

On the Lipschitz Continuity of Spectral Bands of Harper-Like and Magnetic Schrödinger Operators

Horia D. Cornean

Abstract. We show for a large class of discrete Harper-like and continuous magnetic Schrödinger operators that their band edges are Lipschitz continuous with respect to the intensity of the external constant magnetic field. We generalize a result obtained by Bellissard (Commun Math Phys 160:599–613, 1994), and give examples in favor of a recent conjecture of G. Nenciu.

1. Introduction and the Main Results

Harper-like operators. Let $\Gamma \subset \mathbb{R}^2$ be a (possibly irregular) lattice which has the property that there exists an injective map $F : \Gamma \mapsto \mathbb{Z}^2$ such that $|F(\gamma) - \gamma| < 1/2$. The Hilbert space is $l^2(\Gamma)$.

The elements of the canonical basis in $l^2(\Gamma)$ are denoted by $\{\delta_{\mathbf{x}}\}_{\mathbf{x} \in \Gamma}$, where $\delta_{\mathbf{x}}(\mathbf{y}) = 1$ if $\mathbf{y} = \mathbf{x}$ and zero otherwise. In the discrete case, to any bounded self-adjoint operator $H \in B(l^2(\Gamma))$ it corresponds a bounded and symmetric kernel $H(\mathbf{x}, \mathbf{x}') = \langle H\delta_{\mathbf{x}'}, \delta_{\mathbf{x}} \rangle = \overline{H(\mathbf{x}', \mathbf{x})}$. We will extensively use the Schur–Holmgren upper bound for the norm of a self-adjoint operator:

$$\|H\| \leq \sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} |H(\mathbf{x}, \mathbf{x}')|. \quad (1.1)$$

Denote by $\langle \mathbf{x} - \mathbf{x}_0 \rangle^\alpha = [1 + (\mathbf{x} - \mathbf{x}_0)^2]^\frac{\alpha}{2}$, $\alpha \geq 0$. We define \mathcal{C}^α to be the set of bounded and self-adjoint operators $H \in B(l^2(\Gamma))$ which have the property that their kernels obey a weighted Schur–Holmgren type estimate:

$$\|H\|_{\mathcal{C}^\alpha} := \sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} \langle \mathbf{x} - \mathbf{x}' \rangle^\alpha |H(\mathbf{x}, \mathbf{x}')| < \infty. \quad (1.2)$$

We also define the space \mathcal{H}^α which contains bounded and self-adjoint operators H which obey:

$$\|H\|_{\mathcal{H}^\alpha} := \sup_{\mathbf{x}' \in \Gamma} \left\{ \sum_{\mathbf{x} \in \Gamma} \langle \mathbf{x} - \mathbf{x}' \rangle^{2\alpha} |H(\mathbf{x}, \mathbf{x}')|^2 \right\}^{\frac{1}{2}} < \infty. \tag{1.3}$$

The flux of a unit magnetic field orthogonal to the plane through a triangle generated by \mathbf{x}, \mathbf{x}' and the origin is given by:

$$\varphi(\mathbf{x}, \mathbf{x}') := -\frac{1}{2} (x_1 x'_2 - x_2 x'_1) = -\varphi(\mathbf{x}', \mathbf{x}). \tag{1.4}$$

Note the important additive identity:

$$\begin{aligned} \varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{y}, \mathbf{x}') &= \varphi(\mathbf{x}, \mathbf{x}') + \varphi(\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{x}'), \\ |\varphi(\mathbf{x} - \mathbf{y}, \mathbf{y} - \mathbf{x}')| &\leq \frac{1}{2} |\mathbf{x} - \mathbf{y}| |\mathbf{y} - \mathbf{x}'|. \end{aligned} \tag{1.5}$$

Let $K \in \mathcal{C}^0$. Let its kernel be $K(\mathbf{x}, \mathbf{x}')$. We are interested in a family of Harper-like operators $\{K_b\}_{b \in \mathbb{R}}$ given by the kernels $e^{ib\varphi(\mathbf{x}, \mathbf{x}')} K(\mathbf{x}, \mathbf{x}')$. Clearly, $\{K_b\}_{b \in \mathbb{R}} \subset \mathcal{C}^0$. The usual Harper operator lives in $l^2(\mathbb{Z}^2)$, and its generating kernel has the form $K(\mathbf{x}, \mathbf{x}') = k(\mathbf{x} - \mathbf{x}')$ where $k(\mathbf{x})$ equals 1 if $|\mathbf{x}| = 1$, and 0 otherwise.

In Lemma 2.1, we will show that $\mathcal{H}^\alpha \subset \mathcal{C}^0$ if $\alpha > 1$. Now here is the first main result of our paper:

Theorem 1.1. *Let $\alpha > 3$ and $K \in \mathcal{H}^\alpha$. Construct the corresponding family of Harper-like operators $\{K_b\}_{b \in \mathbb{R}}$. Then we have:*

- i. *The resolvent set $\rho(K_b)$ is stable; more precisely, if $\text{dist}(z, \sigma(K_{b_0})) \geq \epsilon > 0$ then there exist $\delta > 0$ and $\eta > 0$ such that $\text{dist}(z, \sigma(K_b)) \geq \eta$ whenever $|b - b_0| < \delta$.*
- ii. *Define $E_+(b) := \sup \sigma(K_b)$ and $E_-(b) := \inf \sigma(K_b)$. Then E_\pm are Lipschitz functions of b .*
- iii. *Let $\alpha > 4$. Assume that K_{b_0} has a gap in the spectrum of the form $(e_-(b_0), e_+(b_0))$, where $e_\pm(b_0) \in \sigma(K_{b_0})$ are the gap edges. Then as long as the gap is not closing by varying b in a closed interval I containing b_0 , the operator K_b will have a gap $(e_-(b), e_+(b))$ whose edges are Lipschitz functions of b on I .*

Remark. Denoting by $\delta b = b - b_0$, then according to our notations we have that $K_b = (K_{b_0})_{\delta b}$. It means that it is enough to prove spectral stability and Lipschitz properties near $b_0 = 0$.

We can complicate the setting by allowing the generating kernel to depend on b .

Corollary 1.2. *Assume that the generating kernel $K(\mathbf{x}, \mathbf{x}'; b)$ obeys all the spatial localization conditions of Theorem 1.1, uniformly in $b \in \mathbb{R}$. Moreover, assume that it also satisfies an extra condition:*

$$\sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} |K(\mathbf{x}, \mathbf{x}'; b) - K(\mathbf{x}, \mathbf{x}'; b_0)| \leq C |b - b_0|, \quad |b - b_0| \leq 1. \tag{1.6}$$

Consider the family $\{K_b\}_{b \in \mathbb{R}}$ generated by $e^{ib\varphi(\mathbf{x}, \mathbf{x}')}K(\mathbf{x}, \mathbf{x}'; b)$. Then Theorem 1.1 holds true for K_b .

Continuous Schrödinger operators. Let us consider the operator in $L^2(\mathbb{R}^2)$

$$H(b) := (\mathbf{p} - b\mathbf{a})^2 + V, \quad \mathbf{p} = -i\nabla_{\mathbf{x}}, \quad \mathbf{a}(\mathbf{x}) = (-x_2/2, x_1/2), \quad b \in \mathbb{R}. \quad (1.7)$$

where we assume that the scalar potential V is smooth and bounded together with all its derivatives on \mathbb{R}^2 . This very strong condition is definitely not necessary for the result given below, but it simplifies the presentation. For the same reason we formulate the result only near $b_0 = 0$.

Theorem 1.3. *Assume that the spectrum of $H(0)$ has a finite and isolated spectral band σ_0 , where $\sigma_0 = [s_-(0), s_+(0)]$. Then if $|b|$ is small enough, σ_0 will evolve into a still isolated spectral island $\sigma_b \subset \sigma(H(b))$. Denote by $s_-(b) := \inf \sigma_b$ and $s_+(b) := \sup \sigma_b$. Then these edges are Lipschitz at $b = 0$, i.e. there exists a constant C such that $|s_{\pm}(b) - s_{\pm}(0)| \leq C|b|$.*

Remark. We do not exclude the appearance of gaps inside σ_b . Moreover, the formulation of this result is slightly different from the one we gave in the discrete case. Here we look at the edges of a finite part of the spectrum, and not at the edges of a gap. In the discrete case both formulations are equivalent. However, our proof does not work in the continuous case if σ_0 is infinite.

1.1. Previous Results and Open Problems

Spectrum stability is a fundamental issue in perturbation theory. It is well known that if W is relatively bounded to H_0 , then the spectrum of $H_\lambda = H_0 + \lambda W$ is at a Hausdorff distance of order $|\lambda|$ from the spectrum of H_0 . But this is in general not true for perturbations which are not relatively bounded. And the magnetic perturbation coming from a constant field is not relatively bounded, neither in the discrete nor in the continuous case.

With the notable exception of a recent paper by Nenciu [29], all previous results on the discrete case we are aware of deal with the situation in which $\Gamma = \mathbb{Z}^2$ and the generating kernel obeys $K(\mathbf{x}, \mathbf{x}') = k(\mathbf{x} - \mathbf{x}')$, where k is sufficiently fast decaying at infinity. Maybe the first proof of spectral stability of Harper operators is due to Elliott [12]. The result is refined in [6] where it is shown that the gap boundaries are $\frac{1}{3}$ -Hölder continuous in b . Later results by Avron et al. [2,3], Helffer and Sjöstrand [17,18], and Haagerup and Rørdam [15] pushed the exponent up to $\frac{1}{2}$. In fact they prove more, they show that the Hausdorff distance between spectra behaves like $|b - b_0|^{\frac{1}{2}}$. These results are optimal in the sense that the Hölder constant is independent of the length of the eventual gaps, and it is known that these gaps can close down precisely like $|b - b_0|^{\frac{1}{2}}$ near rational values of b_0 [16,18]. Note that Nenciu [29] proves a similar result for a much larger class of Harper-like operators. Many other spectral properties of Harper operators can be found in a paper by Herrmann and Janssen [19].

In the continuous case, the stability of gaps was first shown by Avron and Simon [1], and Nenciu [28]. Nenciu's result implicitly gives a $\frac{1}{2}$ -Hölder

continuity in b for the Hausdorff distance between spectra. Then in [5] the Hölder exponent of gap edges was pushed up to $\frac{2}{3}$.

The first proof of Lipschitz continuity of gap edges for Harper-like operators was given by Bellissard [4] (later on Kotani [22] extended his method to more general regular lattices and dimensions larger than two). The configuration space is $\Gamma = \mathbb{Z}^2$ and the generating kernel is of the form $K(\mathbf{x}, \mathbf{x}') = k(\mathbf{x} - \mathbf{x}'; b)$, where $k(\mathbf{x}; b)$ decays polynomially in $|\mathbf{x}|$ and is allowed to depend smoothly on b . This extra-dependence is not central for our discussion, so we will consider that k is b independent. Bellissard’s innovative idea uses in an essential way that the Harper operators generated by translation invariant and fast decaying kernels $k(\mathbf{x} - \mathbf{x}')$ can be written as linear combinations of magnetic translations:

$$\begin{aligned}
 K_b &= \sum_{\gamma \in \mathbb{Z}^2} k(\gamma) W_b(\gamma), \\
 [W_b(\gamma)\psi](\mathbf{x}) &= e^{ib\varphi(\mathbf{x}, \gamma)} \psi(\mathbf{x} - \gamma), \\
 W_b(\gamma)W_b(\gamma') &= e^{ib\varphi(\gamma, \gamma')} W_b(\gamma + \gamma').
 \end{aligned}$$

Bellissard’s crucial observation was that the C^* algebra $\mathcal{A}_{b_0+\delta}$ generated by $\{W_{b_0+\delta}(\gamma)\}_{\gamma \in \mathbb{Z}^2}$ is isomorphic with a sub-algebra of $\mathcal{A}_{b_0} \otimes \mathcal{A}_\delta$ which is generated by $\{W_{b_0}(\gamma) \otimes W_\delta(\gamma)\}_{\gamma \in \mathbb{Z}^2}$. Thus one can construct an operator $\tilde{K}_{b_0+\delta}$ which is isospectral with $K_{b_0+\delta}$. The new operator lives in the space $l^2(\mathbb{Z}^2) \otimes L^2(\mathbb{R})$, and $\tilde{K}_{b_0} = K_{b_0} \otimes \text{Id}$. It turns out that it is more convenient to study the spectral edges of the new operator. The reason is that the singularity induced by the magnetic perturbation is hidden in the extra-dimension. But the proof breaks down in case of irregular lattices or if the generating kernel $K(\mathbf{x}, \mathbf{x}')$ is not just a function of $\mathbf{x} - \mathbf{x}'$.

Coming back to our proof, its crucial ingredient consists in expressing the magnetic phases with the help of the heat kernel of a *continuous Schrödinger operator*, see (5.8)–(5.12). Moreover, the proof in the discrete case also works for continuous kernels living on \mathbb{R}^2 and not just on lattices. This is what we use in the last step of the proof of Theorem 1.3 dealing with continuous magnetic Schrödinger operators.

A limitation of our method consists in the fact that the phases $\varphi(\mathbf{x}, \mathbf{x}')$ are generated by a constant magnetic field. A more general discrete problem was formulated by Nenciu [29] where he proposed to replace the explicit formulas in (1.4) and (1.5) with more general real and antisymmetric phases obeying $\phi(\mathbf{x}, \mathbf{x}') = \overline{\phi(\mathbf{x}', \mathbf{x})} = -\phi(\mathbf{x}', \mathbf{x})$ and

$$|\phi(\mathbf{x}, \mathbf{y}) + \phi(\mathbf{y}, \mathbf{x}') + \phi(\mathbf{x}', \mathbf{x})| \leq \text{area } \Delta(\mathbf{x}, \mathbf{y}, \mathbf{x}')$$

where $\Delta(\mathbf{x}, \mathbf{y}, \mathbf{x}')$ is the triangle generated by the three points. These phases appear very naturally in the continuous case, see [8, 9, 20, 23–27], where it is shown that if $\mathbf{a}(\mathbf{x})$ is the transverse gauge generated by a globally bounded magnetic field $|b(\mathbf{x})| \leq 1$, then $\phi(\mathbf{x}, \mathbf{x}')$ can be chosen to be the path integral of $\mathbf{a}(\mathbf{x})$ on the segment linking \mathbf{x}' with \mathbf{x} . This is the same as the magnetic flux of b through the triangle generated by \mathbf{x}, \mathbf{x}' and the origin.

Using a completely different proof method, Nenciu shows among other things in [29] that the gap edges are Lipschitz up to a logarithmic factor, and he conjectures that they are actually Lipschitz. His method relies on the theory of almost convex functions, and the result provided by this technique is optimal in the sense that it cannot be improved in order to get rid of the logarithm. A new idea would be necessary in order to prove Nenciu’s Lipschitz conjecture.

Our current paper supports this conjecture because it provides examples of phases not coming from a constant magnetic field which still generate Lipschitz gap edges. Let us show this here.

Consider an irregular lattice $\Gamma \subset \mathbb{R}^2$ which is a local deformation of \mathbb{Z}^2 , that is there exists a bijective map $F : \Gamma \rightarrow \mathbb{Z}^2$ such that $|F(\gamma) - \gamma| < \frac{1}{2}$. Define the phases $\tilde{\varphi}(\mathbf{x}, \mathbf{x}') := \varphi(F^{-1}(\mathbf{x}), F^{-1}(\mathbf{x}'))$ where φ is given by (1.4).

Choose any self-adjoint operator $K \in B(l^2(\mathbb{Z}^2))$ given by a kernel $K(\mathbf{x}, \mathbf{x}')$ sufficiently fast decaying outside the diagonal. The same operator can be seen in $B(l^2(\Gamma))$ given by $\tilde{K}(\gamma, \gamma') := K(F(\gamma), F(\gamma'))$. Thus the operator K_b generated by $K_b(\mathbf{x}, \mathbf{x}') := e^{ib\tilde{\varphi}(\mathbf{x}, \mathbf{x}')} K(\mathbf{x}, \mathbf{x}')$ is unitary equivalent with an operator in $B(l^2(\Gamma))$ with a kernel

$$\tilde{K}_b(\gamma, \gamma') := e^{ib\varphi(\gamma, \gamma')} \tilde{K}(\gamma, \gamma').$$

In this case, we know from Theorem 1.1 that the edges of the spectral gaps of \tilde{K}_b and thus K_b will have a Lipschitz behavior. But the general case remains open.

2. Proof of Theorem 1.1

This section is dedicated to the proof of our first theorem. Parts of this proof will be later on adapted to the continuous case in Theorem 1.3.

2.1. Proof of (i)

Let us start by showing the existence of natural embeddings of \mathcal{C}^α ’s in \mathcal{H}^α ’s given by the following short lemma:

Lemma 2.1. *Let $H \in \mathcal{H}^\alpha$ with $\alpha > 1$. Then $H \in \mathcal{C}^\beta$ with $\beta < \alpha - 1$. In particular, if $\alpha > 3$ then the kernel $\langle \mathbf{x} - \mathbf{x}' \rangle^2 |H(\mathbf{x}, \mathbf{x}')|$ obeys a Schur–Holmgren estimate and thus defines a bounded operator.*

Proof. Choose some small enough $\epsilon > 0$ such that $\alpha > \beta + 1 + \epsilon$. We write:

$$\langle \mathbf{x} - \mathbf{x}' \rangle^\beta |H(\mathbf{x}, \mathbf{x}')| \leq \langle \mathbf{x} - \mathbf{x}' \rangle^{-1-\epsilon} \langle \mathbf{x} - \mathbf{x}' \rangle^\alpha |H(\mathbf{x}, \mathbf{x}')|$$

and see that the Cauchy–Schwarz inequality gives

$$\|H\|_{\mathcal{C}^\beta} \leq C_{\alpha, \beta} \|H\|_{\mathcal{H}^\alpha} . \tag{2.1}$$

□

Another technical estimate to be proved in “Appendix” claims that if H has a kernel which is localized near the diagonal, then the resolvent’s kernel will also have such a localization.

Proposition 2.2. *Let $H \in \mathcal{C}^\alpha$, with $\alpha > 0$. Let $z \in \rho(H)$. Then for every $0 \leq \alpha' < \alpha$ we have $(H - z)^{-1} \in \mathcal{H}^{\alpha'}$, and there exists a constant C independent of z such that*

$$\begin{aligned} & \| (H - z)^{-1} \|_{\mathcal{H}^{\alpha'}} \\ & \leq C (1 + \|H\|_{\mathcal{C}^{\alpha+1}}) \left(\frac{1}{\{\text{dist}(z, \sigma(H))\}^{\alpha+2}} + \frac{1}{\text{dist}(z, \sigma(H))} \right). \end{aligned} \tag{2.2}$$

Remark. This proposition is related to what specialists in von Neumann algebras would call dual action, see [30–32]. Stronger localization results have been earlier obtained by Jaffard [21], later generalized by Gröchenig and Leinert [14]. We choose for completeness to give an elementary proof in “Appendix”; our proof also highlights the uniformity in $z \in \rho(H)$.

Now let us start the proof of (i). Constants only depending on ϵ will be named C_ϵ even though they might have different values.

Remember that it is enough to prove the stability result near $b_0 = 0$. Let $K \in \mathcal{H}^\alpha$ with $\alpha > 3$. Lemma 2.1 gives us some $\beta > 2$ such that $K \in \mathcal{C}^\beta$. Proposition 2.2 says that $(K - z)^{-1} \in \mathcal{H}^{\beta'}$ with some $2 < \beta' < \beta$, while Lemma 2.1 insures that there exists $\gamma > 1$ such that $(K - z)^{-1} \in \mathcal{C}^\gamma$.

Denote by $G(\mathbf{x}, \mathbf{x}'; z)$ the kernel of $(K - z)^{-1}$. From (2.1) and (2.2), we obtain a constant C_ϵ such that:

$$\sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} \langle \mathbf{x} - \mathbf{x}' \rangle |G(\mathbf{x}, \mathbf{x}'; z)| \leq C_\epsilon \quad \text{if } \text{dist}(z, \sigma(K)) \geq \epsilon. \tag{2.3}$$

Define the operator $S_b(z)$ to be the one corresponding to the kernel $e^{ib\varphi(\mathbf{x}, \mathbf{x}')} G(\mathbf{x}, \mathbf{x}'; z)$. Using the Schur–Holmgren criterion we can write

$$\|S_b(z)\| \leq C_\epsilon, \quad b \in \mathbb{R}, \quad \text{dist}(z, \sigma(K)) \geq \epsilon.$$

Using (1.5) we can write:

$$(K_b - z)S_b(z) =: 1 + T_b(z), \tag{2.4}$$

where $T_b(z)$ is given by the kernel

$$e^{ib\varphi(\mathbf{x}, \mathbf{x}')} \sum_{\mathbf{y} \in \Gamma} (e^{ib\varphi(\mathbf{x}-\mathbf{y}, \mathbf{x}'-\mathbf{y})} - 1) K(\mathbf{x}, \mathbf{y}) G(\mathbf{y}, \mathbf{x}'; z). \tag{2.5}$$

Note that

$$|e^{ib\varphi(\mathbf{x}-\mathbf{y}, \mathbf{x}'-\mathbf{y})} - 1| \leq |b| |\varphi(\mathbf{x} - \mathbf{y}, \mathbf{x}' - \mathbf{y})| \leq \frac{|b|}{2} |\mathbf{x} - \mathbf{y}| |\mathbf{y} - \mathbf{x}'|. \tag{2.6}$$

Then for any $f \in l^2(\Gamma)$ with compact support we can write:

$$|T_b(z)f|(\mathbf{x}) \leq |b| \sum_{\mathbf{y} \in \Gamma} |\mathbf{x} - \mathbf{y}| |K(\mathbf{x}, \mathbf{y})| |\mathbf{y} - \mathbf{x}'| |G(\mathbf{y}, \mathbf{x}'; z)| |f(\mathbf{x}')| \tag{2.7}$$

and after applying the Schur–Holmgren criterion we get:

$$\|T_b(z)\| \leq |b| \|K\|_{\mathcal{C}^1} \|(K - z)^{-1}\|_{\mathcal{C}^1} \leq |b| C_\epsilon.$$

Thus if $|b|$ is small enough, $\|T_b(z)\| \leq 1/2$ whenever $\text{dist}(z, \sigma(K)) \geq \epsilon$. Now if $\text{Im}z \neq 0$ we know that $K_b - z$ is invertible, and from (2.4) we conclude that there exists a constant C_ϵ such that

$$\begin{aligned} (K_b - z)^{-1} &= S_b(z) (1 + T_b(z))^{-1}, \\ \|(K_b - z)^{-1}\| &\leq C_\epsilon \quad \text{whenever } |b| \leq b_\epsilon \text{ and } \text{dist}(z, \sigma(K)) \geq \epsilon, \end{aligned} \tag{2.8}$$

uniformly in the imaginary part of z . This means that $\text{dist}(z, \sigma(K_b)) \geq \frac{1}{C_\epsilon} > 0$ whenever $|b| \leq b_\epsilon$ and $\text{dist}(z, \sigma(K)) \geq \epsilon$, and the proof of (i) is over. \square

2.2. Proof of (ii)

As before, we only need to consider $b_0 = 0$. We give the proof just for the upper spectral limit E_+ , since the argument for E_- is similar.

2.2.1. Reduction to Localized Operators. We start with an abstract lemma.

Lemma 2.3. *Let $M(b)$ and $N(b)$ be two families of bounded and self-adjoint operators on some Hilbert space \mathcal{H} , such that $\|M(b) - N(b)\| \leq C |b|$ if $|b| \leq 1$. Then:*

$$|\sup \sigma(M(b)) - \sup \sigma(N(b))| \leq \|M(b) - N(b)\| \leq C |b|, \quad |b| \leq 1, \tag{2.9}$$

and a similar estimate holds for the infimum of their spectra. In particular, if $\sup \sigma(N(b))$ is Lipschitz at $b = 0$ then the same is true for $\sup \sigma(M(b))$.

Remark. Note the important thing that we do not require from $M(b)$ and $N(b)$ to converge in norm to $M(0) = N(0)$ when b tends to zero.

Proof. For every $\psi \in \mathcal{H}$ with $\|\psi\| = 1$ we can write

$$\langle M(b)\psi, \psi \rangle \leq \langle N(b)\psi, \psi \rangle + \|M(b) - N(b)\| \leq \sup \sigma(N(b)) + \|M(b) - N(b)\|$$

which means that $\sup \sigma(M(b)) - \sup \sigma(N(b)) \leq \|M(b) - N(b)\|$. By interchanging $M(b)$ with $N(b)$ we obtain the inequality:

$$|\sup \sigma(M(b)) - \sup \sigma(N(b))| \leq \|M(b) - N(b)\|. \tag{2.10}$$

A similar argument shows the same estimate for the infimum of the spectra. Regarding the Lipschitz property, we use that $\sup \sigma(M(0)) = \sup \sigma(N(0))$ and then we apply the triangle inequality:

$$\begin{aligned} &|\sup \sigma(M(b)) - \sup \sigma(M(0))| \\ &\leq |\sup \sigma(N(b)) - \sup \sigma(N(0))| + \|M(b) - N(b)\| \leq C |b|. \end{aligned} \tag{2.11}$$

\square

Getting back to our theorem, we now want to reduce the problem to operators with kernels supported near the diagonal. Denote by χ the characteristic function of the interval $[0, 1]$. Denote by \widehat{K}_b the operator given by the kernel $\widehat{K}_b(\mathbf{x}, \mathbf{x}') := \chi(\sqrt{|b|}|\mathbf{x} - \mathbf{x}'|)K(\mathbf{x}, \mathbf{x}')$ and by \widetilde{K}_b the operator given by $\widetilde{K}_b(\mathbf{x}, \mathbf{x}') := \chi(\sqrt{|b|}|\mathbf{x} - \mathbf{x}'|)e^{ib\varphi(\mathbf{x}, \mathbf{x}')}K(\mathbf{x}, \mathbf{x}')$.

Since $K \in \mathcal{H}^\alpha$ with $\alpha > 3$, according to Lemma 2.1 we have the bound:

$$\sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} |\mathbf{x} - \mathbf{x}'|^2 |K(\mathbf{x}, \mathbf{x}')| = \|K\|_{\mathcal{C}^2} < \infty. \tag{2.12}$$

Via the Schur–Holmgren criterion we obtain:

$$\begin{aligned} & \max\{\|K - \widehat{K}_b\|, \|K_b - \widetilde{K}_b\|\} \\ & \leq \sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} \left[1 - \chi(\sqrt{b}|\mathbf{x} - \mathbf{x}'|) \right] |K(\mathbf{x}, \mathbf{x}')| \leq |b| \|K\|_{C^2}. \end{aligned} \tag{2.13}$$

Using Lemma 2.3 for the pair K and \widehat{K}_b we obtain $|E_+(0) - \sup(\sigma(\widehat{K}_b))| \leq |b| \|K\|_{C^2}$. The same lemma for the pair K_b and \widetilde{K}_b gives $|E_+(b) - \sup(\sigma(\widetilde{K}_b))| \leq |b| \|K\|_{C^2}$. Then the triangle inequality leads to:

$$|E_+(b) - E_+(0)| \leq 2|b| \|K\|_{C^2} + |\sup(\sigma(\widetilde{K}_b)) - \sup(\sigma(\widehat{K}_b))|. \tag{2.14}$$

Thus we have reduced the problem to the study of the spectral edges of \widetilde{K}_b and \widehat{K}_b .

2.2.2. Study of the Operators with Cut-Off. Clearly, $\widetilde{K}_b(\mathbf{x}, \mathbf{x}') = e^{ib\varphi(\mathbf{x}, \mathbf{x}')} \widehat{K}_b(\mathbf{x}, \mathbf{x}')$. Without loss, assume that $b > 0$. Take $\psi \in l^2(\Gamma)$ with compact support and compute (use (5.10) in the second equality):

$$\begin{aligned} \langle \widetilde{K}_b \psi, \psi \rangle &= \sum_{\mathbf{x}, \mathbf{x}' \in \Gamma} e^{ib\varphi(\mathbf{x}, \mathbf{x}')} \widehat{K}_b(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}') \overline{\psi(\mathbf{x})} \\ &= \int_{\mathbf{R}^2} d\mathbf{y} \sum_{\mathbf{x}, \mathbf{x}' \in \Gamma} \psi(\mathbf{x}') \overline{\psi(\mathbf{x})} \frac{4\pi \sinh(2bt)}{b} \widehat{K}_b(\mathbf{x}, \mathbf{x}') \\ &\quad \times \exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2bt)} \right] G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t). \end{aligned} \tag{2.15}$$

Now denote by $A_b(t)$ the operator with kernel

$$\begin{aligned} A_b(\mathbf{x}, \mathbf{x}'; t) &:= \widehat{K}_b(\mathbf{x}, \mathbf{x}') \exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2bt)} \right] \\ &= K(\mathbf{x}, \mathbf{x}') \times \exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2bt)} \right] \chi(\sqrt{b}|\mathbf{x} - \mathbf{x}'|). \end{aligned}$$

The crucial observation is that Eq. (2.15) leads to:

$$\begin{aligned} \langle \widetilde{K}_b \psi, \psi \rangle &= \int_{\mathbf{R}^2} d\mathbf{y} \langle A_b(t) G_b(\mathbf{y}, \cdot; t) \psi, G_b(\mathbf{y}, \cdot; t) \psi \rangle \frac{4\pi \sinh(2bt)}{b} \\ &\leq \sup \sigma(A_b(t)) \frac{4\pi \sinh(2bt)}{b} \int_{\mathbf{R}^2} d\mathbf{y} \|G_b(\mathbf{y}, \cdot; t) \psi\|^2 \\ &= \sup \sigma(A_b(t)) \frac{4\pi \sinh(2bt)}{b} \int_{\mathbf{R}^2} d\mathbf{y} \sum_{\mathbf{x} \in \Gamma} |G_b(\mathbf{y}, \mathbf{x}; t)|^2 |\psi(\mathbf{x})|^2 \\ &= \sup \sigma(A_b(t)) \|\psi\|^2, \end{aligned} \tag{2.16}$$

where in the last line we used (5.12). It means that $\sup \sigma(\widetilde{K}_b) \leq \sup(\sigma(A_b(t)))$ for all t . Now let us show that the operator $A_b(t) - \widehat{K}_b$ has a norm proportional

with b if t is large enough (say $t = b^{-1}$). Indeed, we can write

$$\begin{aligned}
 |A_b(\mathbf{x}, \mathbf{x}'; b^{-1}) - \widehat{K}_b(\mathbf{x}, \mathbf{x}')| &\leq |K(\mathbf{x}, \mathbf{x}')| \chi(\sqrt{b}|\mathbf{x} - \mathbf{x}'|) \left(\exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2)} \right] - 1 \right) \\
 &\leq |K(\mathbf{x}, \mathbf{x}')| \chi(\sqrt{b}|\mathbf{x} - \mathbf{x}'|) \frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2)} \exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2)} \right]
 \end{aligned} \tag{2.17}$$

and on the support of χ we can bound the above difference with:

$$|A_b(\mathbf{x}, \mathbf{x}'; b^{-1}) - \widehat{K}_b(\mathbf{x}, \mathbf{x}')| \leq \text{const } b |\mathbf{x} - \mathbf{x}'|^2 |K(\mathbf{x}, \mathbf{x}')|. \tag{2.18}$$

The right-hand side defines an operator whose norm behaves like b . Thus (2.16) and (2.18) imply:

$$\sup \sigma(\widetilde{K}_b) \leq \sup \sigma(A_b(b^{-1})) \quad \text{and} \quad \|A_b(b^{-1}) - \widehat{K}_b\| \leq C b. \tag{2.19}$$

Using (2.10) for the pair $A_b(b^{-1})$ and \widehat{K}_b we arrive at:

$$\sup \sigma(\widetilde{K}_b) \leq \sup \sigma(\widehat{K}_b) + C b. \tag{2.20}$$

We now want to change places between \widetilde{K}_b and \widehat{K}_b in the above inequality, which would lead to $\sup \sigma(\widehat{K}_b) \leq \sup \sigma(\widetilde{K}_b) + C b$ and thus:

$$|\sup \sigma(\widetilde{K}_b) - \sup \sigma(\widehat{K}_b)| \leq C b,$$

which together with (2.14) would imply:

$$|E_+(b) - E_+(0)| \leq C b, \quad b \geq 0.$$

The key step in the proof of (2.20) was (2.15). Since $\widehat{K}_b(\mathbf{x}, \mathbf{x}') = e^{-ib\varphi(\mathbf{x}, \mathbf{x}')} \widetilde{K}_b(\mathbf{x}, \mathbf{x}')$ we can write (use (5.11) in the second line):

$$\begin{aligned}
 \langle \widehat{K}_b \psi, \psi \rangle &= \sum_{\mathbf{x}, \mathbf{x}' \in \Gamma} e^{-ib\varphi(\mathbf{x}, \mathbf{x}')} \widetilde{K}_b(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}') \overline{\psi(\mathbf{x})} \\
 &= \int_{\mathbf{R}^2} d\mathbf{y} \sum_{\mathbf{x}, \mathbf{x}' \in \Gamma} \psi(\mathbf{x}') \overline{\psi(\mathbf{x})} \frac{4\pi \sinh(2bt)}{b} \widetilde{K}_b(\mathbf{x}, \mathbf{x}') \\
 &\quad \times \exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2bt)} \right] G_b(\mathbf{x}', \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}; t).
 \end{aligned} \tag{2.21}$$

Now everything will work as before, because the phase $e^{ib\varphi(\mathbf{x}, \mathbf{x}')}$ changes neither the localization nor the \mathcal{C}^2 norm of the operators. The proof for the upper spectral edges is over.

The proof for the lower spectral edges is based on an estimate which is very similar with (2.16), in which we reverse the inequality and show that $\inf \sigma(\widetilde{K}_b) \geq \inf \sigma(A_b(t))$ for all t . We give no further details.

2.3. Proof of (iii)

The idea is to reduce the problem to the previous case. Again it is enough to consider $b_0 = 0$ and $b > 0$ small enough. Assume that K has a gap in its spectrum of the form (e_-, e_+) , with $e_{\pm} \in \sigma(K)$. Then due to (i) we know that if b is small enough the gap will survive: we can choose a positively oriented

circle L in the complex plane containing $\Sigma_+(b) := \sigma(K_b) \cap [e_+(b), \infty)$ such that

$$\text{dist}(z, \sigma(K_b)) \geq \eta > 0 \quad \text{whenever } z \in L \text{ and } 0 < b < b_\eta.$$

The orthogonal projector P_b corresponding to $\Sigma_+(b)$ can be written as a Riesz integral and we have:

$$P_b := \frac{i}{2\pi} \int_L (K_b - z)^{-1} dz, \quad K_b P_b = \frac{i}{2\pi} \int_L z (K_b - z)^{-1} dz, \quad b \geq 0. \quad (2.22)$$

If we consider $K_b P_b$ as an operator living on the whole space $l^2(\Gamma)$, then its spectrum is given by the union $\{0\} \cup \Sigma_+(b)$. If we choose $\lambda := 1 + \sup \sigma(K)$, then for b small enough the operator $D_b := K_b P_b - \lambda P_b$ will have $\inf \sigma(D_b) = e_+(b) - \lambda \leq -1/2$. Thus $e_+(b) = \lambda + \inf \sigma(D_b)$, hence $e_+(b)$ is Lipschitz at $b = 0$ if $\inf \sigma(D_b)$ has the same property. This is what we prove next:

Lemma 2.4. *Let $D_b = K_b P_b - \lambda P_b$ with $\lambda := 1 + \sup \sigma(K)$. Then there exists $b_1 > 0$ small enough and a constant $C > 0$ such that for every $0 < b < b_1$ we have $|\inf \sigma(D_b) - \inf \sigma(D_0)| \leq C b$.*

Proof. Remember that we imposed $\alpha > 4$. We have that $\|K_b\|_{\mathcal{H}^\alpha} = \|K\|_{\mathcal{H}^\alpha} < \infty$ for all b . According to Lemma 2.1, there exists $\beta > 3$ such that $\|K_b\|_{\mathcal{C}^\beta} = \|K\|_{\mathcal{C}^\beta} < \infty$. Then if b is smaller than some constant only depending on L , Proposition 2.2 tells us that $(K_b - z)^{-1} \in \mathcal{H}^{\beta'}$ for some $3 < \beta' < \beta$, for all $z \in L$ and $\sup_{z \in L} \|(K_b - z)^{-1}\|_{\mathcal{H}^{\beta'}} \leq C$. Thus both P_b and D_b belong to $\mathcal{H}^{\beta'}$ with $\beta' > 3$ if b is small enough. More precisely, there exists $b_2 > 0$ sufficiently small such that

$$\max\{\|P_b\|_{\mathcal{H}^{\beta'}}, \|D_b\|_{\mathcal{H}^{\beta'}}\} \leq C, \quad 0 \leq b \leq b_2. \quad (2.23)$$

If $G(\mathbf{x}, \mathbf{x}'; z)$ is the integral kernel of $(K - z)^{-1}$, then we introduced at point (i) the operator $S_b(z)$ given by the kernel $e^{ib\varphi(\mathbf{x}, \mathbf{x}')} G(\mathbf{x}, \mathbf{x}'; z)$. Using (2.8) we can write:

$$\sup_{z \in L} \|(K_b - z)^{-1} - S_b(z)\| \leq C b, \quad (2.24)$$

provided b is small enough. Denoting by D_0 the operator given by the integral kernel

$$D_0(\mathbf{x}, \mathbf{x}') := \frac{i}{2\pi} \int_L (z - \lambda) G(\mathbf{x}, \mathbf{x}'; z) dz$$

and by $(D_0)_b$ the operator generated by $e^{ib\varphi(\mathbf{x}, \mathbf{x}')} D_0(\mathbf{x}, \mathbf{x}')$, then using (2.24) we arrive at the estimate:

$$\|D_b - (D_0)_b\| \leq C b \quad \text{whenever } 0 \leq b < b_2. \quad (2.25)$$

It follows from Lemma 2.3 that $\inf \sigma(D_b)$ is Lipschitz at $b = 0$ if $\inf \sigma((D_0)_b)$ has the same property. But for the operator $(D_0)_b$ we can apply point (ii), and the proof is over. \square

3. Proof of Corollary 1.2

In order to keep the notation simple, we will only consider $b_0 = 0$. Here the generating kernel $K(\mathbf{x}, \mathbf{x}'; b)$ depends on b and (1.6) at $b_0 = 0$ reads as:

$$\sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} |K(\mathbf{x}, \mathbf{x}'; b) - K(\mathbf{x}, \mathbf{x}'; 0)| \leq C |b|, \quad |b| \leq 1. \tag{3.1}$$

Let us introduce the family \tilde{K}_b where their kernels are given by $e^{ib\varphi(\mathbf{x}, \mathbf{x}')} K(\mathbf{x}, \mathbf{x}'; 0)$. Clearly, $K_0 = \tilde{K}_0$. Moreover, (3.1) implies that $\|K_b - \tilde{K}_b\| \leq C |b|$ around $b = 0$. We know that Theorem 1.1 (ii) applies for \tilde{K}_b around $b = 0$, so the only thing we have left is to extend it to K_b . From Lemma 2.3 we immediately conclude that $\sup \sigma(K_b)$ and $\inf \sigma(K_b)$ are Lipschitz at $b = 0$.

The spectral stability of K_b can be shown with the same strategy as the one used in (2.3)–(2.8). The operator $S_b(z)$ must be constructed starting from the kernel of $(K_0 - z)^{-1}$ which gets multiplied with the phase $e^{ib\varphi(\mathbf{x}, \mathbf{x}')}$. When we act with $K_b - z$ on $S_b(z)$ as in (2.4), we obtain an extra term which enters in $T_b(z)$, which is $(K_b - \tilde{K}_b)S_b(z)$. This error is again proportional with $|b|$ if z is at some distance from the spectrum of K_0 . Thus (2.8) holds again.

For the case of gaps, the proof is identical with the case independent of b . □

4. Proof of Theorem 1.3

There are important similarities between the proof strategies in the discrete and continuous cases. Although the stability of the resolvent set of $H(b)$ is known, we will sketch a short proof which will also provide some ingredients for the proof of the Lipschitz behavior of the band edges.

4.1. Stability of Gaps

Assume that $M \subset \rho(H(0))$ is a compact set and $\text{dist}(M, \sigma(H(0))) > 0$. The resolvent $(H(0) - z)^{-1}$ is an integral operator given by an integral kernel $Q_0(\mathbf{x}, \mathbf{x}'; z)$.

The singularities of $Q_0(\mathbf{x}, \mathbf{x}'; z)$ are the same as in the case of the free Laplacian and there exists some $\delta > 0$ and $C_M < \infty$ such that uniformly in $\mathbf{x} \neq \mathbf{x}'$ [13, 11]:

$$\begin{aligned} \sup_{z \in M} |Q_0(\mathbf{x}, \mathbf{x}'; z)| &\leq C_M (1 + |\ln(|\mathbf{x} - \mathbf{x}'|)|) e^{-\delta|\mathbf{x} - \mathbf{x}'|}, \\ \sup_{z \in M} |\nabla_{\mathbf{x}} Q_0(\mathbf{x}, \mathbf{x}'; z)| &\leq C_M \left(1 + \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) e^{-\delta|\mathbf{x} - \mathbf{x}'|}. \end{aligned} \tag{4.1}$$

In particular we have the following Schur–Holmgren type property:

$$\sup_{z \in M} \sup_{\mathbf{x}' \in \mathbb{R}^2} \int_{\mathbb{R}^2} |Q_0(\mathbf{x}, \mathbf{x}'; z)| d\mathbf{x} \leq C(M) < \infty. \tag{4.2}$$

This allows us to define for every $z \in M$ a bounded operator $S_b(z)$ whose integral kernel is given by:

$$S_b(\mathbf{x}, \mathbf{x}'; z) := e^{ib\varphi(\mathbf{x}, \mathbf{x}')} Q_0(\mathbf{x}, \mathbf{x}'; z), \quad \sup_{z \in M} \|S_b(z)\| \leq C(M) < \infty. \quad (4.3)$$

Define $T_b(z)$ to be the operator with the integral kernel:

$$T_b(\mathbf{x}, \mathbf{x}'; z) := be^{ib\varphi(\mathbf{x}, \mathbf{x}')} \{2i\mathbf{a}(\mathbf{x} - \mathbf{x}') \nabla_{\mathbf{x}} Q_0(\mathbf{x}, \mathbf{x}'; z) + b|\mathbf{a}(\mathbf{x} - \mathbf{x}')|^2 Q_0(\mathbf{x}, \mathbf{x}'; z)\}. \quad (4.4)$$

The kernel $T_b(\mathbf{x}, \mathbf{x}'; z)$ is bounded because the inequality $|\mathbf{a}(\mathbf{x} - \mathbf{x}')| \leq |\mathbf{x} - \mathbf{x}'|$ compensates the local singularities of $\nabla_{\mathbf{x}} Q_0(\mathbf{x}, \mathbf{x}'; z)$ and $Q_0(\mathbf{x}, \mathbf{x}'; z)$ when $|\mathbf{x} - \mathbf{x}'|$ is small, while when $|\mathbf{x} - \mathbf{x}'|$ is large we have the exponential decay which comes into play. In fact, using (4.1) we see that the kernel $T_b(\mathbf{x}, \mathbf{x}'; z)$ obeys a Schur-Holmgren estimate. We get:

$$\sup_{z \in M} \|T_b(z)\| \leq C(M) |b|, \quad |b| \leq 1. \quad (4.5)$$

Note the important identity valid on Schwartz functions:

$$\{-i\nabla_{\mathbf{x}} - b\mathbf{a}(\mathbf{x})\}e^{ib\varphi(\mathbf{x}, \mathbf{x}')} = e^{ib\varphi(\mathbf{x}, \mathbf{x}')} \{-i\nabla_{\mathbf{x}} - b\mathbf{a}(\mathbf{x} - \mathbf{x}')\}. \quad (4.6)$$

Let us note that $S_b(z)$ leaves the Schwartz space invariant and for such two functions f and g we have (using (4.6)):

$$\begin{aligned} & \langle \{(\mathbf{p} - b\mathbf{a})^2 + V - z\} S_b(z) f, g \rangle \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{ib\varphi(\mathbf{x}, \mathbf{x}')} (\{-i\nabla_{\mathbf{x}} - b\mathbf{a}(\mathbf{x} - \mathbf{x}')\}^2 + V(\mathbf{x}) - z) \\ & \quad \times Q_0(\mathbf{x}, \mathbf{x}'; z) f(\mathbf{x}') \overline{g(\mathbf{x})} \, d\mathbf{x} d\mathbf{x}' \\ &= \langle f, g \rangle + \langle T_b(z) f, g \rangle. \end{aligned} \quad (4.7)$$

The operator $H(b)$ is essentially self-adjoint on the Schwartz space, and after a density argument we conclude that the range of $S_b(z)$ is contained in the domain of $H(b)$ and $(H(b) - z)S_b(z) = 1 + T_b(z)$. Now there exists $b_1 > 0$ small enough such that if $|b| \leq b_1$ we have $\sup_{z \in M} \|T_b(z)\| \leq 1/2$ [see (4.5)]. Then after a standard argument we conclude

$$\begin{aligned} & (H(b) - z)^{-1} = S_b(z)(1 + T_b(z))^{-1}, \\ & \sup_{z \in M} \|(H(b) - z)^{-1}\| \leq C_M, \quad |b| \leq b_1. \end{aligned} \quad (4.8)$$

This means that the gaps in the spectrum of $H(0)$ are preserved. In particular, for every $\epsilon > 0$ there exists $b_2(\epsilon) > 0$ such that:

$$s_-(0) - \epsilon \leq s_-(b) \leq s_+(b) \leq s_+(0) + \epsilon \quad \text{whenever } |b| \leq b_2(\epsilon). \quad (4.9)$$

Choose a positively oriented circle L isolated from $\sigma(H(0))$ such that L completely contains the finite band σ_0 . Then if $|b|$ is small enough L will completely contain σ_b and remain separated from $\sigma(H(b))$.

4.2. The Reduction to Harper-like Operators

As in the discrete case, we construct the Riesz integrals

$$P_b := \frac{i}{2\pi} \int_L (H(b) - z)^{-1} dz, \quad K(b) := H(b)P_b = \frac{i}{2\pi} \int_L z(H(b) - z)^{-1} dz.$$

The operator $H(b)P_b$ seen in the whole space $L^2(\mathbb{R}^2)$ will have the spectrum $\sigma_b \cup \{0\}$. Fix $\lambda_+ := 1 - s_-(0)$. If $|b| \leq b_2(\frac{1}{2})$ [see (4.9)], then we know that $s_+(b) + \lambda_+ \geq s_-(b) + \lambda_+ \geq \frac{1}{2} > 0$. It means that

$$s_+(b) + \lambda_+ = \sup \sigma\{H(b)P_b + \lambda_+P_b\}.$$

Similarly, choosing $\lambda_- := -1 - s_+(0)$ we have $s_-(b) + \lambda_- < -\frac{1}{2} < 0$ hence

$$s_-(b) + \lambda_- = \inf \sigma\{H(b)P_b + \lambda_-P_b\}.$$

In other words, the band edges s_{\pm} will be Lipschitz at $b = 0$ if the spectral edges of the operators

$$K_{\pm}(b) := H(b)P_b + \lambda_{\pm}P_b = \frac{i}{2\pi} \int_L (z + \lambda_{\pm})(H(b) - z)^{-1} dz$$

have the same property. Note that the operator $K_{\pm}(0)$ has an integral kernel given by:

$$K_{\pm}(0)(\mathbf{x}, \mathbf{x}') = \frac{i}{2\pi} \int_L (z + \lambda_{\pm})Q_0(\mathbf{x}, \mathbf{x}'; z) dz, \quad |K_{\pm}(0)(\mathbf{x}, \mathbf{x}')| \leq Ce^{-\delta|\mathbf{x}-\mathbf{x}'|}, \tag{4.10}$$

where the local singularity at $\mathbf{x} = \mathbf{x}'$ disappears due to the integral with respect to z .

Now using (4.8), (4.3) and (4.5) we have:

$$\left\| K_{\pm}(b) - \frac{i}{2\pi} \int_L (z + \lambda_{\pm})S_b(z) dz \right\| \leq C |b|. \tag{4.11}$$

According to Lemma 2.3, the spectral edges of $K_{\pm}(b)$ are Lipschitz at $b = 0$ if the same property is true for

$$(K_{\pm}(0))_b := \frac{i}{2\pi} \int_L (z + \lambda_{\pm})S_b(z) dz.$$

This notation wants to highlight the fact that $(K_{\pm}(0))_b$ is given by the integral kernel

$$(K_{\pm}(0))_b(\mathbf{x}, \mathbf{x}') := e^{ib\varphi(\mathbf{x}, \mathbf{x}')} K_{\pm}(0)(\mathbf{x}, \mathbf{x}').$$

At this point we are in a situation which is completely similar to the discrete case, with the difference that the Hilbert space is $L^2(\mathbb{R}^2)$ and the sums over Γ have to be replaced by integrals. The unperturbed kernel $K_{\pm}(0)(\mathbf{x}, \mathbf{x}')$ has an exponential localization.

We can mimic the proof of Theorem 1.1 (ii) and conclude that the spectral edges of $(K_{\pm}(0))_b$ are Lipschitz at $b = 0$, and we are done. \square

Acknowledgements

H.C. acknowledges support from the Danish F.N.U. grant *Mathematical Physics*. The author is deeply indebted to Gheorghe Nenciu for his encouragements and for many fruitful discussions.

5. Appendix

5.1. Proof of Proposition 2.2

Denote by $G(\mathbf{x}, \mathbf{x}'; z)$ the integral kernel of $(H - z)^{-1}$. If $\alpha' = 0$ we have

$$\sum_{\mathbf{x} \in \Gamma} |G(\mathbf{x}, \mathbf{x}'; z)|^2 = \|(H - z)^{-1} \delta_{\mathbf{x}'}\|^2 \leq \frac{1}{\{\text{dist}(z, \sigma(H))\}^2}$$

uniformly in \mathbf{x}' , an estimate which is in fact much better than (2.2). So from now on we may assume that $0 < \alpha' < \alpha$.

For $\mathbf{k} \in \mathbb{R}^2$ define the unitary multiplication operator $U_{\mathbf{k}}$ by $(U_{\mathbf{k}}f)(\mathbf{x}) = e^{i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x})$. Define the family of isospectral operators $H_{\mathbf{k}} = U_{\mathbf{k}} H U_{\mathbf{k}}^*$, with integral kernels given by $H_{\mathbf{k}}(\mathbf{x}, \mathbf{x}') = e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} H(\mathbf{x}, \mathbf{x}')$. We need the following technical result:

Lemma 5.1. *Let H be an element of C^α . Let n be the integer part of α . Then the mapping*

$$\mathbb{R}^2 \ni \mathbf{k} \mapsto H_{\mathbf{k}} \in B(l^2(\Gamma))$$

is n times continuously differentiable in the norm topology. Moreover, any n 'th order mixed partial derivative of $H_{\mathbf{k}}$ is $\alpha - n$ Hölder continuous at $\mathbf{k} = 0$ in the norm topology.

Proof. Assume that $\mathbf{k} = (k_1, k_2)$. The integral kernel of $H_{\mathbf{k}}$ is $e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} H(\mathbf{x}, \mathbf{x}')$. Let n be the integer part of α . Then $H_{\mathbf{k}}$ is n times differentiable in the norm topology with respect to k_j , $j \in \{1, 2\}$, and its n 'th mixed partial derivative $\partial_{k_1}^m \partial_{k_2}^{n-m} H_{\mathbf{k}}$ is given by the integral kernel $i^n (x_1 - x'_1)^m (x_2 - x'_2)^{n-m} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} H(\mathbf{x}, \mathbf{x}')$. This integral kernel defines a bounded operator because $|(x_1 - x'_1)^m (x_2 - x'_2)^{n-m}| \leq \langle \mathbf{x} - \mathbf{x}' \rangle^n$ and then we can use (1.2).

For the Hölder continuity statement, we use the estimate $|e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} - 1| \leq 2^{1-\beta} |\mathbf{k}|^\beta |\mathbf{x} - \mathbf{x}'|^\beta$ which holds for every $0 \leq \beta \leq 1$. □

Now let $z \in \rho(H)$. Denote by $G_{\mathbf{k}}(\mathbf{x}, \mathbf{x}'; z)$ the integral kernel of $(H_{\mathbf{k}} - z)^{-1}$. Due to the identity $U_{\mathbf{k}}(H - z)^{-1}U_{\mathbf{k}}^* = (H_{\mathbf{k}} - z)^{-1}$ we have:

$$G_{\mathbf{k}}(\mathbf{x}, \mathbf{x}'; z) = e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} G(\mathbf{x}, \mathbf{x}'; z). \tag{5.1}$$

Let us denote by n the integer part of α . We can suppose that $n \geq 1$ since the case $0 < \alpha < 1$ is covered by the argument below.

From the identity

$$(H_{\mathbf{k}'} - z)^{-1} - (H_{\mathbf{k}} - z)^{-1} = -(H_{\mathbf{k}'} - z)^{-1} [H_{\mathbf{k}'} - H_{\mathbf{k}}] (H_{\mathbf{k}} - z)^{-1} \tag{5.2}$$

and from Lemma 5.1 we conclude that the map

$$\mathbb{R}^2 \ni \mathbf{k} \mapsto (H_{\mathbf{k}} - z)^{-1} \in B(l^2(\Gamma))$$

is continuous in the norm topology, and also differentiable. We have:

$$D_{\mathbf{k}}(H_{\mathbf{k}} - z)^{-1} = -(H_{\mathbf{k}} - z)^{-1}[D_{\mathbf{k}}H_{\mathbf{k}}](H_{\mathbf{k}} - z)^{-1}. \tag{5.3}$$

Using this identity at $\mathbf{k} = 0$ in (5.1) leads to:

$$(\mathbf{x} - \mathbf{x}')G(\mathbf{x}, \mathbf{x}'; z) = -\langle (H - z)^{-1}[D_{\mathbf{k}}H_{\mathbf{k}}]_{\mathbf{k}=0}(H - z)^{-1}\delta_{\mathbf{x}'}, \delta_{\mathbf{x}} \rangle$$

which gives:

$$\|(H - z)^{-1}\|_{\mathcal{H}^1} \leq C (1 + \|H\|_{\mathcal{C}^1}) \left(\frac{1}{\text{dist}(z, \sigma(H))^2} + \frac{1}{\text{dist}(z, \sigma(H))} \right).$$

This is true because we have the pointwise bound

$$\begin{aligned} \langle \mathbf{x} - \mathbf{x}' \rangle^{2\alpha} &\leq (1 + |x_1 - x'_1| + |x_2 - x'_2|)^{2\alpha} \leq (3 \max\{1, |x_1 - x'_1|, |x_2 - x'_2|\})^{2\alpha} \\ &\leq 3^{2\alpha} + \sum_{j=1}^2 3^{2\alpha} |x_j - x'_j|^{2\alpha}. \end{aligned} \tag{5.4}$$

By induction we obtain the following rough estimate:

$$\|(H - z)^{-1}\|_{\mathcal{H}^n} \leq C_n (1 + \|H\|_{\mathcal{C}^n}^n) \left(\frac{1}{\text{dist}(z, \sigma(H))^{n+1}} + \frac{1}{\text{dist}(z, \sigma(H))} \right). \tag{5.5}$$

Now let us assume that $n < \alpha < n + 1$. The integral kernel of the n 'th partial derivative of $(H_{\mathbf{k}} - z)^{-1}$ with respect to k_1 is given by $i^n e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} (x_1 - x'_1)^n G(\mathbf{x}, \mathbf{x}'; z)$. Moreover, using (5.3) and Lemma 5.1 we conclude that the operator $\partial_{k_1}^n (H_{\mathbf{k}} - z)^{-1}$ is $\alpha - n$ Hölder continuous at $\mathbf{k} = 0$. Let $\mathbf{k} = (k_1, 0)$. We also have the identity:

$$\begin{aligned} &i^n (e^{ik_1(x_1 - x'_1)} - 1)(x_1 - x'_1)^n G(\mathbf{x}, \mathbf{x}'; z) \\ &= \langle [\partial_{k_1}^n (H_{k_1} - z)^{-1} - \partial_{k_1}^n (H_{k_1} - z)^{-1}|_{\mathbf{k}=0}] \delta_{\mathbf{x}'}, \delta_{\mathbf{x}} \rangle. \end{aligned}$$

If $|k_1| \leq 1$ the following norm estimate holds true according to Lemma 5.1:

$$\begin{aligned} &\|[\partial_{k_1}^n (H_{k_1} - z)^{-1} - \partial_{k_1}^n (H_{k_1} - z)^{-1}|_{\mathbf{k}=0}]\| \\ &\leq C |k_1|^{\alpha - n} (1 + \|H\|_{\mathcal{C}^{\alpha+1}}^n) \left(\frac{1}{\text{dist}(z, \sigma(H))^{n+2}} + \frac{1}{\text{dist}(z, \sigma(H))} \right). \end{aligned} \tag{5.6}$$

Choose $n < \alpha' < \alpha < n + 1$. Then the following integral converges in norm and defines a bounded operator:

$$\tilde{H} := \int_0^\infty \frac{1}{k_1^{1+\alpha'-n}} [\partial_{k_1}^n (H_{k_1} - z)^{-1} - \partial_{k_1}^n (H_{k_1} - z)^{-1}|_{\mathbf{k}=0}] dk_1.$$

Its integral kernel is given by

$$\tilde{G}(\mathbf{x}, \mathbf{x}'; z) := i^n (x_1 - x'_1)^n G(\mathbf{x}, \mathbf{x}'; z) \int_0^\infty \frac{1}{k_1^{1+\alpha'-n}} (e^{ik_1(x_1 - x'_1)} - 1) dk_1.$$

Assuming without loss of generality that $x_1 - x'_1 \neq 0$, and by a change of variable $s = k_1 |x_1 - x'_1|$ we obtain:

$$\begin{aligned} \tilde{G}(\mathbf{x}, \mathbf{x}'; z) &= |x_1 - x'_1|^{\alpha' - n} (x_1 - x'_1)^n G(\mathbf{x}, \mathbf{x}'; z) \\ &\quad \times \int_0^\infty \frac{1}{s^{1 + \alpha' - n}} i^n (e^{is \operatorname{sign}(x_1 - x'_1)} - 1) ds. \end{aligned}$$

Notice that the above integral only has two possible values C_\pm both different from zero, depending on the sign of $x_1 - x_1$. Since $\tilde{G}(\mathbf{x}, \mathbf{x}'; z) = \langle \tilde{H} \delta_{\mathbf{x}'}, \delta_{\mathbf{x}} \rangle = C_\pm(x_1, x'_1) |x_1 - x'_1|^{\alpha' - n} (x_1 - x'_1)^n G(\mathbf{x}, \mathbf{x}'; z)$ with $|C_\pm(x_1, x'_1)| \geq C$ it follows that

$$\sup_{\mathbf{x}' \in \Gamma} \sum_{\mathbf{x} \in \Gamma} |x_1 - x'_1|^{2\alpha'} |G(\mathbf{x}, \mathbf{x}'; z)|^2 \leq C^{-2} \|\tilde{H}\|^2.$$

This argument can be repeated for the other coordinate and bound the l^2 norm of $\langle \cdot - \mathbf{x}' \rangle^{\alpha'} G(\cdot, \mathbf{x}'; z)$ using (5.4). The proof of Proposition 2.2 is over.

5.2. A Few Identities from the Continuous Case

We list here a few well known facts about the continuous two dimensional magnetic Schrödinger operator with constant magnetic field equal to b in $L^2(\mathbb{R}^2)$:

$$H_b = (\mathbf{p} - b\mathbf{a}(\mathbf{x}))^2, \quad \mathbf{p} = -i\nabla_{\mathbf{x}}, \quad \mathbf{a}(\mathbf{x}) = (-x_2/2, x_1/2). \tag{5.7}$$

The integral kernel of the semi-group e^{-tH_b} is denoted with $G_b(\mathbf{x}, \mathbf{x}'; t)$ and is given by the following explicit formula:

$$G_b(\mathbf{x}, \mathbf{x}'; t) = e^{ib\varphi(\mathbf{x}, \mathbf{x}')} \frac{b}{4\pi \sinh(bt)} \exp \left[-\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(bt)} \right] =: e^{ib\varphi(\mathbf{x}, \mathbf{x}')} \tilde{G}_b(\mathbf{x}, \mathbf{x}'; t). \tag{5.8}$$

The semigroup property insures the following identity:

$$G_b(\mathbf{x}, \mathbf{x}'; 2t) = \int_{\mathbb{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t) d\mathbf{y}. \tag{5.9}$$

Then we can write:

$$\begin{aligned} e^{ib\varphi(\mathbf{x}, \mathbf{x}')} &= \frac{1}{\tilde{G}_b(\mathbf{x}, \mathbf{x}'; 2t)} \int_{\mathbb{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t) d\mathbf{y} \\ &= \frac{4\pi \sinh(2bt)}{b} \exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2bt)} \right] \int_{\mathbb{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}'; t) d\mathbf{y}. \end{aligned} \tag{5.10}$$

Taking the complex conjugation in both sides gives:

$$e^{-ib\varphi(\mathbf{x}, \mathbf{x}')} = \frac{4\pi \sinh(2bt)}{b} \exp \left[\frac{b|\mathbf{x} - \mathbf{x}'|^2}{4 \tanh(2bt)} \right] \int_{\mathbb{R}^2} G_b(\mathbf{y}, \mathbf{x}; t) G_b(\mathbf{x}', \mathbf{y}; t) d\mathbf{y}. \tag{5.11}$$

Again the semi-group property gives that:

$$\begin{aligned} \frac{b}{4\pi \sinh(2bt)} &= G_b(\mathbf{x}, \mathbf{x}; 2t) = \int_{\mathbf{R}^2} G_b(\mathbf{x}, \mathbf{y}; t) G_b(\mathbf{y}, \mathbf{x}; t) \, d\mathbf{y} \\ &= \int_{\mathbf{R}^2} |G_b(\mathbf{y}, \mathbf{x}; t)|^2 \, d\mathbf{y} \end{aligned} \quad (5.12)$$

which is clearly \mathbf{x} independent.

References

- [1] Avron, J.E., Simon, B.: Stability of gaps for periodic potentials under variation of a magnetic field. *J. Phys. A Math. Gen.* **18**, 2199–2205 (1985)
- [2] Avron, J., van Mouche, P.H.M., Simon, B.: On the measure of the spectrum for the almost Mathieu operator. *Commun. Math. Phys.* **132**, 103–118 (1990)
- [3] Avron, J., van Mouche, P.H.M., Simon, B.: On the measure of the spectrum for the almost Mathieu operator. *Erratum Commun. Math. Phys.* **139**, 215 (1991)
- [4] Bellissard, J.: Lipschitz Continuity of Gap Boundaries for Hofstadter-like Spectra. *Commun. Math. Phys.* **160**, 599–613 (1994)
- [5] Briet, P., Cornean, H.D.: Locating the spectrum for magnetic Schrödinger and Dirac operators. *Comm. Partial Differ. Equ.* **27**(5–6), 1079–1101 (2002)
- [6] Choi, M.D., Elliott, G.A., Yui, N.: Gauss polynomials and the rotation algebra. *Invent. Math.* **99**, 225–246 (1990)
- [7] Cornean, H.D.: On the magnetization of a charged Bose gas in the canonical ensemble. *Commun. Math. Phys.* **212**(1), 1–27 (2000)
- [8] Cornean, H.D., Nenciu, G.: On eigenfunction decay for two dimensional magnetic Schrödinger operators. *Commun. Math. Phys.* **192**, 671–685 (1998)
- [9] Cornean, H.D., Nenciu, G.: Two-dimensional magnetic Schrödinger operators: width of mini bands in the tight binding approximation. *Ann. Henri Poincaré* **1**(2), 203–222 (2000)
- [10] Cornean, H.D., Nenciu, G., Pedersen, T.G.: The Faraday effect revisited: general theory. *J. Math. Phys.* **47**(1), 013511 (2006)
- [11] Cornean, H.D., Nenciu, G.: The Faraday effect revisited: Thermodynamic limit. *J. Funct. Anal.* **257**(7), 2024–2066 (2009)
- [12] Elliott, G.: Gaps in the spectrum of an almost periodic Schrodinger operator. *C. R. Math. Rep. Acad. Sci. Canada* **4**, 255–259 (1982)
- [13] Germinet, F., Klein, A.: Operator kernel estimates for functions of generalized Schrödinger operators. *Proc. Am. Math. Soc.* **131**, 911–920 (2003)
- [14] Gröchenig, K., Leinert, M.: Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Trans. AMS* **358**, 2695–2711 (2006)
- [15] Haagerup, U., Rørdam, M.: Perturbations of the rotation C^* -algebras and of the Heisenberg commutation relation. *Duke Math. J.* **77**, 627–656 (1995)
- [16] Helffer, B., Kerdelhue, P., Sjöstrand, J.: *Mémoires de la SMF. Série 2* **43**, 1–87 (1990)
- [17] Helffer, B., Sjöstrand, J.: *Equation de Schrödinger avec champ magnétique et équation de Harper. Springer Lect. Notes Phys.* **345**, 118–197 (1989)

- [18] Helffer, B., Sjöstrand, J.: Analyse semi-classique pour l'équation de Harper. II. *Bull. Soc. Math. France* **117**(4), 40 (1990)
- [19] Herrmann, D.J.L., Janssen, T.: On spectral properties of Harper-like models. *J. Math. Phys.* **40**(3), 1197 (1999)
- [20] Iftimie, V., Măntoiu, M., Purice, R.: Magnetic Pseudodifferential Operators. *Publ. Res. Inst. Math. Sci.* **43**(3), 585–623 (2007)
- [21] Jaffard, S.: Propriétés des matrices 'bien localisées' près de leur diagonale et quelques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **7**(5), 461–476 (1990)
- [22] Kotani, M.: Lipschitz continuity of the spectra of the magnetic transition operators on a crystal lattice. *J. Geom. Phys.* **47**(2–3), 323–342 (2003)
- [23] Lein, M., Măntoiu, M., Richard, S.: Magnetic pseudodifferential operators with coefficients in C^* -algebras. <http://arxiv.org/abs/0901.3704v1> (2009)
- [24] Măntoiu, M., Purice, R.: Strict deformation quantization for a particle in a magnetic field. *J. Math. Phys.* **46**(5), 052105 (2005)
- [25] Măntoiu, M., Purice, R.: The magnetic Weyl calculus. *J. Math. Phys.* **45**(4), 1394–1417 (2004)
- [26] Măntoiu, M., Purice, R., Richard, S.: Spectral and propagation results for magnetic Schrödinger operators; a C^* -algebraic framework. *J. Funct. Anal.* **250**(1), 42–67 (2007)
- [27] Nenciu, G.: On asymptotic perturbation theory for quantum mechanics: almost invariant subspaces and gauge invariant magnetic perturbation theory. *J. Math. Phys.* **43**(3), 1273–1298 (2002)
- [28] Nenciu, G.: Stability of energy gaps under variation of the magnetic field. *Lett. Math. Phys.* **11**, 127–132 (1986)
- [29] Nenciu, G.: On the smoothness of gap boundaries for generalized Harper operators. In: *Advances in Operator Algebras and Mathematical Physics. Theta Series in Advanced Mathematics*, vol. 5, pp. 173–182. Theta, Bucharest (2005). arXiv:math-ph/0309009v2
- [30] Takai, H.: Dualité dans les produits croisés de C^* -algèbres. *C. R. Acad. Sci. Paris Sér. A* **278**, 1041–1043 (1974)
- [31] Takai, H.: On a duality for crossed products of C^* -algebras. *J. Funct. Anal.* **19**, 25–39 (1975)
- [32] Takesaki, M.: Duality for crossed products and the structure of von Neumann algebras of type III. *Acta Math.* **131**, 249–310 (1973)

Horia D. Cornean
Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7G
9220 Aalborg, Denmark
e-mail: cornean@math.aau.dk

Communicated by Jean Bellissard.

Received: December 3, 2009.

Accepted: June 24, 2010.