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High Frequency Dispersive Estimates in Dimension Two

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Abstract. We prove dispersive estimates at high frequency in dimension two for both the wave and the Schrödinger groups for a very large class of real-valued potentials.

1. Introduction and statement of results

The purpose of this note is to prove dispersive estimates at high frequency for the wave group $e^{it\sqrt{G}}$ and the Schrödinger group e^{itG}, where G denotes the self-adjoint realization of the operator $-\Delta + V$ on $L^2(\mathbf{R}^2)$ and V is a real-valued potential which decays at infinity in a way that G has no real resonances nor eigenvalues in an interval $[a_0, +\infty)$, $a_0 > 0$. In fact, we are looking for as large as possible class of potentials for which we have dispersive estimates similar to those we do for the free operator G_0. Hereafter G_0 denotes the self-adjoint realization of the operator $-\Delta$ on $L^2(\mathbf{R}^2)$. It turns out that in dimension two one can get such dispersive estimates at high frequency for potentials satisfying

$$\sup_{y\in\mathbf{R}^2} \int_{\mathbf{R}^2} \frac{|V(x)|dx}{|x-y|^{1/2}} \le C < +\infty\,. \tag{1.1}$$

Clearly, (1.1) is fulfilled for potentials $V \in L^\infty(\mathbf{R}^2)$ satisfying

$$|V(x)| \le C\langle x\rangle^{-\delta}\,, \quad \forall x \in \mathbf{R}^2\,, \tag{1.2}$$

with constants $C > 0$, $\delta > 3/2$. Given any $a > 0$, set $\chi_a(\sigma) = \chi_1(\sigma/a)$, where $\chi_1 \in C^\infty(\mathbf{R})$, $\chi_1(\sigma) = 0$ for $\sigma \le 1$, $\chi_1(\sigma) = 1$ for $\sigma \ge 2$. Our first result is the following

Theorem 1.1. *Let V satisfy* (1.1)*. Then, there exists a constant $a_0 > 0$ so that for every $a \geq a_0$, $0 < \epsilon \ll 1$, $2 \leq p < +\infty$, we have the estimates*

$$\|e^{it\sqrt{G}} G^{-3/4-\epsilon} \chi_a(G)\|_{L^1 \to L^\infty} \leq C_\epsilon |t|^{-1/2}, \qquad t \neq 0, \tag{1.3}$$

$$\|e^{it\sqrt{G}} G^{-3\alpha/4} \chi_a(G)\|_{L^{p'} \to L^p} \leq C |t|^{-\alpha/2}, \qquad t \neq 0, \tag{1.4}$$

where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$.

The estimate (1.3) is proved in [2] under the assumption (1.2). Moreover, if in addition one supposes that G has no strictly positive resonances, it is shown in [2] that (1.3) holds for any $a > 0$ still under (1.2). In dimension three an analogue of (1.3) is proved in [2, 4] for potentials satisfying (1.2) with $\delta > 2$, and extended in [3] to a large subset of potentials satisfying

$$\sup_{y \in \mathbf{R}^3} \int_{\mathbf{R}^3} \frac{|V(x)| dx}{|x - y|} \leq C < +\infty. \tag{1.5}$$

In dimensions $n \geq 4$ there are very few results. In [1], an analogue of (1.3) is proved for potentials belonging to the Schwartz class, while in [12] dispersive estimates with a loss of $(n-3)/2$ derivatives are obtained for potentials satisfying (1.2) with $\delta > (n+1)/2$. Recently, in [8] dispersive estimates at low frequency have been proved in dimensions $n \geq 4$ for a very large class of potentials, provided zero is neither an eigenvalue nor a resonance.

Our second result is the following

Theorem 1.2. *Let V satisfy* (1.1)*. Then, there exists a constant $a_0 > 0$ so that for every $a \geq a_0$, we have the estimate*

$$\|e^{itG} \chi_a(G)\|_{L^1 \to L^\infty} \leq C |t|^{-1}, \quad t \neq 0. \tag{1.6}$$

Note that the estimate (1.6) (for any $a > 0$) is proved in [10] for potentials satisfying (1.2) with $\delta > 2$. An one dimensional analogue of (1.6) is proved in [6] for potentials $V \in L^1$. In dimension three an analogue of (1.6) (for any $a > 0$) is proved in [11] for potentials satisfying (1.5) with $C > 0$ small enough, and in [5] for potentials $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$, $0 < \epsilon \ll 1$, not necessarily small. In dimensions $n \geq 4$, an analogue of (1.6) (for any $a > 0$) is proved in [7] for potentials satisfying (1.2) with $\delta > n$ as well as the condition $\widehat{V} \in L^1$. This result has been recently extended in [9] to potentials satisfying (1.2) with $\delta > n - 1$ and $\widehat{V} \in L^1$. Note also the work [13], where an analogue of (1.6) (for any $a > 0$) with a loss of $(n-3)/2$ derivatives is obtained for potentials satisfying (1.2) with $\delta > (n+2)/2$. In [9] dispersive estimates at low frequency have been also proved in dimensions $n \geq 4$ for a very large class of potentials, provided zero is neither an eigenvalue nor a resonance.

To prove Theorem 1.1 we use the same idea we have already used in [8] to prove low frequency dispersive estimates in dimensions $n \geq 4$. The key point is the following estimate which holds in all dimensions $n \geq 2$:

$$h\int_{-\infty}^{\infty} \|Ve^{it\sqrt{G_0}}\psi(h^2G_0)f\|_{L^1}dt \leq \gamma_n C_n(V)h^{-(n-3)/2}\,\|f\|_{L^1}\ , \quad h>0\,, \tag{1.7}$$

where $\psi \in C_0^\infty((0,+\infty))$, $\gamma_n > 0$ is a constant independent of V, h and f, and

$$C_n(V) := \sup_{y\in\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|V(x)|dx}{|x-y|^{(n-1)/2}} < +\infty\,. \tag{1.8}$$

Our approach is based on the observation that if

$$C_n(V)h^{-(n-3)/2} \ll 1\,, \tag{1.9}$$

then (1.7) implies (under reasonable assumptions on the potential) a similar estimate for the perturbed wave group, namely

$$h\int_{-\infty}^{\infty} \|Ve^{it\sqrt{G}}\psi(h^2G)f\|_{L^1}dt \leq \widetilde{C}_n(V)h^{-(n-3)/2}\,\|f\|_{L^1}\ . \tag{1.10}$$

When $n = 3$, (1.9) is fulfilled for small potentials and all h, when $n \geq 4$, (1.9) is fulfilled for large h (i.e. at low frequency) without extra restrictions on the potential, while for $n = 2$, (1.9) is fulfilled for small h (i.e. at high frequency) again without restrictions on the potential others than (1.1). Note that (1.10) may hold without (1.9). Indeed, when $n = 3$, (1.10) is proved in [5] for potentials $V \in L^{3/2-\epsilon} \cap L^{3/2+\epsilon}$ and all $h > 0$, and then used to prove the three dimensional analogue of (1.6). In the present paper we adapt this approach to the case of dimension two, and show that (1.6) follows from (1.10) for potentials satisfying (1.1) only, provided the parameter a is taken large enough (see Section 3).

Note finally that it would be quite irrealistic to expect that the dispersive estimates above could hold for potentials satisfying (1.1) at frequencies smaller than the critical level a_0. In fact, in this range of frequencies one could hardly do better than the already existing results. Recall that for frequencies belonging to an interval $[\varepsilon, a_0]$, $0 < \varepsilon \ll 1$, dispersive estimates for the wave and the Schrödinger groups have been proved respectively in [2] for potentials satisfying (1.2) with $\delta > 3/2$ and in [10] for potentials satisfying (1.2) with $\delta > 2$. For small frequencies in $[0, \varepsilon]$ dispersive estimates for the Schrödinger group have been proved in [10] for potentials satisfying (1.2) with $\delta > 3$ supposing additionally that zero is neither an eigenvalue nor a resonance.

2. Proof of Theorem 1.1

Let $\psi \in C_0^\infty((0,+\infty))$ and set

$$\Phi(t;h) = e^{it\sqrt{G}}\psi(h^2G) - e^{it\sqrt{G_0}}\psi(h^2G_0)\,.$$

We will first show that (1.3) and (1.4) follow from the following

Proposition 2.1. *Let V satisfy* (1.1)*. Then, there exist positive constants C and h_0 so that for $0 < h \le h_0$ we have*

$$\|\Phi(t;h)\|_{L^1\to L^\infty} \le Ch^{-1}|t|^{-1/2}\,, \quad t \neq 0\,. \tag{2.1}$$

Writing

$$\sigma^{-3/4-\epsilon}\chi_a(\sigma) = \int_0^{a^{-1}} \psi(\sigma\theta)\theta^{-1/4+\epsilon}d\theta\,,$$

where $\psi(\sigma) = \sigma^{1/4-\epsilon}\chi_1'(\sigma) \in C_0^\infty((0,+\infty))$, and using (2.1) we get

$$\begin{aligned}\|e^{it\sqrt{G}}G^{-3/4-\epsilon}\chi_a(G) - e^{it\sqrt{G_0}}G_0^{-3/4-\epsilon}\chi_a(G_0)\|_{L^1\to L^\infty} \\ \le \int_0^{a^{-1}} \|\Phi(t;\sqrt{\theta})\|_{L^1\to L^\infty}\theta^{-1/4+\epsilon}d\theta \\ \le C|t|^{-1/2}\int_0^{a^{-1}} \theta^{-3/4+\epsilon}d\theta \le C|t|^{-1/2}\,,\end{aligned} \tag{2.2}$$

provided a is taken large enough. Clearly, (1.3) follows from (2.2) and the fact that it holds for G_0. To prove (1.4), observe that an interpolation between (2.1) and the trivial bound

$$\|\Phi(t;h)\|_{L^2\to L^2} \le C$$

yields

$$\|\Phi(t;h)\|_{L^{p'}\to L^p} \le Ch^{-\alpha}|t|^{-\alpha/2}\,, \quad t \neq 0\,, \tag{2.3}$$

for every $2 \le p \le +\infty$, p' and α being as in Theorem 1.1. Now we write

$$\sigma^{-3\alpha/4}\chi_a(\sigma) = \int_0^{a^{-1}} \psi(\sigma\theta)\theta^{-1+3\alpha/4}d\theta\,,$$

and use (2.3) to obtain (for $0 < \alpha \le 1$)

$$\begin{aligned}\|e^{it\sqrt{G}}G^{-3\alpha/4}\chi_a(G) - e^{it\sqrt{G_0}}G_0^{-3\alpha/4}\chi_a(G_0)\|_{L^{p'}\to L^p} \\ \le \int_0^{a^{-1}} \|\Phi(t;\sqrt{\theta})\|_{L^{p'}\to L^p}\theta^{-1+3\alpha/4}d\theta \\ \le C|t|^{-\alpha/2}\int_0^{a^{-1}} \theta^{-1+\alpha/4}d\theta \le C|t|^{-\alpha/2},\end{aligned} \tag{2.4}$$

provided a is taken large enough. Now, (1.4) follows from (2.4) and the fact that it holds for G_0.

Proof of Proposition 2.1. We will first prove the following

Lemma 2.2. *Let V satisfy* (1.1)*. Then, there exist positive constants C and h_0 so that for $0 < h \le h_0$ we have*

$$\|\psi(h^2G) - \psi(h^2G_0)\|_{L^1\to L^1} \le Ch^{1/2}\,. \tag{2.5}$$

Proof. We will make use of the formula

$$\psi(h^2G) = \frac{2}{\pi}\int_{\mathbf{C}} \frac{\partial\widetilde{\varphi}}{\partial\bar{z}}(z)(h^2G - z^2)^{-1} zL(dz)\,, \tag{2.6}$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbf{C}$, $\widetilde{\varphi} \in C_0^\infty(\mathbf{C})$ is an almost analytic continuation of $\varphi(\lambda) = \psi(\lambda^2)$ supported in a small complex neighbourhood of $\operatorname{supp}\varphi$ and satisfying

$$\left|\frac{\partial\widetilde{\varphi}}{\partial\bar{z}}(z)\right| \le C_N |\operatorname{Im} z|^N\,, \quad \forall N \ge 1\,.$$

For $\pm\operatorname{Im}\lambda \ge 0$, $\operatorname{Re}\lambda > 0$, set

$$R_0^\pm(\lambda) = (G_0 - \lambda^2)^{-1}\,, \quad R^\pm(\lambda) = (G - \lambda^2)^{-1}\,.$$

We have the identity

$$R^\pm(\lambda)\big(1 + VR_0^\pm(\lambda)\big) = R_0^\pm(\lambda)\,. \tag{2.7}$$

It is well known that the kernels of the operators $R_0^\pm(\lambda)$ are given in terms of the zero order Hankel functions by the formula

$$\big[R_0^\pm(\lambda)\big](x,y) = \pm i4^{-1} H_0^\pm(\lambda|x-y|)\,.$$

Moreover, the functions $H_0^\pm$ satisfy the bound

$$|H_0^\pm(\lambda)| \le C|\lambda|^{-1/2} e^{-|\operatorname{Im}\lambda|}\,, \quad |\lambda| \ge 1\,, \quad \pm\operatorname{Im}\lambda \ge 0\,, \tag{2.8}$$

while near $\lambda = 0$ they are of the form

$$H_0^\pm(\lambda) = a_{0,1}^\pm(\lambda) + a_{0,2}^\pm(\lambda)\log\lambda\,, \tag{2.9}$$

where $a_{0,j}^\pm$ are analytic functions. In particular, we have

$$|H_0^\pm(\lambda)| \le C|\lambda|^{-1/2}\,, \quad \operatorname{Re}\lambda > 0\,, \quad \pm\operatorname{Im}\lambda \ge 0\,. \tag{2.10}$$

Using these bounds we will prove the following

Lemma 2.3. *Let V satisfy* (1.1)*. Then, there exist constants $C > 0$ and $0 < h_0 \le 1$ so that for $z \in \mathbf{C}_\varphi^\pm := \{z \in \operatorname{supp}\widetilde{\varphi},\ \pm\operatorname{Im} z \ge 0\}$, we have the estimates*

$$\|VR_0^\pm(z/h)\|_{L^1\to L^1} \le Ch^{1/2}\,, \qquad 0 < h \le 1\,, \tag{2.11}$$

$$\|VR^\pm(z/h)\|_{L^1\to L^1} \le Ch^{1/2}\,, \qquad 0 < h \le h_0\,, \tag{2.12}$$

$$\|R_0^\pm(z/h)\|_{L^1\to L^1} \le Ch^2|\operatorname{Im} z|^{-2}\,, \quad 0 < h \le 1,\ \operatorname{Im} z \ne 0\,, \tag{2.13}$$

$$\|R^\pm(z/h)\|_{L^1\to L^1} \le Ch^2|\operatorname{Im} z|^{-2}\,, \quad 0 < h \le h_0,\ \operatorname{Im} z \ne 0\,. \tag{2.14}$$

Proof. Applying Schur's lemma and using (1.1) and (2.10), we get that the norm in the left-hand side of (2.11) is upper bounded by

$$\sup_{y\in\mathbf{R}^2}\int_{\mathbf{R}^2} |V(x)||H_0^\pm(z|x-y|/h)|dx \le Ch^{1/2}\sup_{y\in\mathbf{R}^2}\int_{\mathbf{R}^2}\frac{|V(x)dx}{|x-y|^{1/2}} \le C'h^{1/2}\,.$$

Similarly, the norm in the left-hand side of (2.13) is upper bounded by

$$\sup_{y\in\mathbf{R}^2}\int_{\mathbf{R}^2}|H_0^\pm(z|x-y|/h)|dx = c_2h^2\int_0^\infty \sigma|H_0^\pm(z\sigma)|d\sigma\,.$$

Since $\widetilde{C}^{-1}\le|z|\le\widetilde{C}$, $\pm\mathrm{Im}\,z>0$, we have $|H_0^\pm(z\sigma)|\le C|\sigma|^{-1/2}e^{-\sigma|\mathrm{Im}\,z|}$ for $\sigma\ge1$. We obtain

$$\int_1^\infty \sigma|H_0^\pm(z\sigma)|d\sigma \le C\int_1^\infty \sigma^{1/2}e^{-\sigma|\mathrm{Im}\,z|}d\sigma\le C|\mathrm{Im}\,z|^{-2}\,.$$

For $0<\sigma\le1$, we use $|H_0^\pm(z\sigma)|\le C|\log|z\sigma||\le C|\sigma|^{-1}$ to obtain

$$\int_0^1 \sigma|H_0^\pm(z\sigma)|d\sigma\le C\,,$$

which clearly implies (2.13).

To prove (2.12) and (2.14) we will make use of the identity (2.7). It follows from (2.11) that there exists a constant $0<h_0\le1$ so that for $0<h\le h_0$ the operator $1+VR_0^\pm(z/h)$ is invertible on L^1 with an inverse satisfying

$$\left\|\left(1+VR_0^\pm(z/h)\right)^{-1}\right\|_{L^1\to L^1}\le C\,,\quad z\in\mathbf{C}_\varphi^\pm\,,\tag{2.15}$$

with a constant $C>0$ independent of z and h. Clearly, (2.12) follows from (2.11) and (2.15), while (2.14) follows from (2.13) and (2.15). □

To prove (2.5) we rewrite the identity (2.7) in the form

$$R^\pm(z/h)-R_0^\pm(z/h) = -R_0^\pm(z/h)VR_0^\pm(z/h)\left(1+VR_0^\pm(z/h)\right)^{-1}\,,$$

and hence, using Lemma 2.3 and (2.15), we get

$$\begin{aligned}\|h^{-2}R^\pm(z/h)-h^{-2}R_0^\pm(z/h)\|_{L^1\to L^1}&\le Ch^{1/2}|\mathrm{Im}\,z|^{-2}\,,\\ 0<h\le h_0\,,\quad z\in\mathbf{C}_\varphi^\pm\,,&\quad \mathrm{Im}\,z\ne0\,.\end{aligned}\tag{2.16}$$

It is easy now to see that (2.5) follows from (2.6) and (2.16). □

We will now derive (2.1) from the following

Proposition 2.4. *Let V satisfy* (1.1). *Then, there exist positive constants C and h_0 so that we have, for $0\le s\le1/2$, $f,g\in L^1$,*

$$\|e^{it\sqrt{G_0}}\psi(h^2G_0)f\|_{L^\infty}\le Ch^{-3/2}|t|^{-1/2}\|f\|_{L^1}\,,\quad h>0\,,\quad t\ne0\,,\tag{2.17}$$

$$\begin{aligned}&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^\infty |t|^s|x-y|^{-s}\left|Ve^{it\sqrt{G_0}}\psi(h^2G_0)f(x)\right||g(y)|\,dtdxdy\\ &\le Ch^{-1/2}\|f\|_{L^1}\|g\|_{L^1}\,,\quad h>0\,,\end{aligned}\tag{2.18}$$

$$\begin{aligned}&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^\infty |t|^s\langle|x-y|/h\rangle^{-s}|Ve^{it\sqrt{G}}\psi(h^2G)f(x)||g(y)|dtdxdy\\ &\le Ch^{s-1/2}\|f\|_{L^1}\|g\|_{L^1}\,,\quad 0<h\le h_0\,.\end{aligned}\tag{2.19}$$

As in [13], we introduce the functions $\psi_1 \in C_0^\infty((0,+\infty))$, $\psi_1 = 1$ on $\operatorname{supp}\psi$, $\widetilde{\psi}(\sigma) = \sigma^{1/2}\psi(\sigma)$, $\widetilde{\psi}_1(\sigma) = \sigma^{-1/2}\psi_1(\sigma)$. Set

$$\Phi_2(t;h) := -h\int_0^t \widetilde{\psi}_1(h^2G_0)\sin\left((t-\tau)\sqrt{G_0}\right)Ve^{i\tau\sqrt{G}}\psi(h^2G)d\tau\,.$$

Then using Duhamel's formula for the wave equation

$$e^{it\sqrt{G}} = e^{it\sqrt{G_0}} + i\frac{\sin(t\sqrt{G_0})}{\sqrt{G_0}}\left(\sqrt{G}-\sqrt{G_0}\right) - \int_0^t \frac{\sin\left((t-\tau)\sqrt{G_0}\right)}{\sqrt{G_0}}Ve^{i\tau\sqrt{G}}d\tau\,,$$

multiplying by $\psi_1(h^2G_0)$ on the left and by $\psi(h^2G)$ on the right, we get the identity

$$\begin{aligned}\Phi(t;h) - \Phi_2(t;h) &= \Phi(t;h) + \psi_1(h^2G_0)\\ &\quad\times\left(e^{it\sqrt{G_0}} + i\frac{\sin(t\sqrt{G_0})}{\sqrt{G_0}}(\sqrt{G}-\sqrt{G_0}) - e^{it\sqrt{G}}\right)\psi(h^2G)\\ &= e^{it\sqrt{G}}\psi(h^2G) - e^{it\sqrt{G_0}}\psi(h^2G_0) + \psi(h^2G_0)e^{it\sqrt{G_0}}\psi(h^2G)\\ &\quad - \psi_1(h^2G_0)e^{it\sqrt{G}}\psi(h^2G)\\ &\quad + i\tilde{\psi}_1(h^2G_0)\sin(t\sqrt{G_0})\tilde{\psi}(h^2G)\\ &\quad - i\psi_1(h^2G_0)\sin(t\sqrt{G_0})\psi(h^2G)\,.\end{aligned}$$

By adding $i\psi_1(h^2G_0)\sin(t\sqrt{G_0})\psi(h^2G_0) - i\widetilde{\psi}_1(h^2G_0)\sin(t\sqrt{G_0})\widetilde{\psi}(h^2G_0) = 0$ and using the commuting properties and $\psi = \psi_1\psi$, we obtain

$$\begin{aligned}\Phi_1(t;h) := \Phi(t;h) - \Phi_2(t;h) &= \left(\psi_1(h^2G) - \psi_1(h^2G_0)\right)e^{it\sqrt{G}}\psi(h^2G)\\ &\quad + \psi_1(h^2G_0)\cos\left(t\sqrt{G_0}\right)\left(\psi(h^2G) - \psi(h^2G_0)\right)\\ &\quad + i\widetilde{\psi}_1(h^2G_0)\sin\left(t\sqrt{G_0}\right)\left(\widetilde{\psi}(h^2G) - \widetilde{\psi}(h^2G_0)\right).\end{aligned}\tag{2.20}$$

By Proposition 2.4 and (2.5), we have

$$\begin{aligned}&\|\Phi_1(t;h)f\|_{L^\infty} \\ &\le Ch^{-1}|t|^{-1/2}\|f\|_{L^1} + Ch^{1/2}\|\Phi(t;h)f\|_{L^\infty}\,,\quad t^{1/2}\left|\left\langle\Phi_2(t;h)f,g\right\rangle\right|\\ &\le h\int_0^{t/2}(t-\tau)^{1/2}\left\|\sin\left((t-\tau)\sqrt{G_0}\right)\widetilde{\psi}_1(h^2G_0)g\right\|_{L^\infty}\|Ve^{i\tau\sqrt{G}}\psi(h^2G)f\|_{L^1}d\tau\\ &\quad + h\int_{t/2}^t\left\|V\sin\left((t-\tau)\sqrt{G_0}\right)\widetilde{\psi}_1(h^2G_0)g\right\|_{L^1}\tau^{1/2}\|e^{i\tau\sqrt{G}}\psi(h^2G)f\|_{L^\infty}d\tau\\ &\le Ch^{-1/2}\|g\|_{L^1}\int_{-\infty}^\infty\|Ve^{i\tau\sqrt{G}}\psi(h^2G)f\|_{L^1}d\tau\\ &+\quad h\sup_{t/2\le\tau\le t}\tau^{1/2}\|e^{i\tau\sqrt{G}}\psi(h^2G)f\|_{L^\infty}\int_{-\infty}^\infty\left\|V\sin\left((t-\tau)\sqrt{G_0}\right)\widetilde{\psi}_1(h^2G_0)g\right\|_{L^1}d\tau\\ &\le Ch^{-1}\|g\|_{L^1}\|f\|_{L^1} + Ch^{1/2}\|g\|_{L^1}\sup_{t/2\le\tau\le t}\tau^{1/2}\|e^{i\tau\sqrt{G}}\psi(h^2G)f\|_{L^\infty}\,,\end{aligned}\tag{2.21}$$

for $t > 0$, which clearly implies

$$t^{1/2}\|\Phi_2(t;h)f\|_{L^\infty} \leq Ch^{-1}\|f\|_{L^1} + Ch^{1/2}\sup_{t/2\leq\tau\leq t}\tau^{1/2}\|e^{i\tau\sqrt{G}}\psi(h^2G)f\|_{L^\infty}\,. \tag{2.22}$$

By (2.20)–(2.22), we conclude

$$t^{1/2}\|\Phi(t;h)f\|_{L^\infty} \leq Ch^{-1}\|f\|_{L^1} + Ch^{1/2}t^{1/2}\|\Phi(t;h)f\|_{L^\infty} + Ch^{1/2}\sup_{t/2\leq\tau\leq t}\tau^{1/2}\|\Phi(\tau;h)f\|_{L^\infty}\,. \tag{2.23}$$

Taking h small enough we can absorb the second and the third terms in the right-hand side of (2.23), thus obtaining (2.1). Clearly, the case of $t < 0$ can be treated in the same way. □

Proof of Proposition 2.4. The kernel of the operator $e^{it\sqrt{G_0}}\psi(h^2G_0)$ is of the form $K_h(|x-y|,t)$, where

$$K_h(\sigma,t) = (2\pi)^{-1}\int_0^\infty e^{it\lambda}J_0(\sigma\lambda)\psi(h^2\lambda^2)\lambda d\lambda = h^{-2}K_1(\sigma h^{-1}, th^{-1})\,, \tag{2.24}$$

where $J_0(z) = (H_0^+(z)+H_0^-(z))/2$ is the Bessel function of order zero. It is shown in [13] (Section 2) that K_h satisfies the estimates (for all $\sigma, h > 0$, $t \neq 0$)

$$|K_1(\sigma,t)| \leq C|t|^{-s}\langle\sigma\rangle^{s-1/2}\,, \qquad \forall s \geq 0\,, \tag{2.25}$$

$$|K_h(\sigma,t)| \leq Ch^{-3/2}|t|^{-s}\sigma^{s-1/2}\,, \qquad 0 \leq s \leq 1/2\,. \tag{2.26}$$

Clearly, (2.17) follows from (2.26) with $s = 1/2$. It is easy also to see that (2.18) follows from (1.1) and the following

Lemma 2.5. *For all* $0 \leq s \leq 1/2$, $\sigma, h > 0$, *we have*

$$\int_{-\infty}^\infty \langle t/h\rangle^s\,|K_h(\sigma,t)|\,dt \leq Ch^{-1}\langle\sigma/h\rangle^{s-1/2}\,, \tag{2.27}$$

$$\int_{-\infty}^\infty |t|^s\,|K_h(\sigma,t)|\,dt \leq Ch^{-1/2}\sigma^{s-1/2}\,. \tag{2.28}$$

Proof. Clearly, (2.28) follows from (2.27). It is also clear from (2.24) that it suffices to prove (2.27) with $h = 1$.

When $0 < \sigma \leq 1$, this follows from (2.25). To see that, we can split the integral into two parts: $|t| \leq 1$ and $|t| \geq 1$ and then use (2.25) with s for the first part and $s + 1 + \epsilon$ for the second part. We conclude by using that $\langle\sigma\rangle^{1+\epsilon} \leq 1$ for the bound of the second part. Let now $\sigma \geq 1$. We decompose K_1 as $K_1^+ + K_1^-$, where $K_1^\pm$ is defined by replacing in (2.24) the function J_0 by $H_0^\pm/2$. Recall that $H_0^\pm(z) = e^{\pm iz}b_0^\pm(z)$, where $b_0^\pm(z)$ is a symbol of order $-1/2$ for $z \geq 1$. Using this fact and integrating by parts m times, we get

$$|K_1^\pm(\sigma,t)| \leq C_m\sigma^{-1/2}\langle t\pm\sigma\rangle^{-m}\,. \tag{2.29}$$

By (2.29), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\langle t\rangle^s|K_1^{\pm}(\sigma,t)|dt &\le 2\sigma^s\int_{-\infty}^{\infty}|K_1^{\pm}(\sigma,t)|dt+\int_{-\infty}^{\infty}|t\pm\sigma|^s|K_1^{\pm}(\sigma,t)|dt\\ &\le C_m'\sigma^{s-1/2}\int_{-\infty}^{\infty}\langle t\pm\sigma\rangle^{-m}dt+C_m\sigma^{-1/2}\int_{-\infty}^{\infty}\langle t\pm\sigma\rangle^{s-m}dt\\ &\le C\sigma^{s-1/2}\,,\end{aligned}$$

which clearly implies (2.27) in this case. □

To prove (2.19) we will use the formula

$$e^{it\sqrt{G}}\psi(h^2G)=(i\pi h)^{-1}\int_0^{\infty}e^{it\lambda}\varphi_h(\lambda)\big(R^+(\lambda)-R^-(\lambda)\big)d\lambda\,, \tag{2.30}$$

where $\varphi_h(\lambda)=\varphi_1(h\lambda)$, $\varphi_1(\lambda)=\lambda\psi(\lambda^2)$. Combining (2.30) together with (2.7), we get

$$Ve^{it\sqrt{G}}\psi(h^2G)=(i\pi h)^{-1}\sum_{\pm}\pm\int_{-\infty}^{\infty}VP_h^{\pm}(t-\tau)U_h^{\pm}(\tau)d\tau\,, \tag{2.31}$$

where

$$\begin{aligned}P_h^{\pm}(t)&=\int_0^{\infty}e^{it\lambda}\widetilde{\varphi}_h(\lambda)R_0^{\pm}(\lambda)d\lambda\,,\\ U_h^{\pm}(t)&=\int_0^{\infty}e^{it\lambda}\varphi_h(\lambda)\big(1+VR_0^{\pm}(\lambda)\big)^{-1}d\lambda\,,\end{aligned}$$

where $\widetilde{\varphi}_h(\lambda)=\widetilde{\varphi}_1(h\lambda)$, $\widetilde{\varphi}_1\in C_0^{\infty}((0,+\infty))$ is such that $\widetilde{\varphi}_1=1$ on supp φ_1. The kernel of the operator $P_h^{\pm}(t)$ is of the form $A_h^{\pm}(|x-y|,t)$, where

$$A_h^{\pm}(\sigma,t)=\pm i4^{-1}\int_0^{\infty}e^{it\lambda}\widetilde{\varphi}_h(\lambda)H_0^{\pm}(\sigma\lambda)d\lambda=h^{-1}A_1^{\pm}(\sigma/h,t/h)\,. \tag{2.32}$$

In the same way as in the proof of Lemma 2.5 one can see that the function $A_h^{\pm}$ satisfies the estimate

$$\begin{aligned}\int_{-\infty}^{\infty}|t|^s\left|A_h^{\pm}(\sigma,t)\right|dt\le Ch^{1/2}\sigma^{s-1/2}(1+h^{\epsilon_s}\sigma^{-\epsilon_s})\,,\\ 0\le s\le 1/2\,,\quad 0<h\le 1\,,\end{aligned} \tag{2.33}$$

where $\epsilon_s=0$ if $0\le s<1/2$, $\epsilon_s=\epsilon$ if $s=1/2$.

Clearly, it suffices to prove (2.19) with $s=0$ and $s=1/2$. For these values of s, using (1.1), (2.31) and (2.33), we obtain

$$\begin{aligned}&\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|t|^s\left\langle|x-y|/h\right\rangle^{-s}\left|Ve^{it\sqrt{G}}\psi(h^2G)f(x)\right||g(y)|\,dtdxdy\\ &\quad\le Ch^{-1}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle|x-y|/h\right\rangle^{-s}\left(|t-\tau|^s+|\tau|^s\right)\\ &\qquad\times\left|VP_h^{\pm}(t-\tau)U_h^{\pm}(\tau)f(x)\right||g(y)|\,d\tau dtdxdy\end{aligned}$$

$$
\begin{aligned}
&\le Ch^{-1}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|V(x)|\,\langle|x-y|/h\rangle^{-s}\,(|t-\tau|^s+|\tau|^s)\\
&\quad\times|A_h^{\pm}(|x-x'|,t-\tau)|\,\big|U_h^{\pm}(\tau)f(x')\big|\,|g(y)|\,d\tau dt dx' dx dy\\
&\le Ch^{-1}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}|V(x)|\,\langle|x-y|/h\rangle^{-s}\,|g(y)|\\
&\quad\times\left(\int_{-\infty}^{\infty}|\tau|^s|A_h^{\pm}(|x-x'|,\tau)|d\tau\right)\left(\int_{-\infty}^{\infty}\big|U_h^{\pm}(\tau)f(x')\big|\,d\tau\right)dx'dxdy\\
&\quad+Ch^{-1}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}|V(x)|\,\langle|x-y|/h\rangle^{-s}\,|g(y)|\\
&\quad\times\left(\int_{-\infty}^{\infty}|A_h^{\pm}(|x-x'|,\tau)|d\tau\right)\left(\int_{-\infty}^{\infty}|\tau|^s\,\big|U_h^{\pm}(\tau)f(x')\big|\,d\tau\right)dx'dxdy\\
&\le Ch^{-1/2}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}|V(x)|\,\langle|x-y|/h\rangle^{-s}\,|x-x'|^{s-1/2}\\
&\quad\times\left(1+h^{\epsilon_s}|x-x'|^{-\epsilon_s}\right)|g(y)|\left(\int_{-\infty}^{\infty}\big|U_h^{\pm}(\tau)f(x')\big|\,d\tau\right)dx'dxdy\\
&\quad+Ch^{-1/2}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}|V(x)|\,\langle|x-y|/h\rangle^{-s}\,|x-x'|^{-1/2}\,|g(y)|\\
&\quad\times\left(\int_{-\infty}^{\infty}|\tau|^s\,\big|U_h^{\pm}(\tau)f(x')\big|\,d\tau\right)dx'dxdy:=I_1+I_2\,.
\end{aligned}
\tag{2.34}
$$

To estimate I_1 when $s=1/2$, set $q=(2\epsilon)^{-1}$, $1/p+1/q=1$, and observe that in view of (1.1) we have the bound

$$
\begin{aligned}
&\int_{\mathbf{R}^2}|V(x)|\,\langle|x-y|/h\rangle^{-1/2}\,|x-x'|^{-\epsilon}dx\\
&\qquad\le\left(\int_{\mathbf{R}^2}|V(x)|\,\langle|x-y|/h\rangle^{-p/2}\,dx\right)^{1/p}\left(\int_{\mathbf{R}^2}|V(x)||x-x'|^{-1/2}dx\right)^{1/q}\\
&\qquad\le C_1\left(\int_{\mathbf{R}^2}|V(x)|\,\langle|x-y|/h\rangle^{-1/2}\,dx\right)^{1/p}\\
&\qquad\le C_1h^{1/(2p)}\left(\int_{\mathbf{R}^2}|V(x)||x-y|^{-1/2}dx\right)^{1/p}\le C_2h^{1/2-\epsilon}\,.
\end{aligned}
$$

Thus, we obtain

$$
I_1\le C'h^{s-1/2}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|U_h^{\pm}(\tau)f(x')||g(y)|d\tau dx'dy\,. \tag{2.35}
$$

To estimate I_2 when $s=1/2$, we use the inequality

$$
\langle|x-y|/h\rangle^{-1/2}\,|x-x'|^{-1/2}\le\langle|x'-y|/h\rangle^{-1/2}\,(|x-y|^{-1/2}+|x-x'|^{-1/2})\,.
$$

We get

$$I_2 \le C''h^{-1/2}\sum_{\pm}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|\tau|^s\langle|x'-y|/h\rangle^{-s}|U_h^{\pm}(\tau)f(x')||g(y)|d\tau dx'dy\,. \tag{2.36}$$

On the other hand, by the identity

$$\big(1+VR_0^{\pm}(\lambda)\big)^{-1} = 1 - VR_0^{\pm}(\lambda)\big(1+VR_0^{\pm}(\lambda)\big)^{-1},$$

we obtain

$$U_h^{\pm}(t) = \widehat{\varphi}_h(t) - \int_{-\infty}^{\infty} VP_h^{\pm}(t-\tau)U_h^{\pm}(\tau)d\tau\,. \tag{2.37}$$

Since

$$\widehat{\varphi}_h(t) = h^{-1}\widehat{\varphi}_1(t/h)\,,$$

we have

$$\int_{-\infty}^{\infty}|t|^s|\widehat{\varphi}_h(t)|dt \le Ch^s\,. \tag{2.38}$$

Using (2.37) and (2.38), in the same way as in the proof of (2.34)–(2.36), we obtain with $s=0$ or $s=1/2$,

$$\begin{aligned}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|t|^s\langle|x-y|/h\rangle^{-s}|U_h^{\pm}(t)f(x)||g(y)|dtdxdy &\le Ch^s\|f\|_{L^1}\|g\|_{L^1}\\ &+ Ch^{s+1/2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|U_h^{\pm}(\tau)f(x')||g(y)|d\tau dx'dy\\ &+ Ch^{1/2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|\tau|^s\langle|x'-y|/h\rangle^{-s}|U_h^{\pm}(\tau)f(x')||g(y)|d\tau dx'dy\,.\end{aligned} \tag{2.39}$$

Taking h small enough we can absorb the second and the third terms in the right-hand side of (2.39) and get the estimate

$$\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|t|^s\langle|x-y|/h\rangle^{-s}|U_h^{\pm}(t)f(x)||g(y)|dtdxdy \le C'h^s\|f\|_{L^1}\|g\|_{L^1}\,. \tag{2.40}$$

Now (2.19) follows from (2.34)–(2.36) and (2.40). □

3. Proof of Theorem 1.2

Set

$$\Psi(t;h) = e^{itG}\psi(h^2G) - e^{itG_0}\psi(h^2G_0)\,.$$

As in the previous section, one can derive (1.6) from the following

Proposition 3.1. *Let V satisfy* (1.1). *Then, there exist positive constants C and h_0 so that for $0<h\le h_0$, we have*

$$\|\Psi(t;h)\|_{L^1\to L^\infty} \le Ch^{1/2}|t|^{-1}\,,\quad t\ne 0\,. \tag{3.1}$$

Proof. We will derive (3.1) from (2.19). To this end, we will use the identity

$$e^{it\lambda^2}\varphi(h^2\lambda^2) = \int_{-\infty}^{\infty} e^{i\tau\lambda}\zeta_h(t,\tau)d\tau\,, \tag{3.2}$$

where $\varphi \in C_0^\infty((0,+\infty))$, $\varphi = 1$ on $\operatorname{supp}\psi_1$, the functions ψ and ψ_1 being as in the previous section, and

$$\zeta_h(t,\tau) = (2\pi)^{-1}\int_0^{\infty} e^{it\lambda^2 - i\tau\lambda}\varphi(h^2\lambda^2)d\lambda = h^{-1}\zeta_1(th^{-2},\tau h^{-1})\,. \tag{3.3}$$

We deduce from (3.2) the formula

$$e^{itG}\psi(h^2G) = \int_{-\infty}^{\infty}\zeta_h(t,\tau)e^{i\tau\sqrt{G}}\psi(h^2G)d\tau\,. \tag{3.4}$$

Given any integer $m \geq 0$, integrating by parts m times and using the well known bound

$$\left|\int_{-\infty}^{\infty} e^{it\lambda^2 - i\tau\lambda}\phi(\lambda)d\lambda\right| \leq C|t|^{-1/2}\,, \quad \forall t \neq 0\,, \quad \tau \in \mathbf{R}\,,$$

where $\phi \in C_0^\infty(\mathbf{R})$, one easily obtains the bound

$$|\zeta_1(t,\tau)| \leq C_m|t|^{-m-1/2}\langle\tau\rangle^m\,, \quad \forall t \neq 0\,, \quad \tau \in \mathbf{R}\,. \tag{3.5}$$

By (3.3) and (3.5),

$$|\zeta_h(t,\tau)| \leq C_m h^{2m}|t|^{-m-1/2}\langle\tau/h\rangle^m\,, \quad \forall t \neq 0\,, \quad \tau \in \mathbf{R}\,, \quad h > 0\,, \tag{3.6}$$

for every integer $m \geq 0$, and hence for all real $m \geq 0$. By (2.5), (2.20) and (3.4), we get

$$\begin{aligned}
\left|\left\langle \Psi(t;h)f,g\right\rangle\right| &\leq Ch^{1/2}\|\Psi(t;h)f\|_{L^\infty}\|g\|_{L^1} \\
&+ \int_{-\infty}^{\infty}|\zeta_h(t,\tau)|\left|\left\langle \cos(\tau\sqrt{G_0})\psi_1(h^2G_0)\big(\psi(h^2G) - \psi(h^2G_0)\big)f,g\right\rangle\right|d\tau \\
&+ \int_{-\infty}^{\infty}|\zeta_h(t,\tau)|\left|\left\langle \sin(\tau\sqrt{G_0})\widetilde{\psi}_1(h^2G_0)\big(\widetilde{\psi}(h^2G) - \widetilde{\psi}(h^2G_0)\big)f,g\right\rangle\right|d\tau \\
&+ h\int_{-\infty}^{\infty}\int_0^{\tau}|\zeta_h(t,\tau)|\left|\left\langle Ve^{i\tau'\sqrt{G}}\psi(h^2G)f, \sin\big((\tau-\tau')\sqrt{G_0}\big)\widetilde{\psi}_1(h^2G_0)g\right\rangle\right|d\tau'd\tau\,.
\end{aligned} \tag{3.7}$$

Using (3.6) with $m = 1/2$ and (2.27) with $s = 1/2$ together with (2.5), we obtain that the first integral (and similarly for the second one) in the right-hand side of

(3.7) is upper bounded by

$$\begin{aligned} Ch|t|^{-1}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}\langle\tau/h\rangle^{1/2}\,|K_h^c(|x-y|,\tau)|\,|g(x)| \\ \times\left|\left(\psi(h^2G)-\psi(h^2G_0)\right)f(y)\right|d\tau dxdy \\ \leq C|t|^{-1}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}|g(x)|\left|\left(\psi(h^2G)-\psi(h^2G_0)\right)f(y)\right|dxdy \\ \leq Ch^{1/2}|t|^{-1}\|f\|_{L^1}\|g\|_{L^1}\,, \end{aligned}$$

where $K_h^c(|x-y|,t)$ denotes the kernel of the operator $\cos\left(t\sqrt{G_0}\right)\psi_1(h^2G_0)$. The last term is upper bounded by

$$\begin{aligned} Ch^2|t|^{-1}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}\int_0^{\tau}\left(|\tau'/h|^{1/2}+\left\langle(\tau-\tau')/h\right\rangle^{1/2}\right)\left|K_h^s\left(|x-y|,(\tau-\tau')\right)\right| \\ \times|Ve^{i\tau'\sqrt{G}}\psi(h^2G)f(x)||g(y)|d\tau'd\tau dxdy \\ \leq Ch^{3/2}|t|^{-1}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|\tau|^{1/2}|Ve^{i\tau\sqrt{G}}\psi(h^2G)f(x)||g(y)|d\tau \\ \times\int_{-\infty}^{\infty}|K_h^s(|x-y|,\tau)|d\tau dxdy \\ +Ch^2|t|^{-1}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|Ve^{i\tau\sqrt{G}}\psi(h^2G)f(x)||g(y)|d\tau \\ \times\int_{-\infty}^{\infty}\langle\tau/h\rangle^{1/2}|K_h^s(|x-y|,\tau)|d\tau dxdy \\ \leq Ch^{1/2}|t|^{-1}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|\tau|^{1/2}\langle|x-y|/h\rangle^{-1/2} \\ \times|Ve^{i\tau\sqrt{G}}\psi(h^2G)f(x)||g(y)|d\tau dxdy \\ +Ch|t|^{-1}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{-\infty}^{\infty}|Ve^{i\tau\sqrt{G}}\psi(h^2G)f(x)||g(y)|d\tau dxdy \\ \leq Ch^{1/2}|t|^{-1}\|f\|_{L^1}\|g\|_{L^1}\,, \end{aligned}$$

where $K_h^s(|x-y|,t)$ denotes the kernel of the operator $\sin(t\sqrt{G_0})\widetilde{\psi}_1(h^2G_0)$, and we have used (2.19) together with the fact that the function $K_h^s(\sigma,t)$ satisfies (2.27). Thus, we obtain

$$\left|\left\langle\Psi(t;h)f,g\right\rangle\right|\leq Ch^{1/2}\|\Psi(t;h)f\|_{L^\infty}\|g\|_{L^1}+Ch^{1/2}|t|^{-1}\|f\|_{L^1}\|g\|_{L^1}\,,$$

which clearly implies (3.1), provided h is taken small enough. □

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