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Octonionic Twists for Supermembrane Matrix Models

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Abstract. A certain $G_2 \times U(1)$ invariant Hamiltonian arising from the standard membrane matrix model via conjugating any of the supercharges by a cubic, octonionic, exponential is proven to have a spectrum covering the whole half-axis $\mathbb{R}_+$. The model could be useful in determining a normalizable zero-energy state in the original $SO(9)$ invariant $SU(N)$ matrix model.

1. Introduction

Despite considerable effort [3, 7, 8, 12, 13, 15–19, 21–25, 29–31, 35, 38, 39, 43, 48, 49] during the last decade, and crucial relevance to M-theory [4, 46] membrane theory [5, 14, 20, 45] reduced Yang–Mills theory [2, 6, 9, 11], existence, uniqueness and structure of zero-energy states in $Spin(9) \times SU(N)$ invariant supersymmetric matrix models are not really understood to a degree that one could call satisfactory.

In this paper we consider models with $G_2 \times U(1) \times SU(N)$ symmetry that we obtain by deforming (cp. [10, 36]) the $Spin(9) \times SU(N)$ models, and which we believe to be relevant both from the point of view of deformation theory and possible relations between ground states, as well as because (for the fixed value of the deformation parameter that we take) the Hamiltonian is slightly simpler, and therefore a good testing ground for new approaches.

The model is introduced in Section 2 by deforming the $Spin(9) \times SU(N)$ invariant one via a particular cubic exponential. In Section 3, with the help of various propositions that are proved in Section 4, this model is shown to share a central feature of the original theory, namely that the Hamiltonian, in contrast with the discreteness of the spectrum (cp. [34, 40]) for the purely bosonic theory, has an essential spectrum covering the whole positive axis (cp. [41, 44]). A summary of the results is presented in Section 5. In the appendices some background material is provided, and the deformation we introduce put into a slightly more general context.

2. The deformed model

To find a normalizable state annihilated by the supercharges [1]

$$Q_\beta := \left(i\delta_{\alpha\beta}\frac{i}{2}f_{ABC}z_B\bar{z}_C + i\Gamma^j_{\alpha\beta}\frac{\partial}{\partial x_{jA}} - \frac{1}{2}f_{ABC}x_{jB}x_{kC}\Gamma^{jk}_{\alpha\beta}\right)\lambda_{\alpha A} + \left(2\delta_{\alpha\beta}\frac{\partial}{\partial z_A} - if_{ABC}x_{jB}\bar{z}_C\Gamma^j_{\alpha\beta}\right)\lambda^\dagger_{\alpha A} \quad (1)$$

(and by their hermitian conjugates) is a difficult task; the $(x_{jA})^{A=1,\ldots,N^2-1}_{j=1,\ldots,7}$ and $z_A = x_{8A} + ix_{9A}$ are bosonic coordinates, $(\lambda_{\alpha A})^{A=1,\ldots,N^2-1}_{\alpha=1,\ldots,8}$ Grassmann variables, f_{ABC} totally antisymmetric structure constants of $SU(N)$, $\Gamma^{jk} := \frac{1}{2}[\Gamma^j, \Gamma^k]$, and $(\Gamma^j)_{j=1,\ldots,7}$ (purely imaginary, antisymmetric) matrices satisfying $\{\Gamma^j, \Gamma^k\} = 2\delta^{jk}\mathbf{1}_{8\times 8}$, in a particular representation given by $i\Gamma^j_{\alpha 8} = \delta^j_\alpha$, $i\Gamma^j_{kl} = -c_{jkl}$, totally anti-symmetric octonionic structure constants.

In [26] conjugation by the exponent of

$$g(x) := \frac{1}{6}f_{ABC}x_{jA}x_{kB}x_{lC}(i\Gamma^{jkl})_{\beta\beta} \quad (2)$$

was shown to remove the third term in (1) (extending an observation made in [45]; note that $\sum_j \Gamma^j_{\alpha\beta}\Gamma^{jkl}_{\beta\beta} = \Gamma^{kl}_{\alpha\beta}$ for fixed β, cp. Appendix B). Correspondingly, defining $H_k := \{Q(k), Q(k)^\dagger\} \geq 0$, where (choosing $\beta = 8$)

$$Q(k) := e^{kg(x)}Q_8e^{-kg(x)} = Q_8 + \frac{k}{2}f_{ABC}x_{jB}x_{lC}\Gamma^{jl}_{\alpha 8}\lambda_{\alpha A}\,, \quad (3)$$

gives

$$\begin{aligned} H_k = & -\Delta_{\mathbb{R}^{9(N^2-1)}} + (k-1)^2V_{1\ldots7} + V_{89} + \bar{z}_Af_{ABC}x_{jB}f_{A'B'C}x_{jB'}z_{A'} \\ & + 2f_{AA'E}x_{jE}(\delta_{\alpha8}\delta_{\alpha'j} - \delta_{\alpha j}\delta_{\alpha'8})\lambda_{\alpha A}\lambda^\dagger_{\alpha'A'} \\ & + 2(k-1)f_{EAA'}x_{jE}\left(i\Gamma^j\right)_{ll'}\lambda_{lA}\lambda^\dagger_{l'A'} \\ & + f_{EAA'}z_E\lambda_{\alpha A}\lambda_{\alpha A'} + f_{EAA'}\bar{z}_E\lambda^\dagger_{\alpha A'}\lambda^\dagger_{\alpha A}\,. \end{aligned} \quad (4)$$

The potential terms for the x- resp. z-coordinates are given by

$$V_{1\ldots7} = \frac{1}{2}f_{ABC}x_{jB}x_{lC}f_{AB'C'}x_{jB'}x_{lC'} \quad \text{resp.} \quad V_{89} = \frac{1}{4}f_{ABC}\bar{z}_Bz_Cf_{AB'C'}z_{B'}\bar{z}_{C'}\,.$$

While for large k,

$$\hat{H} := -\Delta_x + V_{1\ldots7} - 2f_{EAA'}x_{jE}c_{jll'}\lambda_{lA}\lambda^\dagger_{l'A'}\,, \quad (5)$$

appears to be the relevant operator (having rescaled $x \to (k-1)^{-1/3}x$) [2] we will, in this note, exclusively study $H_{k=1} =: \tilde{H}$, which is of the form (cp. (4))

$$\tilde{H} = -\Delta_x + H_D + V_{89} + fz\lambda\lambda + f\bar{z}\lambda^\dagger\lambda^\dagger, \quad (6)$$

$$H_D := -\Delta_z + \bar{z}_Af_{ABC}x_{jB}f_{A'B'C}x_{jB'}z_{A'} + 2f_{AA'E}x_{jE}(\delta_{\alpha8}\delta_{\alpha'j} - \delta_{\alpha j}\delta_{\alpha'8})\lambda_{\alpha A}\lambda^\dagger_{\alpha'A'}\,.$$

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[1] cp. Appendix A

[2] This point (and [10] in general) was discussed with B. Durhuus and J. P. Solovej, -which we gratefully acknowledge.

The operator H_D ($= H_D(x) \geq 0$) arises from the second line of (1) alone, and we will heavily use that its spectrum and eigenfunctions are known [27]. The operator $V_{89}+zf\lambda\lambda+\bar{z}f\lambda^\dagger\lambda^\dagger$ appearing in (6) and involving only the z-coordinates will be denoted by K. We also note that, regardless of the choice of k, the bosonic part of H_k (first line in (4)) has a strictly positive and purely discrete spectrum (this is easily proved along the lines of [34, 40]).

The bosonic part of H_D describes two sets of $n := N^2-1$ harmonic oscillators whose frequencies ω_A are the square root of the eigenvalues of the parametrically x-dependent, positive semidefinite frequency matrix

$$S_{AA'}(x) := f_{ABC}x_{jB}f_{A'B'C}x_{jB'}\,, \tag{7}$$

while its fermionic part, $2W_{\alpha A\,\beta B}(x)\lambda_{\alpha A}\lambda^\dagger_{\beta B}$, that is linear in x_{jA}, has eigenvalues arising from those of $2W_{\alpha A\,\beta B}$, which are $\{\pm 2\omega_A(x)\}_{A=1,\ldots,n}$ as well as $6n$ times the eigenvalue zero – altogether leading to the exact zero-energy state(s) of H_D [27]

$$\psi_x = \prod_{A=1}^{n}\sqrt{\frac{\omega_A(x)}{\pi}}\, e^{-\frac{1}{2}\omega_A(x)\bar{z}'_A z'_A}\, e^{(-\omega_1)}_{\alpha_1 A_1}(x)\ldots e^{(-\omega_l)}_{\alpha_l A_l}(x)\lambda_{\alpha_1 A_1}\ldots\lambda_{\alpha_l A_l}\,, \tag{8}$$

(where, $n \leq l \leq 7n$, $\omega_{l>n} := 0$, and we have diagonalized S via $z'_A = R_{AB}(x)z_B$, $S = R^T[\omega_A^2]R$), that involves the eigenvectors $e^{(\omega)}(x)$ of the matrix $W(x)$ corresponding to eigenvalue ω. Excited states of H_D are obtained by acting with the bosonic creation operators (i.e. multiplying ψ_x by the corresponding Hermite polynomials) and/or adding fermions corresponding to positive eigenvalues 2ω (i.e. multiplying ψ_x by $e^{(+\omega)}_{\alpha A}\lambda_{\alpha A}$).

Alternatively, thinking of the coordinate point $x = (x_{jA})$ as a tuple (X_j) of traceless hermitian $N \times N$ matrices, with $X_j = x_{jA}T_A$ in a basis $\{T_A\}$ s.t. $[T_A, T_B] = if_{ABC}T_C$, the matrix (operator) S can also be written as

$$S(x) = \sum_{j=1}^{7}\mathrm{ad}_{X_j}\circ\mathrm{ad}_{X_j} = \big[X_j, [X_j,\ \cdot\ ]\big]\,,$$

acting on $i\cdot\mathfrak{su}(N) \cong \mathbb{R}^n, E \leftrightarrow (e_A)$. In particular, its lowest eigenvalue $\omega^2_{\min}$ is given by

$$\omega^2_{\min} = \min_{e\in S^{n-1}} e_A S_{AB} e_B = \min_{\|E\|=1}\sum_j \|[X_j, E]\|^2\,,$$

where $\|\cdot\|$ here denotes the corresponding norm on $i\cdot\mathfrak{su}(N) \cong \mathbb{R}^n$. Hence, for $N > 2$, $S(x)$ will have zero-modes not only when all matrices X_j commute, but (of qualitative significance) for the *larger* space of configurations where all the X_j are simultaneously block-diagonalizable.

3. Continuity of the spectrum of $\tilde{H}$

In this section we formulate and prove the main theorem of the paper. We will make use of three propositions and one lemma (which are proved in Section 4 in order not to break the flow of the text).

Main theorem. *For any $\lambda \geq 0$ there exists a sequence (Ψ_t) of rapidly decaying smooth $SU(N)$-invariant functions such that $\|\Psi_t\| = 1$ and*

$$\left\|(\tilde{H} - \lambda)\Psi_t\right\| \to 0 \quad \textit{as} \quad t \to \infty \,.$$

It follows that the spectrum of $\tilde{H}$ (when restricted to the physical Hilbert space) consists of the whole non-negative real line.

This is clearly similar to the the case for the original $H_{k=0}$. However, because of the terms that vanish for $H_{k=1}$, together with the convenient structure of the remaining terms noted in the previous section, we are able to construct such a sequence (Ψ_t) explicitly without resorting to the gauge fixing procedure used in [41, 44].

In the following, we write $\tilde{H}$ as

$$\tilde{H} = -\Delta_x + H_D(x) + K \,, \tag{9}$$

where

$$H_D(x) = -4\partial_{\bar{z}} \cdot \partial_z + \bar{z} \cdot S(x) z + 2W(x)\lambda\lambda^\dagger \tag{10}$$

and

$$K(z) = \frac{1}{4} f\bar{z}z f z\bar{z} + f z\lambda\lambda + f\bar{z}\lambda^\dagger\lambda^\dagger \,. \tag{11}$$

We also point out that, since $\tilde{H}$ is an unbounded operator, it is considered to be defined as a differential operator on the Schwartz class $\mathcal{S}$ of smooth functions of rapid decay, and then extends by closure or Friedrichs extension to a self-adjoint operator in $\mathcal{H} = L^2(\mathbb{R}^{9n}) \otimes \mathcal{F}$.

Our candidate for the sequence Ψ_t will be wavefunctions given by the minimal fermion number ground state ψ_x of $H_D(x)$ multiplied by some gauge invariant cut-off function χ_t. Formally, it is convenient to write the Hilbert space $\mathcal{H}$ as a constant fiber direct integral (see [37]) over the x-coordinates,

$$\mathcal{H} = \int_{\mathbb{R}^{7n}}^{\oplus} \hbar \; dx \,,$$

where (writing $dx = d^{7n}x$, $dz = d^n x_8 d^n x_9$ for the integration measures)

$$\hbar := L^2(\mathbb{R}^{2n}) \otimes \mathcal{F} = \int_{\mathbb{R}^{2n}}^{\oplus} \mathcal{F} \; dz$$

is the z-coordinate Hilbert space on which the operator $H_D(x) + K$ acts in each point x. Hence, for any $\Psi(x, z) = \chi(x)\psi_x(z) \in \mathcal{H}$, we have

$$\|\Psi\|^2_{\mathcal{H}} = \int_{\mathbb{R}^{7n}} |\chi(x)|^2 \|\psi_x\|^2_{\hbar} \; dx = \int_{\mathbb{R}^{7n}} |\chi(x)|^2 \int_{\mathbb{R}^{2n}} \|\psi_x(z)\|^2_{\mathcal{F}} \; dz \; dx \,. \tag{12}$$

We also write the ground state (8) of $H_D(x)$ in a more compact notation,

$$\psi_x(z) = \pi^{-\frac{n}{2}} s(x)^{\frac{1}{4}} e^{-\frac{1}{2}\bar{z}\cdot S(x)^{1/2} z} \xi_x \,, \tag{13}$$

where $s := \det S$, and $\xi_x \in \mathcal{F}_n$ (i.e. n fermions) is the normalized fermionic eigenvector satisfying

$$W(x)\lambda\lambda^\dagger \, \xi_x = -\sum_A \omega_A \, \xi_x = -\operatorname{tr}\left(S(x)^{\frac{1}{2}}\right)\xi_x \,. \tag{14}$$

We note the following:

Proposition 1. *ψ_x is smooth (also in x), rapidly decaying, and $SU(N)$-invariant.*

Proposition 2. *$\|\psi_x\|_{\mathfrak{h}} = 1$, and $\||z|^k\psi_x\|_{\mathfrak{h}} \le \frac{C_k}{\omega_{\min}^{k/2}(x)}$, for $k = 1, 2, 4$ and some positive constants C_k.*

Hence, by choosing an appropriate cut-off function χ_t for the x-coordinates such that $\omega_{\min}(x) \to \infty$ as $t \to \infty$, we can make the terms in $K(z)$ arbitrarily small. The following proposition shows that such a choice is indeed possible.

Proposition 3. *For any $\lambda \ge 0$ and t sufficiently large there exist $SU(N)$-invariant cut-off functions $\chi_t \in C_0^\infty(\mathbb{R}^{7n})$ such that $\forall x \in \operatorname{supp}\chi_t$*

$$\omega_{\min}(x) \ge c_1 t \,, \quad c_2 t \le |x| \le c_3 t \,, \tag{15}$$

and, as $t \to \infty$,

$$\|\chi_t\|_{L^2(\mathbb{R}^{7n})} = 1 \,, \quad \|\partial_{jA}\chi_t\|_{L^2(\mathbb{R}^{7n})} \le c_4 \,, \quad \|(-\Delta_{\mathbb{R}^{7n}} - \lambda)\chi_t\|_{L^2(\mathbb{R}^{7n})} \to 0 \,, \tag{16}$$

where (here, and in the following) $c_{k=1,2,3,\ldots}$ are some positive constants.

As a final preparation before proving the main theorem, we state the following lemma which ensures that also certain derivatives tend to zero.

Lemma 4. *$\|\partial_{jA}\psi_x\|_{\mathfrak{h}} \le \frac{c_5}{\omega_{\min}(x)}$ and $\|\partial_{jA}^2\psi_x\|_{\mathfrak{h}} \le \frac{c_6}{\omega_{\min}^2(x)}$ on $\operatorname{supp}\chi_t$.*

3.1. Proof of the main theorem

Motivated by the expression (9) and the above preparations, we define

$$\Psi_t(x,z) := \chi_t(x)\psi_x(z) \,,$$

where $\chi_t \in C_0^\infty(\mathbb{R}^{7n})$ is chosen according to Proposition 3. We note that Ψ_t is in the domain of $\tilde{H}$ and by (12) has $\|\Psi_t\| = 1$. Acting with $\tilde{H}$ on $\Psi_t(x,z)$, we obtain

$$\tilde{H}\Psi_t = -\psi_x\Delta_x\chi_t - 2\sum_{j,A}\partial_{jA}\chi_t\partial_{jA}\psi_x - \sum_{j,A}\chi_t\partial_{jA}^2\psi_x + \chi_t K(z)\psi_x \,, \tag{17}$$

(where we used the fact that $H_D(x)\psi_x = 0$). Subtracting $\lambda\Psi_t$ from this equation and using Propositions 2, 3 and Lemma 4 (and that any operator on $\mathcal{F}$ is bounded) to estimate the norms of the terms on the r.h.s. as $t \to \infty$, we find

$$\|\psi_x(-\Delta_x - \lambda)\chi_t\|^2 = \|(-\Delta_x - \lambda)\chi_t\|^2_{L^2(\mathbb{R}^{7n})} \to 0\,,$$

$$\|\partial_{jA}\chi_t\,\partial_{jA}\psi_x\|^2 \leq \int |\partial_{jA}\chi_t|^2 \left(\frac{c_5}{\omega_{\min}}\right)^2 dx \leq \frac{c_7}{t^2}\|\partial_{jA}\chi_t\|^2_{L^2(\mathbb{R}^{7n})} \leq \frac{c_8}{t^2} \to 0\,,$$

$$\|\chi_t\partial^2_{jA}\psi_x\|^2 \leq \int |\chi_t|^2 \left(\frac{c_6}{\omega^2_{\min}}\right)^2 dx \leq \frac{c_9}{t^4}\|\chi_t\|^2_{L^2(\mathbb{R}^{7n})} \to 0\,,$$

and

$$\begin{aligned}\|\chi_t K(z)\psi_x\|^2 &\leq \int |\chi_t|^2 (c_{10}\||z|^4\psi_x\|_{\mathcal{h}} + c_{11}\||z|\psi_x\|_{\mathcal{h}})^2 dx \\ &\leq \int \left(\frac{c_{10}C_4}{\omega^2_{\min}} + \frac{c_{11}C_1}{\omega^{1/2}_{\min}}\right)^2 |\chi_t|^2 dx \leq \frac{c_{12}}{t}\|\chi_t\|^2_{L^2(\mathbb{R}^{7n})} \to 0\,.\end{aligned}$$

Hence, $\|(\tilde{H} - \lambda)\Psi_t\| \to 0$ as $t \to \infty$.

It follows that, for any $\lambda \geq 0$, the operator $\tilde{H} - \lambda$ does not have a bounded inverse. Together with $\tilde{H} \geq 0$ from supersymmetry, this proves the theorem. □

4. Proofs

Here we present detailed proofs of the propositions and lemma that were stated in the previous section.

4.1. Proof of Proposition 1

It is obvious from (13) that $\psi_x \in \mathcal{S}(\mathbb{R}^{2n}) \otimes \mathcal{F}_n =: \mathcal{S}_n$. Smoothness in x for the scalar (bosonic) part of ψ_x follows from our requirement that $\omega_{\min}(x) > 0$, i.e. $s > 0$ for every x we consider. As for the fermionic part ξ_x, smoothness follows by considering $\mathcal{F}_n$ as a real space of dimension $\binom{8n}{n}$ and, for each point x, viewing ξ_x as the (up to sign) unique normalized eigenvector of the linear map $\xi \mapsto W(x)\lambda\lambda^\dagger\xi$ with eigenvalue $-\sum_A \omega_A(x)$. (A consistent choice of sign can be made because we will only be working on orientable subsets of $\mathbb{R}^{7n}$.) Smoothness of ξ_x now follows from smoothness of $W(x)$ and the implicit function theorem. Also note that any x-derivatives $\partial_{jA}\psi_x$, $\partial_{jA}\partial_{kB}\psi_x$, etc. still lie in $\mathcal{S}_n$.

ψ_x is $SU(N)$-invariant (covariant) in the sense that $\tilde{R}\psi_{Rx}(Rz) = \psi_x(z)$, where R (resp. $\tilde{R}$) $\in SU(N) \hookrightarrow SO(n)$ (resp. $Spin(\mathcal{F}_n)$). This follows from the uniqueness of ψ_x at each point x and covariance of the operator $H_D(x)$, i.e. $U_R H_D(x) U_R^\dagger = H_D(R^T x)$, where U denotes the corresponding unitary representation of $SU(N)$ on $\mathcal{h}$. □

4.2. Proof of Proposition 2

Since $\psi_x(z)$ is Gaussian in the z-coordinates, the evaluation of the moments $\||z|^k\psi_x\|_{\hbar}^2 = \langle |z|^{2k}\rangle_{\psi_x}$ is straightforward. We find that $\|\psi_x\|_{\hbar}^2 = \|\xi_x\|_{\mathcal{F}}^2 = 1$,

$$\||z|\psi_x\|_{\hbar}^2 = \sum_A \frac{1}{\omega_A},$$

$$\big\||z|^2\psi_x\big\|_{\hbar}^2 = \frac{3}{2}\sum_A \frac{1}{\omega_A^2} + \frac{1}{2}\left(\sum_A \frac{1}{\omega_A}\right)^2, \quad \text{and}$$

$$\begin{aligned}\big\||z|^4\psi_x\big\|_{\hbar}^2 = {} & k_1\sum_A \frac{1}{\omega_A^4} + k_2\left(\sum_A \frac{1}{\omega_A^3}\right)\left(\sum_A \frac{1}{\omega_A}\right) + k_3\left(\sum_A \frac{1}{\omega_A^2}\right)^2 \\ & + \ k_4\left(\sum_A \frac{1}{\omega_A^2}\right)\left(\sum_A \frac{1}{\omega_A}\right)^2 + k_5\left(\sum_A \frac{1}{\omega_A}\right)^4\end{aligned}$$

for some combinatorial factors $k_1, \dots, k_5$. For example, the evaluation of $\||z|^2\Psi_x\|_{\hbar}^2$ goes as follows

$$\begin{aligned}\big\||z|^2\psi_x\big\|_{\hbar}^2 &= \langle |z|^4\rangle_{\psi_x} \\ &= \pi^{-n}s^{\frac{1}{2}}\int \left(|u|^2 + |v|^2\right)^2 e^{-u\cdot S^{1/2}u}e^{-v\cdot S^{1/2}v}d^n u d^n v \\ &= 2\pi^{-\frac{n}{2}}s^{\frac{1}{4}}\int |u|^4 e^{-u\cdot S^{1/2}u}d^n u \ + \ 2\pi^{-n}s^{\frac{1}{2}}\left(\int |u|^2 e^{-u\cdot S^{1/2}u}d^n u\right)^2 \\ &= 2\pi^{-\frac{n}{2}}s^{\frac{1}{4}}\sum_{A,B}\int \tilde{u}_A^2\tilde{u}_B^2 e^{-\sum_C \omega_C\tilde{u}_C^2}d^n\tilde{u} \\ &\quad + \ 2\pi^{-n}s^{\frac{1}{2}}\left(\sum_A\int \tilde{u}_A^2 e^{-\sum_C \omega_C\tilde{u}_C^2}d^n\tilde{u}\right)^2 \\ &= 2\sum_A \frac{1\cdot 3}{2^2\omega_A^2} + 4\sum_{A<B}\frac{1}{2\omega_A}\frac{1}{2\omega_B} + 2\left(\sum_A \frac{1}{2\omega_A}\right)^2 \\ &= \frac{3}{2}\sum_A \frac{1}{\omega_A^2} + \frac{1}{2}\left(\sum_A \frac{1}{\omega_A}\right)^2 = \frac{3}{2}\operatorname{tr} S^{-1} + \frac{1}{2}\left(\operatorname{tr} S^{-\frac{1}{2}}\right)^2,\end{aligned}$$

where we diagonalized $S = R^T[\omega_A]^2 R$ (at the point x) and put $\tilde{u} := Ru$. Hence,

$$\||z|\psi_x\|_{\hbar} \le \sqrt{\frac{n}{\omega_{\min}}}, \quad \big\||z|^2\psi_x\big\|_{\hbar} \le \frac{\sqrt{n(n+3)/2}}{\omega_{\min}}, \quad \text{and} \quad \big\||z|^4\psi_x\big\|_{\hbar} \le \frac{C_4}{\omega_{\min}^2}$$

for some positive constant C_4. □

4.3. Proof of Proposition 3

We divide the proof into two steps. First, we show that the conditions (15) can be satisfied on some subset $D_t \subseteq \mathbb{R}^{7n}$. Then we construct a function χ_t with support on D_t which also satisfies the conditions (16).

4.3.1. Construction of the set D_t. We start by finding an explicit point $\hat{x}$ where $\omega_{\min}(\hat{x}) > 0$. A basis for the Lie algebra $i \cdot \mathfrak{su}(N)$ of traceless hermitian $N \times N$-matrices is given by $N-1$ diagonal ones, h_k, together with the off-diagonal

$$e_{ij} := E_{ij} + E_{ji}\,, \quad f_{ij} := i(E_{ij} - E_{ji})\,,$$

($1 \leq i < j \leq N$), where E_{ij} denotes the standard basis of matrices. For any diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ we have

$$[\Lambda, e_{ij}] = -i(\lambda_i - \lambda_j) f_{ij} \quad \text{and} \quad [\Lambda, f_{ij}] = i(\lambda_i - \lambda_j) e_{ij}\,. \tag{18}$$

Let e.g. $\hat{X}_1 := \mathrm{diag}(m, m-1, \dots, -m+1, -m)$ (or any other traceless diagonal matrix with all entries different), and $\hat{X}_2 := \sum_{i<j} e_{ij}$. Now, take any fixed

$$E = \sum_k \alpha_k h_k + \sum_{i<j} \beta_{ij} e_{ij} + \sum_{i<j} \gamma_{ij} f_{ij} \in i\mathfrak{su}(N)$$

and require E to commute with both $\hat{X}_1$ and $\hat{X}_2$. Then, by (18), $[\hat{X}_1, E] = 0$ implies $\beta_{ij} = \gamma_{ij} = 0$, i.e. E must be diagonal, $E = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. Again, by (18), $[E, \hat{X}_2] = 0$ implies $\sum_{i<j} (\lambda_i - \lambda_j) f_{ij} = 0$, i.e. all λ_i are equal. Tracelessness then implies that $E = 0$. Hence,

$$\|[\hat{X}_1, E]\|^2 + \|[\hat{X}_2, E]\|^2 > 0$$

for all $E \neq 0$, and since S^{n-1} is compact it also follows that

$$\omega^2_{\min}(\hat{x}) \geq \min_{\|E\|=1} \left(\|[\hat{X}_1, E]\|^2 + \|[\hat{X}_2, E]\|^2 \right) =: c > 0\,,$$

where $\hat{x} = (\hat{x}_1, \hat{x}_2, x_3, \dots, x_7) \in \mathbb{R}^{7n}$, and $\hat{x}_1 \leftrightarrow \hat{X}_1$, $\hat{x}_2 \leftrightarrow \hat{X}_2$ as usual.

Now, consider the map $F \colon \mathbb{R}^n \times \mathbb{R}^n \times S^{n-1} \to \mathbb{R}_+$,

$$F(x_1, x_2, e) := \|[X_1, E]\|^2 + \|[X_2, E]\|^2\,.$$

We know from the above that $F(\hat{x}_1, \hat{x}_2, \cdot\,) \geq c$. Furthermore, note that for any $R \in SU(N) \hookrightarrow SO(n)$, since $F(Rx_1, Rx_2, e) = F(x_1, x_2, R^T e)$, we have $F(R\hat{x}_1, R\hat{x}_2, \cdot\,) \geq c > 0$ as well. Then, because F is continuous and S^{n-1} compact, there exists an $\epsilon_R > 0$ such that $F(x_1, x_2, \cdot\,) \geq c/2$ for all x_1, x_2 in the balls $B_{\epsilon_R}(R\hat{x}_1)$ and $B_{\epsilon_R}(R\hat{x}_2)$, respectively. Also, by compactness of $SU(N)$, there is an $\epsilon > 0$ such that

$$F(x_1, x_2, \cdot\,) \geq \frac{c}{2} \quad \forall (x_1, x_2) \in B_\epsilon(R\hat{x}_1) \times B_\epsilon(R\hat{x}_2) \quad \forall R \in SU(N)\,.$$

Therefore, defining

$$D_1 := \bigcup_{R \in SU(N)} B_\epsilon(R\hat{x}_1) \times B_\epsilon(R\hat{x}_2) \times B_1(0)^5 \ \subseteq\ \mathbb{R}^{7n}\,,$$

we find that $\omega_{\min}(x) \geq c/2$ for all $x \in D_1$. Furthermore, we have

$$(|\hat{x}_1| - \epsilon)^2 + (|\hat{x}_2| - \epsilon)^2 \leq |x|^2 \leq (|\hat{x}_1| + \epsilon)^2 + (|\hat{x}_2| + \epsilon)^2 + 5$$

on D_1. By rescaling this set (note that F is homogeneous of degree 2), $D_t := tD_1$, we reach the conditions (15). It is also useful to note that $\omega_{\min} \leq \omega_{\max} = \|S^{\frac{1}{2}}\|_{\mathrm{op}} \leq c_{13}|x|$ (where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm).

4.3.2. Construction of the function χ_t. We set

$$\chi_t(x_1, \ldots, x_7) := \mu_t(x_1, x_2)\eta_t(x_3) \ldots \eta_t(x_6)\zeta_t(x_7)\,,$$

where μ_t, η_t and ζ_t are to be defined below.

Given some spherically symmetric bump function $\eta \in C_0^\infty(\mathbb{R}^n)$ with support on the unit ball $B_1(0)$ and unit L^2-norm, $\|\eta\|_{L^2(\mathbb{R}^n)} = 1$, we define $\eta_t(x) := t^{-n/2}\eta(x/t)$ so that $\operatorname{supp}\eta_t \subseteq B_t(0)$, $\|\eta_t\|_{L^2(\mathbb{R}^n)} = 1$, and $\|\partial^\alpha\eta_t\|_{L^2(\mathbb{R}^n)} = t^{-|\alpha|}\|\partial^\alpha\eta\|_{L^2(\mathbb{R}^n)}$ for any partial derivative multi-index α.

The function ζ_t is chosen to be asymptotically a gauge invariant solution to the Helmholtz equation, namely we take (for the case $\lambda = 0$ we instead take $\zeta_t := \eta_t$)

$$\zeta_t(x) := A_t\rho_t(x)h(x)\,,$$

where $A_t := \|\rho_t h\|^{-1}_{L^2(\mathbb{R}^n)}$, ρ_t is a cut-off function $\rho_t(x) := \rho(x/t)$ such that $\rho \in C_0^\infty(\mathbb{R}^n)$ is spherically symmetric, $0 \leq \rho \leq 1$, $\rho = 1$ on $B_{1/2}(0)$ and $\rho = 0$ outside $B_1(0)$, and

$$h(x) := \frac{1}{\lambda^{\frac{1}{2}}|x|}\sin_{n-3}(\lambda^{\frac{1}{2}}|x|)$$

satisfies $(\Delta + \lambda)h = 0$ (see [42]), with

$$\sin_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2(3+p)4(5+p)\cdots 2k(2k+1+p)} = c_p x^{(1-p)/2} J_{(1+p)/2}(x)\,.$$

Since the Bessel functions J_k behave asymptotically as [1]

$$J_k(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\left(x - \frac{\pi}{4}(2k+1)\right) + O\left(\frac{1}{x}\right)\right),$$

one finds

$$\int_{B_R(0)} h^2 dx = c_{14}\int_0^R J^2_{(n-2)/2}(\lambda^{\frac{1}{2}}r)r\,dr = c_{15}R + o(R)$$

and

$$\int_{B_R(0)} |\partial_A h|^2 dx \leq c_{16}\int_0^R J^2_{n/2}(\lambda^{\frac{1}{2}}r)r\,dr \leq c_{17} + c_{18}R\,.$$

Hence, $A_t \leq c_{19}/t^{1/2} \to 0$, $t \to \infty$, and

$$\begin{aligned}\|\partial_A\zeta_t\|_{L^2(\mathbb{R}^n)} &\leq A_t\big(\|(\partial_A\rho_t)h\|_{L^2(\mathbb{R}^n)} + \|\rho_t\partial_A h\|_{L^2(\mathbb{R}^n)}\big)\\ &\leq \frac{c_{19}}{t^{1/2}}\left(\frac{c_{20}}{t}(c_{15}t)^{1/2} + (c_{17} + c_{18}t)^{1/2}\right) \to c_{21}\,.\end{aligned}$$

Furthermore, $(\sum_A \partial_A^2 + \lambda)\zeta_t = A_t(\Delta\rho_t)h + 2A_t \sum_A(\partial_A\rho_t)(\partial_A h)$, so that

$$\|(-\Delta - \lambda)\zeta_t\|_{L^2(\mathbb{R}^n)} \le \frac{c_{19}}{t^{1/2}} \left(\frac{c_{22}}{t^2}(c_{15}t)^{1/2} + 2n\frac{c_{20}}{t}(c_{17} + c_{18}t)^{1/2} \right) \to 0\,.$$

Lastly, we set $\mu_t(x_1, x_2) := t^{-n}\|\mu\|^{-1}_{L^2(\mathbb{R}^{2n})}\mu(x_1/t, x_2/t)$, where

$$\mu(x_1, x_2) := \int_{R\in SU(N)} \eta_\epsilon(x_1 - R\hat{x}_1)\eta_\epsilon(x_2 - R\hat{x}_2)\; d\mu_H(R)$$

and μ_H denotes some Haar measure on $SU(N)$. Then $\mu_t \in C_0^\infty(\mathbb{R}^{2n})$, $\operatorname{supp}\mu_t \times B_t(0)^5 \subseteq D_t$, $\|\mu_t\|_{L^2(\mathbb{R}^{2n})} = 1$, $\|\partial^\alpha\mu_t\|_{L^2(\mathbb{R}^{2n})} \le c_\alpha/t^{|\alpha|}$, and $\mu_t(Rx_1, Rx_2) = \mu_t(x_1, x_2)$ for all $R \in SU(N)$.

Hence, $\chi_t \in C_0^\infty(\mathbb{R}^{7n})$ is $SU(N)$-invariant, $\operatorname{supp}\chi_t \subseteq D_t$, and

$$\|\chi_t\|^2_{L^2(\mathbb{R}^{7n})} = \|\mu_t\|^2_{L^2(\mathbb{R}^{2n})}\|\eta_t\|^8_{L^2(\mathbb{R}^n)}\|\zeta_t\|^2_{L^2(\mathbb{R}^n)} = 1\,.$$

Furthermore, as $t \to \infty$,

$$\|\partial_{jA}\chi_t\|_{L^2(\mathbb{R}^{7n})} = \begin{cases} \|\partial_{jA}\mu_t\|_{L^2(\mathbb{R}^{2n})} \le c_{23}/t \to 0\,, & j = 1, 2 \\ \|\partial_A\eta_t\|_{L^2(\mathbb{R}^n)} \le c_{24}/t \to 0\,, & j = 3, 4, 5, 6 \\ \|\partial_A\zeta_t\|_{L^2(\mathbb{R}^n)} \le c_{21}\,, & j = 7 \end{cases}$$

and

$$\begin{aligned}\|(-\Delta_x - \lambda)\chi_t\|_{L^2(\mathbb{R}^{7n})} \le \|\Delta_{(x_1,x_2)}\mu_t\|_{L^2(\mathbb{R}^{2n})} &+ \sum_{j=3}^{6}\|\Delta_{x_j}\eta_t\|_{L^2(\mathbb{R}^n)} \\ &+ \ \|(\Delta_{x_7} - \lambda)\zeta_t\|_{L^2(\mathbb{R}^n)} \ \to 0\,. \end{aligned}$$

□

4.4. Proof of Lemma 4

Here, we will denote the partial derivatives $\partial_{jA}\psi_x$, $\partial^2_{jA}\psi_x$ by ψ'_x and ψ''_x, respectively. Since ψ_x is a zero-energy state of H_D, we have in particular that $\partial_{jA}(H_D\psi_x) = 0$, which can be equivalently written as

$$-H_D\psi'_x = \big(\bar{z}\cdot S'(x)z + 2W'(x)\lambda\lambda^\dagger\big)\psi_x =: \Phi_x\,.$$

In the following we will need an estimate on the norm of $\Phi_x \in \mathcal{S}_n$. We have, using Propositions 2 and 3,

$$\begin{aligned}\|\Phi_x\|_{\mathfrak{h}} &\le \|\bar{z}\cdot S'(x)z\psi_x\|_{\mathfrak{h}} + 2\left\|W'(x)\lambda\lambda^\dagger\psi_x\right\|_{\mathfrak{h}} \\ &\le \|S'(x)\|_{\mathrm{op}(\mathbb{R}^n)}\||z|^2\psi_x\|_{\mathfrak{h}} + 2\left\|W'(x)\lambda\lambda^\dagger\right\|_{\mathrm{op}(\mathcal{F})}\|\psi_x\|_{\mathfrak{h}} \\ &\le c_{25}|x|\frac{C_2}{\omega_{\min}(x)} + c_{26} \le c_{27}\frac{t}{t} + c_{26} =: c_5\,.\end{aligned}$$

Now, we note that $\Phi_x \in (\ker\bar{H}_D)^\perp =: \mathcal{P}_+$ where $\bar{H}_D$ denotes the self-adjoint extension of H_D. To see this, consider any $\Phi_0 \in \ker\bar{H}_D$. We have

$$\langle\Phi_x, \Phi_0\rangle_{\mathfrak{h}} = -\langle\bar{H}_D\psi'_x, \Phi_0\rangle_{\mathfrak{h}} = -\langle\psi'_x, \underbrace{\bar{H}_D\Phi_0}_{=0}\rangle_{\mathfrak{h}} = 0\,.$$

Therefore, since the lowest eigenvalue of $\bar{H}_D$ on $\mathcal{P}_+$ is $2\omega_{\min} > 0$, we have $\|\bar{H}_D|_{\mathcal{P}_+}^{-1}\|_{\mathrm{op}(\hslash)} \leq \frac{1}{2\omega_{\min}}$ and

$$\|\psi_x'\|_{\hslash} = \|\bar{H}_D^{-1}\Phi_x\|_{\hslash} \leq \|\bar{H}_D|_{\mathcal{P}_+}^{-1}\|_{\mathrm{op}(\hslash)}\|\Phi_x\|_{\hslash} \leq \frac{c_5}{\omega_{\min}} .$$

Similarly, taking the second derivative we have $\partial_{jA}^2(H_D\psi_x) = 0$, i.e.

$$-H_D\psi_x'' = \bar{z}\cdot S''(x)z\psi_x + 2\big(\bar{z}\cdot S'(x)z + 2W'(x)\lambda\lambda^\dagger\big)\psi_x' =: \tilde{\Phi}_x .$$

Just as above, we see that $\tilde{\Phi}_x \in \mathcal{P}_+ \cap \mathcal{S}_n$, but we will also need an estimate on $\||z|^2\psi_x'\|_{\hslash}$. For this, we recall from the proof of Proposition 2 that

$$\langle |z|^2\psi_x, |z|^2\psi_x\rangle_{\hslash} = \frac{3}{2}\operatorname{tr} S^{-1} + \frac{1}{2}\left(\operatorname{tr} S^{-\frac{1}{2}}\right)^2 =: T(x) ,$$

Note that T is smooth for $s > 0$ and homogeneous of degree -2, because S is homogeneous of degree 2. It follows that, if $x = re$ with $e \in S^{7n-1}$, then $T''(x) = r^{-4}T''(e)$ and

$$2\,\mathrm{Re}\,\langle |z|^2\psi_x'', |z|^2\psi_x\rangle_{\hslash} \;+\; 2\left\||z|^2\psi_x'\right\|_{\hslash}^2 \;=\; T''(x) \leq \frac{1}{r^4}\sup_{K_0}|T''| ,$$

where $K_0 := S^{7n-1} \cap \{x \in \mathbb{R}^{7n} : s(x) \geq (c_1/c_3)^{2n}\}$ is compact and we have used that $\omega_{\min}/|x| \geq c_1/c_3$ by Proposition 3. Hence,

$$\left\||z|^2\psi_x'\right\|_{\hslash}^2 \leq \frac{c_{28}}{|x|^4} + \|\psi_x''\|_{\hslash}\left\||z|^4\psi_x\right\|_{\hslash} \leq \frac{c_{29}}{\omega_{\min}^4}\left(1 + \omega_{\min}^2\|\psi_x''\|_{\hslash}\right) ,$$

so that

$$\begin{aligned}
\|\psi_x''\|_{\hslash} &= \|H_D^{-1}\tilde{\Phi}_x\|_{\hslash} \\
&\leq \frac{1}{\omega_{\min}}\left(c_{30}\||z|^2\psi_x\|_{\hslash} + c_{31}\|S'\|_{\mathrm{op}}\||z|^2\psi_x'\|_{\hslash} + c_{32}\|\psi_x'\|_{\hslash}\right) \\
&\leq \frac{1}{\omega_{\min}}\left(\frac{c_{30}C_2}{\omega_{\min}} + \frac{c_{33}|x|}{\omega_{\min}^2}\left(1 + \omega_{\min}^2\|\psi_x''\|_{\hslash}\right)^{\frac{1}{2}} + \frac{c_{32}c_5}{\omega_{\min}}\right)
\end{aligned}$$

and thus, $\omega_{\min}^2\|\psi_x''\|_{\hslash} \leq c_6$ for some constant c_6. □

5. Summary

We have introduced $G_2 \times U(1) \times SU(N)$ invariant matrix models as deformations of the standard $Spin(9) \times SU(N)$ invariant models by conjugating a supercharge with a cubic, octonionic, exponential. Furthermore, similarly to what has been shown for the original models, we have proved that the spectrum of the corresponding Hamiltonian $\tilde{H}$ covers the whole positive half-axis by finding sequences of states contradicting existence of a bounded inverse to the operator $\tilde{H} - \lambda$ for any $\lambda \geq 0$. However, contrary to the case for the original models, we have constructed such sequences explicitly, *without* fixing the gauge. Making use of the convenient structure of terms appearing in $\tilde{H}$, we could configure the states to annihilate some terms, while, related to having the possibility of making the lowest eigenvalue of a certain frequency matrix S arbitrarily large, other terms could be made arbitrarily small – using a gauge invariant asymptotic solution to the Helmholtz equation, with support on a set of matrices that are not simultaneously block-diagonalizable.

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Appendix A

In this appendix we give notation and conventions used in the paper (cp. e.g. [28]).

The supermembrane matrix theory is a quantum mechanical model with $\mathcal{N} = 16$ supersymmetries, $SU(N)$ gauge invariance and $Spin(9)$ symmetry. The theory involves real bosonic variables x_{sA} (coordinates) and real fermionic ones $\theta_{\alpha A}$ (Majorana spinors) with $s = 1, \ldots, 9$, $\alpha = 1, \ldots 16$ and $A = 1, \ldots, N^2 - 1$ - spatial, spinor and color indices respectively. The corresponding supercharges and the Hamiltonian of the model are

$$\begin{aligned} \mathcal{Q}_\alpha &= \left(p_{sA}\gamma^s_{\alpha\beta} + \frac{1}{2} f_{ABC} x_{sB} x_{tC} \gamma^{st}_{\alpha\beta} \right) \theta_{\beta A} \,, \quad \gamma^{st} = \frac{1}{2}[\gamma^s, \gamma^t] \,, \\ H &= p_{sA} p_{sA} + \frac{1}{2}(f_{ABC} x_{sB} x_{tC})^2 + i f_{ABC} \gamma^s_{\alpha\beta} \theta_{\alpha A} \theta_{\beta B} x_{sC} \,, \\ \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} &= \delta_{\alpha\beta} H + 2\gamma^s_{\alpha\beta} x_{sA} J_A \,, \quad J_A = f_{ABC} \left(x_{sB} p_{sC} - \frac{i}{2} \theta_{\alpha B} \theta_{\alpha C} \right) . \end{aligned} \tag{19}$$

Here p_{sA} are momenta conjugate to x_{sA}, $[x_{sA}, p_{tB}] = i\delta_{st}\delta_{AB}$, γ^s are 16×16 dimensional, real matrices s.t. $\{\gamma^s, \gamma^t\} = 2\delta^{st}\mathbf{1}_{16\times 16}$, $\theta_{\alpha A}$ are Grassmann numbers s.t. $\{\theta_{\alpha A}, \theta_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}$, and f_{ABC} are $SU(N)$ structure constants (real, antisymmetric). The operators are defined on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{9(N^2-1)}) \otimes \mathcal{F}$, where $\mathcal{F}$ is the irreducible representation of θ's, while the physical (gauge invariant) Hilbert space consists of states $|\psi\rangle$ satisfying $J_A|\psi\rangle = 0$ which corresponds to the Gauss law in unreduced $\mathcal{N} = 1$ super Yang–Mills theory.

Such singlet constraint is an essential requirement for the model to be supersymmetric which is apparent in Eq. (19). However, the necessity of the constraint follows also from simply counting the fermionic and bosonic degrees of freedom. Let us consider the Fock space formulation of the model. For the case at hand there are $9(N^2 - 1)$ bosonic degrees of freedom, however there are $\frac{16}{2}(N^2 - 1)$ fermionic ones. The mismatch is equal to $N^2 - 1$, which is exactly the number of constraints coming from the Gauss law.

There are many ways in which one can single out 8 out of 16 fermions (which is required in order to obtain an irreducible Fock representation $\mathcal{F}$). We will follow the convention in [45] and introduce complex spinor variables $\lambda_{\alpha A} := \frac{1}{\sqrt{2}}(\theta_{\alpha A} + i\theta_{8+\alpha\, A})$ i.e.[3]

$$\theta_{\alpha A} = \frac{1}{\sqrt{2}}(\lambda_{\alpha A} + \lambda^{\dagger}_{\alpha A})\,, \quad \theta_{\alpha+8\, A} = \frac{1}{i\sqrt{2}}(\lambda_{\alpha A} - \lambda^{\dagger}_{\alpha A})\,.$$

We then also split the coordinates x_{sA} into $(x_{jA}, z_A, \bar{z}_A)$ where $z_A = x_{8A} + ix_{9A}$ and $j = 1, \ldots, 7$.

After this is done the $Spin(9)$ symmetry of (19) is not explicit, however now an arbitrary wavefunction $\Psi(x, z, \bar{z})$ can be written as

$$\Psi(x, z, \bar{z}) = \psi + \psi_{\alpha A}\lambda_{\alpha A} + \frac{1}{2!}\psi_{\alpha A\,\beta B}\lambda_{\alpha A}\lambda_{\beta B} + \ldots\,,$$

with $\psi_{\alpha_1 A_1 \ldots \alpha_l A_l}$ complex-valued and square integrable. The above sum is finite and truncates when the number of fermions is more than $8(N^2-1)$.[4]

It now follows that the Hamiltonian (19) can be written in terms of non-hermitian ("cohomology") charges $Q_\alpha := \frac{1}{\sqrt{2}}(\mathcal{Q}_\alpha + i\mathcal{Q}_{8+\alpha})$,

$$\begin{aligned} Q_\beta = &\left(i\delta_{\alpha\beta}\frac{i}{2} f_{ABC} z_B \bar{z}_C + i\Gamma^j_{\alpha\beta}\frac{\partial}{\partial x_{jA}} - \frac{1}{2} f_{ABC} x_{jB} x_{kC} \Gamma^{jk}_{\alpha\beta} \right) \lambda_{\alpha A} \\ &+ \left(2\delta_{\alpha\beta}\frac{\partial}{\partial z_A} - i f_{ABC} x_{jB} \bar{z}_C \Gamma^j_{\alpha\beta} \right) \lambda^{\dagger}_{\alpha A}\,, \end{aligned} \tag{20}$$

so that, on the physical Hilbert space,

$$\{Q_\alpha, Q^{\dagger}_\beta\} = \delta_{\alpha\beta} H\,, \quad \{Q_\alpha, Q_\beta\} = 0\,, \quad \{Q^{\dagger}_\alpha, Q^{\dagger}_\beta\} = 0\,.$$

Here, Γ^j are 8×8, purely imaginary, antisymmetric matrices satisfying $\{\Gamma^j, \Gamma^k\} = 2\delta^{jk}\mathbf{1}_{8\times 8}$. We have chosen the following representation of γ^s matrices

$$\gamma^j = \begin{bmatrix} 0 & i\Gamma^j \\ -i\Gamma^j & 0 \end{bmatrix}, \quad \gamma^8 = \begin{bmatrix} 0 & \mathbf{1}_{8\times 8} \\ \mathbf{1}_{8\times 8} & 0 \end{bmatrix}, \quad \gamma^9 = \begin{bmatrix} \mathbf{1}_{8\times 8} & 0 \\ 0 & -\mathbf{1}_{8\times 8} \end{bmatrix},$$

implying

$$\gamma^{jk} = \begin{bmatrix} \Gamma^{jk} & 0 \\ 0 & \Gamma^{jk} \end{bmatrix}, \qquad \gamma^{j8} = \begin{bmatrix} i\Gamma^j & 0 \\ 0 & -i\Gamma^j \end{bmatrix},$$

$$\gamma^{j9} = \begin{bmatrix} 0 & -i\Gamma^j \\ -i\Gamma^j & 0 \end{bmatrix}, \qquad \gamma^{89} = \begin{bmatrix} 0 & -\mathbf{1}_{8\times 8} \\ \mathbf{1}_{8\times 8} & 0 \end{bmatrix},$$

[3]Other choices of 8 fermions are possible, e.g. Majorana–Weyl spinors (see [47]). From now on the spinor indices $\alpha, \beta, \ldots$ run from 1 to 8.

[4]Note that in this notation $\lambda_{\alpha A}$ is a fermionic creation operator while $\lambda^{\dagger}_{\alpha A}$ fermionic annihilation operator.

and

$$\gamma^{jkl} = \begin{bmatrix} 0 & i\Gamma^{jkl} \\ -i\Gamma^{jkl} & 0 \end{bmatrix}, \qquad \gamma^{jk8} = \begin{bmatrix} 0 & \Gamma^{jk} \\ \Gamma^{jk} & 0 \end{bmatrix},$$
$$\gamma^{jk9} = \begin{bmatrix} \Gamma^{jk} & 0 \\ 0 & -\Gamma^{jk} \end{bmatrix}, \qquad \gamma^{j89} = \begin{bmatrix} i\Gamma^{j} & 0 \\ 0 & i\Gamma^{j} \end{bmatrix},$$

where $\gamma^{st} := \frac{1}{2}[\gamma^s, \gamma^t]$, $\gamma^{stu} := \frac{1}{6}(\gamma^s[\gamma^t, \gamma^u] + \text{cycl.})$ and Γ^{jk}, Γ^{jkl} respectively.

It is here where the octonions enter, in choosing the representation $i\Gamma^j_{\alpha 8} = \delta^j_\alpha$, $i\Gamma^j_{kl} = -c_{jkl}$ with totally antisymmetric octonionic structure constants.[5] This is also natural from the view of representation theory of Clifford algebras since the representations of Γ^j are uniquely given by left or right multiplication on the octonion algebra (see e.g. [32]). Furthermore, because the automorphism group of the octonions is given by the exceptional group G_2 (which is also the subgroup of *Spin*(7) fixing a chosen spinor index), the deformed Hamiltonians H_k, $\hat{H}$, and $\tilde{H}$ will be G_2 invariant.

Appendix B

Starting from the 9-dimensional Fierz identity (see e.g. [2])

$$\gamma^s_{\alpha\beta}\gamma^{st}_{\alpha'\beta'} + \gamma^s_{\alpha'\beta}\gamma^{st}_{\alpha\beta'} + \gamma^s_{\alpha\beta'}\gamma^{st}_{\alpha'\beta} + \gamma^s_{\alpha'\beta'}\gamma^{st}_{\alpha\beta} = 2(\delta_{\alpha\alpha'}\gamma^t_{\beta\beta'} - \delta_{\beta\beta'}\gamma^t_{\alpha\alpha'})\,,$$

which holds for all $t = 1, \ldots, 9$, $\alpha, \alpha', \beta, \beta' = 1, \ldots, 16$, and using the representation in Appendix A with $\alpha, \alpha', \beta' = 1, \ldots, 8$, $\beta = 9, \ldots, 16$ (then redefining $\beta := \beta - 8$), we obtain the corresponding 7-dimensional Fierz identity

$$\Gamma^j_{\alpha\beta}\Gamma^{jk}_{\alpha'\beta'} + \Gamma^j_{\alpha'\beta}\Gamma^{jk}_{\alpha\beta'} = \delta_{\alpha\beta}\Gamma^k_{\alpha'\beta'} + \delta_{\alpha'\beta}\Gamma^k_{\alpha\beta'} - \delta_{\alpha\beta'}\Gamma^k_{\alpha'\beta} - \delta_{\alpha'\beta'}\Gamma^k_{\alpha\beta} - 2\delta_{\alpha\alpha'}\Gamma^k_{\beta\beta'}$$

for all $k = 1, \ldots, 7$, $\alpha, \alpha', \beta, \beta' = 1, \ldots, 8$. From this identity it follows that

$$\Gamma^j_{\alpha\beta}\Gamma^{jk}_{\alpha'\beta'} - \Gamma^j_{\alpha'\beta'}\Gamma^{jk}_{\alpha\beta} = -2(\delta_{\alpha\alpha'}\Gamma^k_{\beta\beta'} + \delta_{\beta\beta'}\Gamma^k_{\alpha\alpha'} - \delta_{\alpha'\beta}\Gamma^k_{\alpha\beta'} + \delta_{\alpha\beta'}\Gamma^k_{\alpha'\beta})\,.$$

Multiplying this equation with $\Gamma^l_{\beta'\dot{\beta}}$, summing over β', and taking $\alpha' = \beta = \dot{\beta}$ to be fixed, we obtain

$$\Gamma^j_{\alpha\dot{\beta}}\Gamma^{jkl}_{\dot{\beta}\dot{\beta}} = \Gamma^{kl}_{\alpha\dot{\beta}}\,.$$

Appendix C

In this appendix we consider deformed Hamiltonians from a more general viewpoint and show how one could be led to the particular deformation considered in this paper.

[5]Explicitly, $c_{ijk} = +1$ for $(ijk) = (123), (165), (246), (435), (147), (367), (257)$.

Let us consider the algebra of $\mathcal{N} > 1$ supersymmetric quantum mechanics, $\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = \delta_{\alpha\beta} H$, and the corresponding cohomology supercharges [6]

$$Q_{\alpha\beta} := \frac{1}{\sqrt{2}}(\mathcal{Q}_\alpha + i\mathcal{Q}_\beta)\,, \quad Q^\dagger_{\alpha\beta} = \frac{1}{\sqrt{2}}(\mathcal{Q}_\alpha - i\mathcal{Q}_\beta)\,.$$

We have

$$\{Q_{\alpha\beta}, Q_{\mu\nu}\} = 0\,, \quad \{Q_{\alpha\beta}, Q^\dagger_{\mu\nu}\} = \delta_{(\alpha\beta)(\mu\nu)} H\,.$$

The deformed Hamiltonian $H_{\alpha\beta}(k) := \{Q_{\alpha\beta}(k), Q^\dagger_{\alpha\beta}(k)\}$ (no sum over α,β) given by deformed cohomology supercharges $Q_{\alpha\beta}(k) := e^{kg(x)} Q_{\alpha\beta} e^{-kg(x)}$, where $k \in \mathbb{R}$, and $g(x)$ is some operator s.t. $[\mathcal{Q}_\beta, g(x)]$ commutes with $g(x)$, becomes

$$H_{\alpha\beta}(k) = H - 2ik\Big\{\mathcal{Q}_\alpha, \big[\mathcal{Q}_\beta, g(x)\big]\Big\} - 2k^2\big[\mathcal{Q}_\beta, g(x)\big]^2\,.$$

Substituting the supercharges (19) for the particular model considered here, we obtain

$$\begin{aligned} H_{\alpha\beta}(k) = H &+ k^2\big(\partial_{sA} g(x)\big)^2 + k\gamma^{st}_{\alpha\beta}\big(\partial_{sA} g(x) p_{tA} - \partial_{tA} g(x) p_{sA}\big) \\ &- \ k(\gamma^{st}\gamma^u)_{\alpha\beta} f_{ABC} x_{sA} x_{tB} \partial_{uC} g(x) \\ &+ \ 2ik\partial_{sA}\partial_{tB} g(x) \gamma^s_{\alpha\alpha'}\gamma^t_{\beta\beta'}\theta_{\alpha' A}\theta_{\beta' B}\,. \end{aligned}$$

Now, say we are interested in a particular deformation where $g(x)$ is cubic in x (so that $(\partial g)^2$ is quartic). Because γ^{stu} is totally antisymmetric, a natural choice is

$$g(x) = \frac{1}{6} f_{ABC} x_{sA} x_{tB} x_{uC} \gamma^{stu}_{\alpha\beta}\,,$$

with $\alpha < \beta$. Taking e.g. $(\alpha, \beta) = (8, 16)$ and choosing the representation of γ^s matrices as in Appendix A, we find that

$$g(x) = \frac{1}{6} f_{ABC} x_{jA} x_{kB} x_{lC} i\Gamma^{jkl}_{8,8} = \frac{1}{6} c_{jkl} f_{ABC} x_{jA} x_{kB} x_{lC}\,,$$

and that $H_{8,16}(k)$ becomes precisely H_k in (4).

References

[1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, New York, 1968.

[2] M. Baake, P. Reinicke, V. Rittenberg, *Fierz identities for real clifford algebras and the number of supercharges*, J. Math. Phys. **26** (1985), 1070.

[3] V. Bach, J. Hoppe, D. Lundholm, *Dynamical symmetries in supersymmetric matrix models*, Documenta Math. **13** (2008), 103–116, `arXiv:0706.0355`

[4] T. Banks, W. Fischler, S. Shenker and L. Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev. **D55** (1997), 6189, `arXiv:hep-th/9610043`.

[6] We consider the $(\alpha\beta)$ as distinct pairs of indices with α and β in disjoint subsets of the index set. This is a common construction of non-hermitian charges involving the complex structure i, but other variations are possible; see e.g. [33] and references therein.

[5] E. Bergshoeff, E. Sezgin, P. K. Townsend, *Properties of the eleven-dimensional supermembrane theory*, Ann. Phys. **185** (1988), 330–368.

[6] J. D. Bjorken, *Elements of quantum chromodynamics*, SLAC-PUB-2372 (1979).

[7] M. Bordemann, J. Hoppe, R. Suter, *Zero Energy States for $SU(N)$: A Simple Exercise in Group Theory* ?, `arXiv:hep-th/9909191`.

[8] L. Boulton, M. P. Garcia del Moral, A. Restuccia, *Discreteness of the spectrum of the compactified $D=11$ supermembrane with nontrivial winding,* Nucl. Phys. **B671** (2003), 343–358, `arXiv:hep-th/0211047`.

[9] M. Claudson and M. B. Halpern, *Supersymmetric ground state wave functions*, Nucl. Phys. **B250** (1985), 689.

[10] L. Erdös, D. Hasler, J. P. Solovej, *Existence of the $D0-D4$ bound state: A detailed proof*, Annales Henri Poincare **6** (2005), 247–267, `arXiv:math-ph/0407020`.

[11] R. Flume, *On quantum mechanics with extended supersymmetry and nonabelian Gauge constraints*, Ann. Phys. **164** (1985) 189.

[12] J. Fröhlich, G. M. Graf, D. Hasler, J. Hoppe, S.-T. Yau, *Asymptotic form of zero energy wave functions in supersymmetric matrix models,* Nucl. Phys. **B567** (2000), 231–248, `arXiv:hep-th/9904182`.

[13] J. Fröhlich, J. Hoppe, *On Zero-Mass Ground States in Super-Membrane Matrix Models*, `arXiv:hep-th/9701119`.

[14] J. Goldstone, unpublished.

[15] G. M. Graf, D. Hasler, J. Hoppe, *No zero energy states for the supersymmetric x^2y^2 potential,* Lett. Math. Phys. **60** (2002), 191–196, `arXiv:math-ph/0109032`.

[16] M. B. Green, M. Gutperle, *D-particle bound states and the D-instanton measure*, JHEP **9801** (1998), 005, `arXiv:hep-th/9711107`.

[17] M. B. Halpern, C. Schwartz, *Asymptotic search for ground states of $SU(2)$ matrix theory*, Int. J. Mod. Phys. **A13** (1998), 4367–4408, `arXiv:hep-th/9712133`.

[18] D. Hasler, J. Hoppe, *Asymptotic Factorisation of the Ground-State for $SU(N)$-invariant Supersymmetric Matrix-Models*, `arXiv:hep-th/0206043`.

[19] D. Hasler, J. Hoppe, *Zero Energy States of Reduced Super Yang–Mills Theories in $d+1=4,6$ and 10 dimensions are necessarily $Spin(d)$ invariant*, `arXiv:hep-th/0211226`.

[20] J. Hoppe, *Quantum Theory of a Massless Relativistic Surface and a two dimensional bound state problem*, PhD Thesis MIT, (1982), (scanned version available at `http://www.aei-potsdam.mpg.de/$\sim$hoppe`).

[21] J. Hoppe, D. Lundholm, *On the Construction of Zero Energy States in Supersymmetric Matrix Models IV*, `arXiv:0706.0353`.

[22] J. Hoppe, J. Plefka, *The Asymptotic Groundstate of $SU(3)$ Matrix Theory*, `arXiv:hep-th/0002107`.

[23] J. Hoppe, *Asymptotic Zero Energy States for $SU(N$ greater or equal 3)*, `arXiv:hep-th/9912163`.

[24] J. Hoppe, *On the Construction of Zero Energy States in Supersymmetric Matrix Models*, `arXiv:hep-th/9709132`.

[25] J. Hoppe, S.-T. Yau, *Absence of Zero Energy States in Reduced $SU(N)$ 3d Supersymmetric Yang Mills Theory*, `arXiv:hep-th/9711169`.

[26] J. Hoppe, *On the construction of zero energy states in supersymmetric matrix models II,* `arXiv:hep-th/9709217`.

[27] J. Hoppe, *On the construction of zero energy states in supersymmetric matrix models III,* `arXiv:hep-th/9711033`.

[28] J. Hoppe, *Membranes and Matrix Models*, `arXiv:hep-th/0206192`.

[29] V. G. Kac, A. V. Smilga, *Normalized vacuum states in $N = 4$ supersymmetric Yang–Mills quantum mechanics with any gauge group*, Nucl. Phys. **B571** (2000), 515–554, `arXiv:hep-th/9908096`.

[30] A. Konechny, *On asymptotic Hamiltonian for SU(N) matrix theory*, JHEP **9810** (1998), 018, `arXiv:hep-th/9805046`.

[31] W. Krauth, H. Nicolai, M. Staudacher, *Monte Carlo approach to m-theory*, Phys. Lett. **B431** (1998), 31-41, `arXiv:hep-th/9803117`.

[32] H. B. Lawson, M.-L. Michelsohn, *Spin Geometry,* Princeton University Press, Princeton, New Jersey, 1989.

[33] D. Lundholm, *On the geometry of supersymmetric quantum mechanical systems,* J. Math. Phys. **49** (2008), 062101, `arXiv:0710.2881`.

[34] M. Lüscher, *Some analytic results concerning the mass spectrum of Yang–Mills gauge theories on a torus,* Nucl. Phys. **B219** (1983), 233–261.

[35] G. Moore, N. Nekrasov, S. Shatashvili, *D-particle bound states and generalized instantons*, Commun. Math. Phys. **209** (2000), 77–95, `arXiv:hep-th/9803265`.

[36] M. Porrati, A. Rozenberg, *Bound states at threshold in supersymmetric quantum mechanics*, Nucl. Phys. **B515** (1998), 184–202, `arXiv:hep-th/9708119`.

[37] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators*, Academic Press, New York, 1978.

[38] S. Sethi, M. Stern, *D-brane bound states redux*, Commun. Math. Phys. **194** (1998), 675–705, `arXiv:hep-th/9705046`.

[39] S. Sethi, M. Stern, *Invariance theorems for supersymmetric Yang–Mills theories*, Adv. Theor. Math. Phys. **4** (2000), 487–501, `arXiv:hep-th/0001189`.

[40] B. Simon, *Some quantum mechanical operators with discrete spectrum but classically continuous spectrum,* Ann. Phys. **146** (1983), 209–220.

[41] A. Smilga, *Super Yang Mills quantum mechanics and supermembrane spectrum*, *Proc.* 1989 *Trieste Conf.* M. Duff, C. Pope and E. Sezgin (eds.) (Singapore: World Scientific).

[42] M. Trzetrzelewski, *Large N behavior of two dimensional supersymmetric Yang–Mills quantum mechanics*, J. Math. Phys. **48** (2007), 012302, `arXiv:hep-th/0608147`.

[43] M. Trzetrzelewski, *The number of gauge singlets in supersymmetric Yang–Mills quantum mechanics*, Phys. Rev. **D76** (2007), 085012, `arXiv:0708.2946`.

[44] B. de Wit, W. Lüscher, H. Nicolai, *The supermembrane is unstable,* Nucl. Phys. **B320** (1989), 135–159.

[45] D. de Wit, J. Hoppe, H. Nicolai, *On the quantum mechanics of supermembranes*, Nucl. Phys. **B305** [FS23] (1988), 545–581.

[46] E. Witten, *Bound states of strings and p-branes*, Nucl. Phys. **B460** (1996), 335–350, `arXiv:hep-th/9510135`.

[47] J. Wosiek, *On the $SO(9)$ structure of supersymmetric Yang–Mills quantum mechanics* Phys. Lett. **B619** (2005), 171–176, arXiv:hep-th/0503236.

[48] J. Wosiek, *Supersymmetric Yang–Mills quantum mechanics in various dimensions*, Int. J. Mod. Phys. **A20** (2005), 4484–4491, arXiv:hep-th/0410066.

[49] P. Yi, *Witten index and threshold bound states of d-branes*, Nucl. Phys. **B505** (1997), 307–318, arXiv:hep-th/9704098.

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