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On Hawking's Local Rigidity Theorem for Charged Black Holes

Pin Yu

Abstract. We show the existence of the Hawking vector field in a full neighborhood of a local, regular, bifurcate, non-expanding horizon embedded in a smooth Einstein–Maxwell space–time without assuming the underlying space–time is analytic. This extends a result of Friedrich et al. (Commun Math Phys 204:691–707, 1971), which holds in the interior of the black hole region. Moreover, we also show, in the presence of an additional Killing vector field T which is tangent to the horizon and not vanishing on the bifurcate sphere, then space–time must be locally axially symmetric without the analyticity assumption. This axial symmetry plays a fundamental role in the classification theory of stationary black holes.

1. Introduction

Let $(\mathcal{M}, g, F)$ be a smooth and time oriented Einstein–Maxwell space–time of dimension $3+1$ with electromagnetic field F. Let $\mathcal{S}$ be a smoothly embedded space-like 2-sphere in $\mathcal{M}$ and $\mathcal{H}^+, \mathcal{H}^-$ be the corresponding null boundaries of the causal future and the causal past of $\mathcal{S}$. We also assume that both $\mathcal{H}^+$ and $\mathcal{H}^-$ are regular, achronal, null hypersurfaces in a neighborhood $\mathcal{O}$ of $\mathcal{S}$. The triplet $(\mathcal{S}, \mathcal{H}^+, \mathcal{H}^-)$ is called a *local, regular bifurcate horizon* in $\mathcal{O}$. The main result of the paper asserts if $(\mathcal{S}, \mathcal{H}^+, \mathcal{H}^-)$ is non-expanding (see Definition 2.1), then it must be a Killing bifurcate horizon. More precisely, we have the following theorem:

Theorem 1.1. *Given a local, regular, bifurcate, non-expanding horizon $(\mathcal{S}, \mathcal{H}^+, \mathcal{H}^-)$ in a smooth and time oriented Einstein–Maxwell space–time $(\mathcal{O}, g, F)$, there exists a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of S and a non-trivial Killing vector field K in $\mathcal{O}'$, which is tangent to the null generators of $\mathcal{H}^+$ and $\mathcal{H}^-$. Moreover, the Maxwell field F is also invariant with respect to K, i.e., $\mathcal{L}_K F = 0$.*

The vector field K is called the *Hawking vector field* in the literature. Its existence is already known (see [6]) under the assumption that the space–time is real analytic. In the work of Friedrich et al. [5], the authors showed, by solving wave equations, the existence of the Hawking vector field K without the analyticity assumption, but K could only be constructed inside the domain of dependence of $\mathcal{H}^+ \cup \mathcal{H}^-$ due to the fact that the corresponding wave equations are not well-posed outside this region. The new ingredient in this paper is to extend the Hawking vector field K to a full neighborhood of the bifurcate sphere $\mathcal{S}$, without making any additional regularity assumptions on the underlying space–time $(\mathcal{M}, g)$. We shall achieve this goal following an idea of Alexakis et al., who proved a corresponding result for vacuum space-times in [2].

We also prove the following theorem:

Theorem 1.2. *Given a local, regular, bifurcate and non-expanding horizon $(\mathcal{S}, \mathcal{H}^+, \mathcal{H}^-)$ in a smooth and time oriented Einstein–Maxwell space–time $(\mathcal{O}, g, F)$. Assume there is a Killing vector field T tangent to $\mathcal{H}^+ \cup \mathcal{H}^-$ which does not vanish identically on $\mathcal{S}$. Then there is a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of $\mathcal{S}$, such that we can find a rotational Killing vector Z in $\mathcal{O}'$, i.e., the orbits of Z are closed. Moreover, $[Z, T] = 0$. If in addition $\mathcal{L}_T F = 0$, then $\mathcal{L}_Z F = 0$.*

In reality, when one tries to prove global rigidity theorems for stationary space-times, one does not make the non-expanding assumption on the horizon $\mathcal{H}^+ \cup \mathcal{H}^-$, since it is well-known that the non-expansion is a consequence of the fact that the Killing vector field T is tangent to $\mathcal{H}^+ \cup \mathcal{H}^-$, see [6]. So the first theorem will produce a Hawking vector field K in a full neighborhood of $\mathcal{S}$. The rotational vector field given by Theorem 1.2 then can be written as a linear combination of T and K, i.e., there exists a constant λ such that the one parameter group of diffeomorphism on $\mathcal{M}$ generated by the vector field

$$Z = T + \lambda K$$

is a rotation with a period t_0. The part $\mathcal{L}_Z F = 0$ in the theorem follows immediately. In the proof, we will focus on the geometric construction of Z. We show that the period t_0 is exactly the period of rotations generated by T on the bifurcate sphere $\mathcal{S}$, while to determine λ, it suffices to know the action of T and K on a particular null geodesic on $\mathcal{H}^+ \cup \mathcal{H}^-$, see the proof for more details.

Once more, under the restrictive additional assumption of analyticity of the space–time $(\mathcal{M}, g)$, this second theorem is also known for Einstein vacuum space-times. It is usually called *Hawking's rigidity theorem*, see [6], which asserts that under some global causality, asymptotic flatness and connectivity assumptions, a stationary analytic black hole whose horizon is non-degenerate must be axially symmetric. In the smooth category, one can find a proof in [2] based on the idea that, under a suitable conformal rescaling of null generators on the bifurcate sphere, the level sets of the affine parameters of the null generators on the horizon should represent the integrable surface ruled by the closed rotational orbits.

These two theorems play an important role in the classification theory of stationary black holes, since they reduce the classifications to the cases which are covered by the well-known uniqueness theorems for electro-vac black holes in general relativity, see [3,4,6,9,11]. For more a historical account of this issue, we refer the reader to the paper [2].

We now describe the main ideas of the proofs. The first step towards the proof of Theorem 1.1 is to construct the Hawking vector field K. Since K is a Killing vector field, K satisfies the following covariant linear wave equations:

$$\square_g K_\alpha = -R_\alpha{}^\beta K_\beta \tag{1.1}$$

where $R_{\alpha\beta}$ is the Ricci curvature tensor for the Lorentzian metric g. We hope to reconstruct K by solving this wave equation. This is precisely the strategy used in [5]. According to a famous result of Rendall (see [10]), the equation can be solved in the domain of dependence if initial data is prescribed on the characteristic hypersurfaces, see [10] for the proof. So one needs to find the correct initial data for (1.1) on $\mathcal{H}^+ \cup \mathcal{H}^-$. The choice of the data can be rediscovered by the following heuristic argument: because K is Killing, the restriction of K to a geodesic should be a Jacobi field, so one may guess the initial data on $\mathcal{H}^+$ should be the non-trivial parallel Jacobi field $\underline{u}L$ where L is a null geodesic generator on $\mathcal{H}^+$ and $\underline{u}$ is the corresponding affine parameter, i.e., $L(\underline{u}) = 1$; another way to guess the initial data is to compute the explicit formula for the exact Kerr or Kerr–Newman families of black holes. The second step is to extend the vector field K to the bad region, i.e., the region with is not covered by the domain of dependence. While the Cauchy problem for (1.1) is ill-posed on the bad region, solving (1.1) will no longer work. We have to rely on the new techniques used in [2]. A careful calculation shows K also solves an ordinary differential equation (ODE) which is well-posed in the ill-posed region for (1.1). So one can extend K into the bad region by solving this ODE. This is the way we construct K in a full neighborhood of $\mathcal{S}$. Notice that although K is constructed, we still need to check that K is Killing since to derive Eq. (1.1) and the ODE we have already ignored a certain amount of information. One turns to proving the one parameter group ϕ_t generated by K acts isometrically. We need to show that, for each small t, the pull-back metric $\phi_t^* g$ must coincide with g, in view of the fact that they are both solutions of Einstein–Maxwell equations and coincide on $\mathcal{H}^+ \cup \mathcal{H}^-$. Now the Carleman type uniqueness techniques come into play, see also the results of Alexakis [1], Alexakis et al. [2] and Ionescu and Klainerman [7,8].

The paper is organized as follows. In Sect. 2, we construct a canonical null frame associated with the bifurcate horizon $(\mathcal{S}, \mathcal{H}^+, \mathcal{H}^-)$ and derive a set of partial differential equations (PDE) for various geometric quantities, as consequences of the non-expansion condition and the Einstein–Maxwell equations; in Sect. 3, we give a self-contained proof of Theorem 1.1 in the domain of dependence of $\mathcal{H}^+ \cup \mathcal{H}^-$, which is Proposition B.1 in [5]; in Sect. 4, based on the Carleman type estimates proved in [7,8], we extend the Hawking vector field to a full neighborhood of $\mathcal{S}$ which completes the proof of Theorem 1.1; the last section is devoted to a geometric proof of Theorem 1.2.

Notations

The Greek indices $\alpha, \beta, \gamma, \delta, \rho$ ranging from 1 to 4, and the Roman letters a, b, c from 1 to 2; one uses D_α to denote the covariant derivative D_{e_α}; the curvature tensor is defined by $R_{\alpha\beta\gamma\delta} = g(D_\alpha D_\beta e_\gamma - D_\beta D_\alpha e_\gamma, e_\delta)$, where $D_\alpha D_\beta X = D_\alpha(D_\beta X) - D_{D_\alpha e_\beta} X$; repeated indices are always understood as subject to the Einstein summation convention; since during the proof of our main theorems, we will keep shrinking the open neighborhood $\mathcal{O}$ of $\mathcal{S}$, we will keep denoting such neighborhoods by the same $\mathcal{O}$ for simplicity; we will often use the notation $X \lesssim Y$ whenever there exists some constant C so that $X \leq CY, C$ can depend on some given background metrics g and g' and background fields F and F'. To simplify the formulas, without losing information, we will use the $*$ notation. The expression $A * B$ is a linear combination of tensors, each formed by starting with $A \otimes B$, using the metric to take any number of contractions. So $A * B$ should be treated as a quadratic expression in A and B, while the exact numerical coefficients are irrelevant.

2. Preliminaries

We first set up the double null foliation in $\mathcal{O}$. One can choose a smooth future-directed null pair $(L, \underline{L})$ along $\mathcal{S}$ with normalization

$$g(L, L) = g(\underline{L}, \underline{L}) = 0, \quad g(L, \underline{L}) = -1$$

such that L is tangent to $\mathcal{H}^+$ and $\underline{L}$ is tangent to $\mathcal{H}^-$. In a small neighborhood of $\mathcal{S}$, we extend L along the null geodesic generators of $\mathcal{H}^+$ via parallel transport; we also extend $\underline{L}$ along the null geodesic generators of $\mathcal{H}^-$ via parallel transport. So $D_L L = 0$ and $D_{\underline{L}} \underline{L} = 0$. We now define two optical functions u and $\underline{u}$ near $\mathcal{S}$. The function $\underline{u}$ (respectively, u) is defined along $\mathcal{H}^+$ (respectively, $\mathcal{H}^-$) by setting initial value $\underline{u} = 0$ (respectively, $u = 0$) on $\mathcal{S}$ and solving $L(\underline{u}) = 1$ (respectively, $\underline{L}(u) = 1$). Let $\mathcal{S}_{\underline{u}}$ (respectively, $\mathcal{S}_u$) be the level surfaces of $\underline{u}$ (respectively, u) along $\mathcal{H}^+$ (respectively, $\mathcal{H}^-$). We define $\underline{L}$ (respectively, L) on each point of the hypersurface $\mathcal{H}^+$ (respectively, $\mathcal{H}^-$) to be unique, future directed null vector orthogonal to the surface $\mathcal{S}_{\underline{u}}$ (respectively, $\mathcal{S}_u$) passing though that point and such that $g(L, \underline{L}) = -1$. The null hypersurface $\mathcal{H}^-_{\underline{u}}$ (respectively, $\mathcal{H}^+_u$) is defined to be the set of null geodesics initiating on $\mathcal{S}_{\underline{u}} \subset \mathcal{H}^+$ (respectively, $\mathcal{S}_u \subset \mathcal{H}^-$) in the direction of $\underline{L}$ (respectively, L). We require the null hypersurfaces $\mathcal{H}^-_{\underline{u}}$ (respectively, $\mathcal{H}^+_u$) to be the level sets of the function $\underline{u}$ (respectively, u). By this condition, u and $\underline{u}$ are extended into a neighborhood of $\mathcal{S}$ from the null hypersurface $\mathcal{H}^+ \cup \mathcal{H}^-$. Then we can extend both L and $\underline{L}$ into a neighborhood of $\mathcal{S}$ as gradients of the optical functions u and $\underline{u}$:

$$L = -\mathbf{g}^{\mu\nu} \partial_\mu u \partial_\nu, \quad \underline{L} = -\mathbf{g}^{\mu\nu} \partial_\mu \underline{u} \partial_\nu.$$

Since u and $\underline{u}$ are null optical functions, we know

$$g(L, L) = g(\underline{L}, \underline{L}) = 0$$

while $g(L, \underline{L}) = -1$ only holds on the null surface $\mathcal{H}^+ \cup \mathcal{H}^-$. Moreover, we have

$$L(\underline{u}) = 1 \quad \text{on } \mathcal{H}^+ \quad \text{and} \quad \underline{L}(u) = 1 \quad \text{on } \mathcal{H}^-.$$

We define $\mathcal{S}_{u\underline{u}} = \mathcal{H}^+_u \cap \mathcal{H}^-_{\underline{u}}$. Using the null pair $(L, \underline{L})$ one can choose a null frame $\{e_1.e_2, e_3 = \underline{L}, e_4 = L\}$ such that

$$g(e_a, e_b) = \delta_{ab}, \quad g(e_a, e_3) = g(e_a, e_4) = 0, \quad a, b = 1, 2.$$

At each point $p \in \mathcal{S}_{u\underline{u}} \subset \mathcal{O}, e_1, e_2$ form an orthonormal frame along the 2-surface $\mathcal{S}_{u\underline{u}}$. It is easy to see that we have a lot of freedom to choose e_1 and e_2. To make use of this point, we shall modify the frame by Fermi transport later. Recall the null second fundamental forms $\chi, \underline{\chi}$ and torsion ζ are defined on $\mathcal{H}^+ \cup \mathcal{H}^-$ via the given null pair $(L, \underline{L})$:

$$\chi_{ab} = g(D_{e_a} L, e_b), \quad \underline{\chi}_{ab} = g(D_{e_a} \underline{L}, e_b), \quad \zeta_a = g(D_{e_a} L, \underline{L}).$$

The traces of χ and $\underline{\chi}$ are defined by $tr\chi = \delta^{ab}\chi_{ab}$ and $tr\underline{\chi} = \delta^{ab}\underline{\chi}_{ab}$.

Definition 2.1. We say that $\mathcal{H}^+$ is non-expanding if $tr\chi = 0$ on $\mathcal{H}^+$; similarly $\mathcal{H}^-$ is non-expanding if $tr\underline{\chi} = 0$ on $\mathcal{H}^-$. The bifurcate horizon $(\mathcal{S}, \mathcal{H}^+, \mathcal{H}^-)$ is called non-expanding if both $\mathcal{H}^+, \mathcal{H}^-$ are non-expanding.

The non-expanding condition imposes a very strong restriction on the geometry of the Einstein–Maxwell space–time. We recall the Einstein–Maxwell equations:

$$\begin{cases} R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = T_{\alpha\beta} \\ D_{[\alpha}F_{\beta\gamma]} = 0 \\ D^\alpha F_{\alpha\beta} = 0 \end{cases}$$

where $T_{\alpha\beta} = F_\alpha{}^\mu F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}$ is the energy–momentum tensor for the corresponding electromagnetic field. Since the dimension of the underlying manifold is four, the field theory is conformal, i.e., $trT = 0$. So by tracing the first equation in the system, we know the scalar curvature $R = 0$. One then rewrites the system as

$$\begin{cases} R_{\alpha\beta} = F_\alpha{}^\mu F_{\beta\mu} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu} \\ D_{[\alpha}F_{\beta\gamma]} = 0 \\ D^\alpha F_{\alpha\beta} = 0 \end{cases} \tag{2.1}$$

We recall that the positive energy condition is valid for Einstein–Maxwell energy–momentum tensor, i.e., $T(X, Y) \geq 0$ where (X, Y) is an arbitrary pair of future-directed causal vectors. Let

$$\hat{\chi}_{ab} = \chi_{ab} - \frac{1}{2}tr\chi\delta_{ab}$$

be the traceless part of χ, so on $\mathcal{H}^+$, according to the Raychaudhuri equation:

$$L(tr\chi) = -R_{44} - |\hat{\chi}|^2 - \frac{1}{2}(tr\chi)^2,$$

The non-expansion condition on $\mathcal{H}^+$ implies

$$R_{44} + |\hat{\chi}|^2 = 0,$$

Thanks to the positive energy condition, one knows $R_{44} = T_{44} \geq 0$, so

$$R_{44} = 0, \quad \hat{\chi} = 0 \quad \text{on } \mathcal{H}^+.$$

So $\chi = 0$ on $\mathcal{H}^+$. According to the untraced formulation of the Raychaudhuri equation:

$$L(\chi) + \chi^2 + R(-, L)L = 0,$$

we know for all $X \in T\mathcal{H}^+$,

$$R(X, L)L = 0$$

In view of the first equation in (2.1), $R_{44} = 0$ implies $F_{4a} = 0$, and the vanishing of these quantities implies $R_{4a} = 0$. Combined with $R(X, L)L = 0$, we know $R_{4aba} = 0$. To summarize, the non-expansion condition implies, on the null hypersurface $\mathcal{H}^+$, that

$$\begin{cases} \chi = 0 \\ R_{4a} = 0 \\ R_{4aba} = 0 \\ R_{344a} = 0 \\ F_{4a} = 0 \end{cases} \tag{2.2}$$

Similar identities hold on $\mathcal{H}^-$ when one replaces the index 4 by 3 in this set of equations. It is precisely this set of geometric information that we shall use in the proof of our main theorems. Recall also our choice of the frame e_1, e_2 is arbitrary on $\mathcal{H}^+$. Since we know $\chi = 0$, we can make this choice more rigid by using Fermi transport along L, i.e., we first pick a local orthonormal basis on the bifurcate sphere $\mathcal{S}$, then use the Lie transport relation $\mathcal{L}_L e_a = 0$ to get a basis on $\mathcal{S}_{\underline{u}}$ (which is not an orthonormal frame in general), the vanishing of χ on $\mathcal{H}^+$ guarantees $\{e_1, e_2\}$ is still an orthonormal frame. On $\mathcal{H}^+$, by virtue of the frame $\{e_1.e_2, e_3 = \underline{L}, e_4 = L\}$, we have the following Ricci equations:

$$\begin{cases} D_4 L = 0, & D_a L = -\zeta_a L, \\ D_4 \underline{L} = -\zeta_a e_a & D_a \underline{L} = \underline{\chi}_{ab} e_b + \zeta_a \underline{L} \\ D_4 e_a = -\zeta_a L & D_a e_b = \nabla_a e_b + \underline{\chi}_{ab} L \end{cases} \tag{2.3}$$

where $\nabla_a e_b$ is the projection of $D_a e_b$ onto the surface $\mathcal{S}_u$. A corresponding set of equations hold on $\mathcal{H}^-$. We shall use the following lemma:

Lemma 2.2. *On $\mathcal{H}^+$, we have*

$$R(-, L, \underline{L}, -) = -\nabla\zeta - \nabla_4 \underline{\chi} + \zeta \otimes \zeta$$

i.e.,

$$R_{a43b} = -\nabla_a \zeta_b - \nabla_4 \underline{\chi}_{ab} + \zeta_a \zeta_b.$$

where ∇ denotes the projection of D onto $\mathcal{S}_{\underline{u}}$; a corresponding result holds on $\mathcal{H}^-$.

Proof. For $X, Y \in T\mathcal{S}_{\underline{u}}$, by definition, one has

$$\begin{aligned} &R(X, L, \underline{L}, Y) \\ &\quad = g(D_X D_4 \underline{L}, Y) - g(D_4 D_X \underline{L}, Y) - g(D_{D_X L} \underline{L}, Y) + g(D_{D_4 X} \underline{L}, Y) \end{aligned}$$

We replace X and Y by e_a and e_b, so

$$\begin{aligned} R_{a43b} &= g(D_a(\zeta^c e_c), e_b) - g(D_4(\chi_a{}^c e_c + \zeta_a \underline{L}), e_b) + \zeta_a g(D_4 \underline{L}, e_b) \\ &\quad + g(D_{\nabla_4 e_a - \zeta_a L} \underline{L}, e_b) \\ &= -(\nabla\zeta)(e_a, e_b) - g(D_4(\chi_a{}^c e_c), e_b) + g(D_{\nabla_4 e_a} \underline{L}, e_b) - \zeta_a g(D_4 \underline{L}, e_b) \\ &= -\nabla_a \zeta_b - \nabla_4 \underline{\chi}_{ab} + \zeta_a \zeta_b. \end{aligned}$$

This completes the proof. □

3. The Hawking vector Field Inside the Black Hole

We define the following four regions $\mathcal{I}^{++}, \mathcal{I}^{--}, \mathcal{I}^{+-}$ and $\mathcal{I}^{-+}$:

$$\begin{aligned} \mathcal{I}^{++} &= \{p \in \mathcal{O} | u(p) \geq 0 \text{ and } \underline{u}(p) \geq 0\}, \\ \mathcal{I}^{--} &= \{p \in \mathcal{O} | u(p) \leq 0 \text{ and } \underline{u}(p) \leq 0\}, \\ \mathcal{I}^{+-} &= \{p \in \mathcal{O} | u(p) \geq 0 \text{ and } \underline{u}(p) \leq 0\}, \\ \mathcal{I}^{-+} &= \{p \in \mathcal{O} | u(p) \leq 0 \text{ and } \underline{u}(p) \geq 0\}. \end{aligned} \tag{3.1}$$

In this section, we prove the following proposition:

Proposition 3.1. *Under the assumptions of Theorem 1.1, in a small neighborhood $\mathcal{O}$ of $\mathcal{S}$, there exists a smooth Killing vector field K in $\mathcal{O} \cap (\mathcal{I}^{++} \cup \mathcal{I}^{--})$ such that*

$$K = \underline{u} L - u \underline{L} \quad \text{on } (\mathcal{H}^+ \cup \mathcal{H}^-) \cap \mathcal{O}.$$

Moreover, $\mathcal{L}_K F = 0$ and $[\underline{L}, K] = -\underline{L}$.

The region $\mathcal{O} \cap (\mathcal{I}^{++} \cup \mathcal{I}^{--})$ is the domain of dependence of $\mathcal{H}^+ \cup \mathcal{H}^-$. As we mentioned in the introduction, by using the Newman–Penrose formalism, the first part of the proposition is shown by Friedrich et al. [5]. For the sake of completeness, we provide a direct proof without using Newman–Penrose formalism. As mentioned in the introduction, we consider the following characteristic initial value problem

$$\begin{cases} \Box_g K_\alpha = -R_\alpha{}^\beta K_\beta, \\ \quad K = \underline{u} L - u \underline{L} \quad \text{on } (\mathcal{H}^+ \cup \mathcal{H}^-) \cap \mathcal{O}. \end{cases} \tag{3.2}$$

According to [10], this system of equation is well-posed in $\mathcal{O} \cap (\mathcal{I}^{++} \cup \mathcal{I}^{--})$. A smooth vector field K is now constructed by solving (3.2) in the domain of dependence of $\mathcal{H}^+ \cup \mathcal{H}^-$. To show K is indeed a Killing vector field, one has to show the deformation tensor of K

$$\pi_{\alpha\beta} = \mathcal{L}_K g = D_\alpha K_\beta + D_\beta K_\alpha$$

is zero in $\mathcal{O} \cap (\mathcal{I}^{++} \cup \mathcal{I}^{--})$.

Since K solves (3.2), by commuting derivatives, we know the deformation tensor $\pi_{\alpha\beta}$ solves the following covariant wave equation:

$$\Box_g \pi_{\alpha\beta} = -2R^\rho{}_{\alpha\beta}{}^\delta \pi_{\rho\delta} + R_{\alpha\rho}\pi^\rho{}_\beta + R_{\beta\rho}\pi^\rho{}_\alpha - 2\mathcal{L}_K R_{\alpha\beta}$$

The geometric part of Einstein–Maxwell equations (2.1) provides

$$\begin{aligned}\mathcal{L}_K R_{\alpha\beta} &= \mathcal{L}_K T_{\alpha\beta}\\ &= F_\alpha{}^\rho \mathcal{L}_K F_{\beta\rho} + F_\beta{}^\rho \mathcal{L}_K F_{\alpha\rho} - \pi_{\rho\delta} F_\alpha{}^\rho F_\beta{}^\delta\\ &\quad -\frac{1}{4}\pi_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g_{\alpha\beta}F^{\mu\nu}\mathcal{L}_K F_{\mu\nu} + \frac{1}{2}g_{\alpha\beta}\pi_{\rho\delta}F^\delta{}_\gamma F^{\rho\gamma}\end{aligned}$$

Combined with the electromagnetic part of the Einstein–Maxwell equations (2.1), this formula allows one to derive a set of PDEs satisfied by $\mathcal{L}_K F_{\alpha\beta}$:

$$\begin{cases} D_{[\alpha}\mathcal{L}_K F_{\beta\gamma]} = 0\\ D^\alpha \mathcal{L}_K F_{\alpha\beta} = \pi_{\alpha\gamma}D^\gamma F^\alpha{}_\beta + \frac{1}{2}(D_\alpha \pi_{\beta\gamma} + D_\beta \pi_{\alpha\gamma} - D_\gamma \pi_{\alpha\beta})\end{cases}$$

Put all the equations together, one concludes that $\pi_{\alpha\beta}$ and $\mathcal{L}_K F_{\alpha\beta}$ solve the characteristic initial value problem for the following closed symmetric hyperbolic system:

$$\begin{cases} \Box_g \pi_{\alpha\beta} = -2R^\rho{}_{\alpha\beta}{}^\delta \pi_{\rho\delta} + R_{\alpha\rho}\pi^\rho{}_\beta + R_{\beta\rho}\pi^\rho{}_\alpha\\ \qquad -2(F_\alpha{}^\rho \mathcal{L}_K F_{\beta\rho} + F_\beta{}^\rho \mathcal{L}_K F_{\alpha\rho} - \pi_{\rho\delta}F_\alpha{}^\rho F_\beta{}^\delta)\\ \qquad +\frac{1}{2}\pi_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + g_{\alpha\beta}F^{\mu\nu}\mathcal{L}_K F_{\mu\nu} - g_{\alpha\beta}\pi_{\rho\delta}F^\delta{}_\gamma F^{\rho\gamma}\\ D_{[\alpha}\mathcal{L}_K F_{\beta\gamma]} = 0\\ D^\alpha \mathcal{L}_K F_{\alpha\beta} = \pi_{\alpha\gamma}D^\gamma F^\alpha{}_\beta + \frac{1}{2}(D_\alpha \pi_{\beta\gamma} + D_\beta \pi_{\alpha\gamma} - D_\gamma \pi_{\alpha\beta})\end{cases} \tag{3.3}$$

So to show $\pi_{\alpha\beta} = 0$ and $\mathcal{L}_K F = 0$ in $\mathcal{O}$, it suffices to show

$$\pi_{\alpha\beta} = 0 \quad \mathcal{L}_K F = 0 \quad \text{on } \mathcal{H}^+ \cup \mathcal{H}^-. \tag{3.4}$$

We only check (3.4) on $\mathcal{H}^+$; on $\mathcal{H}^-$, the argument is exactly the same. In view of the expression of $K = \underline{u}L$ on $\mathcal{H}^+$ (since $u = 0$ on $\mathcal{H}^+$) and (2.3), it is easy to show

$$\begin{cases} D_a K_b = D_4 K_a = D_a K_4 = D_4 K_4 = 0, \quad D_4 K_3 = -1\\ D_c D_a K_b = D_4 D_a K_b = D_b D_4 K_a = D_4 D_4 K_a = D_a D_b K_4 = 0,\\ D_4 D_a K_4 = D_a D_4 K_4 = D_4 D_4 K_4 = D_4 D_4 K_3 = D_a D_4 K_3 = 0.\end{cases} \tag{3.5}$$

We see immediately that as long as a component of $\pi_{\alpha\beta}$ does not involve the direction $\underline{L}$, it is zero. More precisely,

$$\pi_{ab} = \pi_{4a} = \pi_{44} = 0 \quad \text{on } \mathcal{H}^+ \tag{3.6}$$

To prove the remaining components of π vanish, we need to make serious use of (3.2) to get derivatives in the $\underline{L}$ direction. Equation (3.2) gives

$$D_3 D_4 K_\beta + D_4 D_3 K_\beta = \sum_{a=1}^{2} D_a D_a K_\beta + R_\beta{}^\rho K_\rho$$

Combined with the curvature identity $D_3 D_4 K_\beta - D_4 D_3 K_\beta = -R_{34\beta}{}^\rho K_\rho$, we have

$$2D_4D_3K_\beta = \sum_{a=1}^{2} D_aD_aK_\beta + R_{\beta\rho}K^\rho + R_{34\beta\rho}K^\rho \tag{3.7}$$

Claim 3.2. *We have* $D_3K_4 = 1, D_aD_3K_4 = D_4D_3K_4 = 0.$

Proof. We set $\beta = 4$ in (3.7), it is easy to check the left hand side of (3.7) is

$$2D_4D_3K_4 = 2L(D_3K_4)$$

while the right-hand side is 0 by (2.2). So $L(D_3K_4) = 0$ on $\mathcal{H}^+$. It implies the value of D_3K_4 on $\mathcal{H}^+$ is determined by its value on $\mathcal{S}$ which is 1. The other two identities can also be proved in a similar way. □

In particular, Claim 3.2 implies $\pi_{34} = 0$.

Claim 3.3. *We have* $D_aK_3 = \underline{u}\zeta_a$ *and* $D_3K_a = -\underline{u}\zeta_a$.

Proof. The first identity in the claim is easy to check by direct computations; we now prove the second one. We first show that

$$L(\zeta_a) = 0. \tag{3.8}$$

We use (2.2):

$$L(\zeta_a) = L(g(D_aL,\, \underline{L})) = g(D_aL, D_4\, \underline{L}) + g(D_4D_aL,\, \underline{L}) \tag{3.9}$$

$$= g(D_4D_aL,\, \underline{L}) = R_{4a43} = 0 \tag{3.10}$$

We now set $\beta = b$ in (3.7), this implies $D_4D_3K_b = 0$, in view of $D_3K_4 = 1$, we can show

$$L(D_3K_a) = -\zeta_a$$

Combined with (3.8), this shows $D_3K_a = -\underline{u}\zeta_a$. □

In particular, Claim 3.3 implies $\pi_{3a} = 0$.

Claim 3.4. *We have* $\pi_{33} = 2D_3K_3 = 0.$

Proof. To prove the claim, one needs help from the Einstein–Maxwell equations (2.1). Since $F_{4a} = 0$, we have

$$\underline{L}(R_{44}) = \underline{L}(F_{4a}^2) = 2F_{4a}\, \underline{L}(F_{4a}) = 0$$

which implies

$$\underline{L}(R_{4aa4}) = 0 \tag{3.11}$$

Recall the second Bianchi identity

$$D_4R_{3aa4} + D_3R_{a4a4} + D_aR_{43a4} = 0 \tag{3.12}$$

A simple computation with the help (2.2) and (3.11) shows the last two terms in (3.12) are zero. So we have

$$L(R_{3aa4}) = D_4R_{3aa4} = 0. \tag{3.13}$$

We compute $L(tr\underline{\chi})$ along $\mathcal{H}^+$:

$$L(tr\underline{\chi}) = L(g(D_a\, \underline{L}, e_a)) = g(D_4D_a\, \underline{L}, e_a) + g(D_a\, \underline{L}, D_4e_a)$$

$$= R_{4a3a} + g(D_aD_4\, \underline{L}, e_a) + |\zeta|^2 = R_{4a3a} + |\zeta|^2 - g(D_a\zeta^\sharp, e_a)$$

where $\zeta^\sharp$ is the dual vector field of ζ. In view of (3.13) and (3.8), we have

$$\begin{aligned} & L(L(tr\underline{\chi})) \\ & \quad = -L(g(D_a\zeta^\sharp, e_a)) = -g(D_4 D_a \zeta^\sharp, e_a) = -R_{4aba}\zeta^b - g(D_a D_4 \zeta^\sharp, e_a) \\ & \quad = -g(D_a D_4(\zeta^b e_b), e_a) = -\zeta^b g(D_a(\zeta_b L), e_a) \end{aligned}$$

This shows

$$L(L(tr\underline{\chi})) = 0. \tag{3.14}$$

Now we are ready to prove the claim. We set $\beta = 3$ in (3.7), so

$$\begin{aligned} 2D_4D_3K_3 &= D_aD_aK_3 + R_{3\rho}K^\rho + R_{343\rho}K^\rho = D_aD_aK_3 + \underline{u}R_{34} + \underline{u}R_{3434} \\ &= D_aD_aK_3 + \underline{u}R_{3aa4} \end{aligned}$$

By Lemma 2.2, we have

$$\begin{aligned} R_{3aa4} &= -\nabla_a\zeta_a - \nabla_4\underline{\chi}_{aa} + \zeta_a^2 = -(\nabla_{e_a}\zeta)(e_a) - \nabla_4(tr\underline{\chi}) + |\zeta|^2 \\ &= -div\zeta + \zeta(\nabla_a e_a) - L(tr\underline{\chi}) + |\zeta|^2 \end{aligned}$$

We also compute

$$D_aD_aK_3 = \underline{u}(div\zeta - \zeta(\nabla_a e_a) - |\zeta|^2) + tr\underline{\chi}$$

The previous computations showed

$$2D_4D_3K_3 = tr\underline{\chi} - \underline{u}L(tr\underline{\chi})$$

So in view of (3.14)

$$L(D_4D_3K_3) = -\underline{u}L(L(tr\underline{\chi})) = 0 \tag{3.15}$$

On $\mathcal{S}$, it is easy to see $D_4D_3K_3 = 0$, so $D_4D_3K_3 = 0$ on $\mathcal{H}^+$, which once again implies $D_3K_3 = 0$ by solving transport equations along L. □

So we proved $\pi_{\alpha\beta} = 0$ on $\mathcal{H}^+$. One still needs to show $\mathcal{L}_K F_{\alpha\beta} = 0$.

Claim 3.5. *We have the following identities*:

$$D_aF_{4b} = D_4F_{4b} = D_4F_{ab} = D_4F_{43} = 0 \tag{3.16}$$

Proof. We will use (2.2) repeatedly:

$$D_aF_{4b} = (D_aF)(L\otimes e_b) = e_a(F_{4b}) - F(D_aL\otimes e_b) - F(L\otimes D_ae_b) = 0.$$

Same argument shows $D_4F_{4b} = 0$. We use Bianchi identity:

$$D_4F_{ab} = -D_aF_{b4} - D_bF_{4a} = 0.$$

We now use the divergence free equation in Einstein–Maxwell equations (2.1):

$$D^\alpha F_{\alpha 4} = 0 \Rightarrow D_aF_{a4} - D_4F_{34} = 0$$

so $D_4F_{34} = D_aF_{a4} = 0$. □

Claim 3.6. *On $\mathcal{H}^+$, we have*

$$\mathcal{L}_K F = 0 \tag{3.17}$$

Proof. Recall that

$$\mathcal{L}_K F_{\alpha\beta} = D_K F_{\alpha\beta} + g^{\rho\delta} D_\alpha K_\delta F_{\rho\beta} + g^{\rho\delta} D_\beta K_\delta F_{\alpha\rho}.$$

We show each component of $\mathcal{L}_K F$ vanishes on $\mathcal{H}^+$. The following three terms are relatively easy:

$$\begin{aligned}
\mathcal{L}_K F_{ab} &= D_K F_{ab} + g^{\rho\delta} D_a K_\delta F_{\rho b} + g^{\rho\delta} D_b K_\delta F_{a\rho} = \underline{u} D_4 F_{ab} = 0.\\
\mathcal{L}_K F_{a4} &= D_K F_{a4} + g^{\rho\delta} D_a K_\delta F_{\rho 4} + g^{\rho\delta} D_4 K_\delta F_{a\rho} = \underline{u} D_4 F_{a4} = 0.\\
\mathcal{L}_K F_{43} &= D_K F_{43} + g^{\rho\delta} D_4 K_\delta F_{\rho 3} + g^{\rho\delta} D_3 K_\delta F_{4\rho}\\
&= D_K F_{43} - D_4 K_3 F_{43} - D_3 K_4 F_{43} = \underline{u} D_4 F_{43} - \pi_{43} F_{43} = 0
\end{aligned}$$

We need some preparation to show that the most difficult term $\mathcal{L}_K F_{3a}$ vanishes. From the electromagnetic part of the Einstein–Maxwell equations (2.1), we have

$$D_4 F_{3b} - D_3 F_{4b} + D_b F_{43} = 0$$

and

$$-(D_4 F_{3b} + D_3 F_{4b}) + D_a F_{ab} = 0$$

So one has

$$2 D_4 F_{3b} = D_a F_{ab} - D_b F_{43} \tag{3.18}$$

Apply the vector field L on (3.18), we have

$$\begin{aligned}
&2L(D_4 F_{3b})\\
&\quad = L(D_a F_{ab}) - L(D_b F_{43}) = D_4 D_a F_{ab} - D_4 D_b F_{43}\\
&\quad = (D_a D_4 F_{ab} - R_{4aa}{}^{\rho} F_{\rho b} - R_{4ab}{}^{\rho} F_{a\rho}) - (D_b D_4 F_{43} - R_{4b4}{}^{\rho} F_{\rho 3} - R_{4a3}{}^{\rho} F_{4\rho})\\
&\quad = D_a D_4 F_{ab} - D_b D_4 F_{43}\\
&\quad = [e_a(D_4 F_{ab}) - D_4 F(D_a e_a, e_b) - D_4 F(e_a, D_a e_b)]\\
&\qquad -[e_b(D_4 F_{43}) - D_4 F(D_b L, \underline{L}) - D_4 F(L, D_a \underline{L})].
\end{aligned}$$

In view of (2.3), one concludes that

$$L(D_4 F_{3b}) = 0 \tag{3.19}$$

Now we compute

$$\begin{aligned}
\mathcal{L}_K F_{3b} &= D_K F_{3b} + g^{\rho\delta} D_3 K_\rho F_{\delta b} + g^{\rho\delta} D_b K_\rho F_{3\delta}\\
&= \underline{u} D_4 F_{3b} + D_3 K_a F_{ab} - D_3 K_4 F_{3b} - D_b K_3 F_{34}\\
&= \underline{u} D_4 F_{3b} - \underline{u} \zeta_a F_{ab} - F_{3b} - \underline{u} \zeta_b F_{34}
\end{aligned}$$

In particular, this shows $\mathcal{L}_K F_{3b} = 0$ on $\mathcal{S}$. Notice that $L(F_{4b}) = L(F_{ab}) = L(F_{43}) = 0$, now apply L on $\mathcal{L}_K F_{3b}$, so we have

$$\begin{aligned}
&L(\mathcal{L}_K F_{3b})\\
&\quad = L(\underline{u} D_4 F_{3b}) - L(\underline{u} \zeta_a F_{ab}) - L(F_{3b}) - L(\underline{u} \zeta_b F_{34})\\
&\quad \overset{(3.19)}{=} D_4 F_{3b} - \zeta_a F_{ab} - [D_4 F_{3b} + F(D_4 \underline{L}, E_b) + F(\underline{L}, D_4 e_b)] - \zeta_b F_{34} = 0
\end{aligned}$$

Now solving this ODE on $\mathcal{H}^+$ completes the proof. □

Remark 3.7. It follows from the previous computation that $D_\alpha \pi_{\beta\gamma} = 0$ on $\mathcal{H}^+$. In fact, $D_a \pi_{\alpha\beta} = 0$ and $D_4 \pi_{\alpha\beta} = 0$ trivially comes from the fact that $\pi_{\alpha\beta} = 0$ on $\mathcal{H}^+$; to see $D_3 \pi_{\alpha\beta} = 0$, we need to investigate the first equation in (3.3), in view of the facts that $\pi_{\alpha\beta} = 0$ and $\mathcal{L}_K F = 0$, this gives

$$D_4 D_3 \pi_{\alpha\beta} + D_3 D_4 \pi_{\alpha\beta} = 0.$$

Combined with the curvature identity

$$D_4 D_3 \pi_{\alpha\beta} - D_3 D_4 \pi_{\alpha\beta} = -R_{34\alpha}{}^{\rho} \pi_{\rho\beta} - R_{34\beta}{}^{\rho} \pi_{\alpha\rho} = 0,$$

it gives $L(D_3 \pi_{\alpha\beta})$=0. Once more, by solving an ODE on $\mathcal{H}^+, D_3 \pi_{\alpha\beta} = 0$ follows from the fact that $D_3 \pi_{\alpha\beta}$ vanishes on $\mathcal{S}$.

We now prove the last statement of Proposition 3.1: $[\,\underline{L}, K] = -\,\underline{L}$ in the domain of dependence. We claim that

$$\begin{cases} D_3 W = -D_W \,\underline{L} \quad \text{where } W = [\,\underline{L}, K] + \,\underline{L}, \\ \quad W = 0 \quad \text{on } \mathcal{H}^+ \cap \mathcal{O}. \end{cases} \tag{3.20}$$

Since K is Killing vector field, for arbitrary vector fields X and Y, we have

$$\mathcal{L}_K (D_X Y) = D_X (\mathcal{L}_K Y) + D_{\mathcal{L}_K X} Y.$$

Therefore,

$$\begin{aligned} D_3 W &= D_3(-\mathcal{L}_K \,\underline{L} + \,\underline{L}) = -D_3(\mathcal{L}_K \,\underline{L}) = -(\mathcal{L}_K (D_3 \,\underline{L}) - D_{\mathcal{L}_K \,\underline{L}} \,\underline{L}) \\ &= D_{\mathcal{L}_K \,\underline{L}} \,\underline{L} = -D_{[\,\underline{L}, K] + \,\underline{L}} \,\underline{L} = -D_W \,\underline{L} \end{aligned}$$

It remains to show $W = 0$ on $\mathcal{H}^+$.

$$W = D_3 K - D_K \,\underline{L} + \,\underline{L} = D_3 K - \underline{u} D_4 \,\underline{L} + \,\underline{L}$$

Since we have already computed the components $D_3 K_\alpha$, it is straightforward to show that $W = 0$ on $\mathcal{H}^+$. By solving the ODE (3.20), we get $W = 0$ in the domain of dependence. This completes the proof.

4. The Hawking Vector Field Outside the Black Hole

In the previous section we constructed the Hawking vector field K inside the black hole region. To extend K to a full neighborhood of the bifurcate sphere $\mathcal{S}$, because the characteristic initial value problem is not well-posed on the full neighborhood as we explained in the introduction, we need to rely on a completely different strategy. The idea is, instead of solving a hyperbolic system, we now solve the ODE $[\,\underline{L}, K] = -\,\underline{L}$ for K. This ODE is well-posed in the complement of the domain of dependence since $\underline{L}$ is transversal to $\mathcal{H}^+$. That is the construction of K. One has to show that K constructed in this way is a Killing vector field. Let ϕ_t be the one parameter family of diffeomorphisms generated by K. When t is small, both of (g, F) and $(\phi_t^* g, \phi_t^* F)$ solve the Einstein–Maxwell equations and they coincide on $\mathcal{H}^+ \cup \mathcal{H}^-$. To prove K is Killing, it suffices to show (g, F) and $(\phi_t^* g, \phi_t^* F)$ coincide in a full neighborhood of $\mathcal{S}$.

We first define a vector field K' by setting $K' = \underline{u} L$ on $\mathcal{H}^+ \cap \mathcal{O}$ and solving the ODE $[\,\underline{L}, K'] = -\,\underline{L}$. The vector field K' is defined and smooth in a full neighborhood $\mathcal{O}$ of $\mathcal{S}$ (since $\underline{L} \neq 0$ on $\mathcal{S}$). Moreover, in view of Proposition 3.1, K' coincides with K in $\mathcal{I}^{++} \cup \mathcal{I}^{--}$. Thus $K := K'$ defines the desired extension. This construction is summarized in the following lemma:

Lemma 4.1. *There exists a smooth extension of the vector field K to a full neighborhood $\mathcal{O}$ of S such that*

$$[\underline{L}, K] = -\underline{L} \quad \text{in } \mathcal{O}. \tag{4.1}$$

Let $g_t = \phi_t^* g$ and $\underline{L}_t = (\phi_{-t})_* \underline{L}$. In view of (4.1) of K, we know that $\frac{d}{dt}\underline{L}_t = -\underline{L}_t$. This implies that

$$\underline{L}_t = e^{-t}\,\underline{L}. \tag{4.2}$$

Let D^t be the Levi–Civita connection of g_t, by the tensorial nature, we know that $D^t_{\underline{L}_t}\underline{L}_t = D_{\underline{L}}\underline{L} = 0$, (4.2) implies that $0 = D^t_{\underline{L}_t}\underline{L}_t = e^{-2t} D^t_{\underline{L}}\underline{L}$. This proves the following lemma

Lemma 4.2. *Assume K is the smooth vector field constructed in* (4.1) *and D^t the Levi–Civita connection of the metric $\phi_t^* g$. We have*

$$D^t_{\underline{L}}\underline{L} = 0 \quad \text{in } \mathcal{O}. \tag{4.3}$$

To summarize, let $F_t = \phi_t^* F$, then we have a family of metrics and two forms (g_t, F_t) which solves the Einstein–Maxwell equations (2.1). Moreover, they coincide in the domain of dependence of $\mathcal{H}^+ \cup \mathcal{H}^-$ and for each t, one has $D^t_{\underline{L}}\underline{L} = 0$. So Theorem 1.1 is a consequence of the following uniqueness statement:

Proposition 4.3. *Assume in a full neighborhood $\mathcal{O}$ of $\mathcal{S}$, g' is a smooth Lorentzian metric and F' is a smooth two form, such that (g', F') solves Einstein–Maxwell equations* (2.1)*. If*

$$\begin{cases} g' = g \qquad \text{and} \quad F' = F \quad \text{in } (\mathcal{I}^{++} \cup \mathcal{I}^{--}) \cap \mathcal{O}, \\ D'_{\underline{L}}\underline{L} = 0 \qquad \text{in } \mathcal{O}, \end{cases}$$

where D' denotes the Levi–Civita connection of the metric g'. Then $g' = g$ and $F' = F$ in a full neighborhood $\mathcal{O}' \subset \mathcal{O}$ of $\mathcal{S}$.

The corresponding proposition for Einstein vacuum space-times was first proved in [1]. A simplified version can be found in [2]. In [7], the authors proved uniqueness results for covariant semi-linear wave equations of a fixed metric. But for the uniqueness at the level of metrics, since the corresponding PDEs are quasi-linear, one has to couple the system with a system of ODEs to recover the semi-linearity. In this section, we use this idea to prove uniqueness for the full curvature tensor and the electromagnetic field. Since the metric is uniquely determined by the curvature tensor, that will prove Proposition 4.3.

Proof. We first derive a system of covariant wave equations for the full curvature tensor $R_{\alpha\beta\gamma\delta}$ of the metric g and $F_{\alpha\beta}$. Recall the second Bianchi identity:

$$D_\alpha R_{\beta\gamma\rho\delta} + D_\beta R_{\gamma\alpha\rho\delta} + D_\gamma R_{\alpha\beta\rho\delta} = 0 \tag{4.4}$$

Taking the divergence of (4.4) and commuting derivatives, we have

$$\begin{aligned}
&D^{\alpha} D_{\alpha} R_{\beta\gamma\rho\delta} \\
&\quad = -[D^{\alpha}, D_{\beta}] R_{\gamma\alpha\rho\delta} - [D^{\alpha}, D_{\gamma}] R_{\alpha\beta\rho\delta} - D_{\beta} D^{\alpha} R_{\gamma\alpha\rho\delta} - D_{\gamma} D^{\alpha} R_{\alpha\beta\rho\delta} \\
&\quad = R^{\alpha}{}_{\beta\gamma\mu} R^{\mu}{}_{\alpha\rho\delta} + R^{\alpha}{}_{\beta\alpha\mu} R_{\gamma}{}^{\mu}{}_{\rho\delta} + R^{\alpha}{}_{\beta\rho\mu} R_{\gamma\alpha}{}^{\mu}{}_{\delta} + R^{\alpha}{}_{\beta\delta\mu} R_{\gamma\alpha\rho}{}^{\mu} \\
&\qquad + R^{\alpha}{}_{\gamma\alpha\mu} R^{\mu}{}_{\beta\rho\delta} + R^{\alpha}{}_{\gamma\beta\mu} R_{\alpha}{}^{\mu}{}_{\rho\delta} + R^{\alpha}{}_{\gamma\rho\mu} R_{\alpha\beta}{}^{\mu}{}_{\delta} + R^{\alpha}{}_{\gamma\delta\mu} R_{\alpha\beta\rho}{}^{\mu} \\
&\qquad + D_{\beta} D_{\delta} R_{\rho\gamma} + D_{\gamma} D_{\rho} R_{\delta\beta} - D_{\beta} D_{\rho} R_{\delta\gamma} - D_{\gamma} D_{\delta} R_{\rho\beta}
\end{aligned}$$

Schematically, we can write the previous formula as

$$\Box_g R_{\alpha\beta\gamma\delta} = (R * R)_{\alpha\beta\gamma\delta} + D_{\gamma} D_{\delta} R_{\alpha\beta} \tag{4.5}$$

where the last term denotes a linear combination of certain components of the Hessian of the Ricci curvature. In this expression, only the structure of the terms is important, the exact numerical coefficients are irrelevant. To have a closed system, we need to compute the Hessian of the Ricci tensor. By the gravitational part of (2.1), we have the following schematically expression:

$$\begin{aligned}
&D_{\gamma} D_{\delta} R_{\alpha\beta} \\
&\quad = F_{\beta\mu} D_{\gamma} D_{\delta} F_{\alpha}{}^{\mu} + F_{\alpha}{}^{\mu} D_{\gamma} D_{\delta} F_{\beta\mu} + D_{\delta} F_{\alpha}{}^{\mu} D_{\gamma} F_{\beta\mu} + D_{\gamma} F_{\alpha}{}^{\mu} D_{\delta} F_{\beta\mu} \\
&\qquad - \frac{1}{2} g_{\alpha\beta} (F^{\mu\nu} D_{\gamma} D_{\delta} F_{\mu\nu} + D_{\gamma} F_{\mu\nu} D_{\delta} F^{\mu\nu}) \\
&\quad = (F * D^2 F)_{\alpha\beta\gamma\delta} + (DF * DF)_{\alpha\beta\gamma\delta}
\end{aligned}$$

Plugging this into (4.5), we have

$$\Box_g R_{\alpha\beta\gamma\delta} = (R * R)_{\alpha\beta\gamma\delta} + (F * D^2 F)_{\alpha\beta\gamma\delta} + (DF * DF)_{\alpha\beta\gamma\delta} \tag{4.6}$$

This equation involves two derivatives of F. In principle, the electromagnetic part of the Einstein–Maxwell equations (2.1) controls only one derivative of F through the second order system:

$$\begin{aligned}
D^{\alpha} D_{\alpha} F_{\beta\gamma} &= -D^{\alpha} D_{\beta} F_{\gamma\alpha} - D^{\alpha} D_{\gamma} F_{\alpha\beta} \\
&= -[D^{\alpha}, D_{\beta}] F_{\gamma\alpha} - [D^{\alpha}, D_{\gamma}] F_{\alpha\beta} - D_{\beta} D^{\alpha} F_{\gamma\alpha} - D_{\gamma} D^{\alpha} F_{\alpha\beta} \\
&= R^{\alpha}{}_{\beta\gamma\mu} F^{\mu}{}_{\alpha} + R^{\alpha}{}_{\beta\alpha\mu} F_{\gamma}{}^{\mu}
\end{aligned}$$

which can be expressed in the following schematic form:

$$\Box_g F_{\alpha\beta} = (R * F)_{\alpha\beta} \tag{4.7}$$

Since for the Einstein–Maxwell equations, the electromagnetic part of is almost decoupled from the gravitational part, we can actually control second derivatives of F by a cost of one derivative on the curvature tensor $R_{\alpha\beta\gamma\delta}$. More precisely, we apply the covariant derivative D_{ρ} on the second equation of (2.1):

$$\begin{aligned}
d(D_{\delta} F)_{\alpha\beta} &= D_{\alpha} D_{\delta} F_{\beta\gamma} + D_{\beta} D_{\delta} F_{\gamma\alpha} + D_{\gamma} D_{\delta} F_{\alpha\beta} \\
&= [D_{\alpha}, D_{\delta}] F_{\beta\gamma} + [D_{\beta}, D_{\delta}] F_{\gamma\alpha} + [D_{\gamma}, D_{\delta}] F_{\alpha\beta} + D_{\rho} (D_{[\alpha} F_{\beta\gamma]}) \\
&= -R_{\alpha\delta\beta\mu} F^{\mu}{}_{\gamma} - R_{\alpha\delta\gamma\mu} F_{\beta}{}^{\mu} - R_{\beta\delta\gamma\mu} F^{\mu}{}_{\alpha} - R_{\beta\delta\alpha\mu} F_{\gamma}{}^{\mu} \\
&\quad - R_{\gamma\delta\alpha\mu} F^{\mu}{}_{\beta} - R_{\gamma\delta\beta\mu} F_{\alpha}{}^{\mu}
\end{aligned}$$

where d stands for the exterior derivative on two forms. Schematically, this gives

$$D_{[\alpha}(DF)_{\beta\gamma]} = (R * F)_{\alpha\beta\gamma}$$

Similarly, we have

$$D^\alpha(DF)_{\alpha\beta} = (R * F)_\beta$$

Applying covariant derivative on these last two equations gives

$$\Box_g(DF)_{\alpha\beta} = (R * DF)_{\alpha\beta} + (DR * F)_{\alpha\beta} \tag{4.8}$$

We summarize (4.6), (4.7) and (4.8) in the following system of equations

$$\begin{cases} \Box_g R_{\alpha\beta\gamma\delta} = (R * R)_{\alpha\beta\gamma\delta} + (F * D^2F)_{\alpha\beta\gamma\delta} + (DF * DF)_{\alpha\beta\gamma\delta} \\ \Box_g F_{\alpha\beta} = (R * F)_{\alpha\beta} \\ \Box_g(DF)_{\alpha\beta} = (R * DF)_{\alpha\beta} + (DR * F)_{\alpha\beta} \end{cases} \tag{4.9}$$

We have a corresponding system of equations for $R'_{\alpha\beta\gamma\delta}$ and $F'_{\alpha\beta}$.

We shall prove Proposition 4.3 in a neighborhood $\mathcal{O}(p)$ of a point $p \in \mathcal{S}$. To do so, we first introduce a fixed coordinate system (x_k) for $k = 1, 2, 3, 4$. We emphasize that the coordinate (x_k) is chosen for both metrics g and g'. In the proof we shall keep shrinking the neighborhoods of p; to simplify notations we keep denoting such neighborhoods by $\mathcal{O}(p)$.

We now fix the null frame $\{e_1, e_2, e_3 = \underline{L}, e_4 = L\}$ on the null hypersurface $\mathcal{H}^+ \cap \mathcal{O}(p)$. We use two different Levi-Civita connections to parallel transport the given null frame along $\underline{L}$:

$$\begin{cases} D_{\underline{L}} v_\alpha = 0 \quad \text{with} \quad v_\alpha = e_\alpha \quad \text{on} \quad \mathcal{H}^+ \cap \mathcal{O}(p) \\ D'_{\underline{L}} v'_\alpha = 0 \quad \text{with} \quad v'_\alpha = e_\alpha \quad \text{on} \quad \mathcal{H}^+ \cap \mathcal{O}(p) \end{cases} \tag{4.10}$$

The frames $\{v_\alpha\}$ and $\{v'_\alpha\}$ are smoothly defined in $\mathcal{O}(p)$. We will express all the geometric quantities in these frames. Let $g_{\alpha\beta} = g(v_\alpha, v_\beta), g'_{\alpha\beta} = g'(v'_\alpha, v'_\beta)$. Since $D_{\underline{L}} v_\alpha = D'_{\underline{L}} v'_\alpha = 0$, we know $\underline{L}(g_{\alpha\beta}) = \underline{L}(g'_{\alpha\beta}) = 0$, so $g_{\alpha\beta} = g'_{\alpha\beta}$. We define

$$h_{\alpha\beta} = g_{\alpha\beta} = g'_{\alpha\beta} \quad \underline{L}(h_{\alpha\beta}) = 0 \quad \text{in } \mathcal{O}(p). \tag{4.11}$$

We define the Christoffel symbols, curvature tensors and their differences

$$\begin{aligned} &\Gamma^\gamma_{\alpha\beta} = g(D_{v_\alpha} v_\beta, v_\gamma), \quad \Gamma'^\gamma_{\alpha\beta} = g'(D'_{v'_\alpha} v'_\beta, v'_\gamma), \quad \delta\Gamma^\gamma_{\alpha\beta} = \Gamma'^\gamma_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta} \\ &R_{\alpha\beta\gamma\delta} = g(R(v_\alpha, v_\beta)v_\gamma, v_\delta), R'_{\alpha\beta\gamma\delta} = g'(R'(v'_\alpha, v'_\beta)v'_\gamma, v'_\delta), \\ &\delta R_{\alpha\beta\gamma\delta} = R'_{\alpha\beta\gamma\delta} - R_{\alpha\beta\gamma\delta} \end{aligned}$$

We also introduce δR and δF to denote the collections of all the $\delta R_{\alpha\beta\gamma\delta}$'s and $\delta\Gamma^\gamma_{\alpha\beta}$'s. One observes that $\Gamma^\gamma_{3\beta} = \Gamma'^\gamma_{3\beta} = \delta\Gamma^\gamma_{3\beta} = 0$. In view of $D_{\underline{L}} v_\alpha = 0$, we drive a system of ODEs for $\Gamma^\gamma_{\alpha\beta}$ and $\Gamma'^\gamma_{\alpha\beta}$:

$$\begin{aligned} \underline{L}(\Gamma^\gamma_{\alpha\beta}) = {} & \underline{L}(g(D_{v_\alpha} v_\beta, v_\gamma)) = g(D_{v_3} D_{v_\alpha} v_\beta, v_\gamma) + g(D_{v_\alpha} v_\beta, D_{v_3} v_\gamma) \\ = {} & R_{3\alpha\beta\gamma} + g(D_{[v_3, v_\alpha]} v_\beta, v_\gamma) + g(D_{v_\alpha} v_\beta, D_{v_3} v_\gamma) \\ = {} & R_{3\alpha\beta\gamma} + \Gamma^\rho_{3\alpha}\Gamma^\gamma_{\rho\beta} - \Gamma^\rho_{\alpha 3}\Gamma^\gamma_{\rho\beta} + g_{\rho\delta}\Gamma^\delta_{\alpha\beta}\Gamma^\rho_{3\gamma} \end{aligned}$$

Schematically, we have

$$\underline{L}(\Gamma^{\gamma}_{\alpha\beta}) = R_{3\alpha\beta\gamma} + (\Gamma * \Gamma)^{\gamma}_{\alpha\beta} \tag{4.12}$$

Similarly, we have

$$\underline{L}(\Gamma'^{\gamma}_{\alpha\beta}) = R'_{3\alpha\beta\gamma} + (\Gamma' * \Gamma')^{\gamma}_{\alpha\beta} \tag{4.13}$$

Taking the difference of (4.12) and (4.13), one has

$$\begin{aligned}\underline{L}(\delta\Gamma^{\gamma}_{\alpha\beta}) &= \delta R_{3\alpha\beta\gamma} + (\Gamma' * \Gamma' - \Gamma * \Gamma)^{\gamma}_{\alpha\beta}\\ &= \delta R_{3\alpha\beta\gamma} + (\Gamma' * \delta\Gamma)^{\gamma}_{\alpha\beta} + (\Gamma * \delta\Gamma)^{\gamma}_{\alpha\beta}\end{aligned}$$

Since the smooth metrics g and g' are given, so their corresponding Christoffel symbols are also fixed in the given coordinate (x_k). One treat these objects as given backgrounds data, in particular, one can bound these objects in L^∞-norm, so we have:

$$|\,\underline{L}(\delta\Gamma)| \lesssim |\delta\Gamma| + |\delta R|. \tag{4.14}$$

Now we also need to express the frames $\{v_\alpha\}$ and $\{v'_\alpha\}$ in terms of the fixed coordinate vector fields ∂_k relative to our local coordinates x_k. We define

$$v_\alpha = v^k_\alpha \partial_k, \quad v'_\alpha = v'^k_\alpha \partial_k, \quad (\delta v)^k_\alpha = v'^k_\alpha - v^k_\alpha$$

Consider $[v_3, v_\alpha] = -D_{v_\alpha} v_3 = -\Gamma^\beta_{\alpha 3} v_\beta = -\Gamma^\beta_{\alpha 3} v^k_\beta \partial_k$, it implies

$$v^j_3 \partial_j(v^k_\alpha) - v^j_\alpha \partial_j(v^k_3) = -\Gamma^\beta_{\alpha 3} v^k_\beta$$

so we have

$$L(v^k_\alpha) = \partial_j(v^k_3) v^j_\alpha - \Gamma^\beta_{\alpha 3} v^k_\beta \tag{4.15}$$

Similarly, we have

$$L(v'^k_\alpha) = \partial_j(v'^k_3) v'^j_\alpha - \Gamma'^\beta_{\alpha 3} v'^k_\beta \tag{4.16}$$

Notice that $\partial_j(v^k_3) = \partial_j(v'^k_3)$ are fixed functions (since $v_3 = v'_3 = \underline{L}$), so by taking the difference, we have the following differential inequality

$$\underline{L}(\delta v) \lesssim |\delta\Gamma| + |\delta v|. \tag{4.17}$$

where we use δv to denote the collection of all the $(\delta v)^k_\alpha$'s. We can also apply coordinate derivatives ∂_k to (4.12), (4.13), (4.15) and (4.16), after taking the differences, one has the following estimates

$$|\,\underline{L}(\partial\delta\Gamma)| \lesssim |\delta\Gamma| + |\partial\delta\Gamma| + |\delta R| + |\partial\delta R|. \tag{4.18}$$

$$|\,\underline{L}(\partial\delta v)| \lesssim |\delta\Gamma| + |\partial\delta\Gamma| + |\delta v| + |\partial\delta v|. \tag{4.19}$$

Now we derive a set covariant of wave equations for δR and $\delta F, \delta DF$ (where δF is defined in an obvious way). In view of (4.9), when we take the differences, the following terms may cause trouble

$$(\Box_g - \Box_{g'})R, \quad (\Box_g - \Box_{g'})F \quad \text{and} \quad (\Box_g - \Box_{g'})DF$$

The estimates are the same for all of them, to illustrate the idea, we now deal with the first one. Since $g_{\alpha\beta} = g'_{\alpha\beta}$, it is easy to see it has the following form

$$|(\Box_g - \Box_{g'})R| \lesssim |\delta\Gamma| + |\partial\delta\Gamma|.$$

Similar relations hold for the other terms. Together with (4.14), (4.17), (4.18) and (4.19), we have the following system of differential inequalities:

$$\begin{cases} |\underline{L}(\delta\Gamma)| \lesssim |\delta\Gamma| + |\delta R|, \\ |\underline{L}(\partial\delta\Gamma)| \lesssim |\delta\Gamma| + |\partial\delta\Gamma| + |\delta R| + |\partial\delta R|, \\ |\underline{L}(\delta v)| \lesssim |\delta\Gamma| + |\delta v|, \\ |\underline{L}(\partial\delta v)| \lesssim |\delta\Gamma| + |\partial\delta\Gamma| + |\delta v| + |\partial\delta v|, \\ |\Box_g\delta R| \lesssim |\delta R| + |\delta F| + |\delta DF| + |\partial\delta DF| + |\delta\Gamma| + |\partial\delta\Gamma|, \\ |\Box_g\delta F| \lesssim |\delta R| + |\delta F| + |\delta\Gamma| + |\partial\delta\Gamma|, \\ |\Box_g\delta DF| \lesssim |\delta R| + |\delta DF| + |\partial\delta R| + |\delta F| + |\delta\Gamma| + |\partial\delta\Gamma|. \end{cases} \tag{4.20}$$

Since in $\mathcal{I}^{++}\cup\mathcal{I}^{--}, g=g'$ and $F=F'$, on the bifurcate horizon $\mathcal{H}^+\cup\mathcal{H}^-$, the following functions $\delta\Gamma, \partial\delta\Gamma, \delta v, \partial\delta v, \delta R, \delta F$ and δDF vanish to infinite order. We now show that they vanish completely in a full neighborhood of $\mathcal{S}$. It is an immediate consequence of the following uniqueness theorem, based on the Carleman estimates developed in [7], due to Alexakis [1], see also Lemma 4.4 of [2].

Proposition 4.4. *Assume $G_i, H_j : \mathcal{O}(p) \to \mathbb{R}$ are smooth functions, $i = 1,\ldots,I, j = 1,\ldots,J$. Let $G = (G_1,\ldots,G_I), H = (H_1,\ldots,H_J)$, $\partial G = (\partial_1 G_1, \partial_2 G_1, \partial_3 G_1, \partial_4 G_1, \ldots, \partial_4 G_I)$ and assume that in $\mathcal{O}(p)$,*

$$\begin{cases} |\Box_{\mathbf{g}} G| \lesssim |G| + |\partial G| + |H|; \\ |\underline{L}(H)| \lesssim |G| + |\partial G| + |H|. \end{cases}$$

Assume that $G = 0$ and $H = 0$ on $(\mathcal{H}^+\cup\mathcal{H}^-)\cap\mathcal{O}(p)$. Then, there exists a neighborhood $\mathcal{O}'(p) \subset \mathcal{O}(p)$ of x_0 such that $G = 0$ and $H = 0$ in $(\mathcal{I}^{+-}\cup\mathcal{I}^{-+})\cap\mathcal{O}'(p)$.

In particular, applied to (4.20), Proposition 4.4 implies $\delta R = 0$ and $\delta F = 0$ in a full neighborhood of $\mathcal{S}$, which shows that the vector field K is Killing in a full neighborhood of $\mathcal{S}$ and $\mathcal{L}_K F = 0$. ☐

Remark 4.5. The vector field K is time-like in $\mathcal{O}\cap(\mathcal{I}^{+-}\cup\mathcal{I}^{-+})$ which follows directly from the fact that $\underline{L}(g(K,K)) \geq 0$.

5. The Rotational Killing Vector Field

We prove Theorem 1.2 in this section. In addition to the Hawking vector field K constructed in Theorem 1.1, we assume $(\mathcal{O}, g, F)$ has another Killing vector field T which preserves F (i.e., $\mathcal{L}_K F = 0$) such that it is tangent to $\mathcal{H}^+\cup\mathcal{H}^-$ and it is not identically zero on $\mathcal{S}$. We will find a constant λ, such that $Z = T + \lambda K$ is a rotational Killing vector field, i.e., all the orbits of Z are closed circles.

We study the action of T on the bifurcate sphere $\mathcal{S}$. Because T is a smooth vector field tangent to the bifurcate horizon $\mathcal{H}^+\cup\mathcal{H}^-$, it must be tangent to $\mathcal{S}$. Since T is not identically zero, thanks to Lemma A.1 in "Appendix", the restriction of the metric g on $\mathcal{S}$ is rotational symmetric. In particular, the vector field $X = T|_{\mathcal{S}}$ on $\mathcal{S}$ has a period t_0 and has two zeroes. One fixes one of the zeroes and denotes it by $p \in \mathcal{S}$. Let γ^+ be the null geodesic emanating from p on $\mathcal{H}^+$; similarly, we define γ^-. On γ^+, we define a function $\lambda(\underline{u}) = \frac{g(T,\underline{L})}{g(K,\underline{L})}$ which measures the projection of T in the K direction. The key observation is

Claim 5.1. *On $\gamma^+, \lambda(\underline{u})$ is a constant.*

Proof. We show that $[T, L]$ is parallel to L on $\mathcal{H}^+$, i.e., there is a function $f : \mathcal{H}^+ \to \mathbb{R}$, such that

$$[T, L] = fL.$$

Since both vectors are tangent to $\mathcal{H}^+$, so is $[T, L]$. It suffices to show $g([T, L], e_a) = 0$.

$$\begin{aligned} g([T, L], e_a) &= g(D_T L, e_a) - g(D_L T, e_a) \overset{Killing}{=} g(D_T L, e_a) + g(D_a T, L) \\ &= g(D_T L, e_a) - g(T, D_a L) = \chi(T, e_a) - \chi(e_a, T) = 0. \end{aligned}$$

We claim that $L(f) = 0$ which follows from the

$$0 = \mathcal{L}_T(D_L L) = D_{\mathcal{L}_T L} L + D_L(\mathcal{L}_T L) = D_{fL} L + D_L(fL) = L(f)L.$$

When one restricts f to γ^+, this implies that $f(\underline{u}) = f(p)$ is a constant. On the bifurcate sphere $\mathcal{S}$, we can compute

$$f = fL(\underline{u}) = [T, L](\underline{u}) = -L(T(\underline{u}))$$

so on γ^+

$$T(\underline{u}) = -f(p)\underline{u}.$$

We turn to $L(\lambda(\underline{u}))$:

$$L(\lambda(\underline{u})) = L\left(\frac{g(T, \underline{L})}{g(K, \underline{L})}\right) = -L\left(\frac{T(\underline{u})}{\underline{u}}\right) = L(f) = 0.$$

This shows $\lambda(\underline{u})$ is a constant on γ^+. $\square$

Remark 5.2. One can construct $\lambda(u)$ on γ^- and show that $\lambda(\underline{u}) = \lambda(u) = \lambda(p)$ is a constant.

Now we can define the rotational vector field Z:

Claim 5.3. *Let $\lambda = -f(p)$, then $Z = T + \lambda K$ is a rotational vector field with period t_0.*

Proof. Since $K = 0$ on $\mathcal{S}, Z|_{\mathcal{S}} = T|_{\mathcal{S}}$ has the same period t_0. We denote ψ_t the one parameter isometry group generated by Z on $\mathcal{O}$. To show Z is rotational, it suffices to show $\psi_{t_0} = id$.

We study the action of ψ_t on the null geodesic γ^+. For all t, since p is a fixed point of ψ_t and ψ_t is an isometry, we know that $\psi_t(\gamma) \subset \gamma$ is a reparametrization of γ^+. In particular, it implies $Z|_{\gamma^+}$ is proportional to $K|_{\gamma^+}$. In view of the definition of λ, we know that $Z|\gamma^+ = 0$ since we have subtracted the corresponding portion of K from T. So $\psi_t|_{\gamma^+} = id$. In particular, $\psi_{t_0}|_{\gamma^+} = id$.

We turn to the action of ψ_{t_0} on the full tangent space of p. The previous argument shows $(\psi_{t_0})_* L = L$; the same considerations on γ^- shows $(\psi_{t_0})_* \underline{L} = \underline{L}$. We also know that $(\psi_{t_0})_* e_a = e_a$ because p is a zero on T on $\mathcal{S}$ and K vanishes on S. $(\psi_{t_0})_*$ is the identity map on the tangent space of p. We can use Lemma A.2 in "Appendix" to conclude that ψ_{t_0} is the identity in a small neighborhood of p. Finally, we can use the compactness of $\mathcal{S}$ and the standard open-closed argument on $\mathcal{S}$ to conclude ψ_{t_0} is the identity map in a small neighborhood of $\mathcal{S}$. $\square$

To finish the proof of Theorem 1.2, we have:

Claim 5.4. $[Z, K] = 0$.

Proof. It suffices to show $[T, K] = 0$. Since both K and T are Killing, in view of the fact that all the Killing vector fields on a manifold form a Lie algebra under $[-, -]$, we know that $W = [T, K]$ is also Killing, so it solves the following equation:

$$\square_g W_\alpha = -R_\alpha{}^\beta W_\beta \tag{5.1}$$

We show that

$$W = 0 \quad \text{on} \quad \mathcal{H}^+ \cup \mathcal{H}^-.$$

It is an easy consequence of the calculations in the proof of Claim 5.1:

$$W = [T, K] = [T, \underline{u}L] = \underline{u}[T, L] + T(\underline{u})L = \underline{u}fL - \underline{u}fL = 0$$

By solving (5.1), we know that $W = 0$ in $\mathcal{I}^{++} \cup \mathcal{I}^{--}$. To show $W = 0$ in a full neighborhood of $\mathcal{S}$, once again we have to use Proposition 4.4 in the straightforward way. This completes the proof. $\square$

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Appendix A: Two Lemmas on Geometry

Lemma A.1. *Assume h is a Riemannian metric on the topological sphere $\mathcal{S}^2$ which admits a non-trivial Killing vector field X, then $(\mathcal{S}^2, h)$ is a Riemannian warped product $([0,1], dr^2) \times_{\phi(r)} (\mathcal{S}^1, d\sigma^2)$. In particular, each orbit of X is closed and has a common period t_0.*

Proof. First, we observe that, if X is non-trivial, then the set $Z(X)$, which consists all zeroes of X, is discrete. It follows from the fact that, the zero locus of a Killing vector field is a disjoint union of totally geodesic sub-manifolds each of even dimension. Since we are on a surface, the zeroes must be discrete. In particular, since the $\mathcal{S}^2$ is compact, X has only finite many zeroes.

The second observations is that, for each zero p of $X, ind_X(p)$ the index of X at p is either 1 or -1. It following from the fact that, X induces an isometry on $T_p\mathcal{S}^2$, which is a 2D rotation. So its index must be 1 or -1.

Now we can apply the Poincaré–Hopf index theorem:

$$\sum_{p\in Z(X)} ind_X(p) = \chi(\mathcal{S}^2) = 2.$$

The previous observation imply that the cardinal number $|Z(X)| \geq 2$. We can pick up two points $p, q \in Z(X)$. Now let us fix a minimal geodesic $\gamma(t)$ between p and q. Let ϕ_t be the flow generated by X. Since on T_pM, $(\phi_t)_*$ is a rotation, it has a period t_0. Let $x \neq p, q$ be a point on γ. We show that the orbit of x under ϕ_t is a closed non-degenerate circle, more precisely, it is the image $\{\phi_t(x)|t \in [0, t_0)\}$. It trivially holds when x is close to either p or q, i.e., in the normal coordinate of p or q, since it will stay on the geodesic sphere which is a circle around either p or q. Since γ is minimal and $X(q) = 0$, so $\phi_t(\gamma)$ is also a minimal geodesic between p and q. When t varies, $\phi_t(\gamma)$ sweeps the whole $\mathcal{S}^2$, we know that all points except q is in the normal coordinate of p, so the orbit x is closed. This finishes the proof of the lemma. □

Lemma A.2. *Assume (M, g) is a Lorentzian manifold, $\phi : M \to M$ is an isometry and $p \in M$ is a fixed point of ϕ. If $\phi_{*p} = id$, then $\phi = id$ locally around p.*

Proof. In Riemannian geometry, this is easy since we have the concept of length; in our case, the difficulty comes from the fact that on the light-cone, the length is not well defined. But the proposition holds inside the light-cone since we can consider maximal time-like geodesics. Since locally the light-cone is the boundary of the future of the point p, the identity map can be continued to the boundary. □

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Pin Yu
Department of Mathematics
Princeton University
Princeton, NJ 08544, USA
e-mail: pinyu@math.princeton.edu

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