{ABCD} = 0. \tag{12b}$$

Crucial for our applications is that the spinors E_{ABCD} and B_{ABCD} can be expressed in terms of quantities intrinsic to a hypersurface $\mathcal{S}$. More precisely, if $\Omega_{AB} = 0$, one has that

$$E_{ABCD} = -r_{(ABCD)} + \frac{1}{2}\Omega_{(AB}{}^{PQ}\Omega_{CD)PQ} - \frac{1}{6}\Omega_{ABCD}K, \tag{13a}$$

$$B_{ABCD} = -\mathrm{i}\, D^Q{}_{(A}\Omega_{BCD)Q}, \tag{13b}$$

where r_{ABCD} is the space spinor counterpart of the Ricci tensor of the intrinsic metric of the hypersurface $\mathcal{S}$.

3.7. Space Spinor Expressions in Cartesian Coordinates

In some occasions it will be necessary to give spinorial expressions in terms of Cartesian or asymptotically Cartesian frames and coordinates. For this we make use of the spatial Infeld-van der Waerden symbols $\sigma^i{}_{\mathbf{AB}}$, $\sigma_i{}^{\mathbf{AB}}$. Given x^i, $\xi_i \in \mathbb{R}^3$ we shall follow the convention that

$$x^{\mathbf{AB}} \equiv \sigma_i{}^{\mathbf{AB}} x^i, \quad \xi_{\mathbf{AB}} \equiv \sigma^i{}_{\mathbf{AB}} \xi_i,$$

with

$$x^{\mathbf{AB}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -x^1 + \mathrm{i}x^2 & x^3 \\ x^3 & x^1 + \mathrm{i}x^2 \end{pmatrix}, \quad \xi_{\mathbf{AB}} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\xi_1 - \mathrm{i}\xi_2 & \xi_3 \\ \xi_3 & \xi_1 - \mathrm{i}\xi_2 \end{pmatrix}. \tag{14}$$

4. Killing Spinor Data

In this section we review some aspects of the space spinor decomposition of the Killing spinor equation (1). A first analysis along these lines was first carried out in [23]. The current presentation is geared towards the construction of geometric invariants.

4.1. General Observations

Given a symmetric spinor κ_{AB} (not necessarily a Killing spinor), it will be convenient to define the following spinors:

$$\xi \equiv \nabla^{PQ} \kappa_{PQ}, \tag{15a}$$

$$\xi_{BF} \equiv \frac{3}{2} \nabla_{(F}{}^{D} \kappa_{B)D}, \tag{15b}$$

$$\xi_{ABCD} \equiv \nabla_{(AB} \kappa_{CD)}, \tag{15c}$$

$$\xi_{AA'} \equiv \nabla^{B}{}_{A'} \kappa_{AB}, \tag{15d}$$

$$H_{A'ABC} \equiv 3\nabla_{A'(A} \kappa_{BC)}, \tag{15e}$$

$$S_{AA'BB'} \equiv \nabla_{AA'} \xi_{BB'} + \nabla_{BB'} \xi_{AA'}. \tag{15f}$$

We will use this notation throughout the rest of the paper. Clearly, for a Killing spinor one has

$$H_{A'ABC} = 0, \quad S_{AA'BB'} = 0.$$

The spinors ξ, ξ_{AB} and ξ_{ABCD} arise in the space spinor decomposition of the spinors $H_{A'ABC}$ and $\xi_{AA'}$. To see this, let $\tau^{AA'}$ denote, as in Sect. 3, the spinorial counterpart of a timelike vector with normalisation $\tau_{AA'}\tau^{AA'} = 2$. Some manipulations show that

$$\xi_{AA'} = \frac{1}{2}\tau_{AA'}\xi - \frac{2}{3}\tau^{B}{}_{A'}\xi_{AB} + \frac{1}{2}\tau^{B}{}_{A'}\nabla\kappa_{AB}, \tag{16a}$$

$$H_{A'ABC} = \tau_{A'(A}\xi_{BC)} + \frac{3}{2}\tau_{A'(A}\nabla\kappa_{BC)} - 3\tau_{A'}{}^{D}\xi_{ABCD}. \tag{16b}$$

Furthermore, the spinors ξ, ξ_{AB} and ξ_{ABCD} correspond to the irreducible components of $\nabla_{AB}\kappa_{CD}$ so that one can write:

$$\nabla_{AB}\kappa_{CD} = \xi_{ABCD} - \frac{1}{3}\epsilon_{A(C}\xi_{D)B} - \frac{1}{3}\epsilon_{B(C}\xi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi. \tag{17}$$

Using the commutator (11b) for vacuum, one can obtain equations for the derivatives of ξ and ξ_{AB}—these will be used systematically in the sequel. The irreducible components of the derivative $\nabla_{AB}\xi_{CD}$ are given by

$$\nabla^{AB}\xi_{AB} = -\frac{1}{2}K\xi + \frac{3}{4}\Omega^{ABCD}\xi_{ABCD} + \frac{1}{2}\Omega^{AB}\xi_{AB} - \frac{3}{4}\Omega^{AB}\nabla\kappa_{AB}, \tag{18a}$$

$$\begin{aligned}\nabla^{C}{}_{(A}\xi_{B)C} = {} & \nabla_{AB}\xi + \frac{3}{2}\Psi_{ABCD}\kappa^{CD} - \frac{2}{3}K\xi_{AB} - \frac{1}{2}\Omega_{ABCD}\xi^{CD} \\ & -\frac{3}{2}\xi_{(A}{}^{CDF}\Omega_{B)CDF} - \frac{3}{2}\nabla^{CD}\xi_{ABCD} - \frac{1}{2}\Omega_{AB}\xi + \frac{1}{2}\Omega_{(A}{}^{C}\xi_{B)C} \\ & +\frac{3}{4}\Omega^{CD}\xi_{ABCD} - \frac{3}{2}\Omega_{(A}{}^{C}\nabla\kappa_{B)C},\end{aligned} \tag{18b}$$

$$\begin{aligned}\nabla_{(AB}\xi_{CD)} = {} & 3\Psi_{F(ABC}\kappa_{D)}{}^{F} + K\xi_{ABCD} - \frac{1}{2}\Omega_{ABCD}\xi + \Omega_{(ABC}{}^{F}\xi_{D)F} \\ & -\frac{3}{2}\Omega^{PQ}{}_{(AB}\xi_{CD)PQ} + 3\nabla^{Q}{}_{(A}\xi_{BCD)Q} + \frac{1}{2}\Omega_{(AB}\xi_{CD)} \\ & -\frac{3}{2}\Omega^{F}{}_{(A}\xi_{BCD)F} + \frac{3}{2}\Omega_{(AB}\nabla\kappa_{CD)}.\end{aligned} \tag{18c}$$

We note the appearance of the term $\nabla_{AB}\xi$ in (18b). Thus, there is no independent equation for the derivative of ξ.

Finally, we consider the equations for the second-order derivatives of ξ. For the sake of simplicity, we restrict our attention to the case when $\Omega_{AB}=0$ so that $K_{ABCD}=K_{CDAB}$. For notational purposes we define $\Omega_{ABCDEF} \equiv \nabla_{(AB}\Omega_{CDEF)}$. One finds

$$\begin{aligned}&\nabla^{AB}\nabla_{AB}\xi \\ &\quad = -\frac{1}{6}K^2\xi - \frac{1}{2}\Omega^{ABCD}\Omega_{ABCD}\xi + 3\Psi_{A}{}^{CDF}\Omega_{BCDF}\kappa^{AB} + \xi_{AB}\nabla^{AB}K \\ &\quad +\frac{3}{4}\hat{\Psi}^{ABCD}\xi_{ABCD} - \frac{9}{4}\Psi^{ABCD}\xi_{ABCD} + 2K\Omega^{ABCD}\xi_{ABCD} \\ &\quad -\frac{15}{4}\Omega^{ABFH}\Omega^{CD}{}_{FH}\xi_{ABCD} + \frac{9}{2}\Omega^{ABCD}\nabla^{F}{}_{D}\xi_{ABCF} \\ &\quad +\frac{3}{2}\nabla^{AB}\nabla^{CD}\xi_{ABCD},\end{aligned} \tag{19a}$$

$$\nabla^{C}{}_{(A}\nabla_{B)C}\xi = \frac{1}{2}\Omega_{ABCD}\nabla^{CD}\xi - \frac{1}{3}K\nabla_{AB}\xi, \tag{19b}$$

$$\begin{aligned}&\nabla_{(AB}\nabla_{CD)}\xi \\ &\quad = -4K\Psi_{(ABC}{}^{E}\kappa_{D)E} + \frac{1}{2}\hat{\Psi}_{ABCD}\xi - \frac{5}{2}\Psi_{ABCD}\xi - \frac{2}{3}\hat{\Psi}_{(ABC}{}^{E}\xi_{D)E} \\ &\quad -\frac{10}{3}\Psi_{(ABC}{}^{E}\xi_{D)E} + \Omega_{ABCDEL}\xi^{EL} + \frac{4}{3}K^2\xi_{ABCD} + 3\Omega_{EFL(ABC}\xi_{D)}{}^{ELF} \\ &\quad +3\Psi_{(AB}{}^{EL}\xi_{CD)EL} - \frac{3}{2}\xi_{(A}{}^{ELF}\Omega_{BCD)}{}^{H}\Omega_{ELFH} - 3\Psi_{EL(A}{}^{F}\kappa^{EL}\Omega_{BCD)F} \\ &\quad -\xi^{EL}\Omega_{ELF(A}\Omega_{BCD)}{}^{F} + \frac{2}{3}K\xi_{(A}{}^{E}\Omega_{BCD)E} + \frac{1}{2}\xi^{ELFH}\Omega_{EL(AB}\Omega_{CD)FH}\end{aligned}$$

$$
\begin{aligned}
&-3\Psi_{E(B}{}^{LF}\kappa_A{}^E\Omega_{CD)LF} - 3\Psi_{E(AB}{}^F\kappa^{EL}\Omega_{CD)LF} - \Omega_{ELF(B}\xi_A{}^E\Omega_{CD)}{}^{LF} \\
&-4K\xi_{(AB}{}^{EL}\Omega_{CD)EL} - \frac{1}{2}\xi\Omega_{(AB}{}^{EL}\Omega_{CD)EL} + \frac{3}{2}\xi^{ELFH}\Omega_{E(ABC}\Omega_{D)LFH} \\
&-2\Omega_{E(BC}{}^H\xi_A{}^{ELF}\Omega_{D)LFH} + \frac{1}{4}\xi^{ELFH}\Omega_{ABCD}\Omega_{ELFH} - \frac{1}{3}K\xi\Omega_{ABCD} \\
&+\frac{1}{2}\xi_{(AB}{}^{EL}\Omega_{CD)}{}^{FH}\Omega_{ELFH} + \frac{2}{5}\xi_{(CD}\nabla_{AB)}K + \frac{12}{5}\xi_{E(BCD}\nabla_{A)}{}^E K \\
&-3\Omega_{E(BCD}\nabla_{A)}{}^E\xi - \frac{3}{2}\Omega_{(A}{}^{ELF}\nabla_{CD}\xi_{B)ELF} - \frac{3}{2}\Omega_{F(A}{}^{EL}\nabla_D{}^F\xi_{BC)EL} \\
&-\frac{9}{2}\Omega_{(AB}{}^{EL}\nabla_D{}^F\xi_{C)ELF} - \frac{9}{2}\nabla_{L(D}\nabla_C{}^E\xi_{AB)E}{}^L - \frac{3}{2}\nabla_{L(D}\nabla^{EL}\xi_{ABC)E} \\
&-6K\nabla_{E(D}\xi_{ABC)}{}^E + 3\Omega_{L(AB}{}^E\nabla^{LF}\xi_{CD)EF} - 3\Omega_{(ABC}{}^E\nabla^{LF}\xi_{D)ELF} \\
&-3\kappa^{EL}\nabla_{L(D}\Psi_{ABC)E} + 3\kappa_{(A}{}^E\nabla_D{}^L\Psi_{BC)EL}.
\end{aligned} \tag{19c}
$$

The equations presented in this section have been deduced using the tensor algebra suite `xAct` for `Mathematica`—see [38].

4.2. Propagation of the Killing Spinor Equation

A straightforward consequence of the Killing spinor equation (1) in a vacuum spacetime is that

$$\Box\kappa_{AB} = -\Psi_{ABCD}\kappa^{CD}, \tag{20}$$

where $\Box \equiv \nabla^{AA'}\nabla_{AA'}$. The latter equation is obtained by applying the differential operator $\nabla^{AA'}$ to Eq. (1) and then using the vacuum commutator relation for the spacetime Levi-Civita connection.

The wave equation (20) plays a role in the discussion of the *propagation* of the Killing spinor equation. More precisely, one has the following result—cfr. [23] for further details:

Lemma 7. *Let κ_{AB} be a solution to Eq. (20). Then the corresponding spinor fields $H_{A'ABC}$ and $S_{AA'BB'}$ will satisfy the system of wave equations*

$$\Box H_{A'ABC} = 4\left(\Psi_{(AB}{}^{PQ}H_{C)PQA'} + \nabla_{(A}{}^{Q'}S_{BC)Q'A'}\right), \tag{21a}$$

$$
\begin{aligned}
\Box S_{AA'BB'} = &-\nabla_{AA'}\left(\Psi_B{}^{PQR}H_{B'PQR}\right) - \nabla_{BB'}\left(\Psi_A{}^{PQR}H_{A'PQR}\right) \\
&+2\Psi_{AB}{}^{PQ}S_{PA'QB'} + 2\bar{\Psi}_{A'B'}{}^{P'Q'}S_{AP'BQ'}.
\end{aligned} \tag{21b}
$$

The crucial observation is that the right-hand sides of Eqs. (21a) and (21b) are homogeneous expressions of the unknowns and their first-order derivatives. The hyperbolicity of Eqs. (21a) and (21b) imply the following result—again, cfr. [23] for further details:

Proposition 8. *The development $(\mathcal{M}, g_{\mu\nu})$ of an initial data set for the vacuum Einstein field equations, $(\mathcal{S}, h_{ab}, K_{ab})$, has a Killing spinor in the domain of dependence of $\mathcal{U} \subset \mathcal{S}$ if and only if the following equations are satisfied on $\mathcal{U}$:*

$$H_{A'ABC} = 0, \tag{22a}$$
$$\nabla H_{A'ABC} = 0, \tag{22b}$$
$$S_{AA'BB'} = 0, \tag{22c}$$
$$\nabla S_{AA'BB'} = 0. \tag{22d}$$

4.3. The Killing Spinor Data Equations

The *Killing spinor data* conditions obtained in Proposition 8 can be re-expressed in terms of conditions on the spinor κ_{AB} which are intrinsic to the hypersurface $\mathcal{S}$. For this one uses the split of $\xi_{AA'}$ and $H_{A'ABC}$ given by Eqs. (16a)–(16b). Extensive computations using the `xAct` suite for `Mathematica` render the following result:

Theorem 9. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an initial data set for the Einstein vacuum field equations, where $\mathcal{S}$ is a Cauchy hypersurface. Let $\mathcal{U} \subset \mathcal{S}$ be an open set. The development of the initial data set will then have a Killing spinor in the domain of dependence of $\mathcal{U}$ if and only if*

$$\xi_{ABCD} = 0, \tag{23a}$$
$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0, \tag{23b}$$
$$3\kappa_{(A}{}^{E}\nabla_{B}{}^{F}\Psi_{CD)EF} + \Psi_{(ABC}{}^{F}\xi_{D)F} = 0, \tag{23c}$$

are satisfied on $\mathcal{U}$. The Killing spinor is obtained by evolving (20) with initial data satisfying conditions (23a)–(23c) and

$$\nabla\kappa_{AB} = -\frac{2}{3}\xi_{AB} \tag{24}$$

on $\mathcal{U}$.

Remark 1. Conditions (23a)–(23c) are intrinsic to $\mathcal{U} \subset \mathcal{S}$ and will be referred to as the *Killing spinor initial data equations*. In particular, Eq. (23a), which can be written as

$$\nabla_{(AB}\kappa_{CD)} = 0, \tag{25}$$

will be called the *spatial Killing spinor equation*, whereas (23b) and (23c) will be known as the *algebraic conditions*.

Remark 2. Theorem 9 is an improvement on Proposition 6 of [23] where the interdependence of the equations implied by (22a)–(22d) was not analysed.

Proof. The proof of Theorem 9 consists of a space spinor decomposition of the conditions (22a)–(22d) and of an analysis of the dependencies of the resulting conditions. All calculations are made on $\mathcal{U} \subset \mathcal{S}$.

- *Decomposition of equation* (22a). Splitting $\tau_F{}^{A'}H_{A'ABC}$ into irreducible parts gives that (22a) is equivalent to

$$\xi_{ABCD} = 0, \tag{26a}$$
$$\nabla\kappa_{AB} = -\frac{2}{3}\xi_{AB}. \tag{26b}$$

- *Decomposition of equation* (22b). It follows that

$$\tau_D{}^{A'}\nabla H_{A'ABC} = \nabla(\tau_D{}^{A'}H_{A'ABC}) + H_{A'ABC}K_{DF}\tau^{FA'}.$$

Hence, under the condition (22a), the irreducible parts of $\tau_D{}^{A'}\nabla H_{A'ABC}$ are given by

$$\nabla\xi_{ABCD} = 0, \tag{27a}$$

$$\nabla^2\kappa_{AB} = -\frac{2}{3}\nabla\xi_{AB}. \tag{27b}$$

From the commutator (11a) together with (26a) and (26b) we get

$$\begin{aligned}\nabla\xi_{ABCD} &= \nabla\nabla_{(AB}\kappa_{CD)}\\ &= 2\Psi_{(ABC}{}^F\kappa_{D)F} - \frac{1}{3}\Omega_{(AB}\xi_{CD)} - \frac{1}{3}\Omega_{ABCD}\xi\\ &\quad +\frac{2}{3}\Omega_{(ABC}{}^F\xi_{D)F} - \frac{2}{3}\nabla_{(AB}\xi_{CD)}.\end{aligned}$$

Equation (18c) and again (26a) and (26b) then yield

$$\nabla\xi_{ABCD} = 4\Psi_{(ABC}{}^F\kappa_{D)F}. \tag{28}$$

Using the commutator (11a) one obtains that

$$\begin{aligned}\nabla\xi &= \nabla_{AB}\nabla\kappa^{AB} - \frac{1}{3}K\xi + \frac{2}{3}K^{AB}\xi_{AB} + \frac{2}{3}\Omega^{AB}\xi_{AB}\\ &\quad -\Omega^{ABCD}\xi_{ABCD} - \frac{1}{2}K^{AB}\nabla\kappa_{AB}\end{aligned} \tag{29a}$$

$$\begin{aligned}\nabla\xi_{AB} &= \frac{3}{2}\Psi_{ABCD}\kappa^{CD} - \frac{1}{2}K_{AB}\xi - \frac{1}{3}K\xi_{AB} + \frac{1}{2}K^C{}_{(A}\xi_{B)C}\\ &\quad +\frac{3}{4}K^{CD}\xi_{ABCD} - \frac{1}{2}\xi\Omega_{AB}\\ &\quad -\frac{1}{2}\xi^C{}_{(A}\Omega_{B)C} + \frac{3}{4}\Omega^{CD}\xi_{ABCD} + \frac{3}{2}\xi_{(A}{}^{CDF}\Omega_{B)CDF} + \frac{1}{2}\xi^{CD}\Omega_{ABCD}\\ &\quad -\frac{3}{4}K^C{}_{(A}\nabla\kappa_{B)C} + \nabla_{C(A}\nabla\kappa_{B)}{}^C\end{aligned} \tag{29b}$$

In terms of the normal derivative and the Sen connection, Eq. (20) reads

$$\begin{aligned}\nabla^2\kappa_{AB} &= -2\Psi_{ABCD}\kappa^{CD} - K\nabla\kappa_{AB}\\ &\quad -\frac{2}{3}\nabla_{AB}\xi - \frac{4}{3}\nabla_{C(A}\xi^C{}_{B)} - 2\nabla^{CD}\xi_{ABCD}\\ &\quad +\frac{1}{3}K_{AB}\xi - \frac{2}{3}K^C{}_{(A}\xi_{B)C} + K^{CD}\xi_{ABCD} + \frac{2}{3}\Omega_{AB}\xi\\ &\quad +\frac{4}{3}\xi^C{}_{(A}\Omega_{B)C} + 2\xi_{ABCD}\Omega^{CD}.\end{aligned} \tag{30}$$

It is worth stressing that Eqs. (29a), (29b) and (30) are valid not only on $\mathcal{U}$, but on the spacetime. Hence, it makes sense taking normal derivatives of these equations. Using (29b), (26b) and (26a), the wave equation (20) is seen to imply

$$\nabla^2\kappa_{AB} + \frac{2}{3}\nabla\xi_{AB} = -\Psi_{ABCD}\kappa^{CD} + \frac{4}{9}K\xi_{AB} + \frac{1}{3}\Omega_{AB}\xi + \xi^{C}{}_{(A}\Omega_{B)C}$$
$$+\frac{1}{3}\Omega_{ABCD}\xi^{CD} - \frac{2}{3}\nabla_{AB}\xi - \frac{2}{3}\nabla_{C(A}\xi^{C}{}_{B)}.$$

Using Eqs. (18b), (26b), (26a), the latter equation reduces to (27b). This far we have that for all solutions to (20), the system (22a), (22b) is equivalent to the system (23a), (23b), (24).

- *Decomposition of equation* (22c). Splitting $\tau_C{}^{A'}\tau_D{}^{B'}S_{AA'BB'}$ into irreducible parts yields

$$\nabla_{(AB}\nabla\kappa_{CD)} - \Omega_{ABCD}\xi + \frac{4}{3}K_{(ABC}{}^{F}\xi_{D)F}$$
$$-K_{(ABC}{}^{F}\nabla\kappa_{D)F} - \frac{4}{3}\nabla_{(AB}\xi_{CD)} = 0, \quad (31a)$$

$$2\nabla\xi - \frac{4}{3}K^{AB}\xi_{AB} + K^{AB}\nabla\kappa_{AB} = 0, \quad (31b)$$

$$\frac{4}{3}\nabla_{AB}\xi^{AB} + K\xi - \frac{4}{3}\Omega^{AB}\xi_{AB} + \Omega^{AB}\nabla\kappa_{AB} - \nabla_{AB}\nabla\kappa^{AB} = 0, \quad (31c)$$

$$\frac{1}{2}K_{BD}\xi - \frac{2}{3}K^{A}{}_{(B}\xi_{D)A} + \frac{1}{2}K^{A}{}_{(B}\nabla\kappa_{D)A} - \frac{2}{3}K_{BDAC}\xi^{AC}$$
$$+\frac{1}{2}K_{BDAC}\nabla\kappa^{AC} + \frac{2}{3}\nabla\xi_{BD} - \frac{1}{2}\nabla^2\kappa_{BD} + \nabla_{BD}\xi = 0. \quad (31d)$$

Using Eqs. (18a), (18c), (26a), (26b) and (27b), one sees that Eqs. (31a)–(31c) simplify to

$$\Psi^{F}{}_{(ABC}\kappa_{D)F} = 0, \quad (32a)$$

$$\nabla\xi = K^{AB}\xi_{AB}, \quad (32b)$$

$$\nabla\xi_{BD} = -\frac{1}{2}K_{BD}\xi + K^{A}{}_{(B}\xi_{D)A} + K_{BDAC}\xi^{AC} - \nabla_{BD}\xi, \quad (32c)$$

while Eq. (31d) is seen to be satisfied identically. Furthermore, employing equations (18a), (18b), (29a), (26a), (26b) and (29b) one obtains Eq. (32b) and (32c). Hence, they are a consequence of the commutators, (26b) and (26a). One concludes that for all solutions to (20), the Eqs. (23a), (23b) together with (24) are equivalent to (22a), (22b), (22c).

- *Decomposition of equation* (22d). A straightforward computation shows that

$$\tau_C{}^{A'}\tau_D{}^{B'}\nabla S_{AA'BB'} = \nabla(\tau_C{}^{A'}\tau_D{}^{B'}S_{AA'BB'})$$
$$+K_{CF}S_{AA'BB'}\tau_D{}^{B'}\tau^{FA'} + K_{DF}S_{AA'BB'}\tau_C{}^{A'}\tau^{FB'}.$$

Hence, if condition (22c) holds, the irreducible parts of $\tau_C{}^{A'}\tau_D{}^{B'}\nabla S_{AA'BB'}$ are ∇-derivatives of (31a)–(31d). Using Eq. (27b), these components become

$$\Omega_{ABCD}\nabla\xi + \Omega_{(AB}\nabla\xi_{CD)} - 2\Omega^F{}_{(ABC}\nabla\xi_{D)F} + \frac{2}{3}\xi_{(AB}\nabla\Omega_{CD)}$$
$$-\frac{1}{2}\nabla\kappa_{(AB}\nabla\Omega_{CD)} + \xi\nabla\Omega_{ABCD} + \frac{4}{3}\xi^F{}_{(A}\nabla\Omega_{BCD)F}$$
$$-(\nabla\kappa^F{}_{(A})\nabla\Omega_{BCD)F} + \frac{4}{3}\nabla\nabla_{(AB}\xi_{CD)} - \nabla\nabla_{(AB}\nabla\kappa_{CD)} = 0, \tag{33a}$$
$$2\nabla^2\xi - 2K^{AB}\nabla\xi_{AB} + (\nabla K^{AB})\nabla\kappa_{AB} - \frac{4}{3}\xi^{AB}\nabla K_{AB} = 0, \tag{33b}$$
$$\xi\nabla K + K\nabla\xi - 2\Omega^{AB}\nabla\xi_{AB} - \frac{4}{3}\xi^{AB}\nabla\Omega_{AB} + (\nabla\kappa^{AB})\nabla\Omega_{AB}$$
$$+\frac{4}{3}\nabla\nabla_{AB}\xi^{AB} - \nabla\nabla_{AB}\nabla\kappa^{AB} = 0, \tag{33c}$$
$$\nabla^3\kappa_{BD} + \frac{2}{3}\nabla^2\xi_{BD} = \frac{4}{3}\xi^A{}_{(B}\nabla\kappa_{D)A} + \xi\nabla\kappa_{BD} - \frac{4}{3}\xi^{AC}\nabla K_{BDAC}$$
$$+(\nabla K^A{}_{(B})\nabla\kappa_{D)A} + (\nabla K_{BDAC})\nabla\kappa^{AC} + K_{BD}\nabla\xi - 2K^A{}_{(B}\nabla\xi_{D)A}$$
$$-2K_{BDAC}\nabla^{AC} + 2\nabla^2\xi_{BD} + 2\nabla\nabla_{BD}\xi. \tag{33d}$$

Now, using the commutator (11a), and Eqs. (27b) and (26b) it is easy so see that

$$\nabla\nabla_{AB}\nabla\kappa_{CD} = -\frac{2}{3}\nabla\nabla_{AB}\xi_{CD}. \tag{34}$$

Taking the normal derivative of the spacetime equations (29a)–(29b) and using the relations (34), (18a), (18b), (26a), (26b), (27a) and (27b) one gets

$$\nabla^2\xi = \xi^{AB}\nabla K_{AB} + K^{AB}\nabla\xi_{AB},$$
$$\nabla^2\xi_{AB} = -\frac{1}{2}\xi\nabla K_{AB} - \xi^C{}_{(A}\nabla K_{B)C} + \frac{1}{3}\xi_{AB}\nabla K - \frac{1}{2}K_{AB}\nabla\xi$$
$$+\frac{1}{3}K\nabla\xi_{AB} + K^C{}_{(A}\nabla\xi_{B)C} - \Omega^C{}_{(A}\nabla\xi_{B)C} + \Omega_{ABCD}\nabla\xi^{CD}$$
$$+\xi^C{}_{(A}\nabla\Omega_{B)C} + \xi^{CD}\nabla\Omega_{ABCD} - \nabla\nabla_{AB}\xi.$$

Using these last two equations together with Eqs. (18a), (18c), (26a), (26b), (27a), (27b) and (32b) one finds that the system (33a)–(33d) reduces to

$$4\Psi^F{}_{(ABC}\xi_{D)F} + 6\kappa^F{}_{(A}\nabla\Psi_{BCD)F} = 0, \tag{35a}$$
$$\nabla^3\kappa_{BD} + \frac{2}{3}\nabla^2\xi_{BD} = 0. \tag{35b}$$

Taking the normal derivative of equation (30) and using Eqs. (18b), (26a), (26b), (27a), (27b) and (32b) one gets Eq. (35b). Finally, using the Bianchi equation (12a), one has that Eq. (35a) reduces to

$$3\kappa_{(A}{}^E\nabla_B{}^F\Psi_{CD)EF} + \Psi_{(ABC}{}^F\xi_{D)F} = 0 \tag{36}$$

This completes the proof. □

Remark. Note that the result is independent of K_{AB} and Ω_{AB}.

4.3.1. The Killing Spinor Initial Data Conditions in Terms of the Levi-Civita Connection. It should be stressed that the Killing spinor equations (23a)–(23c) are truly intrinsic to the hypersurface $\mathcal{S}$. This can be more easily seen by expressing the Sen connection, ∇_{AB}, in terms of the intrinsic (Levi-Civita) connection of the hypersurface, D_{AB}, and the second fundamental form K_{ABCD}. One obtains the following completely equivalent set of equations:

$$D_{(AB}\kappa_{CD)} + \Omega_{(ABC}{}^{E}\kappa_{D)E} = 0,$$

$$\Psi_{(ABC}{}^{F}\kappa_{D)F} = 0,$$

$$\begin{aligned}
&3\kappa_{(A}{}^{E}D_{B}{}^{F}\Psi_{CD)EF} - \frac{3}{4}\Psi_{L(ABC}D^{HL}\kappa_{D)H} - \frac{3}{4}\Psi_{L(ABC}D_{D)}{}^{F}\kappa^{L}{}_{F}\\
&\quad+\frac{3}{4}\Psi_{(ABC}{}^{L}\Omega_{D)FHL}\kappa^{FH} + \frac{3}{2}\Psi_{(AB}{}^{HL}\kappa_{C}{}^{F}\Omega_{D)FHL} - \frac{3}{2}\Psi_{FH(A}{}^{L}\Omega_{BCD)L}\kappa^{FH}\\
&\quad+\frac{3}{8}\Psi_{FH(AB}\kappa_{CD)}\Omega^{FH} + \frac{3}{4}\Psi_{FH(AB}\Omega_{CD)}\kappa^{FH} = 0,
\end{aligned}$$

where the last expression was simplified using the first algebraic condition, and the value of the Weyl spinor is expressed in terms of initial data quantities via formulae (13a)–(13b).

4.4. The Integrability Conditions of the Spatial Killing Spinor Equation

For the rest of the paper we assume that the tensor K_{ab} is symmetric—accordingly, $\Omega_{AB} = 0$. The condition $\xi_{ABCD} \equiv \nabla_{(AB}\kappa_{CD)} = 0$ does not immediately give information about the other irreducible components of $\nabla_{AB}\kappa_{CD}$, namely ξ and ξ_{AB}. However, using $\xi_{ABCD} = 0$ and $\Omega_{AB} = 0$ in the relations (18a)–(18c) one finds that $\nabla_{AB}\xi_{CD}$ can be written in terms of $\nabla_{AB}\xi$ and lower order derivatives of κ_{AB}. Furthermore, using $\xi_{ABCD} = 0$ in the relations (19a)–(19c), we see that the second-order derivatives of ξ can be expressed in terms of lower order derivatives of κ_{AB}. This yields the following result which will play a role in the sequel:

Lemma 10. *Assume that* $\nabla_{(AB}\kappa_{CD)} = 0$*, then*

$$\nabla_{AB}\nabla_{CD}\nabla_{EF}\kappa_{GH} = H_{ABCDEFGH},$$

where $H_{ABCDEFGH}$ *is a linear combination of* κ_{AB}*,* $\nabla_{AB}\kappa_{CD}$ *and* $\nabla_{AB}\nabla_{CD}\kappa_{EF}$ *with coefficients depending on* Ψ_{ABCD}*,* $\hat{\Psi}_{ABCD}$ *and* K_{ABCD}*.*

Remark. It is important to point out that the assertion of the lemma is false if $\nabla_{(AB}\kappa_{CD)} \neq 0$.

5. The Approximate Killing Spinor Equation

In what follows we will regard the spatial Killing spinor equation (23a) as the key condition of the Killing spinor initial data equations. Equation (23a) is an overdetermined condition for the 3 (complex) components of the spinor κ_{AB}: not every initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ admits a solution. One would like to deduce a new equation which always has a solution and such that any solution to Eq. (23a) is also a solution to the new equation.

5.1. The Approximate Killing Spinor Operator

Let $\mathfrak{S}_2$ and $\mathfrak{S}_4$ denote, respectively, the spaces of totally symmetric valence 2 and valence 4 spinors. Given ζ_{ABCD}, $\chi_{ABCD} \in \mathfrak{S}_4$, we introduce an inner product in $\mathfrak{S}_4$ via

$$\langle \zeta_{ABCD}, \chi_{EFGH} \rangle = \int_{\mathcal{S}} \zeta_{ABCD} \hat{\chi}^{ABCD} \mathrm{d}\mu,$$

where $\mathrm{d}\mu$ denotes the volume form of the 3-metric h_{ab}. We introduce the spatial Killing spinor operator Φ via

$$\Phi : \mathfrak{S}_2 \to \mathfrak{S}_4, \quad \Phi(\kappa)_{ABCD} = \nabla_{(AB}\kappa_{CD)}.$$

Now, consider the pairing

$$\begin{aligned}\langle \nabla_{(AB}\kappa_{CD)}, \zeta_{EFGH} \rangle &= \int_{\mathcal{S}} \nabla_{(AB}\kappa_{CD)} \hat{\zeta}^{ABCD} \mathrm{d}\mu \\ &= \int_{\mathcal{S}} \nabla_{AB}\kappa_{CD} \hat{\zeta}^{ABCD} \mathrm{d}\mu.\end{aligned}$$

The formal adjoint, Φ^*, of the spatial Killing operator can be obtained from the latter expression by integration by parts. To this end we note the identity

$$\begin{aligned}&\int_{\mathcal{U}} \nabla^{AB}\kappa^{CD}\hat{\zeta}_{ABCD}\mathrm{d}\mu - \int_{\mathcal{U}} \kappa^{AB}\widehat{\nabla^{CD}\zeta_{ABCD}}\mathrm{d}\mu + \int_{\mathcal{U}} 2\kappa^{AB}\Omega^{CDF}{}_{A}\hat{\zeta}_{BCDF}\mathrm{d}\mu \\ &\quad = \int_{\partial\mathcal{U}} n^{AB}\kappa^{CD}\hat{\zeta}_{ABCD}\mathrm{d}S, \qquad (37)\end{aligned}$$

with $\mathcal{U} \subset \mathcal{S}$, and where $\mathrm{d}S$ denotes the area element of $\partial\mathcal{U}$, n_{AB} is the spinorial counterpart of its outward pointing normal, and ζ_{ABCD} is a symmetric spinor. From (37) it follows that

$$\Phi^* : \mathfrak{S}_4 \to \mathfrak{S}_2, \quad \Phi^*(\zeta)_{CD} = \nabla^{AB}\zeta_{ABCD} - 2\Omega^{ABF}{}_{(C}\zeta_{D)ABF}. \qquad (38)$$

Definition. The composition operator $L \equiv \Phi^* \circ \Phi : \mathfrak{S}_2 \to \mathfrak{S}_2$ given by

$$L(\kappa_{CD}) \equiv \nabla^{AB}\nabla_{(AB}\kappa_{CD)} - \Omega^{ABF}{}_{(A}\nabla_{|DF|}\kappa_{B)C} - \Omega^{ABF}{}_{(A}\nabla_{B)F}\kappa_{CD} = 0, \qquad (39)$$

will be called the *approximate Killing spinor operator*, and Eq. (39) the *approximate Killing spinor equation*.

Remark. Note that every solution to the spatial Killing spinor equation (25) is also a solution to Eq. (39).

5.2. Ellipticity of the Approximate Killing Spinor Operator

As a prior step to the analysis of the solutions to the approximate Killing spinor equation (39), we look first at its ellipticity properties.

Lemma 11. *The operator L defined by Eq.* (39) *is a formally self-adjoint elliptic operator.*

Proof. The operator is by construction formally self-adjoint as it is given by the composition of an operator and its formal adjoint. In order to verify ellipticity, it suffices to look at the operator

$$L'(\kappa)_{\mathbf{CD}} \equiv \partial^{\mathbf{AB}}\partial_{(\mathbf{AB}}\kappa_{\mathbf{CD})},$$

corresponding to the principal part of L in some Cartesian spin frame. In the corresponding Cartesian coordinates (x^1, x^2, x^3) one has that

$$\partial_{\mathbf{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\partial_1 - \mathrm{i}\partial_2 & \partial_3 \\ \partial_3 & \partial_1 - \mathrm{i}\partial_2 \end{pmatrix}, \quad \partial^{\mathbf{AB}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\partial_1 + \mathrm{i}\partial_2 & \partial_3 \\ \partial_3 & \partial_1 + \mathrm{i}\partial_2 \end{pmatrix}.$$

In particular, $\partial^{\mathbf{AB}}\partial_{\mathbf{AB}} = \Delta \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$, the flat Laplacian. One notes that

$$\partial^{\mathbf{PQ}}\partial_{(\mathbf{PQ}}\kappa_{\mathbf{AB})} = \frac{1}{6}\partial^{\mathbf{PQ}}\partial_{\mathbf{PQ}}\kappa_{\mathbf{AB}} + \frac{2}{3}\partial^{\mathbf{PQ}}\partial_{\mathbf{P}(\mathbf{A}}\kappa_{\mathbf{B})\mathbf{Q}} + \frac{1}{6}\partial^{\mathbf{PQ}}\partial_{\mathbf{AB}}\kappa_{\mathbf{PQ}}.$$

Now, writing

$$\kappa_0 \equiv \kappa_{00}, \quad \kappa_1 \equiv \kappa_{01}, \quad \kappa_2 \equiv \kappa_{11},$$

one has that L' can be expressed in matricial form as $A^{ij}\partial_i\partial_j u$, where

$$A^{ij}\partial_i\partial_j \equiv \frac{1}{12} \times \begin{pmatrix} 7\Delta - \partial_3^2 & -2\partial_1\partial_3 & \partial_2^2 - \partial_1^2 & 0 & -2\partial_2\partial_3 & -2\partial_1\partial_2 \\ -\partial_1\partial_3 & 6\Delta + 2\partial_3^2 & \partial_1\partial_3 & \partial_2\partial_3 & 0 & \partial_2\partial_3 \\ \partial_2^2 - \partial_1^2 & 2\partial_1\partial_3 & 7\Delta - \partial_3^2 & 2\partial_1\partial_2 & -2\partial_2\partial_3 & 0 \\ 0 & 2\partial_2\partial_3 & 2\partial_1\partial_2 & 7\Delta - \partial_3^2 & -2\partial_1\partial_3 & \partial_2^2 - \partial_1^2 \\ -\partial_2\partial_3 & 0 & -\partial_2\partial_3 & -\partial_1\partial_3 & 6\Delta + 2\partial_3^2 & \partial_1\partial_3 \\ -2\partial_1\partial_2 & \partial_2\partial_3 & 0 & \partial_2^2 - \partial_1^2 & 2\partial_1\partial_3 & 7\Delta - \partial_3^2 \end{pmatrix}, \tag{40}$$

and

$$u \equiv \begin{pmatrix} \mathrm{Re}(\kappa_0) \\ \mathrm{Re}(\kappa_1) \\ \mathrm{Re}(\kappa_2) \\ \mathrm{Im}(\kappa_0) \\ \mathrm{Im}(\kappa_1) \\ \mathrm{Im}(\kappa_2) \end{pmatrix}. \tag{41}$$

The symbol, $l(\xi_i)$, of the operator given by (40) is then given by replacing ∂_i with $\xi_i \in \mathbb{R}^3$. One finds that

$$\det l(\xi_i) = \frac{1}{36}\left((\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2\right)^6,$$

so that $\det l(\xi_i) = 0$ if and only if $\xi_i = 0$. Accordingly, the operator $L = \Phi^* \circ \Phi$ is elliptic. □

5.3. A Variational Formulation

We note that the approximate Killing spinor equation arises naturally from a variational principle.

Lemma 12. *The approximate Killing spinor equation* (39) *is the Euler-Lagrange equation of the functional*

$$J = \int_{\mathcal{S}} \nabla_{(AB}\kappa_{CD)} \widehat{\nabla^{AB}\kappa^{CD}}\, d\mu. \tag{42}$$

Proof. This is a direct consequence of the identity (37). □

6. Asymptotically Euclidean Manifolds

After having studied some formal properties of the Killing spinor initial data equations (23a)–(23c),(24), and the approximate Killing spinor equation (39), we proceed to analyse their solvability on asymptotically Euclidean manifolds. In order to do this we introduce some relevant terminology and ancillary results.

6.1. General Assumptions

In what follows, we will be concerned with vacuum spacetimes arising as the development of asymptotically Euclidean data sets. Let $(\mathcal{S}, h_{ab}, K_{ab})$, denote a smooth initial data set for the vacuum Einstein field equations. The pair (h_{ab}, K_{ab}) satisfies on the 3-dimensional manifold $\mathcal{S}$ the vacuum constraint equations

$$-2r - K^a{}_a K^b{}_b + K_{ab}K^{ab} = 0, \tag{43a}$$

$$D^a K_{ab} - D_b K^a{}_a = 0, \tag{43b}$$

where r and D denote, respectively, the Ricci scalar and the Levi-Civita connection of the negative definite 3-metric h_{ab}, while K_{ab} corresponds to the extrinsic curvature of $\mathcal{S}$. The unusual coefficients in the formulae above come from our normalisation of τ^μ. For an *asymptotic end* it will be understood an open set diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$. In what follows, the 3-manifold $\mathcal{S}$ will be assumed to be the union of a compact set and two asymptotically Euclidean ends, i_1, i_2.

6.2. Weighted Sobolev Norms

In order to discuss the decays of the various fields on the 3-manifold $\mathcal{S}$ we make use of weighted Sobolev spaces. In what follows, we follow the ideas of [9] written in terms of the conventions of [3]. Choose an arbitrary point $O \in \mathcal{S}$, and let

$$\sigma(x) \equiv \left(1 + \mathrm{d}(O,x)^2\right)^{1/2},$$

where d denotes the Riemannian distance function on $\mathcal{S}$. The function σ is used to define the following weighted L^2 norm:

$$\|u\|_\delta \equiv \left(\int_{\mathcal{S}} |u|^2 \sigma^{-2\delta-3} \mathrm{d}x \right)^{1/2}, \tag{44}$$

for $\delta \in \mathbb{R}$. In particular, if $\delta = -3/2$ one recovers the usual L^2 norm. Different choices of origin give rise to equivalent weighted norms—as mentioned earlier, the convention of indices used in the definition of the norm (44) follows the one of Bartnik [3]. The fall-off conditions of the various fields will be expressed in terms of weighted Sobolev spaces H^s_δ consisting of functions for which the norm

$$\|u\|_{s,\delta} \equiv \sum_{0 \leq |\alpha| \leq s} \|D^\alpha u\|_{\delta - |\alpha|} < \infty,$$

with s a non-negative integer, and where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multiindex, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. We say that $u \in H^\infty_\delta$ if $u \in H^s_\delta$ for all s. We will say that a spinor or a tensor belongs to a function space if its norm does. For instance, the notation $\zeta_{AB} \in H^s_\delta$ is a shorthand notation for $(\zeta_{AB}\hat{\zeta}^{AB} + \zeta_A{}^A \hat{\zeta}_B{}^B)^{1/2} \in H^s_\delta$.

We will make use of the following result:

Lemma 13. *Let $u \in H^\infty_\delta$. Then u is smooth (i.e. C^∞) over $\mathcal{S}$ and has a fall off at infinity such that $D^l u = o(r^{\delta - |l|})$.*

The smoothness of u follows from the Sobolev embedding theorems. The proof of the behaviour at infinity of u can be found in [3]—cfr. Theorem 1.2 (iv)—while the decay for the derivatives follows from the definition of the weighted Sobolev norms.

Remark. Here, r is a radial coordinate on the asymptotic end—see the next section for details.

We also note the following multiplication lemma—cfr. e.g. Theorem 5.6 in [9].

Lemma 14. *Let $u \in H^\infty_{\delta_1}$, $v \in H^\infty_{\delta_2}$. Then*

$$uv \in H^\infty_{\delta_1 + \delta_2 + \varepsilon}, \quad \varepsilon > 0.$$

Notation. We will often write $u = o_\infty(r^\delta)$ for $u \in H^\infty_\delta$ at an asymptotic end.

For the present applications we will require a somehow finer multiplication lemma concerning the behaviour at infinity. For this we exploit the fact that we are working with smooth functions. More precisely

Lemma 15. *Let $u = o_\infty(r^{\delta_1})$, $v = o_\infty(r^{\delta_2})$ and $w = O(r^\gamma)$. Then*

$$uv = o(r^{\delta_1 + \delta_2}), \quad uw = o(r^{\delta_1 + \gamma}).$$

Proof. Let $\partial\mathcal{S}_r$ denote the surfaces of constant r. For sufficiently large r (so that one is in an asymptotic end), the surface $\partial\mathcal{S}_r$ has the topology of the 2-sphere. Now, the functions u, v are continuous and the surfaces $\partial\mathcal{S}_r$ are compact. Therefore, for sufficiently large r the functions

$$f(r) \equiv \max_{\partial\mathcal{S}_r} |ur^{-\delta_1}|, \quad g(r) \equiv \max_{\partial\mathcal{S}_r} |vr^{-\delta_2}|,$$

are finite and well defined. Furthermore $r^{\delta_1}|u| \leq f(r)$, $r^{\delta_2}|v| \leq g(r)$. By construction, one has that $f(r) = o(1)$ and $g(r) = o(1)$—that is, f, $g \to 0$ for $r \to \infty$. One also has that $|wr^{-\gamma}|$ is bounded by a constant C. Hence,

$$\begin{aligned}|uv| &\leq f(r)g(r)r^{\delta_1+\delta_2} = o(r^{\delta_1+\delta_2}),\\ |uw| &\leq f(r)r^{\delta_1}|w| \leq Cf(r)r^{\delta_1+\gamma} = o(r^{\delta_1+\gamma}),\end{aligned}$$

from where the desired result follows. □

Remark. The lemmas extend to symmetric spatial spinors with even number of indices by the Cauchy–Schwartz inequality.

6.3. Decay Assumptions

As mentioned earlier, our analysis will be restricted to initial data sets $(\mathcal{S}, h_{ab}, K_{ab})$ with two asymptotic ends. Without loss of generality one of the ends will be denoted by the subscript/superscript $+$ on the relevant objects, while those of the other end by $-$. Often, when no confusion arises the subscript/superscript will be dropped.

Remark. We do not need to assume any topological restriction apart from paracompactness, orientability and the requirement of two asymptotically flat ends. Hence, we can have an arbitrary number of handles. For black holes, this means that we can handle Misner-type data with several black holes [39].

The standard assumption for asymptotic flatness is that on each end it is possible to introduce asymptotically Cartesian coordinates $x^i_\pm$ with $r = ((x^1_\pm)^2 + (x^2_\pm)^2 + (x^3_\pm)^2)^{1/2}$, such that the intrinsic metric and extrinsic curvature of $\mathcal{S}$ satisfy

$$h_{ij} = -\delta_{ij} + o_\infty(r^{-1/2}), \tag{45a}$$

$$K_{ij} = o_\infty(r^{-3/2}). \tag{45b}$$

Note that the decay conditions (45a) and (45b) allow for data containing non-vanishing linear and angular momentum. For the purposes of our analysis, it will be necessary to have a bit more information about the behaviour of leading terms in h_{ij} and K_{ij}. More precisely, we will require the initial data to be *asymptotically Schwarzschildean* in some suitable sense. For example, in [2] the assumptions

$$h_{ij} = -\left(1 + 2m_\pm r^{-1}\right)\delta_{ij} + o_\infty(r^{-3/2}), \tag{46a}$$

$$K_{ij} = o_\infty(r^{-5/2}), \tag{46b}$$

have been used. This class of data can be described as *asymptotically non-boosted Schwarzschildean*. Here, we consider a more general class of data which include boosted Schwarzschild data. Following [6,26] we assume

$$h_{ij} = -\left(1 + \frac{2A_\pm}{r}\right)\delta_{ij} - \frac{\alpha_\pm}{r}\left(\frac{2x_i x_j}{r^2} - \delta_{ij}\right) + o_\infty(r^{-3/2}), \tag{47a}$$

$$K_{ij} = \frac{\beta_\pm}{r^2}\left(\frac{2x_i x_j}{r^2} - \delta_{ij}\right) + o_\infty(r^{-5/2}), \tag{47b}$$

where $\alpha_\pm$ and $\beta_\pm$ are smooth functions on the 2-sphere and $A_\pm$ denotes a constant. The functions α and β are related to each other via the vacuum constraint equations (43a) and (43b). We will later need to be more specific about their particular form. The decay assumption for the metric, Eq. (45a) and hence also (47a), is included in the analysis of [9].

Important for our analysis is that boosted Schwarzschild data are of this form—see [6]. It is noticed that a second fundamental form of the type given by (47b) is, in general, not trace-free:

$$K_i{}^i = \frac{\beta_\pm}{r^2} + o_\infty(r^{-5/2}).$$

Henceforth, we drop the superscripts/subscripts $\pm$ for ease of presentation. If $\pm$ appears in any formula, $+$ is assumed for the $(+)$-end, $-$ for the $(-)$-end. For the $\mp$ sign we assume the opposite.

6.4. ADM Mass and Momentum

The ADM energy, E, and momentum, p_i, at each end are given by the integrals

$$E = \frac{1}{16\pi}\int_{\partial\mathcal{S}_\infty} \delta^{ij}\left(\partial_i h_{jk} - \partial_k h_{ij}\right)\frac{x^k}{r}\mathrm{d}S,$$

$$p_i = \frac{1}{8\pi}\int_{\partial\mathcal{S}_\infty} \left(K_{ij} - K h_{ij}\right)\frac{x^j}{r}\mathrm{d}S,$$

so that the ADM 4-momentum covector is given by $p_\mu = (E, p_i)$. In what follows it will be assumed that p_μ is timelike—that is, $p_\mu p^\mu > 0$. The need of this assumption will become clear in the sequel. From the ADM 4-momentum, we define the constants

$$m \equiv \sqrt{p^\nu p_\nu}, \quad p^2 \equiv E^2 - m^2.$$

6.5. Asymptotically Schwarzschildean Data

Boosted Schwarzschild data sets are initial data for the Schwarzschild spacetime for which $p_i \neq 0$. They satisfy the decay assumptions (47a)–(47b). This type of data satisfy

$$
\begin{aligned}
A &= \frac{m}{\sqrt{1-v^2}}, \\
\alpha &= 2m\left(1+2\frac{(n\cdot v)^2}{1-v^2}\right)\left(1+\frac{(n\cdot v)^2}{1-v^2}\right)^{-1/2} - \frac{2m}{\sqrt{1-v^2}}, \\
\beta &= 2m\frac{n\cdot v}{1-v^2}\left(\frac{3}{2}+\frac{(n\cdot v)^2}{1-v^2}\right)\left(1+\frac{(n\cdot v)^2}{1-v^2}\right)^{-3/2},
\end{aligned}
$$

where $n^i \equiv x^i/r$, $n\cdot v \equiv n^i v_i$, $v^2 \equiv \delta^{ij}v_i v_j$, v_i is a constant 3-covector—cfr. [6], and $m_\pm = m$. Note that if $v_i = 0$ then (47a)–(47b) reduce to (46a)–(46b). It can be checked that

$$
E = \frac{m}{\sqrt{1-v^2}}, \quad p_i = \frac{m v_i}{\sqrt{1-v^2}}.
$$

Rewriting this in terms of (E, p_i), we get

$$
A = E, \quad \alpha = \frac{2m^2 + 4(n\cdot p)^2}{\sqrt{m^2+(n\cdot p)^2}} - 2E, \quad \beta = \frac{(n\cdot p)E(3m^2+2(n\cdot p)^2)}{(m^2+(n\cdot p)^2)^{3/2}}, \tag{48}
$$

where $n\cdot p = n^i p_i = r^{-1}x^i p_i$.

Assumption. In the sequel, we will restrict our analysis to initial data sets which are asymptotically Schwarzschildean to the order given by (47a)–(47b). For any asymptotically flat data that admit ADM 4-momentum, one can compute (E, p_i), and then try to find coordinates that cast the metric and extrinsic curvature into the form (47a)–(47b) with (A, α, β) given by (48) with $m = m_\pm$. If this is possible, we will say that the data are *asymptotically Schwarzschildean*. We expect this to be the case for a large class of data. The initial data sets excluded by this assumption will be deemed pathological. Examples of such pathological cases can be found in [26]. We stress that all data of the form (46a)–(46b) are included in our more general analysis.

The need to restrict our analysis to asymptotically Schwarzschildean data as defined in the previous paragraph will become evident in the sequel, where we need to find an asymptotic solution to the spatial Killing spinor equation.

7. Asymptotic Behaviour of the Spatial Killing Spinors

In this section we discuss in some detail the asymptotic behaviour of solutions to the spatial Killing spinor equation on an asymptotically Euclidean manifold. We begin by studying the asymptotic behaviour of the appropriate Killing spinor in the Kerr spacetime. Then, we will impose the same asymptotics on the approximate Killing spinor on a slice of a much more general spacetime. In what follows, we concentrate our discussion on a particular asymptotic end.

7.1. Asymptotic Form of the Stationary Killing Vector

As seen in Sect. 2, the Killing spinor of the Kerr spacetime gives rise to its stationary Killing vector ξ^μ. It will be assumed that the spacetime is such that $p_\mu = (E, p_i)$ is timelike at each asymptotic end. If this is the case, then

$p^\mu/\sqrt{p^\nu p_\nu}$ gives the asymptotic direction of the stationary Killing vector at each end—see, e.g. [4]. Let

$$m \equiv \sqrt{p^\nu p_\nu}, \quad p^2 \equiv E^2 - m^2.$$

Recall now, that ξ and ξ_{AB} denote the lapse and shift of the spinorial counterpart, $\xi^{AA'}$, of the Killing vector ξ^μ. One finds that for non-boosted initial data sets of the form (46a)–(46b), one has in terms of the asymptotic Cartesian coordinates and spin frame, that

$$\xi = \pm\sqrt{2} + o_\infty(r^{-1/2}), \quad \xi_{\mathbf{AB}} = o_\infty(r^{-1/2}).$$

The factor of $\sqrt{2}$ arises due to the particular normalisations used in the space spinor formalism. This particular form of the asymptotic behaviour of the Killing vector has been discussed in [2].

Now consider the more general case given by (47a)–(47b). Again, adopting asymptotically Cartesian coordinates, we extend p_i to a constant covector field on the asymptotic end. In terms of the associated asymptotically Cartesian spin frame, we then define $p_{\mathbf{AB}} \equiv \sigma^i{}_{\mathbf{AB}} p_i$. One finds that

$$\xi = \pm\frac{\sqrt{2}E}{m} + o_\infty(r^{-1/2}), \quad \xi_{\mathbf{AB}} = \pm\frac{\sqrt{2}p_{\mathbf{AB}}}{m} + o_\infty(r^{-1/2}). \tag{49}$$

We see that the conditions (49) are well defined even if we do not have a Killing vector in the spacetime. Hence, for the general case when the metric satisfies (47a)–(47b) and the ADM 4-momentum is well defined, we can still impose the asymptotics (49) for our approximate Killing spinor. We will, however, need to assume that the functions in the metric are given by (48). Otherwise, we will not be able to assume $\xi_{ABCD} \in H^\infty_{-3/2}$, as we will do in the next section. We will later see that this condition is important for the solvability of the elliptic equation (39).

7.2. Asymptotic Form of the Spatial Killing Spinor

In the sequel, given an initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ satisfying the decay conditions (47a)–(47b) with A, α and β given by (48) with $m = m_\pm$, it will be necessary to show that it is always possible to solve the equation

$$\nabla_{(AB}\kappa_{CD)} = o_\infty(r^{-3/2}), \tag{50}$$

order by order without making any further assumptions on the data. A direct calculation allows us to verify that

Lemma 16. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ denote an initial data set for the vacuum Einstein field equations satisfying at each asymptotic end the decay conditions (47a)–(47b) with A, α and β given by (48) and m the ADM mass of the respective end. Then*

$$\kappa_{\mathbf{AB}} = \mp\frac{\sqrt{2}E}{3m}\left(1+\frac{2E}{r}\right)x_{\mathbf{AB}} \pm\frac{2\sqrt{2}}{3m}\left(1+\frac{4E}{r}-\frac{m^2+2(n\cdot p)^2}{\sqrt{m^2+(n\cdot p)^2}r}\right)p_{\mathbf{Q}(\mathbf{A}}x_{\mathbf{B})}{}^{\mathbf{Q}} + o_\infty(r^{-1/2}), \tag{51}$$

with $x_{\mathbf{AB}}$ *as in* (14)*, and* $n\cdot p = r^{-1}x^{\mathbf{AB}}p_{\mathbf{AB}}$ *satisfies equation* (50)*.*

Remark. Formula (51) implies the following expansions for ξ and $\xi_{\mathbf{AB}}$:

$$\xi = \pm\frac{\sqrt{2}E}{m}\mp\frac{\sqrt{2}E(m^2+2(n\cdot p)^2)}{m\sqrt{m^2+(n\cdot p)^2}}r^{-1}+o_\infty(r^{-3/2}), \tag{52a}$$

$$\xi_{\mathbf{AB}} = \pm\left(-\frac{2\sqrt{2}E}{m}+\frac{\sqrt{2}(E^2+4(n\cdot p)^2)}{m\sqrt{m^2+(n\cdot p)^2}}+\frac{mE^2}{\sqrt{2}(m^2+(n\cdot p)^2)^{3/2}}\right)(n\cdot p)r^{-2}x_{\mathbf{AB}} \pm\left(\frac{\sqrt{2}}{m}+\frac{2\sqrt{2}E}{mr}-\frac{2\sqrt{2}(m^2+2(n\cdot p)^2)}{m\sqrt{m^2+(n\cdot p)^2}r}\right)p_{\mathbf{AB}}+o_\infty(r^{-3/2}). \tag{52b}$$

In the case of non-boosted data the expansions (51), (52a) and (52b) reduce to

$$\begin{aligned}\kappa_{\mathbf{AB}} &= \mp\frac{\sqrt{2}}{3}\left(1+\frac{2m}{r}\right)x_{\mathbf{AB}}+o_\infty(r^{-1/2}),\\ \xi &= \pm\sqrt{2}\mp\sqrt{2}mr^{-1}+o_\infty(r^{-3/2}),\\ \xi_{\mathbf{AB}} &= o_\infty(r^{-3/2}),\end{aligned}$$

as discussed in [2].

7.3. Existence and Uniqueness of Spinors with Killing Spinor Asymptotics

In this section we prove that given a spinor κ_{AB} satisfying equation (49) and (50), then the asymptotic expansion (51) is unique up to a translation.

Theorem 17. *Assume that on an asymptotic end of the slice $\mathcal{S}$, one has an asymptotically Cartesian coordinate system such that* (47a)*–*(47b) *hold. Then there exists*

$$\kappa_{\mathbf{AB}} = o_\infty(r^{3/2}), \tag{53}$$

such that

$$\xi_{\mathbf{ABCD}} = o_\infty(r^{-3/2}),\quad \xi_{\mathbf{AB}} = \pm\frac{\sqrt{2}p_{\mathbf{AB}}}{m}+o_\infty(r^{-1/2}), \qquad \xi = \pm\frac{\sqrt{2}E}{m}+o_\infty(r^{-1/2}). \tag{54}$$

The spinor $\kappa_{\mathbf{AB}}$ is unique up to order $o_\infty(r^{-1/2})$, apart from a (complex) constant term.

Remark 1. The complex constant term arising in Theorem 17 contains six real parameters. In the sequel, given a particular choice of asymptotically Cartesian coordinates and frame, we will set this constant term to zero. Note that a change of asymptotically Cartesian coordinates would introduce a similar term containing only three real parameters—which by construction could be removed by a suitable choice of gauge. In what follows, we will use coordinate independent expressions, and therefore, this translational ambiguity will not affect the result.

Remark 2. Note that $\xi_{\mathbf{ABCD}} = o_\infty(r^{-3/2})$ implies $\xi_{ABCD} \in L^2$. The conditions in Theorem 17 are coordinate independent.

Proof. A direct calculation shows that the expansion (51) yields (52a), (52b) and $\xi_{\mathbf{ABCD}} = o_\infty(r^{-3/2})$. Hence, (51) gives a solution of the desired form. In order to prove uniqueness we make use of the linearity of the integrability conditions (18a)–(18c) and (19a)–(19c). Note that the translational freedom gives an ambiguity of a constant term in $\kappa_{\mathbf{AB}}$. Let

$$\mathring{\kappa}_{\mathbf{AB}} \equiv \mp\frac{\sqrt{2}E}{3m}\left(1+\frac{2E}{r}\right)x_{\mathbf{AB}} \pm\frac{2\sqrt{2}}{3m}\left(1+\frac{4E}{r}-\frac{m^2+2(n\cdot p)^2}{\sqrt{m^2+(n\cdot p)^2}r}\right)p_{\mathbf{Q}(\mathbf{A}}x_{\mathbf{B})}{}^{\mathbf{Q}}. \tag{55}$$

Let $\breve{\kappa}_{\mathbf{AB}}$, be an arbitrary solution to the system (49), (50). Furthermore, let $\kappa_{\mathbf{AB}} \equiv \breve{\kappa}_{\mathbf{AB}} - \mathring{\kappa}_{\mathbf{AB}}$. We then have

$$\xi_{ABCD}=o_\infty(r^{-3/2}),\quad \xi_{AB}=o_\infty(r^{-1/2}),\quad \xi=o_\infty(r^{-1/2}),\quad \kappa_{AB}=o_\infty(r^{3/2}).$$

To obtain the desired conclusion we only need to prove that $\kappa_{\mathbf{AB}} = C_{\mathbf{AB}} + o_\infty(r^{-1/2})$, where $C_{\mathbf{AB}}$ is a constant. This is equivalent to $D_{AB}\kappa_{CD} = o_\infty(r^{-3/2})$. Note that we now have coordinate independent statements to prove.

We note that from (47a)–(47b) it follows that

$$K_{ABCD}=o_\infty(r^{-2+\varepsilon}),\quad \Psi_{ABCD}=o_\infty(r^{-3+\varepsilon}),$$

with $\varepsilon > 0$. From (17) and Lemma 14 we have

$$\begin{aligned} &D_{AB}\kappa_{CD}\\ &\quad=\xi_{ABCD}-\frac{1}{3}\epsilon_{A(C}\xi_{D)B}-\frac{1}{3}\epsilon_{B(C}\xi_{D)A}-\frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi-K_{AB(C}{}^{E}\kappa_{D)E}\\ &\quad=o_\infty(r^{-1/2+\varepsilon}).\end{aligned}$$

Integrating the latter yields

$$\kappa_{AB}=o_\infty(r^{1/2+\varepsilon}).$$

The constant of integration is incorporated in the remainder term. Repeating this procedure allows to gain an ε in the decay so that

$$D_{AB}\kappa_{CD}=o_\infty(r^{-1/2}),\quad \kappa_{AB}=o_\infty(r^{1/2}).$$

Estimating all terms in (19a), (19b) and (19c) gives

$$\begin{aligned}
\nabla^{AB}\nabla_{AB}\xi &= \xi_{AB}\nabla^{AB}K + o_\infty(r^{-7/2}) \\
&= o_\infty(r^{-7/2+\varepsilon}), && \text{(56a)}\\
\nabla^{C}{}_{(A}\nabla_{B)C}\xi &= \frac{1}{2}\Omega_{ABCD}\nabla^{CD}\xi - \frac{1}{3}K\nabla_{AB}\xi \\
&= o_\infty(r^{-7/2+\varepsilon}), && \text{(56b)}\\
\nabla_{(AB}\nabla_{CD)}\xi &= +\frac{1}{2}\hat{\Psi}_{ABCD}\xi - \frac{5}{2}\Psi_{ABCD}\xi - \frac{2}{3}\hat{\Psi}_{(ABC}{}^{E}\xi_{D)E} - \frac{10}{3}\Psi_{(ABC}{}^{E}\xi_{D)E} \\
&\quad + \Omega_{ABCDEL}\xi^{EL} + \frac{2}{5}\xi_{(CD}\nabla_{AB)}K - 3\Omega_{E(BCD}\nabla_{A)}{}^{E}\xi \\
&\quad - 3\kappa^{EL}\nabla_{L(D}\Psi_{ABC)E} + 3\kappa_{(A}{}^{E}\nabla_{D}{}^{L}\Psi_{BC)EL} + o_\infty(r^{-7/2}) \\
&= o_\infty(r^{-7/2+\varepsilon}). && \text{(56c)}
\end{aligned}$$

Hence, $\nabla_{AB}\nabla_{CD}\xi = o_\infty(r^{-7/2+\varepsilon})$, and therefore $D_{AB}D_{CD}\xi = o_\infty(r^{-7/2+\varepsilon})$. Integrating this yields $D_{AB}\xi = o_\infty(r^{-5/2+\varepsilon})$. In this step the constants of integration are forced to vanish by the condition $D_{AB}\xi = o_\infty(r^{-3/2})$, which is a consequence of $\xi = o_\infty(r^{-1/2})$. Integrating $D_{AB}\xi = o_\infty(r^{-5/2+\varepsilon})$ and using $\xi = o_\infty(r^{-1/2})$ to remove the constants of integration yields

$$\xi = o_\infty(r^{-3/2+\varepsilon}).$$

Estimating all terms in (18a), (18b) and (18c) yields

$$\begin{aligned}
\nabla^{AB}\xi_{AB} &= o_\infty(r^{-7/2+\varepsilon}), && \text{(57a)}\\
\nabla^{C}{}_{(A}\xi_{B)C} &= \frac{3}{2}\Psi_{ABCD}\kappa^{CD} - \frac{2}{3}K\xi_{AB} - \frac{1}{2}\Omega_{ABCD}\xi^{CD} + \nabla_{AB}\xi + o_\infty(r^{-5/2}) \\
&= o_\infty(r^{-5/2+\varepsilon}), && \text{(57b)}\\
\nabla_{(AB}\xi_{CD)} &= 3\Psi_{E(ABC}\kappa_{D)}{}^{E} - \Omega_{E(ABC}\xi_{D)}{}^{E} + o_\infty(r^{-5/2}) \\
&= o_\infty(r^{-5/2+\varepsilon}). && \text{(57c)}
\end{aligned}$$

Hence, $\nabla_{AB}\xi_{CD} = o_\infty(r^{-5/2+\varepsilon})$, and therefore $D_{AB}\xi_{CD} = o_\infty(r^{-5/2+\varepsilon})$. Integrating and using $\xi_{AB} = o_\infty(r^{-1/2})$ to remove the constants of integration yields

$$\xi_{AB} = o_\infty(r^{-3/2+\varepsilon}).$$

Now,

$$\begin{aligned}
&D_{AB}\kappa_{CD} \\
&\quad = \xi_{ABCD} - \frac{1}{3}\epsilon_{A(C}\xi_{D)B} - \frac{1}{3}\epsilon_{B(C}\xi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi - K_{AB(C}{}^{E}\kappa_{D)E} \\
&\quad = o_\infty(r^{-3/2+\varepsilon}).
\end{aligned}$$

Integrating the latter we get

$$\kappa_{AB} = C_{AB} + o_\infty(r^{-1/2+\varepsilon}),$$

where C_{AB} is a constant in some frame. To get a frame independent statement one can still use the estimate $\kappa_{AB} = o_\infty(r^\varepsilon)$. Re-evaluating the estimates (56a), (56b) and (56c) yields

$$\begin{aligned}
\nabla^{AB}\nabla_{AB}\xi &= o_\infty(r^{-7/2}),\\
\nabla^{C}{}_{(A}\nabla_{B)C}\xi &= o_\infty(r^{-9/2+\varepsilon}),\\
\nabla_{(AB}\nabla_{CD)}\xi &= o_\infty(r^{-7/2}).
\end{aligned}$$

Hence, one obtains

$$\nabla_{AB}\nabla_{CD}\xi = o_\infty(r^{-7/2}).$$

Integrating as before, we get

$$\xi = o_\infty(r^{-3/2}).$$

Finally, we can reevaluate the estimates (57b) and (57c), to get

$$\begin{aligned}
\nabla^{C}{}_{(A}\xi_{B)C} &= o_\infty(r^{-5/2}),\\
\nabla_{(AB}\xi_{CD)} &= o_\infty(r^{-5/2}).
\end{aligned}$$

Combining this with (57a), we obtain

$$\nabla_{AB}\xi_{CD} = o_\infty(r^{-5/2}).$$

Integrating as before, we get

$$\xi_{AB} = o_\infty(r^{-3/2}).$$

Hence,

$$\begin{aligned}
D_{AB}\kappa_{CD} &= \xi_{ABCD} - \frac{1}{3}\epsilon_{A(C}\xi_{D)B} - \frac{1}{3}\epsilon_{B(C}\xi_{D)A} - \frac{1}{3}\epsilon_{A(C}\epsilon_{D)B}\xi - K_{AB(C}{}^{E}\kappa_{D)E}\\
&= o_\infty(r^{-3/2}),
\end{aligned}$$

from where the result follows. □

From the asymptotic solutions we can obtain a globally defined spinor $\mathring{\kappa}_{AB}$ on $\mathcal{S}$ that will act as a seed for our approximate Killing spinor.

Corollary 18. *There are spinors $\mathring{\kappa}_{AB}$, defined everywhere on $\mathcal{S}$, such that the asymptotics at each end is given by (51), where opposite signs are used at each end. Different choices of $\mathring{\kappa}_{AB}$ can only differ by a spinor in $H^\infty_{-1/2}$.*

Remark. The opposite signs at each end are motivated by looking at the explicit example of standard Kerr data.

Proof. Theorem 17 gives the existence at each end. Smoothly cut off these functions, and paste them together. This gives a smooth spinor $\mathring{\kappa}_{AB}$ defined everywhere on $\mathcal{S}$. Furthermore $\nabla_{(AB}\mathring{\kappa}_{CD)} \in H^\infty_{-3/2}$. □

8. The Approximate Killing Spinor Equation in Asymptotically Euclidean Manifolds

In this section we study the invertibility properties of the approximate Killing spinor operator $L : \mathfrak{S}_2 \to \mathfrak{S}_2$ given by Eq. (39) on a manifold $\mathcal{S}$ which is asymptotically Euclidean in the sense discussed in Sect. 6. In order to do so, we first present some adaptations to our context of results for elliptic equations that can be found in [9,13,35].

8.1. Ancillary Results of the Theory of Elliptic Equations on Asymptotically Euclidean Manifolds

8.1.1. Asymptotic Homogeneity of *L*. Let u be the vector given by Eq. (41). The elliptic operator defined by (39) can be written matricially in the form

$$(A^{ij} + a_2^{ij})D_iD_ju + a_1^iD_iu + a_0u = 0,$$

where A^{ij} corresponds to the matrix associated with the elliptic operator with constant coefficients L' given by Eq. (40), and a_2^{ij}, a_1^j, a_0 are matrix-valued functions such that

$$a_2^{ij} \in H^\infty_{-1/2}, \quad a_1^j \in H^\infty_{-3/2}, \quad a_0 \in H^\infty_{-5/2}.$$

Using the terminology of [9,35] we say that L is an *asymptotically homogeneous elliptic operator*.[3] This is the standard assumption on elliptic operators on asymptotically Euclidean manifolds. It follows from [9], Theorem 6.3 that

Theorem 19. *The elliptic operator*

$$L : H^2_\delta \to H^0_{\delta-2},$$

with δ is not a negative integer is a linear bounded operator with finite dimensional Kernel and closed range.

8.1.2. The Kernel of *L*. We investigate some relevant properties of the Kernel of L. This, in turn, requires an analysis of the Kernel of the operator of the Killing spinor equation (25).

The following is an adaptation to the smooth spinorial setting of an ancillary result from [13].[4]

Theorem 20. *Let $\nu_{A_1B_1\cdots A_pB_p}$ be a C^∞ spinorial field over $\mathcal{S}$ such that*

$$\nabla_{E_{m+1}F_{m+1}}\cdots\nabla_{E_1F_1}\nu_{A_1B_1\cdots A_pB_p} = H_{E_{m+1}F_{m+1}\cdots E_1F_1A_1B_1\cdots A_pB_p}$$

with m, p non-negative integers, and where $H_{E_{m+1}F_{m+1}\cdots E_1F_1A_1B_1\cdots A_pB_p}$ is a linear combination of $\nu_{A_1B_1\cdots A_pB_p}$, $\nabla_{E_1F_1}\nu_{A_1B_1\cdots A_pB_p}, \ldots, \nabla_{E_mF_m}\cdots$

[3] The sharp conditions for a second-order elliptic operator to be asymptotically homogeneous are that

$$a_2^{ij} \in H^\infty_\delta, \quad a_1^i \in H^\infty_{\delta-1}, \quad a_0 \in H^\infty_{\delta-2},$$

for $\delta < 0$. As one sees, our operator L satisfies these conditions with a margin.

[4] The hypotheses in [13] are much weaker than the ones presented here. The adaptation to the smooth setting has been chosen for simplicity.

$\nabla_{E_1F_1}\nu_{A_1B_1\cdots A_pB_p}$ *with coefficients* b_k *where* k *denotes the order of the derivative the coefficient is associated with. If* $b_k \in H^\infty_{\delta_k}$ *with*

$$k-m-1>\delta_k, \quad 0\leq k\leq m$$

and $\nu_{A_1B_1\cdots A_pB_p}\in H^\infty_\beta$*,* $\beta<0$*, then*

$$\nu_{A_1B_1\cdots A_pB_p}=0 \quad \text{on } \mathcal{S}.$$

This last result, together with Lemma 10 allows to show that there are no non-trivial Killing spinor candidates that go to zero at infinity—in [13] an analogous result has been proved for Killing vectors. More precisely,

Proposition 21. *Let* $\nu_{AB}\in H^\infty_{-1/2}$ *such that* $\nabla_{(AB}\nu_{CD)}=0$*. Then* $\nu_{AB}=0$ *on* $\mathcal{S}$*.*

Proof. From Lemma 10 it follows that $\nabla_{AB}\nabla_{CD}\nabla_{EF}\nu_{GH}$ can be expressed as a linear combination of lower order derivatives with smooth coefficients with the proper decay. Thus, Theorem 20 applies with $m=2$ and one obtains the desired result. □

We are now in the position to discuss the Kernel of the approximate Killing spinor operator in the case of spinor fields that go to zero at infinity. The following is the main result of this section:

Proposition 22. *Let* $\nu_{AB}\in H^\infty_{-1/2}$*. If* $L(\nu_{AB})=0$*, then* $\nu_{AB}=0$*.*

Proof. Using the identity (37) with $\zeta_{ABCD}=\nabla_{(AB}\nu_{CD)}$ and assuming that $L(\nu_{CD})=0$, one obtains

$$\int_{\mathcal{S}}\nabla^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}\mathrm{d}\mu=\int_{\partial\mathcal{S}_\infty}n^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}\mathrm{d}S,$$

where ∂S_∞ denotes the sphere at infinity. Assume now, that $\nu_{AB}\in H^\infty_{-1/2}$. It follows that $\nabla_{(AB}\nu_{CD)}\in H^\infty_{-3/2}$ and furthermore, using Lemma 15 that

$$n^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}=o(r^{-2}).$$

The integration of the latter over a finite sphere of sufficiently large radius is of type $o(1)$. Thus one has that

$$\int_{\partial\mathcal{S}_\infty}n^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}\mathrm{d}S=0,$$

from where

$$\int_{\mathcal{S}}\nabla^{AB}\nu^{CD}\widehat{\nabla_{(AB}\nu_{CD)}}\mathrm{d}\mu=0.$$

Therefore, one concludes that

$$\nabla_{(AB}\nu_{CD)}=0.$$

That is, ν_{AB} has to be a spatial Killing spinor. Using Proposition 21 it follows that $\nu_{AB}=0$ on $\mathcal{S}$. □

8.1.3. The Fredholm Alternative and Elliptic Regularity. We will make use of the following adaptation of the Fredholm alternative for second-order asymptotically homogeneous elliptic operators on asymptotically Euclidean manifolds—cfr. [9]:

Theorem 23. *Let A be an asymptotically homogeneous elliptic operator of order 2 with smooth coefficients. Given δ not a negative integer, the equation*

$$A(\zeta_{AB}) = f_{AB}, \quad f_{AB} \in H^0_{\delta-2},$$

has a solution $\zeta_{AB} \in H^2_\delta$ if

$$\int_{\mathcal{S}} f_{AB}\hat{\nu}^{AB}\, d\mu = 0$$

for all ν_{AB} satisfying

$$\nu_{AB} \in H^0_{-1-\delta}, \quad A^*(\nu_{AB}) = 0,$$

where A^* denotes the formal adjoint of A.

In order to assert the regularity of solutions, we will need the following elliptic estimate—see expression (62) in the proof of Theorem 6.3 of [9]:

Theorem 24. *Let A be an asymptotically homogeneous elliptic operator of order 2 with smooth coefficients. Then for any $\delta \in \mathbb{R}$ and any $s \geq 2$ there exists a constant C such that for every $\zeta_{AB} \in H^s_{loc} \cap H^0_\delta$, the following inequality holds:*

$$\|\zeta_{AB}\|_{H^s_\delta} \leq C\left(\|A(\zeta_{AB})\|_{H^{s-2}_{\delta-2}} + \|\zeta_{AB}\|_{H^{s-2}_\delta}\right).$$

Notation. H^s_{loc} denotes the local Sobolev space. That is, $u \in H^s_{loc}$ if for an arbitrary smooth function v with compact support, $uv \in H^s$.

Remark. If A has smooth coefficients, and $A(\zeta_{AB}) = 0$ then it follows that all the H^s_δ norms of ζ_{AB} are bounded by the H^0_δ norm. Thus, it follows that if a solution to $A(\zeta_{AB}) = 0$ exists, it must be smooth—*elliptic regularity.*

8.2. Existence of Approximate Killing Spinors

We are now in the position of providing an existence proof to solutions to Eq. (39) with the asymptotic behaviour discussed in Sect. 7.2.

Theorem 25. *Given an asymptotically Euclidean initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ satisfying the asymptotic conditions (47a)–(47b) and (48), there exists a smooth unique solution to Eq. (39) with asymptotic behaviour at each end given by (51).*

Proof. We consider the Ansatz

$$\kappa_{AB} = \mathring{\kappa}_{AB} + \theta_{AB}, \quad \theta_{AB} \in H^2_{-1/2},$$

with $\mathring{\kappa}$ given by Corollary 18. Substitution into Eq. (39) renders the following equation for the spinor θ_{AB}:

$$L(\theta_{CD}) = -L(\mathring{\kappa}_{CD}). \tag{58}$$

By construction it follows that

$$\nabla_{(AB}\mathring{\kappa}_{CD)} \in H^{\infty}_{-3/2},$$

so that

$$F_{CD} \equiv -L(\mathring{\kappa}_{CD}) \in H^{\infty}_{-5/2}.$$

Using Theorem 23 with $\delta = -1/2$, one concludes that Eq. (58) has a unique solution if F_{AB} is orthogonal to all $\nu_{AB} \in H^{0}_{-1/2}$ in the Kernel of $L^* = L$. Proposition 21 states that this Kernel is trivial. Thus, there are no restrictions on F_{AB} and Eq. (58) has a unique solution as desired. Due to elliptic regularity, any $H^{2}_{-1/2}$ solution to the previous equation is in fact a $H^{\infty}_{-1/2}$ solution—cfr. Lemma 24. Thus, θ_{AB} is smooth. To see that κ_{AB} does not depend on the particular choice of $\mathring{\kappa}_{AB}$, let $\mathring{\kappa}'_{AB}$, be another choice. Let κ'_{AB} be the corresponding solution to (58). Due to Corollary 18, we have $\mathring{\kappa}_{AB} - \mathring{\kappa}'_{AB} \in H^{\infty}_{-1/2}$. Hence, we have $\kappa_{AB} - \kappa'_{AB} \in H^{\infty}_{-1/2}$ and $L(\kappa_{AB} - \kappa'_{AB}) = 0$. According to Proposition 22, $\kappa_{AB} - \kappa'_{AB} = 0$, and the proof is complete. □

The following is a direct consequence of Theorem 25, and will be crucial for obtaining an invariant characterisation of Kerr data:

Corollary 26. *A solution, κ_{AB}, to Eq.* (39) *with asymptotic behaviour given by* (51) *satisfies $J < \infty$ where J is the functional given by Eq.* (42)*.*

Proof. The functional J given by Eq. (42) is the L^2 norm of $\nabla_{(AB}\kappa_{CD)}$. Now, if κ_{AB} is the solution given by Theorem 25, one has that $\nabla_{(AB}\kappa_{CD)} \in H^{0}_{-3/2}$. In Bartnik's conventions one has that

$$\|\nabla_{(AB}\kappa_{CD)}\|_{L^2} = \|\nabla_{(AB}\kappa_{CD)}\|_{H^{0}_{-3/2}} < \infty.$$

The result follows. □

Remark. Again, let κ_{AB} be the solution to Eq. (39) given by Theorem 25. Using the identity (37) with $\zeta_{ABCD} = \nabla_{(AB}\kappa_{CD)}$ one obtains that

$$J = \int_{\partial\mathcal{S}_\infty} n^{AB}\kappa^{CD}\widehat{\nabla_{(AB}\kappa_{CD)}}\mathrm{d}S < \infty.$$

Thus, the invariant J evaluated at the solution κ_{AB} given by Theorem 25 can be expressed as a boundary integral at infinity. A crude estimation of the integrand of the boundary integral does not allow directly to establish its boundedness. This follows, however, from Corollary 26. Hence, the leading order terms of $n_{AB}\kappa_{CD}$ and $\nabla_{(AB}\kappa_{CD)}$ are orthogonal.

For an independent proof of this fact, see Appendix A.

9. The Geometric Invariant

In this section we show how to use the functional (42) and the algebraic conditions (23b) and (23c) to construct the desired geometric invariant measuring the deviation of $(\mathcal{S}, h_{ab}, K_{ab})$ from Kerr initial data. To this end, let κ_{AB} be a solution to Eq. (39) as given by Theorem 25. Furthermore, let $\xi_{AB} \equiv \frac{3}{2}\nabla^P{}_{(A}\kappa_{B)P}$. Define

$$I_1 \equiv \int_{\mathcal{S}} \Psi_{(ABC}{}^F\kappa_{D)F}\hat{\Psi}^{ABCG}\hat{\kappa}^D{}_G \mathrm{d}\mu, \tag{59a}$$

$$\begin{aligned} I_2 \equiv \int_{\mathcal{S}} &\left(3\kappa_{(A}{}^E\nabla_B{}^F\Psi_{CD)EF} + \Psi_{(ABC}{}^F\xi_{D)F}\right) \\ &\times \left(3\hat{\kappa}^{AP}\widehat{\nabla^{BQ}\Psi^{CD}}{}_{PQ} + \hat{\Psi}^{ABCP}\hat{\xi}^D{}_P\right) \mathrm{d}\mu. \end{aligned} \tag{59b}$$

The geometric invariant is then defined by

$$I \equiv J + I_1 + I_2. \tag{60}$$

Remark. It should be stressed that by construction I is coordinate independent and that $I \geq 0$. We also have the following lemma.

Lemma 27. *The geometric invariant given by (60) is finite for an initial data set $(\mathcal{S}, h_{ab}, K_{ab})$ satisfying the decay conditions (47a)–(47b).*

Proof. From Corollary 26 we already have $J < \infty$. From the form of the decay assumptions (47a)–(47b) we have $\Psi_{ABCD} \in H^\infty_{-3+\varepsilon}$, $\varepsilon > 0$. By Lemma 14 and $\kappa_{AB} \in H^\infty_{1+\varepsilon}$ we have

$$\Psi_{(ABC}{}^F\kappa_{D)F} \in H^\infty_{-3/2}.$$

Thus, again one finds that $I_1 < \infty$. A similar argument shows that

$$3\kappa_{(A}{}^E\nabla_B{}^F\Psi_{CD)EF} + \Psi_{(ABC}{}^F\xi_{D)F} \in H^\infty_{-3/2},$$

from where it follows that $I_2 < \infty$. Hence, the invariant (60) is finite and well defined. □

Finally, we are in the position of stating the main result of this article. It combines all the results in the sections 2 to 7.

Theorem 28. *Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an asymptotically Euclidean initial data set for the Einstein vacuum field equations satisfying on each of its two asymptotic ends the decay conditions (47a)–(47b) and (48) with a timelike ADM 4-momentum. Furthermore, assume that $\Psi_{ABCD} \neq 0$ and $\Psi_{ABCD}\Psi^{ABCD} \neq 0$ everywhere on $\mathcal{S}$. Let I be the invariant defined by Eqs. (42), (59a), (59b) and (60), where κ_{AB} is given as the only solution to Eq. (39) with asymptotic behaviour on each end given by (51). The invariant I vanishes if and only if $(\mathcal{S}, h_{ab}, K_{ab})$ is locally an initial data set for the Kerr spacetime.*

Proof. Due to our smoothness assumptions, if $I = 0$ it follows that Eqs. (23a)–(23c) are satisfied on the whole of $\mathcal{S}$. Thus, the development of $(\mathcal{S}, h_{ab}, K_{ab})$ will have, at least in a slab, a Killing spinor. Accordingly, it must be of Petrov type D, N or O on the slab—see Theorem 1. The types N and O are excluded by the assumptions $\Psi_{ABCD} \neq 0$ and $\Psi_{ABCD}\Psi^{ABCD} \neq 0$ on $\mathcal{S}$—by continuity, these conditions will also hold in a suitably small slab. Thus, the development of the data can only be of Petrov type D—at least on a suitably small slab.

Now, from the general theory on Killing spinors, we know that $\xi_{AA'} = \nabla_{A'}{}^{Q}\kappa_{AQ}$ will be, in general, a complex Killing vector. In particular, both the real and imaginary parts of $\xi_{AA'}$ will be real Killing vectors. The Killing initial data for $\xi_{AA'}$ on $\mathcal{S}$ consist of the fields ξ and ξ_{AB} on $\mathcal{S}$ calculated from κ_{AB} using the expressions (15a) and (15b). It can be verified that

$$\xi - \hat{\xi} = o_\infty(r^{-1/2}), \quad \xi_{AB} + \hat{\xi}_{AB} = o_\infty(r^{-1/2}).$$

The latter corresponds to the Killing initial data for the imaginary part of $\xi_{AA'}$. It follows that the imaginary part of $\xi_{AA'}$ goes to zero at infinity. However, there are no non-trivial Killing vectors of this type [4,13]. Thus, $\xi_{AA'}$ is a real Killing vector. This means that the spacetime belongs, at least in a suitably small slab of $\mathcal{S}$, to the generalised Kerr-NUT class. By construction, it tends to a time translation at infinity so that, in fact, it is a stationary Killing vector. By virtue of the decay assumptions (47a)–(47b) the development of the initial data will be asymptotically flat, and it can be verified that the Komar mass of each end coincides with the corresponding ADM mass—these are non-zero by assumption. Hence, Theorem 6 applies and the slab of $\mathcal{S}$ is locally isometric to the Kerr spacetime. □

Corollary 29. *If furthermore, the slice $\mathcal{S}$ is assumed, a priori, to have the same topology as a slice of the Kerr spacetime one has that the invariant I vanishes if and only if $(\mathcal{S}, h_{ab}, K_{ab})$ is an initial data set for the Kerr spacetime.*

Proof. This follows from the uniqueness of the maximal globally hyperbolic development of Cauchy data—see [12]. □

Remark 1. A improvement of Theorem 6 in which no *a priori* restrictions on the Petrov type of the spacetime are made—see the remark after Theorem 6—would allow to remove the conditions $\Psi_{ABCD} \neq 0$ and $\Psi_{ABCD}\Psi^{ABCD} \neq 0$, and thus obtain a stronger characterisation of Kerr data.

Remark 2. It is of interest to analyse whether the same conclusion of the corollary can be obtained without making *a priori* assumptions on the topology of the 3-manifold.

10. Future Prospects

We have seen that one can construct a geometric invariant for a slice with two asymptotically flat ends. A natural extension of this work would be to also allow asymptotically hyperboloidal and asymptotically cylindrical slices. Furthermore, one would like to analyse parts of manifolds in the same way.

In this case we need to find appropriate conditions that can be imposed on κ_{AB} on the boundary of the region we would like to study. A typical scenario would be to study the domain of outer communication for a black hole, or the exterior of a star.

Another natural question to be asked is how the geometric invariant behaves under time evolution. A great part of this problem is to obtain a time evolution of κ_{AB} such that it satisfies (39) on every leaf of the foliation. If the geometric invariant is small, one could instead use (20) as an approximate evolution equation for the approximate Killing spinor. In this case the system (21a), (21b) could be used to gain control over the evolution.

If some type of constancy or monotonicity property could be established for the geometric invariant, this would be a useful tool for studying non-linear stability of the Kerr spacetime and also in the numerical evolutions of black hole spacetimes.

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Appendix A. An Alternative Estimation of the Boundary Integral

In this section we present an alternative argument to show that the boundary integral

$$\int_{\partial\mathcal{S}_r} n^{AB}\kappa^{CD}\widehat{\nabla_{(AB}\kappa_{CD)}}\mathrm{d}S,$$

is finite as $r \to \infty$—cfr. the remark after Corollary 26. For simplicity, we only consider the non-boosted case, so we have

$$\kappa_{AB} = \pm\frac{\sqrt{2}}{3}rn_{AB} + O(1).$$

A similar, but much lengthier argument can be implemented in the boosted case. It is only necessary to consider the finiteness of the integral

$$r\int_{\partial\mathcal{S}_r} n^{AB}n^{CD}\widehat{\nabla_{(AB}\kappa_{CD)}}\mathrm{d}S \quad \text{as } r \to \infty. \tag{61}$$

We begin by investigating the multipole structure of $\xi_{ABCD} \equiv \nabla_{(AB}\kappa_{CD)}$ in an asymptotically flat end $\mathcal{U} \subset \mathcal{S}$. The equation satisfied by ξ_{ABCD} is

$$\nabla^{AB}\xi_{ABCD} - 2\Omega^{ABF}{}_{(C}\xi_{D)ABF} = 0, \tag{62}$$

—see Eq. (39). As $\mathcal{U} \approx (r_0, \infty) \times \mathbb{S}^2$, with $r_0 \in \mathbb{R}$, it will be convenient to work in spherical coordinates. For simplicity, we adopt the point of view that all the angular dependence of the various functions involved is expressed in terms of (spin-weighted) spherical harmonics. Accordingly, we use the differential operators $\eth$, $\bar{\eth} \in \mathrm{T}\mathbb{S}^2$—see, e.g. [41]. Let $\omega_+, \omega_- \in \mathrm{T}^*\mathbb{S}^2$ denote the 1-forms dual to $\eth$ and $\bar{\eth}$:

$$\langle \eth, \omega_+ \rangle = 1, \quad \langle \bar{\eth}, \omega_- \rangle = 1.$$

In addition, we consider $\partial_r \in \mathrm{T}\mathcal{U}$. The operators $\eth, \bar{\eth}$ are extended into $\mathrm{T}\mathcal{U}$ by requiring that

$$[\eth, \partial_r] = [\bar{\eth}, \partial_r] = 0.$$

Again, let $\mathrm{d}r \in \mathrm{T}^*\mathcal{U}$ denote the form dual to ∂_r. One has that

$$\delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j = \mathrm{d}r \otimes \mathrm{d}r + r^2 \left(\omega_+ \otimes \omega_- + \omega_- \otimes \omega_+\right).$$

Now, recalling that

$$h_{ij} = -\left(1 + \frac{2m}{r}\right) \delta_{ij} + o_\infty(r^{-3/2}),$$

we introduce the following frame and coframe:

$$\begin{aligned}
e_{01} &= \left(1 - \frac{m}{r}\right) \partial_r + o_\infty(r^{-3/2}), \quad \sigma^{01} = \left(1 + \frac{m}{r}\right) \mathrm{d}r + o_\infty(r^{-3/2}) \\
e_{00} &= \left(1 - \frac{m}{r}\right) \frac{1}{r} \eth + o_\infty(r^{-5/2}), \quad \sigma^{00} = \left(1 + \frac{m}{r}\right) r\omega_+ + o_\infty(r^{-1/2}) \\
e_{11} &= \left(1 - \frac{m}{r}\right) \frac{1}{r} \bar{\eth} + o_\infty(r^{-5/2}), \quad \sigma^{11} = \left(1 + \frac{m}{r}\right) r\omega_- + o_\infty(r^{-1/2}).
\end{aligned}$$

The fields e_{AB} and σ^{AB} satisfy

$$\langle e_{AB}, \sigma^{CD} \rangle = h_{AB}{}^{CD}, \quad h = h_{ABCD} \sigma^{AB} \otimes \sigma^{CD}.$$

where $h_{ABCD} \equiv -\epsilon_{A(C}\epsilon_{D)B}$. Let μ_{AB} denote a smooth spinorial field. Its covariant derivative $D_{EF}\mu_{AB}$ can be computed using

$$D_{EF}\mu_{AB} = e_{EF}(\mu_{AB}) - \Gamma_{EF}{}^Q{}_A \mu_{QB} - \Gamma_{EF}{}^Q{}_B \mu_{AQ},$$

where $\Gamma_{EF}{}^Q{}_A$ denote the spin coefficients of the frame e_{AB}.

The components of the spinor field ξ_{ABCD} with respect to the frame e_{AB} can be written as

$$\xi_{ABCD} = \xi_0 \epsilon^0_{ABCD} + \xi_1 \epsilon^1_{ABCD} + \xi_2 \epsilon^2_{ABCD} + \xi_3 \epsilon^3_{ABCD} + \xi_4 \epsilon^4_{ABCD},$$

where

$$\epsilon^k_{ABCD} \equiv \epsilon_{(A}{}^{(E}\epsilon_B{}^F \epsilon_C{}^G \epsilon_{D)}{}^{H)_k},$$

where ${}^{(EFGH)_k}$ means that after symmetrisation, k indices are set to 1. In terms of this formalism, Eq. (62) is given by

$$\begin{aligned}
&\epsilon^{AP}\epsilon^{BQ} e_{PQ}(\xi_{ABCD}) - 4\Gamma^{ABQ}{}_{(A}\xi_{BCD)Q} + 2K^{ABQ}{}_{(A}\xi_{BCD)Q} \\
&\quad -2\Omega^{ABQ}{}_{(C}\xi_{D)ABQ} = 0.
\end{aligned} \tag{63}$$

Recalling that by assumption $\xi_{ABCD} = o_\infty(r^{-3/2})$, a lengthy but straightforward calculation shows that (63) implies the equations

$$\partial_r\xi_1 - \frac{1}{r}\bar{\eth}\xi_0 + \frac{1}{6}\frac{1}{r}\eth\xi_2 + \frac{3}{r}\left(1+\frac{m}{r}\right)\xi_1 = o_\infty(r^{-5}), \tag{64a}$$

$$\partial_r\xi_2 + \frac{3}{2}\frac{1}{r}\bar{\eth}\xi_1 + \frac{3}{2}\frac{1}{r}\eth\xi_3 + \frac{3}{r}\left(1+\frac{m}{r}\right)\xi_2 = o_\infty(r^{-5}), \tag{64b}$$

$$\partial_r\xi_3 + \frac{1}{r}\eth\xi_4 - \frac{1}{6}\frac{1}{r}\bar{\eth}\xi_2 + \frac{3}{r}\left(1+\frac{m}{r}\right)\xi_3 = o_\infty(r^{-5}). \tag{64c}$$

A computation shows that

$$n_{(AB}n_{CD)} = \epsilon^2_{ABCD},$$

so that the boundary integral (61) involves only the component ξ_2. Furthermore, only the harmonic $Y_{0,0}$ (monopole) contributes to the integral as ϵ^2_{ABCD} is a constant spinor in our frame. From the Eqs. (64a)–(64c), it follows that the coefficient $\xi_{2;0}$ of ξ_2 associated with the harmonic $Y_{0,0}$ satisfies the ordinary differential equation

$$\left(1-\frac{m}{r}\right)\partial_r\xi_{2;0} + \frac{3}{r}\xi_{2;0} = f(r), \quad f(r) = o_\infty(r^{-5}).$$

Consequently, one has that

$$\xi_{2;0} = \frac{\alpha}{(r-m)^3} + \frac{1}{(r-m)^3}\int r(r-m)^2 f(r)\mathrm{d}r, \quad \alpha\in\mathbb{C}.$$

It follows that

$$\xi_{2;0} = \frac{\alpha}{r^3} + o_\infty(r^{-4}).$$

Using this last expression in the integral (61) and recalling that $\mathrm{d}S = O(r^2)$, it follows that

$$r\int_{\partial\mathcal{S}_r} n^{AB}n^{CD}\nabla_{\widehat{(AB}\kappa_{CD)}}\mathrm{d}S = 4\pi\alpha < \infty.$$

It is worth noting that the constant α contains information of global nature, and it is only known after one has solved the approximate Killing spinor equation.

Appendix B. Tensor Expressions

For many applications, it is useful to have tensor expressions for the invariants. To this end, define the following tensors on $\mathcal{S}$:

$$\kappa_a \equiv \sigma_a{}^{AB}\kappa_{AB}, \quad \zeta\equiv\xi,$$
$$\zeta_a \equiv \sigma_a{}^{AB}\xi_{AB}, \quad \zeta_{ab}\equiv \sigma_a{}^{AB}\sigma_b{}^{CD}\xi_{ABCD},$$
$$C_{ac} \equiv E_{ac} + \mathrm{i}B_{ac}.$$

Here ϵ_{abc}, E_{ac} and B_{ac} are the pull-backs of $\frac{1}{\sqrt{2}}\tau^\mu\epsilon_{\mu\alpha\beta\gamma}$, $\frac{1}{2}\tau^\gamma\tau^\delta C_{\alpha\beta\gamma\delta}$ and $\frac{1}{4}\epsilon_{\mu\nu\gamma\delta}\tau^\beta\tau^\delta C_{\alpha\beta}{}^{\mu\nu}$, respectively. Observe that we are using a negative definite metric. In this section we assume $K_{ab} = K_{ba}$.

The tensorial versions of the Eqs. (15a), (15b), (15c) then read

$$\zeta = D^a\kappa_a,$$
$$\zeta_a = \frac{3}{2\sqrt{2}}\mathrm{i}\epsilon_{abc}D^c\kappa^b - \frac{3}{4}K_{ab}\kappa^b + \frac{3}{4}K_b{}^b\kappa_a,$$
$$\zeta_{ab} = D_{(a}\kappa_{b)} - \frac{1}{3}h_{ab}D^c\kappa_c - \frac{1}{\sqrt{2}}\mathrm{i}\epsilon_{cd(a}K_{b)}{}^d\kappa^c.$$

Note that the spatial Killing spinor equation $\zeta_{ab} = 0$ reduces to the conformal Killing vector equation in the time symmetric case ($K_{ab} = 0$).

Expressed in terms of these tensors the elliptic equation (39) takes the form

$$D^b\zeta_{ab} - \frac{1}{\sqrt{2}}\mathrm{i}\epsilon_{acd}K^{bc}\zeta_b{}^d = 0. \tag{66}$$

Let $\kappa_a \in H^\infty_{3/2}$ be the solution to (66) with the asymptotics

$$\kappa_i = \mp\frac{\sqrt{2}E}{3m}\left(1+\frac{2E}{r}\right)x_i$$
$$\pm\frac{2\mathrm{i}}{3m}\left(1+\frac{4E}{r}-\frac{m^2+2(n\cdot p)^2}{\sqrt{m^2+(n\cdot p)^2}r}\right)\epsilon_i{}^{jk}p_jx_k + o_\infty(r^{-1/2}),$$

at each end, where $p_\mu = (E, p_i)$ is the ADM-4 momentum, $m \equiv \sqrt{p^\mu p_\mu}$, and $n\cdot p = r^{-1}x^ip_i$. The metric and extrinsic curvature are assumed to have the asymptotics (47a) and (47b), respectively.

The integrand in (42) is

$$\mathfrak{J} \equiv \xi_{ABCD}\hat{\xi}^{ABCD} = \zeta_{ab}\bar{\zeta}^{ab}.$$

From the equation

$$\sigma_a{}^{AB}\sigma_b{}^{CD}\Psi_{(ABC}{}^F\kappa_{D)F} = \frac{1}{\sqrt{2}}\mathrm{i}\epsilon_{cd(a}C_{b)}{}^d\kappa^c.$$

we get the integrand for the I_1 part of the invariant

$$\mathfrak{J}_1 \equiv \Psi_{(ABC}{}^F\kappa_{D)F}\hat{\Psi}^{ABCP}\hat{\kappa}^D{}_P$$
$$= -\frac{1}{2}C^{bc}\bar{C}_{bc}\kappa^a\bar{\kappa}_a + \frac{1}{2}C_b{}^c\bar{C}_{ac}\kappa^a\bar{\kappa}^b + \frac{1}{4}C_a{}^c\bar{C}_{bc}\kappa^a\bar{\kappa}^b.$$

In order to discuss the integrand of I_2 we introduce the spinor $\Sigma_{ABCD} \equiv \nabla_{(A}{}^F\Psi_{BCD)F}$ and its tensor equivalent $\Sigma_{ab} = \sigma_a{}^{AB}\sigma_b{}^{CD}\Sigma_{ABCD}$. One finds that

$$0 = \sigma_a{}^{AB}\nabla^{CD}\Psi_{ABCD} = D^bC_{ab} - \frac{\mathrm{i}}{\sqrt{2}}\epsilon_{acd}C^{bc}K_b{}^d,$$
$$\Sigma_{ab} = \frac{\mathrm{i}}{\sqrt{2}}\epsilon_{df(a}D^fC^d{}_{b)} + \frac{1}{2}C^{cd}K_{cd}h_{ab} + C_{ab}K^f{}_f - \frac{3}{2}C^c{}_{(a}K_{b)c}.$$

The integrand for I_2 is given by

$$\begin{aligned}
\mathfrak{I}_2 &= (3\kappa_{(A}{}^{F}\Sigma_{BCD)F} + \Psi_{(ABC}{}^{F}\xi_{D)F})(3\hat{\kappa}^{AP}\hat{\Sigma}^{BCD}{}_{P} + \hat{\Psi}^{ABCP}\hat{\xi}^{D}{}_{P}) \\
&= -\frac{9}{2}\Sigma^{bc}\bar{\Sigma}_{bc}\kappa^{a}\bar{\kappa}_{a} + \frac{9}{2}\Sigma_{b}{}^{c}\bar{\Sigma}_{ac}\kappa^{a}\bar{\kappa}^{b} + \frac{9}{4}\Sigma_{a}{}^{c}\bar{\Sigma}_{bc}\kappa^{a}\bar{\kappa}^{b} + \frac{3}{2}\bar{\Sigma}_{bc}C^{bc}\bar{\kappa}^{a}\zeta_{a} \\
&\quad -\frac{3}{4}\bar{\Sigma}_{ac}C_{b}{}^{c}\bar{\kappa}^{a}\zeta^{b} - \frac{3}{2}\bar{\Sigma}_{bc}C_{a}{}^{c}\bar{\kappa}^{a}\zeta^{b} + \frac{3}{2}\Sigma_{bc}\bar{C}^{bc}\kappa^{a}\bar{\zeta}_{a} - \frac{3}{4}\Sigma_{ac}\bar{C}_{b}{}^{c}\kappa^{a}\bar{\zeta}^{b} \\
&\quad -\frac{3}{2}\Sigma_{bc}\bar{C}_{a}{}^{c}\kappa^{a}\bar{\zeta}^{b} + \frac{1}{2}C^{bc}\bar{C}_{bc}\zeta^{a}\bar{\zeta}_{a} + \frac{1}{2}C_{b}{}^{c}\bar{C}_{ac}\zeta^{a}\bar{\zeta}^{b} + \frac{1}{4}C_{a}{}^{c}\bar{C}_{bc}\zeta^{a}\bar{\zeta}^{b}.
\end{aligned}$$

The complete invariant is given by

$$I = \int_{\mathcal{S}} (\mathfrak{I} + \mathfrak{I}_1 + \mathfrak{I}_2)\mathrm{d}\mu.$$

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