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On the Ultraviolet Problem for the 2D Weakly Interacting Fermi Gas

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Abstract. We prove that the effective potential of the two-dimensional interacting continuous Fermi gas with infrared cutoff is an analytic function of the coupling strength near the origin. This is the starting point to study the infrared problem of the model without putting any ultraviolet cutoff, as usually done in the literature.

1. Introduction

We consider a continuous system of two-dimensional fermions interacting with a smooth integrable potential $\lambda\bar{v}(\vec{x}-\vec{y})$, $\vec{x},\vec{y}\in\mathbb{T}_L^2$, the two-dimensional torus of side L. $\bar{v}(\vec{x})$ is supposed to be bounded, smooth and with finite range. Since the spin will not play any role in this paper, we shall suppose that the fermions are spinless.

In the mathematical physical literature the infrared problem for this system (sometimes called the jellium) is in general studied with an ultraviolet (UV) cutoff, whose introduction is motivated by the remark that the model can not be valid at high energies, see for example [3, 5]. This implies that the results could strongly depend on the cutoff scale, if the system were not stable in the UV region. Indeed, in these papers the interaction is not supposed to satisfy the usual stability condition, which would be sufficient at least to ensure the existence of the thermodynamical limit for the pressure, for any β, and of the correlation functions, for β large enough [8, 12]. Moreover, even in the case of a stable potential, the techniques that are used to study the infrared problem do not allow to use this condition, but are based on the analytic dependence on λ of the effective potential.

The aim of this paper is to prove that the UV cutoff can be safely removed. Note that the dimension plays an essential role; in fact, if $d\geq 3$, it is well known that there are two-body interactions, which do not satisfy the stability condition, for which the pressure can not be defined, even for fermion systems [12]. In agreement with this difficulty, our proof can not be extended to $d > 2$, see remark 3) below.

Let us now define more precisely the model. Given the inverse temperature β, we define $\Lambda = [0, \beta] \times \mathbb{T}_L^2$ and, if $x_0 \in [0, \beta]$ and $\vec{x} \in \mathbb{T}_L^2$, we put $\mathbf{x} = (x_0, \vec{x})$. Moreover, we define $\mathcal{D}_{\beta,L} = \mathcal{D}_\beta \times \mathcal{D}_L$ with $\mathcal{D}_\beta = \{k_0 = \frac{2\pi}{\beta}(n_0 + \frac{1}{2}) \,:\, n_0 \in \mathbb{Z}\}$ and $\mathcal{D}_L = \{\vec{k} = \frac{2\pi}{L}\vec{n} \,:\, \vec{n} \in \mathbb{Z}^2\}$; the elements of $\mathcal{D}_{\beta,L}$ will be denoted by $\mathbf{k} = (k_0, \vec{k})$. Finally, given the particle mass $m > 0$ and the chemical potential μ, we shall define $\varepsilon_0(\vec{k}) = (\vec{k}^2/2m) - \mu$.

We study the UV problem for the system of fermions, by analyzing the *effective potential*

$$V_{eff}(\varphi) = \log \int P(d\psi) e^{-V(\psi+\varphi)}\,, \tag{1.1}$$

where φ is the (Grassmannian) *external field*, $P(d\psi)$ is the (Grassmannian) Gaussian measure of covariance

$$g_{uv}(\mathbf{x}-\mathbf{y}) = \int P(d\psi)\psi^-_{\mathbf{x}}\psi^+_{\mathbf{y}} = \frac{1}{\beta L^2} \sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{\chi(k_0^2 + \varepsilon_0(\vec{k})^2)}{-ik_0 + \varepsilon_0(\vec{k})}\,, \tag{1.2}$$

$\chi(t)$, defined for $t > 0$, being a smooth cutoff function, with a fixed infrared cutoff on scale 1. We can choose it, for example, by putting $\chi(t)$ equal to 1 for $t \geq 2$ and equal to 0 for $t \leq 1$. Finally the interaction term is given by

$$V(\psi) = \lambda \int_{\Lambda\times\Lambda} d\mathbf{x}d\mathbf{y}\, v(\mathbf{x}-\mathbf{y})\; \psi^+_{\mathbf{x}}\psi^+_{\mathbf{y}}\psi^-_{\mathbf{y}}\psi^-_{\mathbf{x}} + \nu \int_\Lambda d\mathbf{x}\psi^+_{\mathbf{x}}\psi^-_{\mathbf{x}}\,, \tag{1.3}$$

$$v(\mathbf{x}-\mathbf{y}) = \delta(x_0 - y_0)\bar{v}(\vec{x}-\vec{y})\,, \tag{1.4}$$

ν being a finite *counterterm*, which is in general introduced to fix the Fermi surface at the free theory value, when the infrared cutoff is lowered [3, 5]; hence it is a constant of order λ, whose precise value is of no importance in this paper. $V_{eff}(\varphi)$ can be formally expanded in powers of φ; we shall write

$$V_{eff}(\varphi) = -\beta L^2 p(\lambda,\nu) + \sum_{k=1}^\infty \int d\underline{\mathbf{x}}\; d\underline{\mathbf{y}}\; W_{2k}(\underline{\mathbf{x}},\underline{\mathbf{y}},\lambda,\nu) \prod_{i=1}^k \varphi^+_{\mathbf{x}_i} \prod_{i=1}^k \varphi^-_{\mathbf{y}_i}\,, \tag{1.5}$$

where we used the notation $\underline{\mathbf{x}} \equiv (\mathbf{x}_1, \ldots, \mathbf{x}_k)$. The *pressure* $p(\lambda,\nu)$ and the *kernels* $W_{2k}(\underline{\mathbf{x}},\underline{\mathbf{y}},\lambda,\nu)$ have well defined formal representations as powers of λ and ν; we shall write:

$$p(\lambda,\nu) = \sum_{n+m\geq 1} p_{n,m}\lambda^n\nu^m\,, \tag{1.6}$$

$$W_{2k}(\underline{\mathbf{x}},\underline{\mathbf{y}},\lambda,\nu) = \sum_{n+m\geq 1} \lambda^n\nu^m W_{2k,n,m}(\underline{\mathbf{x}},\underline{\mathbf{y}})\,.$$

The kernels $W_{2k}(\underline{\mathbf{x}},\underline{\mathbf{y}},\lambda,\nu)$ are finite sums of products of delta functions of the difference between two space or time variables times suitable measurable functions of a subset of the $\underline{\mathbf{x}}$ and $\underline{\mathbf{y}}$ variables, determined by the delta functions; we shall call, with an abuse of notation, L^1 norm of $W_{2k}(\underline{\mathbf{x}}, \underline{\mathbf{y}}, \lambda, \nu)$, and we shall denote by $\int d\underline{\mathbf{x}} d\underline{\mathbf{y}} |W_{2k}(\underline{\mathbf{x}}, \underline{\mathbf{y}}, \lambda, \nu)|$, the sum of the L^1 norms of these measurable functions. The L^1 norm is of course a uniform bound for the Fourier transform of the kernels.

As it is well known, it is difficult to check directly the convergence of the power series in (1.6), because of the UV singularity of the propagator (1.2). Up to now, the convergence has been proved (in a suitable norm) only in the one-dimensional case [2]. However, it is possible to show that, if one expand in Feynman graphs the series coefficients, the L^1 norm of each graph satisfies a $C^{n+m}/(n+m)!$ bound; since the number of graphs is of order $(n+m)!^2$, one gets a $C^{n+m}(n+m)!$ bound for the L^1 norm of the series coefficients [6]. This result is obtained by using the fact that the singular part of the covariance (1.2) is different from zero only if $x_0 - y_0$ or $x_0 - y_0 + \beta$ are positive, see Section 2. This argument works in all dimensions, but we shall prove that, in the two-dimensional case, we can use it in a more efficient way, so proving that the series coefficients indeed satisfy a C^{n+m} bound.

In order to get this result, we introduce in Section 2 a suitable UV regularization of the propagator, depending on a integer parameter N, which diverges as the cutoff is removed, and we add a superscript $^{(N)}$ to all quantities in (1.6), to remind the dependence on N. Then we prove that the series coefficients satisfy bounds good enough to imply the power series convergence of $p^{(N)}(\lambda)$ and bounds on the L^1 norm of $W_{2k}^{(N)}(\underline{\mathbf{x}}, \underline{\mathbf{y}}, \lambda, \nu)$, for λ and ν small enough, uniformly in N. These bounds are based on a multi-scale expansion, which allows us to prove also the convergence, as $N \to \infty$, of $p^{(N)}(\lambda)$, $p_{n,m}^{(N)}$, as well as convergence of $W_{2k,n,m}^{(N)}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ and $W_{2k}^{(N)}(\underline{\mathbf{x}}, \underline{\mathbf{y}}, \lambda, \nu)$ in the L^1 norm, to quantities satisfying (1.6). We could also prove the convergence at non coinciding points of the correlation functions and the fast decaying properties on scale 1 of the effective potential kernels. For simplicity, here we shall only prove the following theorem.

Theorem 1.1. *There are constants C_k, independent of L, β and N, such that*

$$|p_{n,m}^{(N)}| \le C_0^{n+m}\,, \tag{1.7}$$

$$(\beta L^2)^{-1} \int d\underline{\mathbf{x}}\, d\underline{\mathbf{y}}\, |W_{2k,n,m}^{(N)}(\underline{\mathbf{x}}, \underline{\mathbf{y}})| \le C_k^{n+m}\,. \tag{1.8}$$

Moreover, there exist constants $p_{n,m}$ and distributions $W_{2k,n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, with the same structure of $W_{2k,n,m}^{(N)}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ as sum of products of delta functions times measurable functions, which satisfy the same bounds and are such that if λ and ν are small enough,

$$\lim_{N\to\infty} \sum_{n+m\ge 1} \lambda^n \nu^m |p_{n,m}^{(N)} - p_{n,m})| = 0\,, \tag{1.9}$$

$$\lim_{N\to\infty} (\beta L^2)^{-1} \int d\underline{\mathbf{x}}\, d\underline{\mathbf{y}} \sum_{n+m\ge 1} \lambda^n \nu^m |W_{2k,n,m}^{(N)}(\underline{\mathbf{x}}, \underline{\mathbf{y}}) - W_{2k,n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})| = 0\,. \tag{1.10}$$

Remark.

1) The expansion used to prove this theorem could be easily used to prove that the results of [5] and [3] about the normal behavior of the weakly interacting Fermi system up to exponentially small temperatures (close to the expected phase transition) can be obtained without any fixed ultraviolet cutoff.
2) The proof of the theorem is indeed valid even for a system with a fixed UV cutoff on the $\vec{k}$ variables, for example a system of fermions on a fixed lattice. In this case, however, the UV problem is much milder and a much simpler procedure to get the same result through a different multi-scale expansion is described in App. A of [4]. Moreover, in [11] it has been shown that, if there is a UV cutoff on the $\vec{k}$ variables, no scale decomposition is needed, thanks to an improved technique to control fermion determinants.
3) The expansion which allows us to prove the theorem could be applied also to the case of spatial dimension grater than 2. However, if $d > 2$, the bounds (1.7) and (1.8) can not be proved, as we shall explain in Section 4. As we said above, this result should be expected, since the existence of the thermodynamical limit is strictly related with the stability of the potential for $d \geq 3$ [12], so that we can at most make the hypothesis, in agreement with the $C^{n+m}(n+m)!$ bound of the series coefficients (see above), that the perturbative series are Borel summable, if the two-body potential $\bar{v}(\vec{x})$ is stable and $\lambda > 0$. However, our technique is not suitable for getting a result of this type.

The plan of the paper is the following. In Section 2 we discuss the multi-scale decomposition of the covariance, in Section 3 we describe the corresponding expansion of the effective potential and, finally, in Section 4 we prove the theorem.

2. The decomposition of the covariance

Note that

$$g_{uv}(\mathbf{x}) = g(\mathbf{x}) - g_{ir}(\mathbf{x})\,, \tag{2.1}$$

where

$$g(\mathbf{x}) = \frac{1}{\beta L^2} \sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{1}{-ik_0 + \varepsilon_0(\vec{k})}\,, \tag{2.2}$$

$$g_{ir}(\mathbf{x}) = \frac{1}{\beta L^2} \sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \frac{1-\chi(k_0^2+\varepsilon_0(\vec{k})^2)}{-ik_0 + \varepsilon_0(\vec{k})}\,. \tag{2.3}$$

As it is well known, the sum over k_0 in the r.h.s. of (2.2) can be performed and one gets, if $|x_0| \leq \beta$,

$$g(\mathbf{x}) = \frac{1}{L^2} \sum_{\vec{k}\in\mathcal{D}_L} e^{-i\vec{k}\cdot\vec{x}-x_0\varepsilon_0(\vec{k})} \left[\frac{\theta(x_0)}{1+e^{-\beta\varepsilon_0(\vec{k})}} - \frac{1-\theta(x_0)}{1+e^{+\beta\varepsilon_0(\vec{k})}}\right], \tag{2.4}$$

where $\theta(t)$ is the step function, equal to 1 for $t > 0$ and equal to 0 for $t \le 0$ (this choice ensures that $g(0, \vec{x}) = \lim_{x_0 \to 0^-} g(x_0, \vec{x})$, a required condition [10]).

The function $g(\mathbf{x})$, which represents the covariance of the free fermion gas without cutoffs, is antiperiodic in x_0 of period β and periodic of period 2β; then we shall consider it as defined on $\mathbb{T}^1_{2\beta} \times \mathbb{T}^2_L$. This function has a singularity at the points $x_0 = 0, \pm\beta$, which can be described in the following way. Let $h_0(t)$ be a smooth function on $\mathbb{R}$, equal to 1 for $|t| \le 1$ and equal to 0 for $|t| \ge 2$, and let us call $\tilde{h}_0(x_0)$ the function on $\mathbb{T}^1_{2\beta}$, which is equal to $h_0(x_0)$ around $x_0 = 0$ and equal to $h_0(x_0 + \beta)$ around $x_0 = -\beta$. Let us call also $h_1(\vec{x})$ the function on $\mathbb{T}^2_L$, which is equal to $h_0(\vec{x}^2)$ around $\vec{x} = 0$, and $f(\mathbf{x}) = \tilde{h}_0(x_0) h_1(\vec{x})$. One can easily check that

$$f(\mathbf{x}) g(\mathbf{x}) = G(\mathbf{x}) + R_1(\mathbf{x}) , \tag{2.5}$$

where

$$\begin{aligned} G(\mathbf{x}) &= h_0(x_0) h_1(\vec{x}) \theta(x_0) e^{\mu x_0} \delta^{(L)}_{x_0/m}(\vec{x}) \\ &\quad - h_0(x_0 + \beta) h_1(\vec{x}) \theta(x_0 + \beta) e^{\mu(x_0+\beta)} \delta^{(L)}_{(x_0+\beta)/m}(\vec{x}) , \end{aligned} \tag{2.6}$$

$$\begin{aligned} R_1(\mathbf{x}) &= -h_0(x_0) h_1(\vec{x}) \frac{1}{L^2} \sum_{\vec{k} \in \mathcal{D}_L} \frac{e^{-i\vec{k}\cdot\vec{x} - x_0 \varepsilon_0(\vec{k})}}{1 + e^{+\beta\varepsilon_0(\vec{k})}} \\ &\quad + h_0(x_0 + \beta) h_1(\vec{x}) \frac{1}{L^2} \sum_{\vec{k} \in \mathcal{D}_L} \frac{e^{-i\vec{k}\cdot\vec{x} - (x_0+\beta)\varepsilon_0(\vec{k})}}{1 + e^{+\beta\varepsilon_0(\vec{k})}} , \end{aligned} \tag{2.7}$$

and, if $t > 0$, we are defining

$$\delta^{(L)}_t(\vec{x}) = \frac{1}{L^2} \sum_{\vec{k} \in \mathcal{D}_L} e^{-i\vec{k}\cdot\vec{x} - t\vec{k}^2/2} = \sum_{\vec{n} \in \mathbb{Z}^2} \frac{e^{-(\vec{x}+\vec{n}L)^2/(2t)}}{2\pi t} . \tag{2.8}$$

By using (2.1) and (2.5), we can write:

$$g_{uv} = f g_{uv} + (1-f) g_{uv} = f g - f g_{ir} + (1-f) g_{uv} \equiv G + R , \tag{2.9}$$

where $R = R_1 - f g_{ir} + (1-f) g_{uv}$. It is easy to check that $R(\mathbf{x})$ is a smooth function on $\mathbb{T}^1_{2\beta} \times \mathbb{T}^2_L$ and that, given any integer $M > 0$, there exists a constant C_M, independent of L and β, such that, if $d_\beta(x_0, y_0)$ and $d_L(\vec{x}, \vec{y})$ denote the distances on $\mathbb{T}^1_\beta$ (not $\mathbb{T}^1_{2\beta}$) and $\mathbb{T}^2_L$, respectively, then

$$|R(\mathbf{x} - \mathbf{y})| \le \frac{C_M}{1 + d_\beta(x_0 - y_0)^M + d_L(\vec{x} - \vec{y})^M} . \tag{2.10}$$

As it is well known, the decomposition (2.9) of the covariance implies that, if we call $P_R(d\psi)$ and $P_G(d\psi)$ the Gaussian measures of covariance $R(\mathbf{x} - \mathbf{y})$ and

$G(\mathbf{x}-\mathbf{y})$, respectively, then we can rewrite (1.1) in the form

$$V_{eff}(\varphi) = \log \int P_R(d\psi) e^{V_G(\psi+\varphi)}\,, \tag{2.11}$$

$$V_G(\varphi) = \log \int P_G(d\psi) e^{-V(\psi+\varphi)}\,. \tag{2.12}$$

The decaying property (2.10) of $R(\mathbf{x})$ implies, by using standard techniques based on the Gram–Hadamard inequality, that the integration in (2.11) gives no problem. Hence, in order to prove Theorem 1.1, it is sufficient to prove it for the functional $V_G(\varphi)$ of (2.12). We shall do that by a suitable scale decomposition of the covariance, that we are going to describe.

If $\gamma > 1$ is the scaling parameter, we define, for any integer $h \ge 0$,

$$\theta_h(x_0) = \theta(x_0) u(\gamma^h x_0)\,, \quad u(t) = h_0(t) - h_0(\gamma t)\,. \tag{2.13}$$

Since $\sum_{h=0}^\infty \theta_h(x_0) = \theta(x_0) h_0(x_0)$, we have

$$G(\mathbf{x}) = \sum_{h=0}^{\infty} G_h(\mathbf{x})\,, \tag{2.14}$$

where, by (2.6),

$$\begin{aligned} G_h(\mathbf{x}) = {} & \theta_h(x_0) h_1(\vec{x}) e^{\mu x_0} \delta^{(L)}_{x_0/m}(\vec{x}) \\ & - \theta_h(x_0+\beta) h_1(\vec{x}) e^{\mu(x_0+\beta)} \delta^{(L)}_{(x_0+\beta)/m}(\vec{x})\,. \end{aligned} \tag{2.15}$$

It is easy to see that there are two constants A and κ, independent of L, β and h, such that

$$|G_h(\mathbf{x}-\mathbf{y})| \le A\gamma^h e^{-\kappa \gamma^{h/2} d_L(\vec{x},\vec{y})} \big[\theta_h(x_0-y_0) + \theta_h(x_0-y_0+\beta)\big]\,. \tag{2.16}$$

This implies, in particular, that, given two points $\mathbf{x}, \mathbf{y} \in \Lambda$,

$$\begin{aligned} G_h(\mathbf{x}-\mathbf{y}) \ne 0 \Rightarrow {} & \gamma^{-h-1} < x_0 - y_0 < 2\gamma^{-h} \\ & \text{or} \quad -\beta + \gamma^{-h-1} < x_0 - y_0 < -\beta + 2\gamma^{-h}\,, \end{aligned} \tag{2.17}$$

that is $G_h(\mathbf{x}-\mathbf{y})$ can be different from 0 only if the time coordinates x_0 and y_0 are ordered on the interval $[0,\beta]$ thought as a torus, so that x_0 follows y_0 in the positive direction.

Let us now put

$$G_{\le N}(\mathbf{x}) = \sum_{h=0}^{N} G_h(\mathbf{x}) \tag{2.18}$$

and let us call $P_{\le N}(d\psi)$ the gaussian measure with covariance $G_{\le N}(\mathbf{x}-\mathbf{y})$. We shall regularize the functional (2.12), by putting

$$V_G^{(N)}(\varphi) = \log \int P_{\le N}(d\psi) e^{-V(\psi+\varphi)}\,. \tag{2.19}$$

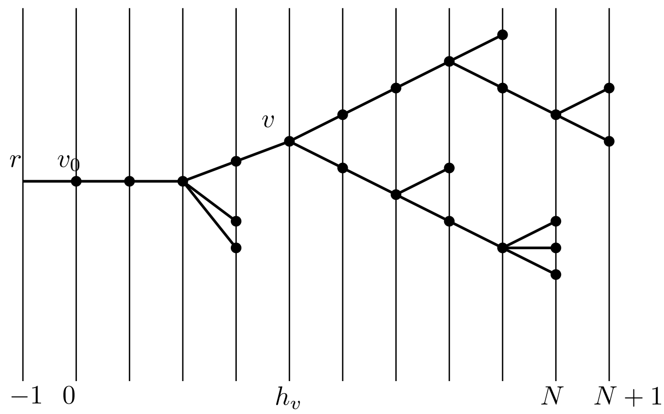

FIGURE 1. An example of tree.

3. The tree expansion

An essential role in our analysis will be played by the tree expansion [1,7], in a form and with notations very similar to those used in [2]; we assume that the reader is enough familiar with this method to allow us to skip many technical details. We start with some definitions and notations.

1) Let us consider the family of all *unlabeled trees* which can be constructed by joining a point r, the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the tree (see Figure 1), so that only one line emerges from the root. The unlabeled trees are partially ordered from the root to the endpoints in the natural way (we shall use the symbol $<$ to denote the order); n will be called the *order* of the unlabeled tree.

   We shall consider also the *labeled trees with cutoff* N (which in general will be simply called trees in the following); they are defined by associating some *labels* with the unlabeled trees, as explained in the following items. We shall denote $\mathcal{T}_n^{(N)}$ the set of labeled trees of order n and cutoff N.
2) Given $\tau \in \mathcal{T}_n^{(N)}$, we associate to each endpoint one of the two terms in the r.h.s. of (1.3) and we shall distinguish the two choices by saying that *the endpoint is of type* λ *or* ν, respectively. We shall call *space-time points* the corresponding integration variables; they will be ordered so that $\mathbf{x}_{2i-1}, \mathbf{x}_{2i}$, $i = 1, \ldots, n_4$, will denote the $2n_4$ space-time points associated to the n_4 endpoints of type λ and $\mathbf{x}_{2n_4+j}$, $j = 1, \ldots, n - n_4$, those associated to the endpoints of type ν. $\underline{\mathbf{x}}$ will be the set of all space-time points. We shall use also the notation $\mathbf{x}_j = (x_{0,j}, \vec{x}_j)$ to denote the time and space components of $\mathbf{x}_j$.
3) We associate to the endpoints $4n_4 + 2n_2 = n_\varphi$ fields, ordered in a fixed arbitrary way; we shall attach a label $f = 1, \ldots, n_\varphi$ to each field to distinguish

them and we shall call I_τ the set of this labels. If $f \in I_\tau$, $\mathbf{x}_f$ and σ_f will denote the space-time and the σ label, respectively, of the corresponding field $\psi^{\sigma_f}_{\mathbf{x}_f}$.

4) We introduce a family of vertical lines, labeled by a *frequency index* h, which takes all the integer values between -1 and $N+1$; the vertical lines are ordered from left to right as the frequency index increases. Furthermore the root must belong to the line with index -1, the endpoints must belong to a line with index ≥ 0 and, finally, any branch point must belong to a vertical line with index larger than -1 and smaller than $N+1$.
5) We call *non trivial vertices* of τ its branch points (this set is empty if $n=1$ and, in this case, there is only one unlabeled tree); we call *trivial vertices* the points where the branches *connecting two non trivial vertices* intersect the family of vertical lines; finally, we call vertices the trivial or non trivial vertices and the endpoints (see the dots in Figure 1).
6) Given a vertex v, we call h_v the frequency index of the vertical line containing it. The first vertex of the tree (having frequency index 0) will be denoted v_0. Given a trivial or non trivial vertex v, s_v will denote the number of lines branching from v (then $s_v = 1$, if v is a trivial vertex). If v is an endpoint and v' is the non trivial vertex immediately preceding it, there is the constraint $h_v = h_{v'} + 1$; this follows from the fact that $G_h(0, \vec{x} - \vec{y}) = 0$.
7) Given a vertex v, we shall call the *cluster of* v the family of space-time points associated to all the endpoints following v, if v is not an endpoint, or v itself, otherwise.
8) Finally, we denote $\mathcal{E}_h$ and $\mathcal{E}^T_h$ the expectation and the truncated expectation, respectively, with respect to the Gaussian measure with covariance G_h.

We can expand the functional (2.19) as

$$V^{(N)}_G(\varphi) = \sum_{k=0}^{\infty} \sum_{n \geq 1} V^{(N)}_{2k,n}(\varphi)\,, \tag{3.1}$$

where $V^{(N)}_{2k,n}(\varphi)$ is the contribution of the terms of order $2k$ in the field and order n in the couplings λ, ν, which will further expanded in the following way. Given a tree $\tau \in \mathcal{T}^{(N)}_n$, we associate to it many different terms, each term being characterized by selecting, for any vertex $v \in \tau$, a subset $P_v \subset I_\tau$, so that the family $\{P_v, v \in \tau\}$ satisfies the following conditions.

1) If v is an endpoint, P_v coincides with the set of fields appearing in the corresponding interaction term.
2) If v is not an endpoint (v not e.p. in the following) and $v^1, \ldots, v^{s_v}$ are the vertices immediately following v, then $P_v \subseteq \bigcup_{i=1}^{s_v} P_{v_i}$ and, if $v > \tilde{v}_0$, the first non trivial vertex of τ, $P_v \neq \emptyset$. Moreover, if we define $Q_{v^i} = P_v \bigcap P_{v^i}$ (so that $P_v = \bigcup_{i=1}^{s_v} Q_{v^i}$), then $P_{v_i} \backslash Q_{v_i} \neq \emptyset$ for any i, if $s_v > 1$.

Let us now define, for any set $P \subset I_\tau$,

$$\varphi(P) = \prod_{f \in P} \varphi^{\sigma_f}_{\mathbf{x}_f}\,, \quad \psi(P) = \prod_{f \in P} \psi^{\sigma_f}_{\mathbf{x}_f}\,. \tag{3.2}$$

One can show that (see [2] for details)

$$V_{2k,n}^{(N)}(\varphi) = \sum_{\tau \in \mathcal{T}_n^{(N)}} \sum_{|P_{v_0}|=2k} \int d\underline{\mathbf{x}}\, \varphi(P_{v_0})\, W(\tau, P_{v_0}, \underline{\mathbf{x}})\,, \tag{3.3}$$

$$W(\tau, P_{v_0}, \underline{\mathbf{x}}) = \Bigg[\prod_{\substack{v>v_0 \\ v \text{ not e.p.}}} \sum_{P_v} \Bigg] W\big(\tau, \{P_v\}, \underline{\mathbf{x}}\big)\,, \tag{3.4}$$

$$\begin{aligned} W\big(\tau, \{P_v\}, \underline{\mathbf{x}}\big) &= (-\nu)^{n_2} \prod_{i=1}^{n_4} \big[-\lambda v(\mathbf{x}_{2i-1} - \mathbf{x}_{2i}) \big] \\ &\quad \cdot \Bigg[\prod_{v \text{ not e.p.}} \frac{1}{s_v!} \mathcal{E}_{h_v}^T \big(\psi(P_{v^1} \backslash Q_{v^1}), \dots, \psi(P_{v^{s_v}} \backslash Q_{v^{s_v}}) \big) \Bigg]\,. \end{aligned} \tag{3.5}$$

Note that there is no explicit dependence on N in $W(\tau, \{P_v\}, \underline{\mathbf{x}})$, so that $V_{2k,n}^{(N)}(\varphi)$ depends on N only because the sum over the trees is restricted to $\mathcal{T}_n^{(N)}$.

4. Proof of Theorem 1.1

By the remark following (2.12), in order to prove the first part of Theorem 1.1, that is the bounds (1.7) and (1.8), it is sufficient to prove that, if we put $\lambda_0 = \max\{|\lambda|, |\nu|\}$,

$$(\beta L^2)^{-1} \int d\underline{\mathbf{x}} \left| \sum_{\tau \in \mathcal{T}_n^{(N)}} \sum_{|P_{v_0}|=2k} W(\tau, P_{v_0}, \underline{\mathbf{x}}) \right| \leq (C_k \lambda_0)^n\,. \tag{4.1}$$

To begin with, we note that, by proceeding as in App. 2 of [2] (where an essential role is played by the Gram–Hadamard inequality, used as in [9]) and by using (2.16), we get

$$\left| \mathcal{E}_h^T \big(\psi(P_1), \dots, \psi(P_s) \big) \right| \leq C^{\sum_j |P_j|} \gamma^{\frac{h}{2} \sum_j |P_j|} \sum_T e^{-\kappa d_T^{(h)}(P_1, \dots, P_s)}\,, \tag{4.2}$$

where T is an *anchored tree graph* between the clusters of space-time points from which the fields labeled by $P_1, \dots, P_s$ emerge; this means that T is a set of lines connecting two points in different clusters, which becomes a tree graph if one identifies all the points in the same cluster. Moreover, if $\mathbf{x}^i$ and $\mathbf{y}^i$, $i = 1, \dots, s$ are the space-time coordinates of the two points connected by the lines belonging to T, we define:

$$d_T^{(h)}(P_1 \dots P_s) = \sum_{i=1}^{s-1} \big(\gamma^h d_\beta(x_0^i, y_0^i) + \gamma^{h/2} d_L(\vec{x}^i, \vec{y}^i) \big)\,. \tag{4.3}$$

Note that, if $s = 1$, the sum over T is void and must be understood as a trivial factor 1.

We can use the inequality (4.2) to bound the r.h.s. of (3.5). We get

$$\big|W\big(\tau,\{P_v\},\underline{\mathbf{x}}\big)\big| \le (C\lambda_0)^n \left[\prod_{i=1}^{n_4} |v(\mathbf{x}_{2i-1}-\mathbf{x}_{2i})|\right] \tag{4.4}$$
$$\cdot \left[\prod_{v \text{ not e.p.}} \frac{1}{s_v!}\gamma^{\frac{h_v}{2}\sum_{i=1}^{s_v}[|P_{v_i}|-|P_v|]} \sum_{T_v} e^{-\kappa d_{T_v}^{(h_v)}(P_{v_1},\ldots,P_{v_{s_v}})}\right],$$

where we used the fact that $\sum_{v \text{ not e.p.}} \sum_{i=1}^{s_v}[|P_{v_i}|-|Q_{v_i}|] \le 2n$.

Let us now observe that, if we select one tree graph T_v for each vertex v, except the endpoints, and we add, for each endpoint of type λ, the line connecting the two corresponding space-time points, we get a spanning tree for the all set of points $\underline{\mathbf{x}}$ associate to τ. We can use this spanning tree (and the corresponding $e^{-\kappa d_{T_v}^{(h_v)}(P_{v_1},\ldots,P_{v_{s_v}})}$ or $v(\mathbf{x}_{2i-1}-\mathbf{x}_{2i})$ factors) to perform the integrations over the space-time points in the usual way, by ordering them in a way suggested by the spanning tree so that each integration involves only one difference of coordinates, except the last one, which gives a volume factor. As shown in App. 3 of [2], the sum over the possible choices of the spanning tree is controlled, up to a C^n constant, by the $1/s_v!$ factors, hence we get:

$$(\beta L^2)^{-1}\int d\underline{\mathbf{x}}\big|W\big(\tau,\{P_v\},\underline{\mathbf{x}}\big)\big| \le (C\lambda_0)^n \|\bar v\|_1^{n_4}$$
$$\cdot \prod_{v \text{ not e.p.}} \gamma^{\frac{h_v}{2}\sum_{i=1}^{s_v}[|P_{v_i}|-|P_v|]}\gamma^{-2h_v(s_v-1)}, \tag{4.5}$$

where $\|\bar v\|_1 = \int d\vec x\, \bar v(\vec x)$. Note that $v(\mathbf{x})$ is not a bounded function, see (1.4); hence, in the previous bound an essential role is played by the fact that all the spanning trees contain all the lines associated to the endpoints of type λ.

Let us now call $n_{4,v}$ and $n_{2,v}$ the number of endpoints of type λ and ν, respectively, following the vertex v, if it is not an endpoint, and n_v their sum. Then, it is easy to check that, if v is not an endpoint, $\sum_{\tilde v\ge v}\sum_{i=1}^{s_{\tilde v}}[|P_{\tilde v_i}|-|P_{\tilde v}|] = 4n_{4,v}+2n_{2,v}-|P_v|$ and that $\sum_{\tilde v\ge v}(s_{\tilde v}-1) = n_{4,v}+n_{2,v}-1$. Then we can rewrite (4.5) as

$$(\beta L^2)^{-1}\int d\underline{\mathbf{x}}\big|W\big(\tau,\{P_v\},\underline{\mathbf{x}}\big)\big| \le (C\lambda_0)^n \prod_{v \text{ not e.p.}} \gamma^{-d(n_{4,v},n_{2,v},|P_v|)}, \tag{4.6}$$

where the *vertex dimension* $d(n_4,n_2,p)$ is defined as

$$d(n_4,n_2,p) = 2(n_4+n_2-1) - \frac{1}{2}(4n_4+2n_2-p) = n_2 - 2 + \frac{p}{2}. \tag{4.7}$$

If, given the tree $\tau \in \mathcal{T}_n^{(N)}$, it turns out that $d(n_{4,v},n_{2,v},|P_v|) > 0$ for any $v > v_0$ which is not an endpoint, then, by proceeding as in §3 of [2], one can perform the sum over the sets $\{P_v, v > v_0\}$ and all the labels of τ of the l.h.s. of (4.1), obtaining a $(C\lambda_0)^n$ bound. Hence, if this property were true for all trees of order n, one could get the bound (4.1), after exchanging the absolute value with

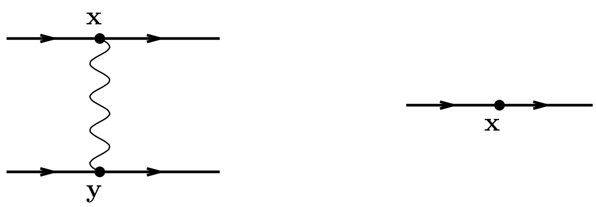

Figure 2. The interaction terms.

the sums in the l.h.s., since the number of unlabeled trees of order n is bounded by 4^n. The dependence on k of the constant C_k is related to the sum over P_{v_0}.

However, one can see immediately that there are trees which contain vertices with zero or negative dimension. Luckily, this can happen only in a few cases:

(a) $|P_v| = 4$, $n_{2,v} = 0$
(b) $|P_v| = 2$, $n_{2,v} = 0, 1$
(c) $|P_v| = 0$, $n_{2,v} = 0, 1, 2$ (this can happen only in the vertices between v_0 and the first non trivial vertex of τ)

Note that this result is strictly related with the space dimension. In fact our procedure works even for $d > 2$, the only difference being that, instead of (4.7), we get $d(n_4, n_2, p) = -n_4(d/2-1) + n_2 - d/2 - 1 + pd/4$, so that, for any value of $|P_v|$, we have vertices with negative dimension, if $n_{4,v}$ is large enough. This makes completely useless the tree expansion. We want now to show that, if $d = 2$, we can improve our bounds, by summing the contributions of some trees in the l.h.s. of (4.1), before exchanging the absolute value with the sums.

Note that, if we expand, given any vertex v, all the truncated expectations in the vertices $\tilde v \geq v$, we get a sum of connected Feynman graphs. These graphs can be drawn in the usual way by representing the two interaction terms of (1.3) as in Figure 2, where the solid lines correspond to fermions and the wiggling lines correspond to the two-body potential; moreover, the arrow on a fermion line coming in (out) the space-time point $\mathbf{x}$ means that the corresponding field has positive (negative) σ-label. It turns out that the graphs must have $|P_v|$ external fermion lines, associated to the field variables in P_v, one internal wiggling line for each endpoint of type λ following v and $\{\sum_{i=1}^{s_{\tilde v}}[|P_{\tilde v_i}| - |P_{\tilde v}|]\}/2$ internal fermion lines, associated to propagators of scales $h_{\tilde v}$, for each $\tilde v \geq v$. Moreover, the set of all fermion lines can be partitioned uniquely in a family of paths, which are simple loops (*fermionic loops* in the following) or open paths connecting two external lines; in all paths the arrows define a precise orientation. However, thanks to the time ordering condition (2.17) (see the remark following it), the value of the graph is 0 if there is a fermionic loop containing a number of space-time points less than $\beta\gamma^{h_v}/2$, as it is easy to check. It follows that, if

$$n_v < n^*_{h_v} \equiv [\beta\gamma^{h_v}/2] \tag{4.8}$$

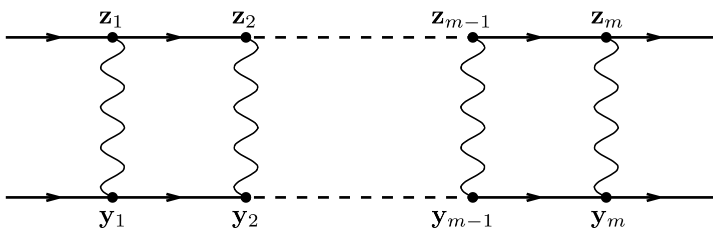

FIGURE 3. A ladder graph.

the value of the graph is exactly zero, if there is some fermionic loop. Moreover, because of the delta function in the time variables of the two-body potential and the fact that the connectivity of the graph can be ensured only through the wiggling lines, the orientation of all open fermion lines must be the same.

The previous considerations imply in particular that, if v satisfies condition (a) above and (4.8), the only graphs giving a non vanishing contribution are those having the structure of a *ladder graph*, that is a graph with two open paths with the same orientation, connected by $n_{4,v}$ non intersecting wiggling lines, see Figure 3. Let us now suppose that v satisfies condition (b) or (c) and (4.8); in this case, it is possible to build graphs without loops only if $n_{4,v} = 0$ and $|P_v| = 2$, so that $n_v = n_{2,v} = 1$, which is impossible if $v > v_0$. Hence a tree τ may give a non vanishing contribution only if there is no trivial or non trivial vertex satisfying (b) or (c), together with (4.8), except the trivial tree with $n_v = n_{2,v} = 1$.

We are now ready to modify our expansion. Given a tree $\tau \in \mathcal{T}_n^{(N)}$ and the sets $\{P_v, v \in \tau\}$, let us consider the family F_τ of all vertices different from v_0, which satisfy condition (a), (4.8) and the further condition that, if $v \in F_\tau$, there is no vertex $\tilde v < v$, except possibly v_0, which also satisfies (a) and (4.8). We want to rebuild our expansion (3.3), by summing all terms associated to trees which are obtained from τ by substituting, for each $v \in F_\tau$, the subtree starting from v with another arbitrary subtree with the same number n_v of endpoints of type λ; we shall call $\mathcal{T}_v^{(N)}$ the family of all such subtrees starting from v. Moreover, we sum also over all the choices of the sets $P_{\tilde v}$, $\tilde v > v$, and we leave unchanged the set P_v, again for each $v \in F_\tau$. In order to describe the result of such operation, we note that, if $v \in F_\tau$ and $m = n_v$,

$$\sum_{\tau^* \in \mathcal{T}_v^{(N)}} \left[\prod_{\substack{\tilde v > v \\ \tilde v \text{ not e.p.}}} \sum_{P_{\tilde v}} \right] \prod_{\substack{\tilde v \ge v \\ \tilde v \text{ not e.p.}}} \frac{1}{s_{\tilde v}!} \mathcal{E}^T_{h_{\tilde v}} \big(\psi(P_{\tilde v^1} \backslash Q_{\tilde v^1}), \ldots, \psi(P_{\tilde v^{s_{\tilde v}}} \backslash Q_{\tilde v^{s_{\tilde v}}}) \big) = \tag{4.9}$$

$$\frac{1}{m!} \mathcal{E}^T_{[h_v, N]} \left(\psi^+_{\mathbf{z}_1} \psi^+_{\mathbf{y}_1} ; \psi^+_{\mathbf{z}_2} \psi^+_{\mathbf{y}_2} \psi^-_{\mathbf{y}_2} \psi^-_{\mathbf{z}_2} ; \ldots , \psi^+_{\mathbf{z}_{m-1}} \psi^+_{\mathbf{y}_{m-1}} \psi^-_{\mathbf{y}_{m-1}} \psi^-_{\mathbf{z}_{m-1}} ; \psi^-_{\mathbf{y}_m} \psi^-_{\mathbf{z}_m} \right) ,$$

where $\mathcal{E}^T_{[h,N]}$ denotes the truncated expectation with respect to the Gaussian measure with covariance $G_{[h,N]} = \sum_{h'=h}^N G_{h'}$ and the fields involved in the truncated expectation are those associated to the endpoints following v, except the fields associate to P_v, that is $\psi^-_{\mathbf{z}_1}$, $\psi^-_{\mathbf{y}_1}$, $\psi^+_{\mathbf{z}_m}$ and $\psi^+_{\mathbf{y}_m}$. This claim easily follows by comparing the multiscale expansion of $\log \int P_{[h_v,N]}(d\psi) \exp V(\psi+\varphi)$ with the ordinary cumulant expansion, $P_{[h,N]}(d\psi)$ being the measure with covariance $G_{[h,N]}$.

Let us now call $\tilde{\mathcal{T}}_n^{(N)}$ the set of trees, whose definition differs from that of $\mathcal{T}_n^{(N)}$ for the following reasons.

1) The number of endpoints is not fixed, but is at most n.
2) Besides the endpoints of type ν and λ (whose number is still denoted by n_2 and n_4, respectively), there are also n_4^* endpoints, to be called of type λ^* and *order* $n_v^* \le n$; if F_τ is the set of endpoints of type λ^*, the following constraint has to be satisfied:
$$n_4 + n_2 + \sum_{v \in F_\tau} n_v^* = n\,. \tag{4.10}$$
3) To each $v \in F_\tau$ we associate a frequency label $h_v \in [1,N]$, satisfying the constraint that $h_v = h_{v'}+1$, if v' is the higher vertex preceding v in the tree.
4) To each $v \in F_\tau$ we associate n_v^* interaction terms of type λ; we shall call $\underline{\mathbf{x}}_v^*$ the set of corresponding $2n_v^*$ space-time points. Moreover, if the set $\underline{\mathbf{x}}_v^*$ is written as in the second line of (4.9) and Figure 3, with $m = n_v^*$, and we put $\underline{\mathbf{x}}_m = \underline{\mathbf{x}}_v^*$, $h = h_v$, we shall define the function

$$\begin{aligned}\tilde{v}^{(N)}_{m,h}(\underline{\mathbf{x}}_m) &= \frac{1}{m!}\prod_{i=1}^m v(\mathbf{y}_i - \mathbf{z}_i) \\ &\cdot \mathcal{E}^T_{[h_v,N]}\Big(\psi^+_{\mathbf{z}_1}\psi^+_{\mathbf{y}_1};\psi^+_{\mathbf{z}_2}\psi^+_{\mathbf{y}_2}\psi^-_{\mathbf{y}_2}\psi^-_{\mathbf{z}_2};\ldots,\psi^+_{\mathbf{z}_{m-1}}\psi^+_{\mathbf{y}_{m-1}}\psi^-_{\mathbf{y}_{m-1}}\psi^-_{\mathbf{z}_{m-1}};\psi^-_{\mathbf{y}_m}\psi^-_{\mathbf{z}_m}\Big)\,.\end{aligned} \tag{4.11}$$

It follows that we can substitute (3.3) with a similar expansion:

$$V^{(N)}_{2k,n}(\varphi) = \sum_{\tau \in \tilde{\mathcal{T}}_n^{(N)}} \sum_{|P_{v_0}|=2k} \int d\underline{\mathbf{x}}\, \varphi(P_{v_0})\, \tilde{W}^{(N)}(\tau, P_{v_0}, \underline{\mathbf{x}})\,, \tag{4.12}$$

$$\tilde{W}^{(N)}(\tau, P_{v_0}, \underline{\mathbf{x}}) = \Bigg[\prod_{\substack{v>v_0\\ v \text{ not e.p.}}} \sum_{P_v}{}^{*}\Bigg] \tilde{W}^{(N)}\big(\tau, \{P_v\}, \underline{\mathbf{x}}\big)\,, \tag{4.13}$$

where $\sum^*_{P_v}$ means that the sum is constrained by the condition $|P_v| \ge 6$, if $n_v < n^*_{h_v}$, and

$$\begin{aligned}\tilde{W}^{(N)}\big(\tau,\{P_v\},\underline{\mathbf{x}}\big) &= (-\nu)^{n_2}\prod_{i=1}^{n_4}\big[-\lambda v(\mathbf{x}_{2i-1}-\mathbf{x}_{2i})\big]\prod_{v\in F_\tau}\Big[(-\lambda)^{n_v^*}\tilde{v}^{(N)}_{n_v^*,h_v}(\underline{\mathbf{x}}_v^*)\Big] \\ &\cdot \Bigg[\prod_{v \text{ not e.p.}} \frac{1}{s_v!}\mathcal{E}^T_{h_v}\big(\psi(P_{v^1}\backslash Q_{v^1}),\ldots,\psi(P_{v^{s_v}}\backslash Q_{v^{s_v}})\big)\Bigg]\,.\end{aligned} \tag{4.14}$$

Note that, if $v \in F_\tau$, the set P_v is not fixed, but has to be chosen in all possible ways among the sets of four fields, a couple with $\sigma(f) = +$ and a couple with $\sigma(f) = -$, that one can select so that the two couples of fields belong to two different interaction terms of type λ, among those associated to v, see item 4) above. For the other vertices, P_v is defined as before.

It is easy to see that $\tilde W^{(N)}(\tau, \{P_v\}, \underline{\mathbf{x}})$ satisfies a bound similar to (4.5). The only difference is that now we do not have a spanning tree and a corresponding decaying factor to perform the integration inside the sets $\underline{\mathbf{x}}_v^*$ associated to the endpoints of type λ^*. However, if we shrink every set $\underline{\mathbf{x}}_v^*$ to a point, we get a spanning tree by choosing, as before, a tree graph T_v or the interaction line for the other vertices; if we use this spanning tree to perform in the usual way the integrations over the differences of coordinates associated to its lines, we are left with the integrations over the sets $\underline{\mathbf{x}}_v^*$, $v \in F_\tau$, with the condition that one of the space-time points from which the four external lines P_v emerge is fixed, in each set. It follows, by observing that, if $v \in F_\tau$, the set P_v can be chosen in $n_v^*(n_v^*-1)$ different ways, and by using the constraint (4.10), that we have to multiply the r.h.s. of (4.5) by $\prod_{v\in F_\tau} ||\tilde v^{(N)}_{n_v^*, h_v}||$, having defined

$$||\tilde v^{(N)}_{m,h}|| = \max_i \sup_{\mathbf{x}_i^*} \int d(\underline{\mathbf{x}}_m \backslash \mathbf{x}_i^*) |\tilde v^{(N)}_{m,h}(\underline{\mathbf{x}}_m)| \,, \tag{4.15}$$

where $\mathbf{x}_i^*$ is one of the four points from which the four external lines of P_v emerge.

On the other hand, the graph expansion of (4.9) gives rise to a sum of $2^{m-2}(m-2)!$ ladder graphs with propagator $G_{[h,N]}$; hence, by using (4.11) and (1.4), we get

$$\begin{aligned} ||\tilde v^{(N)}_{m,h}|| \le 2^m \sup_j \int & \left[\prod_{i=1, i\neq j}^m dz_{i,0} d\vec z_i \right] \left[\prod_{i=1}^m d\vec y_i \right] \left[\prod_{i=1}^m |\bar v(\vec y_i - \vec z_i)| \right] \\ & \cdot \left[\prod_{i=1}^{m-1} |G_{[h,N]}(\mathbf{z}_{i+1} - \mathbf{z}_i)| |G_{[h,N]}(z_{i+1,0} - z_{i,0}, \vec y_{i+1} - \vec y_i)| \right] . \end{aligned} \tag{4.16}$$

By using (2.15) and (2.13), we can easily see that, uniformly in N,

$$\int d\vec x |G_{[h,N]}(x_0, \vec x)| \le C\big[h_0(\gamma^h |x_0|) + h_0(\gamma^h |x_0 + \beta|)\big] \,. \tag{4.17}$$

Therefore, if we integrate in (4.16) first the space coordinates, by using a spanning tree containing all fermion lines and one interaction line, then the time coordinates, by using the fact that $h_0(\gamma^h|x_0|)$ has support in an interval of size γ^{-h}, we get

$$||\tilde v^{(N)}_{m,h}|| \le C^m \gamma^{-h(m-1)} \|\bar v\|_\infty^{m-1} \|\bar v\|_1 \,, \tag{4.18}$$

where $\|\bar v\|_\infty = \sup_{\vec x} |\bar v(\vec x)|$.

The bound (4.18) implies that $\tilde W^{(N)}(\tau, \{P_v\}, \underline{\mathbf{x}})$ satisfies a bound similar to (4.6). We could indeed even improve the bound, by using the $\gamma^{-h(m-1)}$ factor in the r.h.s. of (4.18), but we do not need this improvement. It will be sufficient to use the fact that now the set $\{P_v\}$ has the important property that, if a vertex v has non positive dimension (that is it satisfies either condition (a), (b) or (c) above), then $n_v \ge n^*_{h_v}$, that is n_v has to be very large, if h_v is large. To exploit this property in an efficient way, let us select, given τ, the family R_τ of vertices such that $n_v \ge n^*_{h_v}$ and there is no vertex $\tilde v > v$ with the same property. We note that, for any $\varepsilon > 0$,

$$1 = \prod_{v\in R_\tau} e^{-\varepsilon n^*_{h_v}} e^{\varepsilon n^*_{h_v}} \le e^{\varepsilon n} \prod_{v\in R_\tau} e^{-\frac{\varepsilon\beta}{2}\gamma^{h_v}}\,, \tag{4.19}$$

so that, if we choose $\varepsilon = 2\beta^{-1}$, we get

$$1 \le e^{2\beta^{-1}n}(4e^{-2})^{|R_\tau|} \prod_{v\in R_\tau} \gamma^{-2h_v} \le C^n \prod_{v\in R_\tau} \gamma^{-2h_v}\,. \tag{4.20}$$

By using this bound, we get finally

$$(\beta L^2)^{-1} \int d\underline{\mathbf{x}} \big|\tilde W^{(N)}\big(\tau, \{P_v\}, \underline{\mathbf{x}}\big)\big| \\ \le (C\lambda_0)^n \Bigg[\prod_{v\in R_\tau} \gamma^{-2h_v}\Bigg] \prod_{v \text{ not e.p.}} \gamma^{-d(n_{4,v},n_{2,v},|P_v|)}\,. \tag{4.21}$$

Let us call τ^* the minimal subtree of τ which contains v_0 and the set R_τ of vertices. Our definitions imply that all the vertices of τ which have non positive dimension belong to τ^* and that the elements of R_τ are the endpoints of τ^*. Hence, we can distribute the γ^{-2h_v} factors in the r.h.s. of (4.21) along the paths connecting the vertices in R_τ to v_0, so to increase the dimension of all vertices of τ^* of at least two units. Since $d(n_4, n_2, p)$ is ≥ -1 and grows linearly with p, this is sufficient to control the sum over the set $\{P_v\}$, see remark after (4.7).

In order to complete the proof of Theorem 1.1, we define the constants $p_{n,m}$ and the distributions $W_{2k,n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$ in a way analogous to $p^{(N)}_{n,m}$ and $W^{(N)}_{2k,n,m}(\underline{\mathbf{x}}, \underline{\mathbf{y}})$, that is by using, instead of the expansion (4.12), the similar expansion

$$V_{2k,n}(\varphi) = \sum_{\tau\in\tilde{\mathcal T}_n} \sum_{|P_{v_0}|=2k} \int d\underline{\mathbf{x}}\; \varphi(P_{v_0})\; \tilde W(\tau, P_{v_0}, \underline{\mathbf{x}})\,, \tag{4.22}$$

where $\tilde{\mathcal T}_n = \bigcup_{N\ge 0} \tilde{\mathcal T}^{(N)}_n$ and

$$\tilde W(\tau, P_{v_0}, \underline{\mathbf{x}}) = \Bigg[\prod_{\substack{v>v_0\\ v \text{ not e.p.}}} \sum_{P_v}{}^{*}\Bigg] \tilde W\big(\tau, \{P_v\}, \underline{\mathbf{x}}\big)\,, \tag{4.23}$$

$\tilde W(\tau, \{P_v\}, \underline{\mathbf{x}})$ being the expression obtained from (4.14) by substituting $\tilde v^{(N)}_{n^*_v,h_v}(\underline{\mathbf{x}}^*_v)$ with $\tilde v_{n^*_v,h_v}(\underline{\mathbf{x}}^*_v) = \tilde v^{(\infty)}_{n^*_v,h_v}(\underline{\mathbf{x}}^*_v)$.

In order to prove the bounds (1.9) and (1.10), it is sufficient to prove that there exists a constant δ_N, which goes to 0 as $N \to \infty$, such that

$$a(\beta L^2)^{-1} \int d\underline{\mathbf{x}} \Bigg| \sum_{\tau \in \tilde{\mathcal{T}}_n^{(N)}} \sum_{|P_{v_0}|=2k} \tilde{W}^{(N)}(\tau, P_{v_0}, \underline{\mathbf{x}}) - \sum_{\tau \in \tilde{\mathcal{T}}_n} \sum_{|P_{v_0}|=2k} \tilde{W}(\tau, P_{v_0}, \underline{\mathbf{x}}) \Bigg| \le (C_k \lambda_0)^n \delta_N \,. \quad (4.24)$$

This is indeed an almost immediate consequence of the previous bounds. In fact, we can write the expression inside the modulus in the r.h.s. of (4.24) as the difference between

$$\Delta_{1,N}(\underline{\mathbf{x}}) = \sum_{\tau \in \tilde{\mathcal{T}}_n^{(N)}} \sum_{|P_{v_0}|=2k} \left[\tilde{W}^{(N)}(\tau, P_{v_0}, \underline{\mathbf{x}}) - \tilde{W}(\tau, P_{v_0}, \underline{\mathbf{x}}) \right] \quad (4.25)$$

and

$$\Delta_{2,N}(\underline{\mathbf{x}}) = \sum_{\tau \in \tilde{\mathcal{T}}_n \setminus \tilde{\mathcal{T}}_n^{(N)}} \sum_{|P_{v_0}|=2k} \tilde{W}(\tau, P_{v_0}, \underline{\mathbf{x}}) \,. \quad (4.26)$$

On the other hand, by proceeding as in the proof of (4.18), it is easy to prove that

$$||\tilde{v}_{m,h}^{(N)} - \tilde{v}_{m,h}|| \le C^m \gamma^{-N} \gamma^{-h(m-2)} \|\bar{v}\|_\infty^{m-1} \|\bar{v}\|_1 \,, \quad (4.27)$$

which implies in a simple way that

$$(\beta L^2)^{-1} \int d\underline{\mathbf{x}} |\Delta_{1,N}(\underline{\mathbf{x}})| \le (C_k \lambda_0)^n \gamma^{-N} \,. \quad (4.28)$$

Let us now consider $\Delta_{2,N}(\underline{\mathbf{x}})$. The sum over the trees in the r.h.s. of (4.26) is restricted to trees which have at least one non trivial vertex with frequency index greater than N, say $\bar{v}$. Hence we can extract from the bound of $(\beta L^2)^{-1} \int d\underline{\mathbf{x}}$ $|\tilde{W}(\tau, P_{v_0}, \underline{\mathbf{x}})|$, which is of the form of the r.h.s. in (4.6), a factor $\gamma^{-N/2}$, without affecting the bound of the sum over the trees; in fact this factor can be compensated by lowering of 1/2 the vertex dimension (which is ≥ 1) in all the vertices of the path joining $\bar{v}$ with the root. It follows that

$$(\beta L^2)^{-1} \int d\underline{\mathbf{x}} |\Delta_{2,N}(\underline{\mathbf{x}})| \le (C_k \lambda_0)^n \gamma^{-N/2} \,. \quad (4.29)$$

Therefore, the bound (4.24) is proved with $\delta_N = \gamma^{-N/2}$.

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