Ann. Henri Poincaré 10 (2009), 377–394

1424-0637/020377-18, *published online* May 22, 2009
DOI 10.1007/s00023-009-0412-1

Annales Henri Poincaré

On Confining Potentials and Essential Self-Adjointness for Schrödinger Operators on Bounded Domains in $\mathbb{R}^n$

Gheorghe Nenciu and Irina Nenciu

Abstract. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2-smooth boundary, $\partial\Omega$, of co-dimension 1, and let $H = -\Delta + V(x)$ be a Schrödinger operator on Ω with potential $V \in L^\infty_{loc}(\Omega)$. We seek the weakest conditions we can find on the rate of growth of the potential V close to the boundary $\partial\Omega$ which guarantee essential self-adjointness of H on $C_0^\infty(\Omega)$. As a special case of an abstract condition, we add optimal logarithmic type corrections to the known condition $V(x) \geq \frac{3}{4d(x)^2}$ where $d(x) = \mathrm{dist}(x, \partial\Omega)$. More precisely, we show that if, as x approaches $\partial\Omega$,

$$V(x) \geq \frac{1}{d(x)^2}\left(\frac{3}{4} - \frac{1}{\ln(d(x)^{-1})} - \frac{1}{\ln(d(x)^{-1}) \cdot \ln\ln(d(x)^{-1})} - \cdots\right)$$

where the brackets contain an arbitrary finite number of logarithmic terms, then H is essentially self-adjoint on $C_0^\infty(\Omega)$. The constant 1 in front of each logarithmic term is optimal. The proof is based on a refined Agmon exponential estimate combined with a well-known multidimensional Hardy inequality.

1. Introduction

Consider a particle in a bounded domain Ω in $\mathbb{R}^n$, $n \geq 1$, in the presence of a potential V. At the heuristic level, if $V(x) \to \infty$ as x approaches the boundary $\partial\Omega$, then the particle is confined in Ω and never visits the boundary. At the classical level, this indeed happens when $V(x) \to \infty$ as $x \to \partial\Omega$ (see, e.g. [17, Theorem X.5]). At the quantum level, the problem is much more complicated due to the possibility that the particle tunnels through the infinite potential barrier and "sees" the boundary. The fact that the particle never feels the boundary amounts

We wish to thank F. Gesztesy, A. Laptev, M. Loss and B. Simon for useful comments and suggestions. I.N.'s research was partly supported by the NSF grant DMS 0701026.

to saying that V determines completely the dynamics: there is no need for boundary conditions. At the mathematical level, by Stone's Theorem, the problem is then finding conditions on the rate of growth of $V(x)$ as $x \to \partial\Omega$ which ensure that the Schrödinger operator

$$H = -\Delta + V \tag{1.1}$$

is essentially self-adjoint on $C_0^\infty(\Omega)$. Let us note here that oscillations of the potential could also play a role in the essential self-adjointness problem due to the possibility of coherent reflections by an appropriately chosen sequence of potential barriers (see [17], the Appendix to Chapter X.1). In this paper we will not consider oscillatory potentials, but rather focus on potentials which grow to infinity at the boundary of the domain.

The problem has a long and distinguished history; for details and further references, we send the reader to [6] and [17] and the review papers [4,5,15]. In the 1-dimensional case (say, $\Omega = (0,1)$) there exists a well-developed theory of essential self-adjointness of Sturm-Liouville operators, which is based on limit point/limit circle Weyl type criteria (see e.g. [6,17] and the references therein). In particular if, under appropriate regularity conditions,

$$V(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2}\,, \tag{1.2}$$

where $d(x) = \operatorname{dist}\big(x, \{0,1\}\big)$, then H is essentially self-adjoint on $C_0^\infty(0,1)$. The constant $\frac{3}{4}$ is optimal, in the sense that if for some $\varepsilon > 0$,

$$0 \leq V(x) \leq \left(\frac{3}{4} - \varepsilon\right) \cdot \frac{1}{d(x)^2}\,,$$

near 0 and/or 1, then H is not essentially self-adjoint on $C_0^\infty(0,1)$ (see Theorem X.10 in [17]). Many results have been generalized from one to higher dimensions – see, for example, a comprehensive review of these results in [5]. In particular, if Ω is a bounded domain with C^2 boundary $\partial\Omega$ of codimension 1, and if V satisfies (1.2) as x approaches $\partial\Omega$, with $d(x) = \operatorname{dist}(x, \partial\Omega)$, then H defined as in (1.1) is essentially self-adjoint on $C_0^\infty(\Omega)$. Moreover, Theorem 6.2 in [5] implies that for the case at hand the essential self-adjointness of H is assured by a weaker condition, namely

$$V(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2} - \frac{c}{d(x)} \tag{1.3}$$

with some $c \in \mathbb{R}_+$. This raises the following optimality question: While among power-type growth conditions, $\frac{3}{4} \cdot \frac{1}{d(x)^2}$ is optimal both in the exponent and in the constant, does a growth condition of the type

$$V(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2}\Big(1 - m\big(d(x)\big)\Big)\,, \quad \lim_{t\to 0+} m(t) = 0\,, \quad m(t) \geq 0$$

still imply essential self-adjointness of H? It turns out that this is false – see the counterexample in the proof of Theorem 3. So the question of optimality should be refined to asking whether $\frac{3}{4} \cdot \frac{1}{d(x)^2}$ is the leading term of a (possibly formal) asymptotic expansion near $\partial\Omega$ of a critical potential V_c such that $V \geq V_c$ near $\partial\Omega$ implies essential self-adjointness of H on $C_0^\infty(\Omega)$. This would amount to finding the form and size of sub-leading terms in the asymptotic expansion of V_c.

The main result of this note is the affirmative answer to this optimality question. Namely, we show that for bounded domains Ω in $\mathbb{R}^n$, $n = 1, 2, 3, \ldots$ having C^2 boundary of codimension 1, and for potentials V satisfying

$$V(x) \geq \frac{1}{d(x)^2}\left(\frac{3}{4} - \frac{1}{\ln(d(x)^{-1})} - \frac{1}{\ln(d(x)^{-1}) \cdot \ln\ln(d(x)^{-1})} - \cdots\right) \tag{1.4}$$

as x approaches $\partial\Omega$, the Schrödinger operator H is essentially self-adjoint on $C_0^\infty(\Omega)$, and that the constants 1 in front of each logarithmic term on the right-hand side of (1.4) are optimal (for a precise statement, see Theorem 3).

Two remarks are in order here. The first one is that we are interested in optimality rather than generality. Accordingly, and also in order not to obscure the main ideas of our proofs by technicalities, we consider the simplest case, which is still the most interesting from a physical point of view: a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary of co-dimension 1. In addition, we only consider scalar Schrödinger operators with regular (L^∞_{loc}) potentials, and we consider only what one can think of as the "isotropic" case, i.e. we seek conditions on $V(x)$ which depend only on $d(x)$, and not on the specific point x_0 of the boundary that x approaches, or the direction along which $x \to x_0$. In this setting the proofs are short and elementary. Concerning more general situations, we note here that many results about the essential self-adjointness problem of second order elliptic operators of general form on arbitrary domains in $\mathbb{R}^n$ can be found in [5] (see especially Corollary 3.3 and its applications) – and then one can consider again the above optimality question. At the price of technicalities, one may be able to extend the results of the present note to more general situations, e.g. boundaries with components of higher co-dimension, local singularities of the potential or second order elliptic operators of general form. Reducing the regularity of the boundary $\partial\Omega$ below C^2 seems to require a finer analysis – in particular, of multidimensional Hardy inequalities on domains with less smooth boundaries (see e.g. [7, 8, 16] and references therein for results in this area).

The second remark concerns the method of proof. While the proofs in [5] are based on his theory of semimaximal operators, our method of proof is based on the observation that essential self-adjointness follows (via the fundamental criterion for self-adjointness, see, e.g., [2, 17]) from Agmon type results on exponential decay of eigenfunctions (see [1, Theorem 1.5a]). As stated, the result in [1] does not lead to optimal growth conditions on the potential. One has both to strengthen the exponential decay estimates, and to combine them with multidimensional Hardy inequalities [7]. So our basic technical result is an exponential estimate of Agmon-type – see Theorem 4. Here the point is that our condition (Σ.2) below is strictly weaker than the corresponding condition (3.12) from Brusentsev [5].

The paper is organized as follows. In Section 2 we state the problem and the main results. Section 3 contains the proof of the Agmon-type Theorem 4. While some of the results in this section go back to Agmon [1] and are well-known (e.g. the identity in Lemma 3.2), we give complete proofs for the reader's convenience. Finally Section 4 contains the proofs of Theorems 1 and 2.

2. Main results

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with C^2-smooth boundary, $\partial\Omega$, of co-dimension 1. We consider the function

$$d(x) = \mathrm{dist}(x, \partial\Omega), \quad \text{for} \quad x \in \Omega\,, \tag{2.1}$$

where "dist" denotes the usual, Euclidean distance in $\mathbb{R}^n$. As is well-known (see, for example, the Appendix to Chapter 14 in [13]), d is Lipshitz and differentiable a.e. in Ω. More importantly for us here, there exists a constant

$$d_\Omega > 0 \quad \text{(depending only on the domain } \Omega)$$

such that for $x \in \Omega$ with $d(x) < d_\Omega$, d is twice-differentiable and

$$|\nabla d(x)| \leq 1\,. \tag{2.2}$$

Remark 2.1. Actually $|\nabla d(x)| = 1$ for $x \in \Omega$ with $d(x) < d_\Omega$, see for example [13], or Lemma 6.2 in [5], but in the proofs below we use only (2.2).

In Ω we consider the Schrödinger operator $H = -\Delta + V$ with $V \in L^\infty_{\mathrm{loc}}(\Omega)$, defined on $\mathcal{D}(H) = C_0^\infty(\Omega)$. As explained in the Introduction, we are seeking growth conditions on V close to $\partial\Omega$ ensuring essential self-adjointness of H. These will be given in terms of functions G described below:

Condition (Σ). *A function $G : (0,\infty) \to \mathbb{R}$ is said to satisfy condition (Σ) if it is $C^1(0,\infty)$ and such that:*

(Σ.1) *There exists $d_0 > 0$, $d_0 \leq d_\Omega$, such that*

$$0 \leq G'(t) \leq \frac{1}{t}\,, \quad \text{for} \quad t \in (0, d_0) \quad \text{and}$$
$$G'(t) = 0 \quad \text{for} \quad t \geq d_0\,.$$

(Σ.2) *For any $\rho_0 \leq \frac{d_0}{2}$,*

$$\sum_{n=1}^{\infty} 4^{-n} e^{-2G(2^{-n}\rho_0)} = \infty \tag{2.3}$$

We can now formulate our main result:

Theorem 1. *Consider an open, bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-smooth boundary, and the Schrödinger operator*

$$H = -\Delta + V\,, \tag{2.4}$$

with $V \in L^\infty_{\mathrm{loc}}$ and domain $\mathcal{D}(H) = C_0^\infty(\Omega)$. Assume that there exists a function G satisfying condition (Σ) such that

$$V = V_1 + V_2\,, \tag{2.5}$$

where

$$V_1(x) + \frac{1}{4} \cdot \frac{1}{d(x)^2} \geq G'\big(d(x)\big)^2 \quad \textit{and} \quad V_2 \in L^\infty(\Omega)\,. \tag{2.6}$$

Then H is essentially self-adjoint in $L^2(\Omega)$.

Theorem 1 follows from the fundamental criterion for self-adjointness (see, for example, [2,17]), a multidimensional Hardy inequality [7], and a (refined) Agmon-type exponential estimate (see Theorem 4 in Section 3).

We now turn to various examples of functions G satisfying condition (Σ), and the associated criteria for essential self-adjointness of H in terms of the growth of the potential at the boundary of the domain.

The first, simplest example of a function G satisfying condition (Σ) is the one for which at sufficiently small t:

$$G(t) = \ln t\,,$$

which leads to the classical bound

$$V_1(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2}\,, \quad \text{as} \quad x \to \partial\Omega\,. \tag{2.7}$$

The second example is (again for t sufficiently small)

$$G(t) = \ln t - c \cdot t\,, \quad c \in \mathbb{R}^+\,.$$

This choice of G leads, through (2.6), to

$$V_1(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2} - \frac{\tilde{c}}{d(x)}\,, \quad \text{as} \quad x \to \partial\Omega\,, \tag{2.8}$$

for all $\tilde{c} < 2c$. This is the lower bound obtained by Brusentsev in [5, Theorem 6.2] for the case at hand.

The next example is (again for sufficiently small t) of the form

$$G(t) = \ln t + \int_t f(u)\,du$$

with

$$f(u) \geq 0\,, \quad \lim_{u \to 0} u f(u) = 0\,, \quad \text{and} \quad \lim_{t \to 0} \int_t f(u)\,du < \infty\,. \tag{2.9}$$

This leads to a bound on V of the form

$$V(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2} - \frac{\tilde{c}}{d(x)} \cdot f\big(d(x)\big)\,, \quad \text{as} \quad x \to \partial\Omega\,, \tag{2.10}$$

with f as above and all $\tilde{c} < 2$. Although this result does not appear in an explicit form in [5] it can still be obtained from Corollary 3.3 in [5]. Note that, since we required that $uf(u) \to 0$ as $u \to 0$, the second term $\frac{\tilde{c}}{d(x)} \cdot f(d(x))$ in (2.10) is of lower order than $\frac{1}{d(x)^2}$, and thus does not contradict the optimality of $\frac{3}{4} \cdot \frac{1}{d(x)^2}$.

The last example is our main hierarchy of essential self-adjointness conditions. Let $p \in \mathbb{Z}$, $p \geq 2$, and iteratively define

$$L_1(t) = \ln(1/t)\,, \quad L_p(t) = \ln L_{p-1}(t)\,, \tag{2.11}$$

where each L_p is defined for $t \in (0, e_p^{-1})$ with $e_1 = e$ and $e_p = e^{e_{p-1}}$. Then we have the following result:

Theorem 2. *Consider an open, bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-smooth boundary, and the Schrödinger operator*

$$H = -\Delta + V, \tag{2.12}$$

with $V \in L^\infty_{\mathrm{loc}}$ and domain $\mathcal{D}(H) = C_0^\infty(\Omega)$. Let $p \in \mathbb{Z}$, $p \geq 2$ and assume that

$$V = V_1 + V_2\,, \tag{2.13}$$

where

$$V_1(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2} - \frac{1}{d(x)^2} \sum_{j=2}^{p} \left(\prod_{k=1}^{j-1} L_k\big(d(x)\big) \right)^{-1} - \frac{1}{d(x)} \cdot f\big(d(x)\big)\,, \tag{2.14}$$

for all x with $d(x) < \min(e_p^{-1}, d_\Omega)$, with f satisfying (2.9)*, V_1 bounded from below on Ω, and $V_2 \in L^\infty(\Omega)$.*

Then H is essentially self-adjoint in $L^2(\Omega)$.

Remark 2.2. Let K be a positive constant. Rewriting $V(x)$ as

$$V(x) = \big(V_1(x) + K\big) + \big(V_2(x) - K\big)$$

one sees that, in order to obtain the result above, it is sufficient to prove Theorem 2 with V_1 bounded from below by some (appropriately chosen) positive constant; this is exactly how we prove the theorem – see Section 4.

Note that, for any given $j \geq 2$, each term $\frac{1}{t} \cdot (\prod_{k=1}^{j-1} L_k(t))^{-1}$ is non-integrable, and hence a higher order correction than the integrable term $f(t)$. Further note that the domain on which $\sum_{j \leq p} (\prod_{k=1}^{j-1} L_k(t))^{-1}$ is well defined shrinks to the empty set as $p \to \infty$.

The term $\frac{1}{4} \cdot \frac{1}{d(x)^2}$ in (2.6) comes from the additional "barrier" given by the uncertainty principle of quantum mechanics via the Hardy inequality (see (4.2) below). The fact that Hardy inequalities appear here is not surprising since, as expressions of the uncertainty principle, they play a key role in various aspects of the spectral analysis of Schrödinger and Dirac operators like stability, self-adjointness, etc (see e.g. [10–12, 14] and the references therein). During the last decade a large body of literature about improvements to Hardy inequalities has appeared (see e.g. the references in [3,9,11,18]). In particular, in [3] (under suitable conditions) the following optimal improvement of (4.2) was proved:

$$\int_\Omega |\nabla \varphi(x)|^2\, dx \geq \frac{1}{4} \int_\Omega \frac{|\varphi(x)|^2}{d(x)^2} \left(1 + \sum_{i=1}^{\infty} \prod_{k=1}^{i} X_k^2 \left(\frac{d(x)}{D} \right) \right) dx \tag{2.15}$$

where D is a sufficiently large constant, and $X_k(t)$, $t > 0$ are defined recursively by

$$X_1(t) = (1 - \ln t)^{-1}\,, \quad X_k(t) = X_1\big(X_{k-1}(t)\big)\,.$$

However, this improvement of Hardy's inequality does not lead to an improvement of the result in Theorem 2 (which according to Theorem 3 is already optimal at the level of logarithmic subleading terms). Indeed, at the level of the leading term, as $t \to 0$, $X_k^2 = L_k^{-2}$ and so the contribution of the logarithmic terms in (2.15) can be absorbed in the last (integrable) term on the right-hand side of (2.14).

As we will show in Section 4, the theorem follows from Theorem 1 with the following choice, for sufficiently small t, of G function:

$$G_p(t) = \ln t + \frac{1}{2} \cdot \sum_{j=2}^{p} L_j(t) + \int_t \tilde{f}(u)\, du\,, \tag{2.16}$$

where $\tilde{f}$ also satisfies (2.9).

Our last result is about the optimality of (2.14). With the hypotheses of Theorem 2, it is well-know that the constant $\frac{3}{4}$ in front of the first term on the right-hand side of (2.14) is optimal. We claim that the constant 1 in front of each logarithmic term in the sum above is also optimal, in the following precise sense:

Theorem 3. *Given $p \geq 2$ and a constant $c > 1$, there exist potentials V for which $H = -\Delta + V$ is* not *essentially self-adjoint, and which grow close to the boundary $\partial\Omega$ as*

$$\begin{aligned} V(x) \geq \frac{3}{4} \cdot \frac{1}{d(x)^2} - \frac{1}{d(x)^2} \sum_{j=2}^{p-1} \left(\prod_{k=1}^{j-1} L_k\big(d(x)\big) \right)^{-1} \\ - c \cdot \frac{1}{d(x)^2} \cdot \left(\prod_{k=1}^{p-1} L_k\big(d(x)\big) \right)^{-1} . \end{aligned} \tag{2.17}$$

We end this section with a discussion of condition (Σ) and its relation with condition (3.12) from Corollary 3.3 in [5]. We comment first on condition (Σ.1). Note that (Σ.2) implies that $G(t) \to -\infty$ as $t \to 0$. So $G'(t) \geq 0$ in (Σ.1) only adds that $G(t) \to -\infty$ monotonically which is not a real restriction as far as we are not considering (as already stated in the Introduction) the effect of oscillations of the potential. In fact, if one considers potentials which grow monotonically as $x \to \partial\Omega$ one may impose even a stronger condition that $G'(t)$ is monotonically increasing to ∞ as $t \to 0$. Consider now $G'(t) \leq \frac{1}{t}$ in (Σ.1). This is again harmless (as far as it does not contradict (Σ.2)!) since if $G_1'(t) \geq G_2'(t)$ then Theorem 1 with $G(t) = G_2(t)$ gives a stronger result than with $G(t) = G_1(t)$.

The crucial condition is (Σ.2) and this is to be compared with Brusentsev's condition (3.12) from Corollary 3.3. We show now that Brusentsev's condition (3.12) is (at least for $G(t)$ satisfying (Σ.1)) strictly stronger than (Σ.2). Notice that we have restricted our attention to the situation when his matrix $A \equiv I$. Comparing functions, we see that in Brusentsev's notation the function which determines the growth of the potential at the boundary is $\eta(x)$, and that we are therefore interested in showing that, if

$$\eta(x) = -G\big(d(x)\big)\,, \tag{2.18}$$

satisfies condition (3.12) in [5], then G must satisfy our condition (Σ.2). Condition (3.12) in Brusentsev guarantees that there exists a constant $C > 0$ such that

$$|\nabla\eta(x)| \cdot e^{-\eta(x)} \leq C\,. \tag{2.19}$$

If we recall that for x with $d(x)$ small enough, $|\nabla d(x)| = 1$, then we get from (2.18) and (2.19) that

$$\frac{d}{dt}\, e^{G(t)} = G'(t)e^{G(t)} \leq C\,,$$

for all $0 < t < d_\Omega$. But since $G(t) \to -\infty$ as $t \to 0+$, we can integrate, for all n greater than some fixed integer N_Ω,

$$e^{G(2^{-n}\rho_0)} = \int_0^{2^{-n}\rho_0} G'(t)e^{G(t)}\, dt \leq 2^{-n} \cdot C\rho_0\,.$$

Plugging this into the series from (2.3) we get

$$\sum_{n=1}^{\infty} 4^{-n} e^{-2G(2^{-n}\rho_0)} \geq \sum_{n \geq N_\Omega}^{\infty} 4^{-n} \cdot 4^n (C\rho_0)^{-2} = +\infty\,,$$

thus showing that G satisfies (Σ.2).

Conversely, recall the G_p defined in (2.16). As we will show in Section 4, the function G_p satisfies (Σ). Take now the simplest case $G(t) = G_2(t)$ with $\tilde{f} \equiv 0$ i.e. $G(t) = \ln t + \frac{1}{2}\ln\ln\frac{1}{t}$ for sufficiently small t and set

$$\eta(x) = -G\big(d(x)\big)\,.$$

Then as $t = d(x) \to 0+$

$$|\nabla\eta(x)|e^{-\eta(x)} = G'(t) \cdot e^{G(t)} = \left(\ln\frac{1}{t}\right)^{\frac{1}{2}} \left(1 - \frac{1}{2}\frac{1}{\ln\frac{1}{t}}\right) \to +\infty\,,$$

and hence η does not satisfy condition (3.12) from [5].

3. Agmon-type estimates

Proposition 3.1. *Let ψ be a weak solution of*

$$H\psi = E\psi\,,$$

i.e. $\psi \in H^1_{\mathrm{loc}(\Omega)}$ and satisfies

$$\big\langle \psi, (H-E)\varphi \big\rangle = 0\,, \quad \textit{for every} \quad \varphi \in C_0^\infty(\Omega) \tag{3.1}$$

Let $g \in C^1(\Omega)$ be a real-valued function for which there exists a constant $c > 0$ such that

$$\big\langle \varphi, (H-E)\varphi \big\rangle - \int_\Omega |\varphi(x)|^2 |\nabla g(x)|^2\, dx \geq c\|\varphi\|^2 \tag{3.2}$$

for every $\varphi \in C_0^\infty(\Omega)$.

For $\rho > 0$, small enough, set $\Omega_\rho = \{x \in \Omega \,|\, d(x) > \rho\}$. Then there exists a constant $K = K(c) < \infty$, independent of ρ, such that

$$\int_{\Omega_{2\rho}} |e^{g(x)}\psi(x)|^2\,dx \le \frac{K(c)}{\rho} \int_{\Omega_\rho \setminus \Omega_{2\rho}} \left(\frac{1}{\rho} + |\nabla g(x)|\right) |e^{g(x)}\psi(x)|^2\,dx\,. \tag{3.3}$$

Since this might be of independent interest and the proof is the same, we will actually prove this proposition in a slightly more general context. Indeed, consider the Schrödinger operator with magnetic potential on Ω

$$H = (\vec{p} - \vec{a})^2 + V\,, \quad V \in L^\infty_{\mathrm{loc}}(\Omega)\,, \quad \vec{a} \in C^1_{\mathrm{loc}}(\Omega)\,, \quad \vec{p} = -i\nabla\,, \tag{3.4}$$

defined on $\mathcal{D}(H) = C_0^\infty(\Omega)$ and, for $\varphi, \psi \in W^{1,2}$, the associated quadratic form

$$h[\varphi,\psi] = \int_\Omega \overline{(\vec{p} - \vec{a})\varphi} \cdot (\vec{p} - \vec{a})\psi\,dx + \int_\Omega \bar{\varphi} \cdot V\psi\,dx\,. \tag{3.5}$$

Note that if φ and ψ are both in $C_0^2(\Omega)$, then

$$h[\varphi,\psi] = \int_\Omega \overline{\varphi(x)}\,(H\psi)(x)\,dx\,.$$

One of the main technical ingredients is the following simple identity [1]:

Lemma 3.2. *Let ψ be a weak solution of $H\psi = E\psi$, and let $f = \bar{f} \in C_0^1(\Omega)$. Then*

$$(h - E)[f\psi, f\psi] = \langle \psi, |\vec{\nabla} f|^2\psi \rangle\,. \tag{3.6}$$

Proof. Consider first $f \in C_0^\infty$ and let $\varphi \in C_0^\infty$. Then

$$(h - E)[\varphi, f\psi] = \langle (H - E)\varphi, f\psi \rangle = \langle f(H - E)\varphi, \psi \rangle\,.$$

Since $[f, \vec{p} - \vec{a}] = i\nabla f$ on C_0^∞, we get that

$$[f, H] = \left[f, (\vec{p} - \vec{a})^2\right] = i\big((\vec{p} - \vec{a}) \cdot \nabla f + \nabla f \cdot (\vec{p} - \vec{a})\big),$$

and so, if we remember that ψ is a weak solution,

$$(h - E)[\varphi, f\psi] = \langle [f, H]\varphi, \psi \rangle = \langle \varphi, [H, f]\psi \rangle\,.$$

Since $f\psi \in W_0^{1,2}(\Omega)$ and C_0^∞ is dense in the $W^{1,2}$ topology, the identity above implies that

$$\begin{aligned}(h - E)[f\psi, f\psi] &= \langle \psi, f[H, f]\psi \rangle = \mathrm{Re}\,\langle \psi, f[H, f]\psi \rangle \\ &= \frac{1}{2}\left\langle \psi, \big(f[H, f] - [H, f]f\big)\psi \right\rangle \\ &= \frac{1}{2}\left\langle \psi, \big[f, [H, f]\big]\psi \right\rangle.\end{aligned} \tag{3.7}$$

Finally, a straightforward computation shows that

$$\big[f, [H, f]\big] = -i\big[f, (\vec{p} - \vec{a}) \cdot \nabla f + \nabla f \cdot (\vec{p} - \vec{a})\big] = -i\,(2i\nabla f \cdot \nabla f) = 2|\nabla f|^2\,,$$

which completes the proof. □

Proof of Proposition 3.1. As in [1], we will now choose a function f to plug into the formula (3.6). More precisely, let

$$f = e^g \phi \,,$$

where $g \in C^1(\Omega)$, real-valued, is the function from the statement of the proposition, and $\phi \in C_0^\infty(\Omega)$, $0 \le \phi \le 1$, is a cut-off function,

$$\phi(x) = \begin{cases} 0\,, & x \notin \Omega_\rho \\ 1\,, & x \in \Omega_{2\rho}\,. \end{cases}$$

Taking ϕ of the form $\phi(x) = k(d(x))$ where

$$k(t) = \begin{cases} 0\,, & 0 \le t \le \rho \\ 1\,, & t \ge 2\rho \end{cases}$$

one sees that for ρ small enough (say $\rho < \frac{d_\Omega}{2}$)

$$|\nabla \phi| \le \frac{K_1}{\rho}\,, \tag{3.8}$$

with K_1 an absolute constant. Then

$$|\nabla f|^2 = f^2 |\nabla g|^2 + m\,,$$

where

$$m = 2 f e^g \nabla g \cdot \nabla \phi + e^{2g} |\nabla \phi|^2\,.$$

Estimating directly leads to:

$$\begin{aligned} |\langle \psi, m\psi\rangle| \le \langle \psi, |m|\psi\rangle &= \int_\Omega |\psi|^2 (2e^{2g}\phi |\nabla g|\,|\nabla\phi| + e^{2g}|\nabla\phi|^2)\,dx \\ &\le \frac{K_1}{\rho} \int_{\Omega_\rho \setminus \Omega_{2\rho}} \left|\psi e^g\right|^2 \left(2|\nabla g| + \frac{K_1}{\rho}\right) dx \end{aligned}$$

where in the last inequality we used (3.8), as well as the fact that $\nabla\phi \equiv 0$ on $(\Omega \setminus \Omega_\rho) \bigcup \Omega_{2\rho}$. But now recall that the Agmon condition (3.2) was that

$$(h - E)[\varphi, \varphi] - \int_\Omega |\varphi(x)|^2 |\nabla g(x)|^2\,dx \ge c\|\varphi\|^2\,,$$

with c independent of φ and ρ. Using the density of C_0^∞ in $W_0^{1,2}$, we obtain that

$$(h - E)[f\psi, f\psi] - \langle f\psi, |\nabla g|^2 f\psi\rangle \ge c\|f\psi\|^2\,. \tag{3.9}$$

Since

$$(h - E)[f\psi, f\psi] - \langle f\psi, |\nabla g|^2 f\psi\rangle = \langle \psi, m\psi\rangle\,,$$

we obtain

$$\frac{K_1}{\rho} \int_{\Omega_\rho \setminus \Omega_{2\rho}} \left|\psi e^g\right|^2 \left(2|\nabla g| + \frac{K_1}{\rho}\right) dx \ge |\langle \psi, m\psi\rangle| \ge c \int_\Omega \left|f\psi\right|^2 dx\,,$$

which, if we recall the choice of f made at the beginning of the proof, leads directly to the claim of the proposition. □

Theorem 4. *Consider an open, bounded domain $\Omega \subset \mathbb{R}^n$ with C^2-smooth boundary, and the Schrödinger operator*

$$H = -\Delta + V\,, \tag{3.10}$$

with $V \in L^\infty_{\mathrm{loc}}$ and domain $\mathcal{D}(H) = C_0^\infty(\Omega)$. Assume that there exist $E \in \mathbb{R}$ and $c > 0$ such that

$$\big\langle \varphi, (H-E)\varphi \big\rangle - \int_\Omega |\nabla g(x)|^2 |\varphi(x)|^2 \geq c\|\varphi\|^2\,, \tag{3.11}$$

for all $\varphi \in C_0^\infty(\Omega)$, where $g(x) = G(d(x))$ for some G satisfying condition (Σ).

If ψ is a weak solution of $H\psi = E\psi$, then $\psi \equiv 0$.

Proof. Let $d_0 > 0$ be the constant that appears in condition (Σ) for the function G from the hypothesis. Fix, for the time being, $0 < \rho_0 \leq d_0/2$, and let $\rho > 0$ be such that $2\rho \leq \rho_0$. Then define a "normalized" G function:

$$G_\rho(t) = G(t) - G(\rho)\,,$$

and set

$$g_\rho(x) = G_\rho\big(d(x)\big).$$

Note that for all $x \in \Omega$ we have

$$\nabla g_\rho(x) = G'\big(d(x)\big)\nabla d(x)\,. \tag{3.12}$$

This, together with condition $(\Sigma.1)$ for G, and the fact that $|\nabla d(x)| \leq 1$ for $d(x) < d_\Omega$, implies in particular that

$$|\nabla g_\rho(x)| \leq \frac{1}{d(x)} \quad \text{for} \quad x \in \Omega \setminus \Omega_{d_0/2}\,. \tag{3.13}$$

On the other hand, look at $x \in \Omega_{\rho_0}$. Since

$$\rho_0 \leq d(x)\,,$$

condition $(\Sigma.1)$ implies that

$$g_\rho(x) \geq G_\rho(\rho_0) = G(\rho_0) - G(\rho)\,, \tag{3.14}$$

and so

$$e^{2g_\rho(x)} \geq e^{2G(\rho_0)} \cdot e^{-2G(\rho)}\,, \quad \text{for all} \quad x \in \Omega_{\rho_0}\,. \tag{3.15}$$

Therefore

$$e^{2G(\rho_0)} \cdot e^{-2G(\rho)} \int_{\Omega_{\rho_0}} |\psi(x)|^2\,dx \leq \int_{\Omega_{\rho_0}} |e^{g_\rho(x)}\psi(x)|^2\,dx \leq \int_{\Omega_{2\rho}} |e^{g_\rho(x)}\psi(x)|^2\,dx\,,$$

where we used the fact that $2\rho \leq \rho_0$ and so $\Omega_{\rho_0} \subset \Omega_{2\rho}$. Now note that $\nabla g_\rho = \nabla g$, and so g_ρ satisfies (3.11) with the same E and c as g. In particular, one can apply Proposition 3.1 and obtain

$$\int_{\Omega_{2\rho}} |e^{g_\rho(x)}\psi(x)|^2\,dx \leq \frac{K(c)}{\rho} \int_{\Omega_\rho \setminus \Omega_{2\rho}} \left(\frac{1}{\rho} + \big|\nabla g_\rho(x)\big|\right) |e^{g_\rho(x)}\psi(x)|^2\,dx\,.$$

Since $0 < \rho < 2\rho < \rho_0 \le d_0/2$, it follows that $\Omega_\rho \setminus \Omega_{2\rho} \subset \Omega \setminus \Omega_{d_0/2}$ and so (3.13) implies that

$$\frac{K(c)}{\rho} \int_{\Omega_\rho \setminus \Omega_{2\rho}} \left(\frac{1}{\rho} + |\nabla g_\rho(x)| \right) |e^{g_\rho(x)} \psi(x)|^2 \, dx \le \frac{\tilde{K}(c)}{\rho^2} \int_{\Omega_\rho \setminus \Omega_{2\rho}} |\psi(x)|^2 \, dx \,,$$

where we also used the fact that, for $x \in \Omega_\rho \setminus \Omega_{2\rho}$,

$$g_\rho(x) \le G_\rho(2\rho) = G(2\rho) - G(\rho) = \int_\rho^{2\rho} G'(t) \, dt \le \int_\rho^{2\rho} \frac{1}{t} \, dt = \log 2 \,.$$

Putting it all together, we get that

$$K_2(c, \rho_0) \cdot \rho^2 e^{-2G(\rho)} \int_{\Omega_{\rho_0}} |\psi(x)|^2 \, dx \le \int_{\Omega_\rho \setminus \Omega_{2\rho}} |\psi(x)|^2 \, dx \,. \tag{3.16}$$

for some constant $K_2(c, \rho_0)$.

Now, let $n \ge 1$ be an integer, and set

$$\rho_n = \frac{1}{2^n} \rho_0 \,.$$

So $2\rho_n = \rho_{n-1}$, and we get

$$\bigcup_{n=1}^{M} (\Omega_{\rho_n} \setminus \Omega_{2\rho_n}) = \bigcup_{n=1}^{M} (\Omega_{\rho_n} \setminus \Omega_{\rho_{n-1}}) = \Omega_{\rho_M} \setminus \Omega_{\rho_0} \subset \Omega \,.$$

So using (3.16) successively with $\rho = \rho_n$, $1 \le n \le M$, and summing leads to

$$\rho_0^2 K_2(c, \rho_0) \left(\sum_{n=1}^{M} 4^{-n} e^{-2G(2^{-n}\rho_0)} \right) \int_{\Omega_{\rho_0}} |\psi(x)|^2 \, dx \le \int_\Omega |\psi(x)|^2 \, dx < \infty \,. \tag{3.17}$$

But from condition (Σ.2) we know that the series $\sum_{n \ge 1} 4^{-n} e^{-2G(2^{-n}\rho_0)}$ diverges, and so we find that

$$\int_{\Omega_{\rho_0}} |\psi(x)|^2 \, dx = 0 \,. \tag{3.18}$$

But $\rho_0 > 0$ was arbitrary, and so by taking $\rho_0 \to 0$ it follows that

$$\int_\Omega |\psi(x)|^2 \, dx = 0 \,, \tag{3.19}$$

as claimed. □

4. Proofs of the main theorems

Our strategy in approaching Theorem 1 consists of combining Agmon-type decay estimates for (weak) eigenfunctions (see Theorem 4) with multidimensional Hardy inequalities. More precisely, for H as above, the fundamental criterion for self-adjointness tells us that Theorem 1 can be proved as follows:

Lemma 4.1. *With the hypotheses of Theorem 4, there exists an $E < 0$ such that, for any $\psi \in L^2(\Omega)$, the condition*

$$\big\langle \psi, (H-E)\varphi \big\rangle = 0\,, \quad \textit{for every} \quad \varphi \in C_0^\infty(\Omega) \tag{4.1}$$

implies that $\psi \equiv 0$.

Proof. In view of Theorem 4 the only thing to be proved is that for $\psi \in L^2(\Omega)$, (3.1) implies that ψ is a weak solution of $(H-E)\psi = 0$ i.e. $\psi \in H^1_{loc}(\Omega)$. This is an interior regularity result for elliptic equations and follows from general theory. In our simple setting one can see by elementary means that $\psi \in H^2_{loc}(\Omega)$. Indeed let $\tilde\Omega \subset \Omega$, $\operatorname{dist}(\tilde\Omega, \partial\Omega) > 0$. Then $V \in L^\infty(\tilde\Omega)$ and from $\langle \psi, (H-E)\varphi\rangle = 0$ it follows

$$\big|\big\langle \psi, (-\Delta+1)\varphi\big\rangle\big| = \big|\big\langle \psi, (V-E+1)\varphi\big\rangle\big| \le K_{\tilde\Omega,E}\|\varphi\|$$

for all $\varphi \in C_0^\infty(\tilde\Omega)$, which via the Riesz lemma implies

$$\big\langle \psi, (-\Delta+1)\varphi\big\rangle = \langle \Phi, \varphi\rangle$$

for some $\Phi \in L^2(\tilde\Omega)$. This means that the distribution $(-\Delta+1)\Psi$ on $C_0^\infty(\tilde\Omega)$ is represented by a $L^2(\tilde\Omega)$ function and the proof is finished. □

The following multidimensional Hardy inequality will allow us to complete the proof of our main theorem:

Theorem 5 (Multidimensional Hardy Inequality). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2-smooth boundary. Then there exists a constant $A = A(\Omega) \in \mathbb{R}$ such that*

$$\frac{1}{4}\int_\Omega \frac{|\varphi(x)|^2}{d(x)^2}\,dx \le \int_\Omega |\nabla\varphi(x)|^2\,dx + A\|\varphi\|^2 \tag{4.2}$$

for every $\varphi \in C_0^\infty(\Omega)$.

This particular form of the Hardy inequality in domains in $\mathbb{R}^n$ can be found, for example, in [7].

Now the proof of Theorem 1 follows very quickly.

Proof of Theorem 1. From the fundamental criterion for self-adjointness (via Lemma 4.1) and the Agmon-type Theorem 4, we conclude that what we must show in order to complete the proof is that there exist $E \in \mathbb{R}$, as well as $c > 0$ and a function $g(x) = G(d(x))$ with G satisfying (Σ) such that

$$\big\langle \varphi, (H-E)\varphi\big\rangle - \int_\Omega |\nabla g(x)|^2|\varphi(x)|^2\,dx \ge c\|\varphi\|^2\,, \tag{4.3}$$

for all $\varphi \in C_0^\infty(\Omega)$.

Recall that under the hypotheses of Theorem 1, the potential $V = V_1 + V_2$ with $V_2 \in L^\infty(\Omega)$ and

$$V_1(x) \ge G'\big(d(x)\big)^2 - \frac{1}{4}\cdot\frac{1}{d(x)^2}\,,$$

for some G satisfying (Σ). Using exactly this G to define the g we need, and applying the result of the multidimensional Hardy inequality above, we get that for $E \in \mathbb{R}$

$$\begin{aligned}\big\langle\varphi,(H-E)\varphi\big\rangle-\int_\Omega|\nabla g(x)|^2|\varphi(x)|^2\,dx\\ &\geq\int_\Omega\left(V_1(x)-G'\big(d(x)\big)^2+\frac{1}{4d(x)^2}\right)\cdot|\varphi(x)|^2\,dx\\ &\quad+\big(-\|V_2\|_{L^\infty}-A-E\big)\|\varphi\|^2\\ &\geq\big(-\|V_2\|_{L^\infty}-A-E\big)\|\varphi\|^2\,.\end{aligned}$$

On the way we have used the fact that $|\nabla g(x)|^2 \leq G'(d(x))^2$. So choosing, for example,

$$E=-\|V_2\|_{L^\infty}-A-1 \tag{4.4}$$

leads to (4.3) being satisfied with $c=1$. This is exactly what we needed, and concludes our proof. □

Proof of Theorem 2. As already explained in Section 2, Theorem 2 follows directly from Theorem 1 and a choice of function G which for small t coincides with (see (2.16)):

$$\ln t+\frac{1}{2}\cdot\sum_{j=2}^{p}L_j(t)+\int_t\tilde{f}(u)\,du\,,$$

where we recall that the functions L_j were defined in (2.11), and $\tilde{f}$, which is to be found, must satisfy (2.9).

More precisely, let

$$\mathcal{L}_p(t)=\sum_{k=2}^{p}\left(\prod_{j=1}^{k-1}L_j(t)\right)^{-1} \tag{4.5}$$

defined for $0<t<e_p^{-1}$. Notice that $\lim_{t\to 0}\mathcal{L}_p(t)=0$ and moreover $\frac{\mathcal{L}_p(t)^2}{t}$ is integrable at zero. So if we define

$$\tilde{f}(t)=\begin{cases} f(t)+\frac{1}{4t}\mathcal{L}_p(t)^2\,, & \text{for}\quad 0\leq t\leq e_p^{-1}\\ 0\,, & \text{for}\quad t\geq e_p^{-1}\,,\end{cases} \tag{4.6}$$

then it satisfies (2.9). Let now $h(t)$ be a smooth function with the properties:

$$h(t)=\begin{cases} t\,, & \text{for}\quad 0\leq t\leq\frac{d_0}{2}\\ \frac{3}{4}d_0\,, & \text{for}\quad t\geq d_0\,,\end{cases} \tag{4.7}$$

and $0<h'(t)\leq 1$ for all $0<t<d_0$. Here $d_0\leq\min\{e_p^{-1},d_\Omega\}$ and in addition is sufficiently small such that for $t\in(0,d_0)$

$$1-\frac{1}{2}\mathcal{L}_p(t)-t\tilde{f}(t)\geq\frac{2}{3}\,. \tag{4.8}$$

We claim that

$$G_p(t) = \ln h(t) + \frac{1}{2} \cdot \sum_{j=2}^{p} L_j\big(h(t)\big) + \int_{h(t)} \tilde{f}(u)\, du\,, \tag{4.9}$$

satisfies all the needed conditions.

To check that G_p satisfies (Σ), first note that, for any $k \geq 1$,

$$L_k'(t) = -\frac{1}{t}\left(\prod_{j=1}^{k-1} L_j(t)\right)^{-1},$$

and so, for $t \in (0, d_0)$

$$G_p'(t) = \frac{1}{h(t)} \cdot \left[1 - \frac{1}{2}\mathcal{L}_p\big(h(t)\big) - h(t)\tilde{f}\big(h(t)\big)\right] h'(t)\,, \tag{4.10}$$

while for $t \geq d_0$, $G_p'(t) = 0$. Then $(\Sigma.1)$ follows from (4.10) , (4.8) and the properties of $h(t)$.

To check $(\Sigma.2)$, note that from (4.9) for $t < \frac{d_0}{2}$ (take into account that $d_0 < 1$ and for $t < \frac{d_0}{2}$, $h(t) = t$)

$$e^{-2G_p(2^{-n}\rho_0)} \geq 4^n \rho_0^{-2} e^{-2\int_{0+} \tilde{f}(u)\, du} \cdot \left(\prod_{j=1}^{p-1} L_j(2^{-n}\rho_0)\right)^{-1}.$$

If we define, for $x \in \mathbb{R}$ large enough, the log-log functions $\ln_0(x) = x$, $\ln_k(x) = \ln(\ln_{k-1}(x))$, then note that for all $1 \leq j \leq p-1$ and $n \geq N(\rho_0) = \frac{1}{1-\ln 2} \ln \rho_0^{-1}$ (remember that $2\rho_0 < e_p^{-1}$)

$$L_j(2^{-n}\rho_0) = \ln_j(2^n\rho_0^{-1}) = \ln_{j-1}(n \ln 2 + \ln \rho_0^{-1}) \leq \ln_{j-1} n\,.$$

But then

$$\sum_{n=0}^{\infty} 4^{-n} e^{-2G_p(2^{-n}\rho_0)} \geq \text{const.} \sum_{n=N(\rho_0)}^{\infty} \frac{1}{n \ln(n) \ln_2(n) \cdots \ln_{p-2}(n)} = +\infty\,,$$

where the divergence of the latter series is an elementary consequence of the integral test.

Since $\sup_{t \geq \frac{d_0}{2}} G_p'(t)^2 < \infty$, in view of the remark following Theorem 2, all that remains to be done in order to apply Theorem 1 is to show that for $t \in (0, \frac{d_0}{2})$ (2.14) implies (2.6) with $G(t) = G_p(t)$. Taking into account (4.10) it is sufficient to check that for $t \in (0, \frac{d_0}{2})$:

$$\frac{1}{t^2} - \frac{1}{t^2}\mathcal{L}_p(t) - \frac{1}{t} f(t) \geq \frac{1}{t^2}\left(1 - \frac{1}{2}\mathcal{L}_p(t) - t\tilde{f}(t)\right)^2 \tag{4.11}$$

Doing the algebra one gets the condition

$$-\frac{f(t)}{t} \geq -2\frac{\tilde{f}(t)}{t} + \frac{\mathcal{L}_p(t)^2}{4t^2} + \big(t\tilde{f}(t) + \mathcal{L}_p(t)\big)\frac{\tilde{f}(t)}{t}$$

Now taking into account (4.6) and that from (4.8), for $t \in (0, \frac{d_0}{2})$, $\mathcal{L}_p \leq \frac{2}{3}$, $t\tilde{f}(t) \leq \frac{1}{3}$ one has

$$-2\frac{\tilde{f}(t)}{t} + \frac{\mathcal{L}_p(t)^2}{4t^2} + \big(t\tilde{f}(t) + \mathcal{L}_p(t)\big)\frac{\tilde{f}(t)}{t} \leq -\frac{\tilde{f}(t)}{t} + \frac{\mathcal{L}_p(t)^2}{4t^2} = -\frac{f(t)}{t},$$

and the proof is finished. □

Finally we turn to the proof of our optimality theorem:

Proof of Theorem 3. In order to achieve this, we will work in 1 dimension, on the interval $(0,1)$, and construct such a potential close to 0. In this case, let $\alpha \in \mathbb{R}$ and consider the wave function

$$\psi_{p,\alpha}(x) = x^{-\frac{1}{2}} \cdot \left(\prod_{j=1}^{p-1} L_j(x)\right)^{-\frac{1}{2}} \cdot L_p(x)^{\alpha}. \tag{4.12}$$

First note that $\psi_{p,\alpha}$ grows as $x \to 0+$ for all $\alpha \in \mathbb{R}$, but that

$$\int_{0+} \psi_{p,\alpha}^2(x)\,dx = \infty \quad \Longleftrightarrow \quad \alpha \geq -\frac{1}{2}.$$

A direct calculation shows that

$$\psi''_{p,\alpha}(x) = V_{p,\alpha}(x)\psi_{p,\alpha}(x), \tag{4.13}$$

with

$$\begin{aligned} V_{p,\alpha}(x) = \frac{3}{4}\cdot\frac{1}{x^2} - \frac{1}{x^2}\cdot\sum_{j=1}^{p-1}\left(\prod_{k=1}^{j} L_k(x)\right)^{-1} \\ + \big(2\alpha + o(1)\big)\cdot\frac{1}{x^2}\cdot\left(\prod_{j=1}^{p} L_j(x)\right)^{-1} \end{aligned} \tag{4.14}$$

where the $o(1)$ comes from a sum of terms which are of lower order (in the same spirit as in the previous proof). In this case, they are

$$\begin{aligned} &\frac{1}{4}\cdot\frac{1}{x^2}\cdot\left(\sum_{j=1}^{p-1}\left(\prod_{k=1}^{j} L_k(x)\right)^{-1}\right)^2 + \frac{\alpha^2}{x^2}\cdot\left(\prod_{j=1}^{p} L_j(x)\right)^{-2} \\ &\quad - \frac{\alpha}{x^2}\cdot\left(\sum_{j=1}^{p-1}\left(\prod_{k=1}^{j} L_k(x)\right)^{-1}\right)\cdot\left(\prod_{j=1}^{p} L_j(x)\right)^{-1} \\ &\quad + \frac{1}{2}\cdot\frac{1}{x^2}\sum_{j=1}^{p-1}\sum_{k=1}^{j}\left(\prod_{l=1}^{j} L_l(x)\right)^{-1}\left(\prod_{l=1}^{k} L_l(x)\right)^{-1} \\ &\quad - \frac{\alpha}{x^2}\sum_{k=1}^{p}\left(\prod_{l=1}^{p} L_l(x)\right)^{-1}\left(\prod_{m=1}^{k} L_m(x)\right)^{-1}. \end{aligned}$$

Further note that the other (decreasing at 0+) solution of

$$\phi''_{p,\alpha}(x) = V_{p,\alpha}(x)\phi_{p,\alpha}(x)$$

is given by the usual relation

$$\phi_{p,\alpha}(x) = \psi_{p,\alpha}(x) \cdot \int_0^x \frac{1}{\psi^2_{p,\alpha}(y)}\, dy\,.$$

Since $\psi^{-2}_{p,\alpha}(y) \sim y^{1-\epsilon}$ as $y \to 0+$ for any given $\epsilon > 0$, we see that $\phi_{p,\alpha}(x) \to 0$ as $x \to 0+$, and so in particular $\phi_{p,\alpha}$ and $\psi_{p,\alpha}$ are indeed two independent solutions. But for $\alpha < -\frac{1}{2}$, they are both in $L^2(0+)$ and so we are in the limit-circle case and

$$H_{p,\alpha} = -\Delta + V_{p,\alpha}$$

is *not* essentially self-adjoint on $(0, 1)$. But this is exactly the type of potential we were looking for: given a constant $c > 1$, pick an $\alpha < -\frac{c}{2} < -\frac{1}{2}$. Thus $H_{p,\alpha}$ is not essentially self-adjoint, but for x close enough to the boundary $\partial\Omega$, equation (4.14) together with our choice of α implies that $V_{p,\alpha}$ satisfies (2.17), as claimed in the theorem.

Finally, the potentials $V_{p,\alpha}$ can also be used in several space dimensions to construct counterexamples $\square$

References

[1] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators.* Mathematical Notes, **29**, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[2] N. I. Akhiezer, I. M. Glazman, *Theory of linear operators in Hilbert space. Vol II.* Translated from the third Russian edition by E. R. Dawson. Translation edited by W. N. Everitt. Monographs and Studies in Mathematics, 10. *Pitman (Advanced Publishing Program), Boston, Mass.-London*, 1981.

[3] G. Barbatis, S. Filippas, A. Tertikas, *Series expansions for L^p Hardy inequalities*, Indiana Univ. Math. J. **52** (2003), 171–190.

[4] M. Braverman, O. Milatovic, M. Shubin, *Essential self-adjointness of Schrödinger type operators on manifolds*, Russ. Math. Surveys **57** (2002), 641–692.

[5] A. G. Brusentsev, *Selfadjointness of elliptic differential operators in $L_2(G)$, and correction potentials*, Trans. Moscow Math. Soc. **65** (2004), 31–61.

[6] E. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1955.

[7] B. Davies, *A review of Hardy inequalities*, The Mazćya anniversary collection, Vol. 2 (Rostock, 1998), 55–67, Oper. Theory Adv. Appl., **110**, Birkhäuser, Basel, 1999.

[8] B. Davies, *The Hardy constant*, Quart. J. Math. Oxford (2) **46** (1996), 417–431.

[9] J. Dolbeault, M. Esteban, M. Loss, L. Vega, *An analytical proof of Hardy-like inequalities related to the Dirac operator*, J. Funct. Anal. **216** (2004), 1–21.

[10] M. Esteban, M. Loss, *Self-adjointness via partial Hardy-like inequalities*, in Proceedings of the QMath10 Conference: Mathematical Results in Quantum Mechanics. World Scientific 2008, 41–47.

[11] R. Frank, E. H. Lieb, R. Seiringer, *Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. **21** (2008), 925–950.

[12] F. Gesztesy, *On non-degenerate ground states for Schrödinger operators*, Rep. Math. Phys. **20** (1984), 93–109.

[13] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1983.

[14] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, A. Laptev, J. Tidblom, *Many-particle Hardy inequalities*, J. Lond. Math. Soc. (2) **77** (2008), 99–114.

[15] H. Kalf, U.-V. Schminke, J. Walter, R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, in Lecture Notes in Mathematics **448** (1975), 182–226.

[16] A. Laptev, A. Sobolev, *Hardy inequalities for simply connected planar domains*, arXiv:math/0603362

[17] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness.* Academic Press , New York-London, 1975. xv+361 pp.

[18] A. Tertikas, N. B. Zographopoulos, *Best constants in the Hardy-Rellich inequalities and related improvements*, Adv. Math. **209** (2007), 407–459.

Gheorghe Nenciu
Department of Theoretical Physics and Mathematics
University of Bucharest
P.O. Box MG 11
RO-077125 Bucharest
Romania
and
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
21, Calea Griviţei
RO-010702 Bucharest, Sector 1
Romania
e-mail: Gheorghe.Nenciu@imar.ro

Irina Nenciu
Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL
USA
and
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
21, Calea Griviţei
RO-010702 Bucharest, Sector 1
Romania
e-mail: nenciu@math.uic.edu

Communicated by Claude Alain Pillet.

Submitted: November 18, 2008.

Accepted: January 19, 2009.