Large Normally Hyperbolic Cylinders in a priori Stable Hamiltonian Systems

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Abstract. We prove the existence of normally hyperbolic cylinders in a priori stable Hamiltonian systems the size of which is bounded from below independently of the size of the perturbation. This result should have applications to the study of Arnold's diffusion.

A major problem in dynamical systems consists in studying the Hamiltonian systems on $\mathbb{T}^n \times \mathbb{R}^n$ of the form

$$H(q,p) = h(p) - \epsilon^2 G(t,q,p), \quad (t,q,p) \in \mathbb{T} \times \mathbb{T}^n \times \mathbb{R}^n. \qquad (H)$$

Here ϵ should be considered as a small perturbation parameter, we put a square because the sign of the perturbation will play a role in our discussion. In the unperturbed system ($\epsilon = 0$) the momentum variable p is constant.

We want to study the dynamics of the perturbed system in the neighborhood of a torus $\{p = p_0\}$, corresponding to a resonant frequency. There is no loss of generality in assuming that the frequency is of the form

$$\partial h(p_0) = (\omega, 0) \in \mathbb{R}^m \times \mathbb{R}^r.$$

If the restricted frequency ω is non-resonant in $\mathbb{R}^m$, then it is expected that the averaged system

$$H_a(q,p) = H_a(q_1, q_2, p_1, p_2) = h(p) - \epsilon^2 V(q_2) \qquad (H_a)$$

should locally approximate the dynamics of (H) near $p = p_0 = (p_1^0, p_2^0)$, where $q = (q_1, q_2) \in \mathbb{T}^m \times \mathbb{T}^r$ and $p = (p_1, p_2) \in \mathbb{R}^m \times \mathbb{R}^r$, and where

$$V(q_2) = \int G(t, q_1, q_2, p_0) \mathrm{d}t \, \mathrm{d}q_1.$$

We make the following hypothesis on the averaged system:

Hypothesis 1. *The function h is convex with positive definite Hessian and the averaged potential V has a non-degenerate local maximum at $q_2 = 0$.*

Under Hypothesis 1, the averaged system has an invariant manifold of equations

$$(\partial_{p_2} h = 0, q_2 = 0) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Because h has positive definite Hessian, the equation $\partial_{p_2} h(p_1, p_2) = 0$ is non-singular and it defines a smooth m-dimensional manifold in $\mathbb{R}^n$ which can also be described parametrically by the relation $p_2 = P_2(p_1)$ for some function $P_2 : \mathbb{R}^m \longrightarrow \mathbb{R}^r$. Therefore, the corresponding invariant manifold can be written in a parametric form as

$$\{(q_1, 0, p_1, P_2(p_1)); (q_1, p_1) \in \mathbb{T}^m \times \mathbb{R}^m\},$$

it is a cylinder. Moreover, this manifold is normally hyperbolic in the sense of [12]. It is necessary at this point to precise the terminology. An open manifold will be called *weakly invariant* for a flow if the vector field is tangent at each point to this manifold. It will be called *strongly invariant* if it contains the full orbit of each of its points. A compact strongly invariant manifold is called normally hyperbolic if it is eventually absolutely 1-normally hyperbolic for the time-one flow in the sense of [12], Definition 4.

Definition 1. A weakly invariant open manifold N (for some vector field X) is called normally hyperbolic if there exists:
- A vector field Y on a compact manifold M.
- An embedding $i : U \longrightarrow M$ from a neighborhood U of N into M which conjugates $X_{|U}$ and $Y_{|i(U)}$.
- A normally hyperbolic strongly invariant compact manifold $\tilde{N}$ in M (for the vector field Y) such that $i(N) \subset \tilde{N}$.

Returning to the invariant cylinder of the averaged system, we observe that the open sub-cylinder

$$\{(q_1, 0, p_1, P_2(p_1)); (q_1, p_1) \in \mathbb{T}^m \times \mathbb{R}^m, \|p_1\| < \delta\}, \quad \delta > 0$$

is a normally hyperbolic weakly (and even strongly) invariant open sub-manifold for the averaged system in the sense of Definition 1. From this observation, and from the fact that the full system can be considered locally (near $p = p_0$) as a perturbation of the averaged system, one can prove the existence of a small normally hyperbolic weakly invariant cylinder in the full system, this is well understood. This cylinders can also be seen as the center manifold of a "whiskered" (or partially hyperbolic) torus, which is the continuation in the full systems of the invariant torus

$$\{(t, q_1, 0, p_0), \quad (t, q_1) \in \mathbb{T} \times \mathbb{T}^m\}$$

which exists in the averaged system. The name whiskered comes from the fact that this torus has hyperbolic normal directions, this name (as well as the corresponding object) was introduced by Arnold in [1]. The existence of a whiskered torus in the original system was proved in [18], following earlier works on the persistence of partially hyperbolic KAM tori. It is well understood, see for example [4] that such a torus must be contained in an invariant cylinder which is normally hyperbolic. Proving the existence of whiskered tori

involves KAM theory, which is quite demanding in terms of regularity, while the existence of the invariant cylinder relies on the softer theory of normal hyperbolicity. The idea of embedding whiskered tori into a normally hyperbolic cylinder and to use the theory of normal hyperbolicity in the context of Arnold diffusion is more recent than the paper of Arnold. To the best of our knowledge, it appears first in Moeckel [16]. It was then progressively understood that normally hyperbolic invariant cylinders can be used to produce diffusion even in the absence of whiskered tori.

We described two well-known methods allowing to prove the existence of small normally hyperbolic weakly invariant cylinders in the full system for $\epsilon > 0$. However, the size of the invariant cylinder that has been obtained in the literature is small, meaning that it converges to 0 with ϵ. Our point in the present paper is that a large normally hyperbolic weakly invariant cylinder actually exists:

Theorem 1. *Assume that H is smooth (or at least C^r for a sufficiently large r) and satisfies Hypothesis* 1. *Assume that ω is Diophantine, and fix $\kappa > 0$. Then, there exists an open ball $B \subset \mathbb{R}^m$ containing p_1^0, a neighborhood U of 0 in $\mathbb{T}^r$, a positive number ϵ_0 and, for $\epsilon < \epsilon_0$ two C^1 functions*

$$Q_2^\epsilon : \mathbb{T} \times \mathbb{T}^m \times B \longrightarrow U \subset \mathbb{T}^r \quad and \quad P_2^\epsilon : \mathbb{T} \times \mathbb{T}^m \times B \longrightarrow \mathbb{R}^r$$

such that the annulus

$$A^\epsilon = \{(t, q_1, Q_2^\epsilon(t, q_1, p_1), p_1, P_2^\epsilon(t, q_1, p_1)), \quad (t, q_1, p_1) \in \mathbb{T} \times \mathbb{T}^m \times B\}$$

is weakly invariant for (H) (in the sense that the Hamiltonian vector field is tangent to it). We have $P_2^\epsilon \longrightarrow P_2^0$ uniformly as $\epsilon \longrightarrow 0$, where P_2^0 is the function $(t, q_1, p_1) \longmapsto P_2(p_1)$. Moreover, we have $\|P_2^\epsilon - P_2^0\|_{C^1} \leqslant \kappa$, and $\|Q_2^\epsilon\|_{C^1} \leqslant \kappa/\epsilon$. Each strongly invariant set of (H) (in the sense that it contains the full orbit of each of its points, for example, a whiskered torus) contained in the domain

$$\mathcal{D}^\epsilon := \mathbb{T} \times \mathbb{T}^m \times U \times B \times \{p_2 \in \mathbb{R}^r : \|p_2\| \leqslant \epsilon\}$$

is contained in A^ϵ for $\epsilon < \epsilon_0$. The cylinder A^ϵ is normally hyperbolic and symplectic.

The novelty here is that the ball B does not depend on ϵ. Easy examples show that we cannot expect a fine control of the asymptotic behavior of Q_2^ϵ in terms of the averaged system only except if we restrict to smaller domains depending on ϵ. This asymptotic behavior also depends on the averaged systems at other frequencies. However, the very weak estimates we have are sufficient to describe the restricted dynamics. Let $A_0^\epsilon \subset \mathbb{T}^n \times \mathbb{R}^n$ be the restriction of the invariant annulus to the section $\{t = 0\}$,

$$A_0^\epsilon = \{(q_1, Q_2^\epsilon(0, q_1, p_1), p_1, P_2^\epsilon(0, q_1, p_1)), \quad (q_1, p_1) \in \mathbb{T}^m \times B\},$$

and let $\phi : A_0^\epsilon \longrightarrow \mathbb{T}^n \times \mathbb{R}^n$ be the time-one flow of H (which is well-defined on A_0^ϵ when ϵ is small enough). Then, A_0^ϵ is somewhat invariant for ϕ (although there are some difficulties near the boundary) in a sense that will be given

more precisely below. We define the map $\Phi : \mathbb{T}^m \times B \longrightarrow \mathbb{T}^m \times \mathbb{R}^m$ as the restriction of ϕ to A_0^ϵ seen in coordinates (q_1, p_1), more precisely

$$\Phi(q_1, p_1) = (q_1, p_1) \circ \phi\left(q_1, Q_2^\epsilon(0, q_1, p_1), p_1, P_2^\epsilon(0, q_1, p_1)\right).$$

Note that this map is well-defined on $\mathbb{T}^m \times B$. Let us finally consider an open ball $B_0 \subset \mathbb{R}^m$ which contains p_0 and whose closure is contained in B, and set

$$A_{00}^\epsilon = \{(q_1, Q_2^\epsilon(0, q_1, p_1), p_1, P_2^\epsilon(0, q_1, p_1)), \quad (q_1, p_1) \in \mathbb{T}^m \times B_0\}.$$

Proposition 2. *The map Φ is converging uniformly (when $\epsilon \longrightarrow 0$) on $\mathbb{T}^m \times B_0$ to the map*

$$\Phi_0 : \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \longmapsto \begin{pmatrix} q_1 + \partial_{p_1} h\left(p_1, P_2(p_1)\right) \\ p_1 \end{pmatrix},$$

which gives the unperturbed dynamics on the invariant cylinder of the averaged system. Moreover, we have $\phi(A_{00}^\epsilon) \subset A_0^\epsilon$ when ϵ is small enough. Finally, given $\eta > 0$, we can choose the ball B_0 small enough so that the inequality

$$\|d\Phi - d\Phi_0\|_{C^0} \leqslant \eta$$

holds on $\mathbb{T}^m \times B_0$ when ϵ is small enough.

The frequency map

$$p_1 \longmapsto \Omega_0(p_1) := \partial_{p_1} h\left(p_1, P_2(p_1)\right)$$

has positive torsion in the sense that

$$\partial_{p_1}\Omega_0 = \partial_{p_1}^2 h(p_1, P_2(p_1))$$

is a positive definite symmetric matrix for all $p_1 \in \mathbb{R}^m$. As a consequence, when ϵ is small enough, the restricted map Φ has positive torsion in a neighborhood (independent of ϵ) of $\mathbb{T}^m \times \{p_1^0\}$, in the sense that

$$\partial_{p_1}(q_1 \circ \Phi)_{(q_1, p_1)}\rho_1 \cdot \rho_1 > 0 \quad \forall \rho_1 \in \mathbb{R}^m$$

for all $q_1 \in \mathbb{T}^m$ and $p_1 \in B_0$ provided that B_0 has been chosen small enough. The map Φ is symplectic with respect to the symplectic form obtained by restriction of the ambient symplectic form to A_0^ϵ. It is part of the statement of Theorem 1 that this form is non-degenerate on A_0^ϵ. Note that this symplectic form is not $dq_1 \wedge dp_1$ in general.

In the case $m = 1$, (but for any dimension n) one can combine these results with the existing techniques on the a priori unstable situation, like the variational methods coming from Mather Theory (see [2,14]), developed for the a priori unstable situation in [3,6,7] or more geometric methods such as [11] (the papers [8,18] also treat the a priori unstable situation, but it seems to me at first sight that they require too strong information on the restricted dynamics to be applicable here). One can then hope to obtain, under additional non-degeneracy assumptions, the existence of restricted Arnold diffusion in the following sense: there exists $\delta > 0$ and ϵ_0 such that, for each $\epsilon \in]0, \epsilon_0[$ there exists an orbit $(q_\epsilon(t), p_\epsilon(t))$ with the following property: the image $p_\epsilon(\mathbb{R})$ is not contained in any ball of radius δ in $\mathbb{R}^n$. Once again, the key point here is that δ can be chosen independent of ϵ. Specifying the needed "non-degeneracy

assumptions" will require some further work, but I believe it will not require any method beyond those which are already available.

Of course, finding "global" Arnold diffusion, as announced in [15], that is, orbits wondering in the whole phase space along different resonant lines (or far away along a given resonant line) requires a specific study of relative resonances (when the restricted frequency ω is resonant), where the existence of normally hyperbolic invariant cylinders cannot be obtained by the method used in the present paper.

Let us close this introduction with a remark on uniqueness. In general, there is no uniqueness statement for the normally invariant cylinder we obtain. However, in the case $m = 1$, we can obtain a stronger result: let $[p_1^-, p_1^+] \subset B \subset \mathbb{R}$ be an interval such that both $\Omega_0(p_1^-)$ and $\Omega_0(p_1^+)$ are Diophantine. Then, there exists whiskered tori $\mathbb{T}_-^\epsilon$ and $\mathbb{T}_+^\epsilon$ of dimension 2 in $\mathbb{T} \times \mathbb{T}^n \times \mathbb{R}^n$ which are close to the unperturbed tori

$$T_-^0 = \left\{ (t, q_1, 0, p_1^-, P_2(p_1^-)) : \quad (t, q_1) \in \mathbb{T} \times \mathbb{T} \right\}$$

and

$$T_+^0 = \left\{ (t, q_1, 0, p_1^+, P_2(p_1^+)) : \quad (t, q_1) \in \mathbb{T} \times \mathbb{T} \right\}.$$

The whiskered tori $\mathbb{T}_\pm^\epsilon$ are contained in the annulus A^ϵ. They bound a compact part $A_\leqq^\epsilon$ of A^ϵ which is then strongly invariant in the sense that it contains the full orbit of each of its points. The annulus $A_\leqq^\epsilon$ is then unique in the sense that if $\tilde{A}^\epsilon$ is another normally hyperbolic cylinder given by Theorem 1 (with the same domain B), then it must contain $A_\leqq^\epsilon$. The cylinder $A_\leqq^\epsilon$ is a normally hyperbolic invariant cylinder in the genuine sense. If the interval $[p^-, p^+]$ has been chosen small enough, then the restricted map $\Phi : A_\leqq^\epsilon \longrightarrow A_\leqq^\epsilon$ is a C^1 area preserving twist map (for the appropriate area form). When $m > 1$ one should not expect the same kind of properties, since Arnold diffusion may occur inside the invariant cylinder.

1. Averaging

To apply averaging methods, it is easier to consider the extended phase space

$$(t, e, q, p) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^n \times \mathbb{R}^n$$

where the Hamiltonian flow can be seen as the Hamiltonian flow of the autonomous Hamiltonian function

$$\tilde{H}(t, e, q, p) = h(p) + e - \epsilon^2 G(t, q, p)$$

on one of its energy surfaces, for example $\tilde{H} = 0$. Then, we consider a smooth solution $f(t, q)$ of the Homological equation

$$\partial_t f + \partial_q f \cdot (\omega, 0) = G(t, q, p_0) - V(q_2).$$

Such a solution exists because ω is Diophantine, as can be checked easily by power series expansion. It is unique up to an additive constant. We consider the smooth symplectic diffeomorphism

$$\psi^\epsilon : (t, e, q, p) \longmapsto (t, e + \epsilon^2 \partial_t f(t, q), q, p + \epsilon^2 \partial_q f(t, q))$$

and use the same notation for the diffeomorphism $(t,q,p) \longmapsto (t,q,p + \epsilon^2 \partial_q f(t,q))$. We have

$$\tilde{H} \circ \psi^\epsilon = h(p) + e - \epsilon^2 V(q_2) - \epsilon^2 R(t,q,p) + O(\epsilon^4),$$

where $R(t,q,p) = G(t,q,p) - G(t,q,p_0)$. In other words, by the time-dependent symplectic change of coordinates ψ^ϵ, we have reduced the study of H to the study of the time-dependent Hamiltonian

$$H_1(t,q,p) = h(p) - \epsilon^2 V(q_2) - \epsilon^2 R(t,q,p) + O(\epsilon^4)$$

where $R = O(p - p_0)$. As a consequence, Theorem 1 holds for H if it holds for H_1. More precisely, assume that there exists an invariant cylinder

$$\mathcal{A}^\epsilon = (t, q_1, \mathcal{Q}_2^\epsilon(t,q_1,p_1), p_1, \mathcal{P}_2^\epsilon(t,q_1,p_1))$$

for H_1, with $\|\mathcal{Q}_2^\epsilon\|_{C^1} \leqslant \kappa/2\epsilon$ and $\|\mathcal{P}_2^\epsilon - P_2^0\|_{C^1} \leqslant \kappa/2$. Then, the annulus $A^\epsilon := \psi^\epsilon(\mathcal{A}^\epsilon)$ is invariant for H. Since ψ^ϵ is ϵ^2-close to the identity, while $\|\mathcal{Q}_2^\epsilon\|_{C^1} \leqslant \kappa/2\epsilon$, the annulus A^ϵ has the form

$$A^\epsilon = (t, q_1, Q_2^\epsilon(t,q_1,p_1), p_1, P_2^\epsilon(t,q_1,p_1))$$

for C^1 functions Q_2^ϵ, P_2^ϵ which satisfy $\|Q_2^\epsilon\|_{C^1} \leqslant \kappa/\epsilon$ and $\|P_2^\epsilon - P_2^0\|_{C^1} \leqslant \kappa$. We will prove that Theorem 1 holds for H_1 in Sect. 4. We first expose some useful tools.

2. Normally Hyperbolic Manifolds

We shall now present a version of the classical theory of normally hyperbolic manifolds adapted for our purpose. On $\mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$, let us consider the time-dependent vector field

$$\dot{z} = Z(t,z,x,y)$$
$$\dot{x} = A(z)x$$
$$\dot{y} = -B(z)y.$$

We assume that the function

$$Z : \mathbb{R} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \longrightarrow \mathbb{R}^{n_z}$$

is C^1-bounded in the domain

$$\mathbb{R} \times \mathbb{R}^{n_z} \times \{x \in \mathbb{R}^{n_x} : \|x\| < 1\} \times \{y \in \mathbb{R}^{n_y} : \|y\| < 1\}, \qquad \text{(D)}$$

and that the matrices A and B are C^1-bounded functions of z. Moreover, we assume that there exists constants $a > b > 0$ such that

$$A(z)x \cdot x \geqslant a\|x\|^2, \quad B(z)y \cdot y \geqslant a\|y\|^2$$

for all x, y, z, and such that

$$\|\partial_{(t,z)}Z(t,z,x,y)\| \leqslant b$$

for all (t, z, x, y) belonging to (D). We consider the perturbed vector field

$$
\begin{aligned}
\dot{z} &= Z(t, z, x, y) + R_z(t, z, x, y) \\
\dot{x} &= A(z)x + R_x(t, z, x, y) \\
\dot{y} &= -B(z)y + R_x(t, z, x, y).
\end{aligned}
$$

where $R = (R_z, R_x, R_y)$ is seen as a small perturbation.

Theorem 2. *There exists $\epsilon > 0$ such that, when $\|R\|_{C^1} < \epsilon$, the maximal invariant set of the perturbed vector field contained in the domain* (D) *is a graph of the form*

$$
\{(t, z, X(t, z), Y(t, z)), \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{n_z}\}
$$

where X and Y are C^1 maps. This graph is normally hyperbolic, and it is contained in the domain

$$
\mathbb{R}^{n_z} \times \{x \in \mathbb{R}^{n_x} : \|x\| \leqslant (2/a)\|R\|_{C^0}\} \times \{y \in \mathbb{R}^{n_y} : \|y\| \leqslant (2/a)\|R\|_{C^0}\}.
$$

In other words, we have

$$
\|(X, Y)\|_{C^0} \leqslant (2/a)\|R\|_{C^0}.
$$

The C^1 norm of (X, Y) is converging to zero when the C^1 norm of the perturbation converges to zero.

Proof. The invariant space $\mathbb{R}^{n_z}$ is normally hyperbolic in the sense of [10,12]. As a consequence, the standard theory applies and implies the existence of functions X and Y such that the graph $(t, z, X(t, z), Y(t, z))$ is invariant, normally hyperbolic, and contained in (D). Note that we are slightly outside of the hypotheses of the statements in [12] because our unperturbed manifold is not compact. However, the results actually depend on uniform estimates rather than on compactness (see [9, Appendix B], for example, see also [5]), and we assumed such uniform estimates.

Let us now prove the estimate on (X, Y). We have the inequality

$$
\dot{x} \cdot x \geqslant a\|x\|^2 + x \cdot R_x \geqslant a\|x\|(\|x\| - \|R_x\|_{C^0}/a)
$$

which implies that

$$
\dot{x} \cdot x \geqslant \|x\|\|R_x\|_{C^0}
$$

if

$$
2\|R_x\|_{C^0}/a \leqslant \|x\| \leqslant 1,
$$

hence this domain cannot intersect the invariant graph. Similar considerations show that the domain $2\|R_y\|_{C^0}/a \leqslant \|y\| \leqslant 1$ cannot intersect the graph. $\square$

3. Hyperbolic Linear System

Let us consider the linear Hamiltonian system on $\mathbb{R}^n \times \mathbb{R}^n$ generated by the Hamiltonian

$$H(q, p) = \frac{1}{2}\langle Bp, p\rangle - \frac{1}{2}\langle Aq, q\rangle,$$

where both A and B are positive definite symmetric matrices. We recall that this system can be reduced to

$$G(x, y) = \langle Dx, y\rangle,$$

where D is a positive definite symmetric matrix, by a linear symplectic change of variables $(q, p) \longrightarrow (x, y)$. To do so, we consider the symmetric positive definite matrix

$$L := \left(A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}\right)^{1/2},$$

which is the only symmetric and positive definite solution of the equation $L^2AL^2 = B$. Considering the change of variables

$$x = \frac{1}{\sqrt{2}}(Lp + L^{-1}q); \quad y = \frac{1}{\sqrt{2}}(Lp - L^{-1}q)$$

or equivalently

$$q = \frac{1}{\sqrt{2}}L(x - y); \quad p = \frac{1}{\sqrt{2}}L^{-1}(x + y),$$

an elementary calculation shows that we obtain the desired form for the Hamiltonian in coordinates (x, y), with

$$D = LAL = L^{-1}BL^{-1}.$$

As a consequence, the equations of motions in the new variables take the block-diagonal form

$$\dot{x} = Dx; \quad \dot{y} = -Dy.$$

In the original coordinates (q, p) the stable space (which is the space $x = 0$) is the space $\{(q, -L^2q), q \in \mathbb{R}^n\}$ while the unstable space is $\{(q, L^2q), q \in \mathbb{R}^n\}$.

4. Proof of Theorem 1

We now prove Theorem 1 for the Hamiltonian

$$H_1(t, q, p) = h(p) - \epsilon^2 V(q_2) - \epsilon^2 R(t, q, p) + O(\epsilon^{2+\gamma}),$$

where $R = O(p - p_0)$ and $\gamma > 0$ ($\gamma = 2$ in our situation). We assume that Hypothesis 1 holds. We lift all the angular variables to the universal covering, and see H_1 as a Hamiltonian of the variables

$$(t, q, p) = (t, q_1, q_2, p_1, p_2) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{R}^r$$

which is one-periodic in t, q. We assume that $p_0 = 0$.

We will need some notations. We set $A := \partial^2 V(0)$, it is a symmetric positive definite matrix. We will denote by $B(p_1)$ a matrix which depends

smoothly on p_1, is uniformly positive definite, is constant outside of a neighborhood of $p_1 = 0$ in $\mathbb{R}^m$, and coincides with $\partial^2_{p_2} h(p_1, P_2(p_1))$ in a neighborhood of $p_1 = 0$. We will denote by $\tilde{P}_2(p_1)$ a compactly supported smooth function $\tilde{P}_2 : \mathbb{R}^m \longrightarrow \mathbb{R}^r$ which coincides with P_2 around $p_1 = 0$. Finally, we will denote by $h_0(p_1)$ a smooth compactly supported function which is equal to $h(p_1, P_2(p_1))$ around $p_1 = 0$.

It is useful to introduce two new positive parameters α and δ. We always assume that

$$0 < \epsilon < \delta < \alpha < 1.$$

In the sequel, we shall chose α small, then δ small with respect to α, and work with ϵ small enough with respect to α and δ. The parameter δ represents the size of the normally hyperbolic cylinder we intend to find. We will denote by χ a smooth function of its arguments which may depend (in an unexplicited way) on the parameters ϵ, δ, but which is C^2-bounded, uniformly in ϵ, δ. The notation χ will be used in a similar way when only C^1 bounds are assumed.

Lemma 3. *There exists a smooth Hamiltonian function $H_2(t, q, p)$ (which depends on the parameters ϵ, δ) of the form*

$$H_2 = h_0(p_1) + \frac{1}{2} B(p_1) \cdot (p_2 - \tilde{P}_2(p_1))^2 - \frac{\epsilon^2}{2} A \cdot q_2^2$$
$$+ \epsilon^3 \chi \left(p_1, (p_2 - \tilde{P}_2(p_1))/\epsilon \right) + \epsilon^2 \delta^{3/2} \chi(q_2/\sqrt{\delta})$$
$$+ \epsilon^2 \delta \chi(t, q, p/\delta) + \epsilon^{2+\gamma} \chi(t, q, p)$$

which coincides with H_1 on the domain

$$\left\{ \|q_2\| \leqslant \sqrt{\delta}, \|p_1\| \leqslant \delta, \|p_2 - P_2(p_1)\| \leqslant \epsilon \right\}.$$

Proof. Let us expand the function h with respect to p_2 at the point $P_2(p_1)$:

$$h(p_1, p_2)$$
$$= h(p_1, P_2(p_1)) + \frac{1}{2} \partial^2_{p_2} h(p_1, P_2(p_1)) \cdot (p_2 - P_2(p_1))^2 + \underline{S}(p) \cdot (p_1 - P_2(p_1))^3$$

where $\underline{S}(p)$ is a 3-linear form on $\mathbb{R}^r$ depending smoothly on p. We consider a 3-form $S(p)$ which depends smoothly on p, is compactly supported, and is equal to $\underline{S}(p)$ near $p = 0$. Let $i : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ (for any k) be a compactly supported smooth map which is equal to the identity on the unit ball. Then the function

$$h_0(p_1) + \frac{1}{2} B(p_1) \cdot (p_1 - \tilde{P}_2(p_1))^2 + \epsilon^3 S(p) \cdot \left(i \left[(p_2 - \tilde{P}_2(p_1))/\epsilon \right] \right)^3$$
$$= h_0(p_1) + \frac{1}{2} B(p_1) \cdot (p_1 - \tilde{P}_2(p_1))^2 + \epsilon^3 \chi \left(p_1, (p_2 - \tilde{P}_2(p_1))/\epsilon \right)$$

is equal to h if p belongs to a given neighborhood of 0 (independent of ϵ, δ) and satisfies $\|p_2 - \tilde{P}_2(p_1)\| \leqslant \epsilon$. Similarly, we write

$$V(q_2) = \frac{1}{2} A \cdot q_2^2 + W(q_2) \cdot q_2^3$$

for some 3-linear form $W(q_2)$. It is equal to

$$\frac{1}{2}A \cdot q_2^2 + \delta^{3/2} W(q_2) \cdot \left(i\left(q_2/\sqrt{\delta}\right)\right)^3 = \frac{1}{2}A \cdot q_2^2 + \delta^{3/2}\underline{\chi}\left(q_2/\sqrt{\delta}\right)$$

on $\{\|q_2\| \leqslant \sqrt{\delta}\}$. Finally, we observe that the function $R(t, q, p)$ can be written in the form

$$R(t, q, p) = L(t, q, p) \cdot p$$

and is equal to the function

$$\delta L(t, q, p) \cdot i(p/\delta) = \delta\underline{\chi}(t, q, p/\delta)$$

on $\{\|p\| \leqslant \delta\}$. Collecting all terms proves the Lemma.   □

We will now prove the existence of a normally hyperbolic invariant graph for H_2 contained in the region

$$\{\|q_2\| \leqslant \sqrt{\delta}, \|p_2 - \tilde{P}_2(p_1)\| \leqslant \epsilon\}$$

Its intersection with $\{\|p_1\| < \delta\}$ will give a weakly invariant manifold for H_1 (meaning that the Hamiltonian vector field of H_1 is tangent to it). To simplify the following equations, we set

$$h_2(p) := h_0(p_1) + \frac{1}{2}B(p_1) \cdot (p_2 - \tilde{P}_2(p_1))^2.$$

The Hamiltonian vector field of H_2 can be written

$$\dot{q}_1 = \partial_{p_1} h_2(p) + \epsilon^2 \chi\left(p_1, (p_2 - \tilde{P}_2(p_1))/\epsilon\right) + \epsilon^2 \chi(t, q, p/\delta)$$
$$\dot{p}_1 = 0 + \epsilon^2 \delta \chi(t, q, p/\delta)$$
$$\dot{q}_2 = B(p_1)(p_2 - \tilde{P}_2(p_1)) + \epsilon^2 \chi(p_1, (p_2 - \tilde{P}_2(p_1))/\epsilon) + \epsilon^2 \chi(t, q, p)$$
$$\dot{p}_2 = \epsilon^2 A q_2 + \epsilon^2 \delta \chi(q_2/\sqrt{\delta}) + \epsilon^2 \delta \chi(t, q_1, q_2, p/\delta)$$

recalling the convention that $\chi(.)$ always denotes a C^1 function of its arguments, depending on ϵ and δ, but bounded in C^1 independently of δ and ϵ. Motivated by Sect. 2, we set

$$L(p_1) = \left(A^{-1/2}(A^{1/2}B(p_1)A^{1/2})^{1/2}A^{-1/2}\right)^{1/2},$$

and perform the change of variables $(t, q_1, p_1, q_2, p_2) \longrightarrow (\tau, \theta, r, x, y)$ given by:

$$\tau = \epsilon t, \quad \theta = \epsilon \alpha q_1, \quad r = p_1,$$
$$x = L(p_1)(p_2 - \tilde{P}_2(p_1)) + \epsilon L^{-1}(p_1)q_2, \ y = L(p_1)(p_2 - \tilde{P}_2(p_1)) - \epsilon L^{-1}(p_1)q_2,$$

recalling that α is a fixed positive parameter. Equivalently, this can be written

$$t = \tau/\epsilon, \quad q_1 = \theta/\epsilon\alpha, \quad p_1 = r,$$
$$q_2 = L(r)(x - y)/2\epsilon, \quad p_2 = \tilde{P}_2(r) + L^{-1}(r)(x + y)/2.$$

In the new coordinates, the principal part of the vector field takes the form (denoting $\acute{f}$ for $df/d\tau$)

$$\acute{\theta} = \alpha\Omega(r, x, y), \quad \acute{r} = 0, \quad \acute{x} = D(r)x, \quad \acute{y} = -D(r)y,$$

with

$$\Omega(r, x, y) := \partial_{p_1} h_2 \left(r, \tilde{P}_2(r) + L^{-1}(r)(x + y)/2 \right)$$

and

$$D(r) := L(r)AL(r) = L^{-1}(r)B(r)L^{-1}(r).$$

The equality above holds because $L(r)$ solves the equation $L^2(r)AL^2(r) = B(r)$. Let us write in details the calculations leading to the expressions of $\acute{x} := dx/d\tau$ (the calculation for $\acute{y}$ is similar):

$$
\begin{aligned}
\epsilon\acute{x} = \dot{x} &= L(p_1)\left(\dot{p}_2 - \partial_{p_1}\tilde{P}_2 \cdot \dot{p}_1\right) + \epsilon L^{-1}(p_1)\dot{q}_2 + (\partial_{p_1}L \cdot \dot{p}_1)\left(p_2 - \tilde{P}_2(p_1)\right) \\
&\quad + \epsilon\left(\partial_{p_1}(L^{-1}) \cdot \dot{p}_1\right)q_2 \\
&= \epsilon^2 L(p_1)Aq_2 + \epsilon L^{-1}(p_1)B(p_1)(p_2 - \tilde{P}_2(p_1)) \\
&\quad + \epsilon^2\delta\chi(t, q, p/\delta, x, y) + \epsilon^3\chi(p_1, (x + y)/\epsilon) \\
&\quad + \epsilon^2\delta\chi(q_2/\sqrt{\delta}) + \epsilon^{2+\gamma}\chi(t, q, p, x, y) \\
&= \epsilon L(r)AL(r)(x - y)/2 + \epsilon L^{-1}(r)B(r)L^{-1}(r)(x + y)/2 \\
&\quad + \epsilon^2\delta\chi(\tau/\epsilon, \theta/\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) + \epsilon^2\delta\chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon) \\
&= \epsilon D(r)x + \epsilon^2\delta\chi(\tau/\epsilon, \theta/\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) \\
&\quad + \epsilon^2\delta\chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon).
\end{aligned}
$$

The function $\Omega(r, x, y)$ is C^1-bounded on

$$\{(r, x, y), \quad \|x\| \leqslant 1, \|y\| \leqslant 1\}.$$

We can choose $\alpha < 1$ once and for all in order that the principal part of the vector field satisfies the hypotheses of Theorem 2. The full vector field can be written in the new coordinates, (with the notation $\acute{f} := df/d\tau$):

$$
\begin{aligned}
\acute{\theta} &= \alpha\Omega(r, x, y) + \epsilon^2\chi(\tau/\epsilon, \theta/\alpha\epsilon, r, x/\epsilon, y/\epsilon) \\
\acute{r} &= 0 + \epsilon\delta\chi(\tau/\epsilon, \theta/\alpha\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) \\
\acute{x} &= D(r)x + \epsilon\delta\chi(\tau/\epsilon, \theta/\alpha\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) \\
&\quad + \epsilon\delta\chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon) \\
\acute{y} &= -D(r)y + \epsilon\delta\chi(\tau/\epsilon, \theta/\alpha\epsilon, r/\delta, x/\delta, y/\delta, x/\epsilon, y/\epsilon) \\
&\quad + \epsilon\delta\chi(r/\sqrt{\delta}, x/\sqrt{\delta}\epsilon, y/\sqrt{\delta}\epsilon).
\end{aligned}
$$

In this expression, we observe that the uniform norm of the perturbation is $O(\epsilon\delta)$ while the C^1 norm is $O(\sqrt{\delta})$ (recall that $0 < \epsilon < \delta < 1$). We can apply Theorem 2 and find a unique bounded normally hyperbolic invariant graph

$$(\tau, \theta, X(\tau, \theta, r), r, Y(\tau, \theta, r)).$$

Moreover Theorem 2 also implies that

$$\|(X, Y)\|_{C^0} \leqslant C\epsilon\delta.$$

Because the invariant graph we have obtained is the maximal invariant set contained in the domain $\{\|x\| \leqslant 1, \|y\| \leqslant 1\}$, and since the vector field is ϵ-periodic in t and $\alpha\epsilon$-periodic in q_1, we conclude that the functions X and Y

are ϵ-periodic in t and $\alpha\epsilon$-periodic in q_1. In the initial coordinates, we have an invariant graph

$$(t, q_1, Q_2^\epsilon(t, q_1, p_1), p_1, P_2^\epsilon(t, q_1, p_1))$$

with

$$Q_2^\epsilon(t, q_1, p_1) = L(p_1)\left(X(\epsilon t, \epsilon q_1, p_1) - Y(\epsilon t, \epsilon q_1, p_1)\right)/2\epsilon$$

and

$$P_2^\epsilon(t, q_1, p_1) = \tilde{P}_2(p_1) + L^{-1}(p_1)\left(X(\epsilon t, \epsilon q_1, p_1) + Y(\epsilon t, \epsilon q_1, p_1)\right)/2.$$

The functions Q_2^ϵ and P_2^ϵ are 1-periodic in (t, q_1). The invariant graph we have obtained is normally hyperbolic for the flow of H_2, and its strong stable and strong unstable directions have the same dimension r. It follows from general results on partial hyperbolicity in a symplectic context (see e.g. [13, Proposition 1.8.3][1]) that it is a symplectic manifold. This means that the restriction to the invariant graph of the ambient symplectic form is a symplectic form. Observing that

$$\|Q_2^\epsilon\|_{C^0} \leqslant C\delta, \quad \|P_2^\epsilon\|_{C^0} \leqslant C\epsilon\delta,$$

we infer that the annulus

$$\{(t, q_1, Q_2^\epsilon(t, q_1, p_1), p_1, P_2^\epsilon(t, q_1, p_1)) :$$
$$t \in \mathbb{T}, q_1 \in \mathbb{T}^m, p_1 \in \mathbb{R}^m, \|p_1\| < \delta\} \subset \mathbb{T} \times \mathbb{T}^n \times \mathbb{R}^n$$

is contained in the domain

$$\{\|q_2\|\sqrt{\delta}, \|p_1\| \leqslant \delta, \|p_2 - \tilde{P}_2(p_1)\| \leqslant \epsilon\}$$

where $H_2 = H_1$ provided δ has been chosen small enough. It is thus a weakly invariant cylinder for H_1 i.e. the extended Hamiltonian vector field of H_1 on $\mathbb{T} \times \mathbb{T}^n \times \mathbb{R}^n$ is tangent to this annulus at each point. Orbits may still exit from the cylinder through its boundary. We finish with the estimates on the C^1 norms. Since the C^1 size of the perturbation is $O(\sqrt{\delta})$, we can make it as small as we want by choosing δ small. We can thus assume that $\|(X, Y)\|_{C^1}$ is small, and this implies the desired C^1 estimates on P_2^ϵ and Q_2^ϵ. We have proved Theorem 1 for H_1, we conclude from Sect. 1 that Theorem 1 holds for H.

5. Proof of Proposition 2

Let (q_1, p_1) be given in $\mathbb{T}^m \times B$, and let $(q_1(t), q_2(t), p_1(t), p_2(t))$ be the orbit (under H) of the point

$$(q_1, Q_2^\epsilon(0, q_1, p_1), p_1, P_2^\epsilon(0, q_2, p_2)).$$

---

[1] In this text, the equality of the dimensions of the stable and unstable directions (that obviously holds here) is stated as a conclusion, although it should be taken as an assumption.

We have the Hamilton equations

$$\dot{q}_1(t) = \partial_{p_1} H(t, q_1(t), q_2(t), p_1(t), p_2(t))$$
$$\dot{p}_1(t) = -\partial_{q_1} H(t, q_1(t), q_2(t), p_1(t), p_2(t)).$$

They imply that $\dot{p}_1 = O(\epsilon^2)$, and we conclude that $p_1(t) \in B$ for all $t \in [0,1]$ if $p_1 \in B_0$, provided ϵ is small enough. This implies the inclusion

$$\phi(A_{00}^\epsilon) \subset A_0^\epsilon,$$

and it also implies that

$$(q_1(t), q_2(t), p_1(t), p_2(t)) = (q_1(t), Q_2^\epsilon(t, q_1(t), p_1(t)), p_1(t), P_2^\epsilon(t, q_1(t), p_1(t)))$$

for each $t \in [0,1]$. The Hamilton equations then take the form

$$\dot{q}_1(t) = \partial_{p_1} h\left(p_1(t), P_2^\epsilon(t, q_1(t), p_1(t))\right)$$
$$\qquad - \epsilon^2 \partial_{p_1} G\left(t, q_1(t), Q_2^\epsilon(t, q_1(t), p_1(t)), P_2^\epsilon(t, q_1(t), p_1(t))\right)$$
$$\dot{p}_1(t) = +\epsilon^2 \partial_{q_1} G\left(t, q_1(t), Q_2^\epsilon(t, q_1(t), p_1(t)), P_2^\epsilon(t, q_1(t), p_1(t))\right).$$

The map Φ is thus the time-one flow of the vector field

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \longmapsto \begin{pmatrix} \partial_{p_1} h\left(p_1, P_2^\epsilon(t, q_1, p_1)\right) - \epsilon^2 \partial_{p_1} G\left(t, q_1, Q_2^\epsilon(t, q_1, p_1), P_2^\epsilon(t, q_1, p_1)\right) \\ \epsilon^2 \partial_{q_1} G\left(t, q_1, Q_2^\epsilon(t, q_1, p_1), P_2^\epsilon(t, q_1, p_1)\right) \end{pmatrix}$$

which converges uniformly to the vector field

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \longmapsto \begin{pmatrix} \partial_{p_1} h\left(p_1, P_2(p_1)\right) \\ 0 \end{pmatrix}$$

when $\epsilon \longrightarrow 0$ on $\mathbb{T}^m \times B$. We conclude that Φ is converging uniformly to Φ_0 (as defined in Proposition 2). Moreover, we see that the C^1 distance between these two vector fields is $O(\kappa)$ (κ is a parameter introduced in the statement of Theorem 1), so it can be made arbitrarily small by taking B_0 small enough. The same statement then holds for the time-one flows Φ and Φ_0.

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