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A Cox Process Involved in the Bose–Einstein Condensation

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Abstract. The point process corresponding to the configurations of bosons in standard conditions is a Cox process driven by the square norm of a centered Gaussian process. This point process is infinitely divisible. We point out the fact that this property is preserved by the Bose–Einstein condensation phenomenon and show that the obtained point process after such a condensation occured, is still a Cox process but driven by the square norm of a shifted Gaussian process, the shift depending on the density of the particles. This law provides an illustration of a "super"- Isomorphism Theorem existing above the usual Isomorphism Theorem of Dynkin available for Gaussian processes.

1. Introduction

The point processes corresponding to the spatial configurations of fermions and bosons in standard conditions have been clearly identified (see Macchi [13, 14]) and are usually respectively named fermion point processes and boson point processes. Shirai and Takahashi [16] have given an unified presentation of these two classes of processes by introducing the definition of alpha-permanental (or alpha-determinantal) random point processes. Indeed they have established the existence of random point processes whose Laplace transforms are equal to the power $(-\frac{1}{\alpha})$ of a Fredholm determinant. When $\alpha = -1$ one obtains a determinantal (or a fermion) point process, when $\alpha = 1$ it is a permanental (or a boson) point process.

These results allow to check the property of infinite divisibility of some alpha-permanental point processes for $\alpha > 0$, and in particular of the point process associated to an infinite collection of bosons in an infinite box with a density lower than the Bose–Einstein critical density. We remind that a point process N is infinitely divisible if for every integer k there exists k independent identically distributed point processes $N_{1,k}, N_{2,k}, \dots, N_{k,k}$ such that

$$N \overset{(\text{law})}{=} N_{1,k} + N_{2,k} + \cdots + N_{k,k} . \tag{1.1}$$

Note that a fermion point process can not be infinitely divisible. One can have an intuition of that fact thanks to the "anti-bunching" property of the fermions: two fermions can not be closer than the "correlation length", a given distance whose existence has been assumed theoretically before being recently put in evidence experimentally (see [10] Jeltes et al.). This property makes impossible the realization of (1.1) even for $k = 2$ since two points belonging respectively to the support of $N_{1,2}$ and $N_{2,2}$ can not be closer than the correlation length. The independency of $N_{1,2}$ and $N_{2,2}$ can not afford that. More generally for $\alpha < 0$, alpha-permanental random point processes can not be infinitely divisible.

Coming back to the bosons, one may ask whether a Bose–Einstein condensation would preserve this infinite divisibility property. The answer is affirmative and based on a recent paper of Tamura and Ito [19] who have obtained in a new way the law of the configurations of the particles of an ideal Bosonian gas containing particles in a Bose–Einstein condensation state. We shall analyze in Section 2 their result and show why infinite divisibility is the key to understand the factorization of the Cox process involved in the Bose–Einstein condensation. In Section 3, we show that this factorization provides an illustration of a "super"- Isomorphism Theorem existing above the usual Isomorphism Theorem of Dynkin [3].

2. An infinitely divisible Cox process

Shirai and Takahashi [16] have extended the notions of boson and fermion point processes by introducing the following distributions denoted by $\mu_{\alpha,K}$. The corresponding random point processes are sometimes called alpha-permanental point processes. In the definition below, E is a locally compact Hausdorff space with a countable basis, λ is a nonnegative Radon measure on E, and Q is the space of nonnegative integer-valued Radon measures on E. An operator K on $L^2(E,\lambda)$ is locally bounded if for every compact subset of E, A, the operator P_AKP_A is bounded (P_A denotes the projection from $L^2(E,\lambda)$ to $L^2(A,\lambda)$).

Definition 2.1. For K a locally bounded integral operator on $L^2(E,\lambda)$ and α a fixed number, the distribution $\mu_{\alpha,K}$ on Q satisfies, when it exists

$$\int_Q \mu_{\alpha,K}(d\xi)\exp\big(-\langle\xi,f\rangle\big) = \mathrm{Det}(I+\alpha K_\phi)^{-1/\alpha} \tag{2.1}$$

for every nonnegative measurable function f with compact support on E, K_ϕ stands for the trace class operator defined by

$$K_\phi(x,y) = \sqrt{\phi(x)}K(x,y)\sqrt{\phi(y)}$$

and

$$\phi(x) = 1 - \exp\big(-f(x)\big)\,.$$

The function Det denotes the Fredholm determinant.

For a different presentation of these distributions we refer the reader to the paper of Hough et al. [9]

When $\alpha = 1$, $\mu_{1,K}$ is the distribution of the configurations of a Bosonian gas. Shirai and Takahashi have established sufficient conditions, on the operator K, for the existence of the distribution $\mu_{\alpha,K}$. In particular for K, a locally bounded integral operator on $L^2(E, \lambda)$ and α a fixed positive number, they have shown that if (α, K) satisfies

(B) : *the kernel function of the operator* $J_\alpha = K(I + \alpha K)^{-1}$ *is nonnegative*

then $\mu_{\alpha,K}$ exists and is infinitely divisible.

Note that $\mu_{\alpha,K}$ is infinitely divisible iff $\mu_{n\alpha,K/n}$ exists for every $n \in \mathbb{N}^*$.

In the special case $E = \mathbb{R}^d$, $\alpha = 1$ and J_1 with kernel $J_1(x, y) = \frac{1}{(4\pi\beta)^{d/2}} \exp(-|x - y|^2/4\beta)$ (K is such that $J_1 = K(I + K)^{-1}$), the distribution $\mu_{1,K}$ can be obtained as the limit of the distributions of the positions in $\mathbb{R}^d$ of N identical particles following the Bose–Einstein statistics in a finite box. More precisely, one starts from the following random point measure $\mu_{(L,N)}$ which describes the location of an ideal Bosonian gas, composed of N particles in a volume $V = [-L/2, L/2]^d$ with $d \geq 1$, at a given temperature T

$$\int_{V^N} \mu_{(L,N)}(d\xi) e^{-\langle \xi, f\rangle} = C \int_{V^N} \exp\left(-\sum_{j=1}^{N} f(x_j)\right) per\big(G_L(x_i, x_j)\big)_{1\leq i,j\leq N} dx_1 \dots dx_N$$

where the constant C is equal to $\int_{V^N} per(G_L(x_i, x_j))_{1\leq i,j\leq N} dx_1 \dots dx_N$, G_L denotes the operator $\exp(\beta\Delta_L)$ with $\beta = 1/T$ and Δ_L is the Laplacian under the periodic condition in $L^2(V)$.

As N and V are tending to ∞ with $N/V \to \rho$, $\mu_{(L,N)}$ converges to a limit depending on ρ. Indeed, denoting by ρ_c the critical density $\int_{\mathbb{R}^d} \frac{dx}{(2\pi)^d} \frac{e^{-\beta|x|^2}}{1-e^{-\beta|x|^2}}$ which is finite for $d > 2$, we have

– if $\rho < \rho_c$, then $\mu_{(L,N)}$ converges to μ_{1,K_ρ}, where $K_\rho = \ell(\rho)J_1(I - \ell(\rho)J_1)^{-1}$ and $\ell(\rho)$ is a positive constant depending on ρ.

  This last result provides a justification to the fact that μ_{1,K_ρ} is the distribution of the configurations of an ideal Bosonian gas. The next result is more illuminating; indeed, in the case $d > 2$ and

– if $\rho \geq \rho_c$, then $\mu_{(L,N)}$ converges to a random point process with a distribution ζ given by

$$\int_Q \zeta(d\xi) e^{-\langle \xi, f\rangle} = Det(I + K_\phi)^{-1} \exp\left\{ -(\rho - \rho_c)\left(\sqrt{1 - e^{-f}}, (I + K_\phi)^{-1}\sqrt{1 - e^{-f}}\right)\right\} \quad (2.2)$$

where $K = J_1(I - J_1)^{-1}$.

The physical explanation of this split convergence, actually a phase transition, is due to the fact that when the density of the gas becomes higher than ρ_c, a certain proportion of the particles tends to lower the density by reaching the lowest level of energy. This phenomenon, called the Bose–Einstein condensation, predicted by Einstein in 1925, is intensively studied today especially since this phenomenon has been experimentally obtained (for $d = 3$ of course) in 1995 by a team at JILA. It is interesting to see that the Bose–Einstein condensation phenomenon provides an illustration in the case $d = 3$ of a mathematical physics result available for any dimension d greater than 3.

These results have been established by many authors. In particular they are consequences of the works of Bratteli and Robinson [2] (see Theorem 5.2.32 Chap. 5 p. 69) and of Fichtner and Freudenberg [7]. The way Tamura and Ito have obtained these results in [18] and [19], deserves a special attention because they need neither quantum field theories nor the theory of states on the operator algebras, but mostly an integral formula due to Vere-Jones [21]. Further, Tamura and Ito have actually done more than (2.2). In [19] their proof is based on the following theorem.

Theorem A. *Let K be a locally bounded symmetric integral operator on $L^2(E, \lambda)$ such that $(1, K)$ satisfies condition* **(B)** *and*

$$\int_E J_1(x, y)\lambda(dy) \le 1 \quad \lambda(dx) \ a.e. \tag{2.3}$$

Then for every $r > 0$, there exists a unique random measure with distribution ζ_r on Q such that for every non-negative measurable function f on E

$$\int_Q \zeta_r(d\xi)e^{-\langle\xi,f\rangle} = exp\left\{ - r\left(\sqrt{1 - e^{-f}}, (I + K_\phi)^{-1}\sqrt{1 - e^{-f}}\right)\right\} \tag{2.4}$$

where $(.,.)$ denotes the inner product of $L^2(E, \lambda)$.

Tamura and Ito's result generates several natural questions:

- In (2.2) the distribution of the configurations of the particles is, by Theorem A, the convolution of two distributions: $\mu_{1,K} * \zeta_{\rho-\rho_c}$. It is tempting to imagine that $\mu_{1,K}$ corresponds to the fraction of the particles with level of energy greater than 0 and that $\zeta_{\rho-\rho_c}$ corresponds to the particles that did "coalesce" (i.e. without kinetic energy or similarly in a quantic state equal to 1.) Indeed $\ell(\rho)$ is a continuous function of ρ on $(0, \rho_c]$ that takes the value 1 at ρ_c. Hence the distribution of the configurations of particles with density ρ_c has the distribution $\mu_{1,K}$. The problem is to know whether the configurations of particles with 0 kinetic energy are independent of the configurations of moving particles? In what follows we shall answer this question.
- The assumption of condition (B) in Theorem A, makes $\mu_{1,K}$ infinitely divisible. Since $\zeta_r = (\zeta_{r/n})^{*n}$, ζ_r is infinitely divisible as well. Consequently the distribution ζ given by (2.2) is also infinitely divisible. Moreover, thanks to Theorem A, the distribution ζ exists for any K such that $(1, K)$ satisfies

condition (B) and (2.3). We therefore obtain a family of infinitely divisible distributions – who are they? Their characterization is the theme of Theorem 2.3.

- Besides, in their paper [16] (Theorem 6.12) Shirai and Takahashi have obtained a factorization involving $\mu_{\alpha,K}$ for (α, K) satisfying condition (B) (see (2.13) below). In the case $\alpha = 1$, is this factorization connected to (2.2)? We will show that the answer is affirmative and that they are both direct consequences of the infinite divisibility of $\mu_{\alpha,K}$.

To analyze further the results of Tamura and Ito, we will use the notion of a Cox process.

Definition 2.2. A Cox process is a Poisson point process with a random intensity σ on the space of Radon measures on E. Its distribution Π_σ satisfies therefore

$$\int_Q \Pi_\sigma(d\xi)\exp\big(-\langle\xi, f\rangle\big) = I\!E\left[\exp\left(-\int_E \big(1-e^{-f(x)}\big)\sigma(dx)\right)\right],$$

for every nonnegative measurable function f with compact support on E.

We shall work mostly with Cox processes with random intensity $\psi(x)\lambda(dx)$ where $(\psi(x), x \in E)$ is a positive process such that $I\!E(\psi(x))$ is a locally bounded function of x. Such a Cox process is said to be driven by (ψ, λ). We denote its distribution by $\Pi_{\psi,\lambda}$ or Π_ψ when there is no ambiguity about the measure λ.

If ψ is equal to $\frac{1}{2}\eta^2$ with η real valued centered Gaussian process with covariance $(K(x, y), x, y \in E)$ then

$$\int_Q \Pi_{(\psi,\lambda)}(d\xi)\exp\big(-\langle\xi, f\rangle\big) = I\!E\left[\exp\left(-\int_E (1-e^{-f(x)})\psi(x)\lambda(dx)\right)\right]$$

for every positive function f with compact support. Using the Dominated Convergence Theorem one shows then that $\Pi_{(\psi,\lambda)} = \mu_{\frac{1}{2},K}$. Note that a priori the couple $(\frac{1}{2}, K)$ does not satisfy condition (B) of Shirai and Takahashi.

Remark 2.2.1. The infinite divisibility of a Cox process with distribution Π_ψ is not equivalent to the infinite divisibility of the process ψ. Of course, the infinite divisibility of ψ implies the infinite divisibility of Π_ψ, but the converse is not true. This fact has been stated in 1975 by Kallenberg [11] (Ex. 8.6, p. 58 Chap. 8 – see also Shanbhag and Westcott (1977) [15]). Condition (B) of Shirai and Takahashi allows to put in evidence examples of squared Gaussian processes η^2 which are not infinitely divisible although the Cox process with distribution Π_{η^2} is infinitely divisible. For this purpose, consider the example of the ideal Bose gas, where $J_1(x, y) = \frac{1}{(4\pi\beta)^{d/2}}\exp(-|x-y|^2/4\beta)$. By condition(B), $\mu_{1,K}$ is infinitely divisible. Further note that, $\mu_{1,K} = \Pi_{\frac{1}{2}\eta^2,\lambda} * \Pi_{\frac{1}{2}\eta^2,\lambda}$, where $(\eta_x, x \in \mathbb{R}^d)$ is a centered Gaussian process with covariance $(K(x, y), x, y \in \mathbb{R}^d)$ and λ is the Lebesgue measure on $\mathbb{R}^d$. Hence $\Pi_{\frac{1}{2}\eta^2,\lambda}$ is infinitely divisible. Using now Bapat's characterization of Gaussian processes with infinitely divisible square [1], we can easily choose

$x, y, z \in \mathbb{R}^d$ such that $(\eta_x^2, \eta_y^2, \eta_z^2)$ is not infinitely divisible. Indeed, if $y_i z_i < 0$ for every $1 \le i \le d$ then $|y-z|^2 > |y-x|^2 + |z-x|^2$ for x in $\mathbb{R}^d$ with $|x|$ small enough. The matrix $(K(a,b), a, b \in \{x, y, z\})$ has only positive coefficients and $(K^{-1}(a,b), a, b \in \{x, y, z\})$ has at least one off-diagonal positive coefficient. Consequently this last matrix can not be an M-matrix. In Lemma 2.5, we shall characterize the infinitely divisible Π_ψ where ψ is a nonnegative process.

To state the main result of this section, Theorem 2.3, we need the following notation. Let $(\eta_x, x \in E)$ be a centered Gaussian process with covariance $(K(x,y), x, y \in E)$ and a a point not in E. Extend the process η to $E \cup \{a\}$ by setting $\eta_a = 0$ and $K(a,a) = K(x,a) = K(a,x) = 0$ for all $x \in E$. Let $(\psi_x, x \in E)$ be a centered Gaussian process with covariance $(K(x,y)+1, x, y \in E)$. One can similarly extend it to $E \cup \{a\}$ in the obvious manner, namely, $E(\psi_x \psi_a) = 1$ for all x in E. For a measure λ on E and $\epsilon > 0$, define

$$\lambda_\epsilon = \lambda + \epsilon \delta_a$$

where δ_a is the Dirac measure with mass at a. For a positive random process $(\phi_x, x \in E \cup \{a\})$on $E \cup \{a\}$, we denote by $\Pi_{\phi, \lambda_\epsilon}$ the distribution of a Cox process with random intensity $\phi_x \lambda_\epsilon(dx)$ on $E \cup \{a\}$. Note that for the process η above, we have: $\Pi_{\eta^2, \lambda_\epsilon} = \Pi_{\eta^2, \lambda}$. Without ambiguity Q will denote the space of nonnegative integer-valued Radon measures on $E \cup \{a\}$. With these notation, we are now ready to state Theorem 2.3. Its proof is deferred to the end of this section.

Theorem 2.3. *Let $(\eta_x, x \in E)$ be a centered Gaussian process with covariance $(K(x,y), x, y \in E)$. Let $(\psi_x, x \in E)$ be a centered Gaussian process with covariance $(K(x,y)+1, x, y \in E)$. Assume that the distribution $\Pi_{\frac{1}{2}\eta^2, \lambda}$ is infinitely divisible, then the following five points are equivalent.*

(i) *The distribution $\Pi_{\frac{1}{2}(\eta+c)^2, \lambda}$ is infinitely divisible for every constant c in $\mathbb{R}$.*

(ii) *The distribution $\Pi_{\frac{1}{2}\psi^2, \lambda_\epsilon}$ is infinitely divisible for every $\epsilon > 0$.*

(iii) *For every $r > 0$ there exists a random measure with distribution ν_r on Q such that*

$$\Pi_{\frac{1}{2}(\eta+r)^2, \lambda} = \Pi_{\frac{1}{2}\eta^2, \lambda} \ * \ \nu_r \, . \tag{2.5}$$

Moreover the distribution ν_r satisfies

$$\int_Q \nu_r(d\xi) e^{-\langle \xi, f \rangle} = exp\left\{ -\frac{1}{2} r^2 \left(\sqrt{1-e^{-f}}, (I + K_\phi)^{-1} \sqrt{1-e^{-f}} \right) \right\}, \tag{2.6}$$

where the inner product is taken with respect to the measure λ.

(iv) *For every $\epsilon > 0$ and $r > 0$, there exists a random measure with distribution ν_r on Q satisfying* (2.5) *and* (2.6) *but for the measure λ_ϵ instead of the measure λ.*

(v) *The distribution $\Pi_{\frac{1}{2}(\eta+c)^2, \lambda_\epsilon}$ is infinitely divisible for every constant c in $\mathbb{R}$ and every $\epsilon > 0$.*

Remark 2.3.2. Note that the distribution ν_r of (iii) is not necessarily the distribution of a Cox process. In Section 3, we will see that, this makes precisely the difference between the infinite divisibility of Π_ϕ and the infinite divisibility of ϕ. We shall see also that this remark extends from squared Gaussian processes to nonnegative processes.

We are now in position to analyze further the results of Tamura and Ito. Recall that in the case $\rho > \rho_c$, the case when a Bose–Einstein condensation occurs, the obtained limit ζ is equal to $\mu_{1,K} * \zeta_{\rho-\rho_c}$, where ζ_r is defined by Theorem A. First we note that by Theorem 2.3, the existence of $\zeta_{\rho-\rho_c}$ for every $\rho > \rho_c$ is equivalent to the infinite divisibility of $\mu_{1,K+1}$. To check directly this last property we can, for example, verify that $(1, K+1)$ satisfies condition (B). Indeed we have the following general result which does not require that K be symmetric.

Proposition 2.4. *Let K be an integral operator on $L^2(E,\lambda)$ such that $(1,K)$ satisfies condition* (B) *and*

$$\int_E J_1(y,x)\lambda(dy) \le 1 \quad \lambda(dx) \ a.e. \tag{2.7}$$

and

$$\int_E J_1(x,y)\lambda(dy) \le 1 \quad \lambda(dx) \ a.e. \tag{2.8}$$

Then $(1, K+1)$ satisfies condition (B).

Proof of Proposition 2.4. Set $\overline{J}_1 = (K+1)(I+K+1)^{-1}$. We have to show that $\overline{J}_1$ has a nonnegative kernel. We denote by $\mathbb{1}$ the integral operator on $L^1(E,\lambda)$ with the kernel identically equal to 1. We then have $I+K+\mathbb{1} = (I-J_1)^{-1} + \mathbb{1} = (I+\mathbb{1}(I-J_1))(I-J_1)^{-1}$, which leads to $\overline{J}_1 = (K+\mathbb{1})(I-J_1)\left(I+\mathbb{1}(I-J_1)\right)^{-1}$. Let f be an nonnegative element of $L^2(E,\lambda)$. We set $g = (I+\mathbb{1}(I-J_1))^{-1}f$ and similarly $f = g + \mathbb{1}(I-J_1)g$. Note that $\mathbb{1}(I-J_1)g$ is a constant function that we denote by $c(g)$. We claim that $c(g) \ge 0$. Indeed, $c(g) = \mathbb{1}(I-J_1)(f-c(g))$ and by (2.7) the integral operator $\mathbb{1}(I-J_1)$ has a positive kernel. Hence if $c(g) < 0$ then $f - c(g)$ is a positive function and therefore so is $c(g) = \mathbb{1}(I-J_1)(f-c(g))$, which leads to a contradiction. Thus, $c(g) \ge 0$.

We now have:

$$\overline{J}_1 f = (K+\mathbb{1})(I-J_1)g = J_1 g + \mathbb{1}(I-J_1)g = J_1\big(f-c(g)\big) + c(g) = J_1 f + (I-J_1)c(g)\,.$$

By (2.8) $(I-J_1)c(g) \ge 0$ and thanks to (B), $J_1 f$ is nonnegative. Consequently $\overline{J}_1 f$ is non negative as well. □

Theorem 2.3 and Proposition 2.4 prove Theorem A of Tamura and Ito.

Restricting our attention to the case of ideal Bosonian particles, we see that, with the notations of Theorem 2.3, (2.2) becomes

$$\zeta = \mu_{1,K} * \nu_{\sqrt{2(\rho-\rho_c)}}\,. \tag{2.9}$$

Can we interpret $\nu_{\sqrt{2(\rho-\rho_c)}}$ as the law of the configurations of the particles with 0 kinetic energy and density $\rho-\rho_c$? These particles are at temperature $T = 1/\beta$ and the distribution ζ depends on T. Now imagine that we can lower the temperature T

to 0, we then have $\rho_c \to 0$ and for any positive function f with compact support, we easily obtain

$$\int_Q \zeta(d\xi)e^{-\langle \xi, f\rangle} \longrightarrow \exp\left\{-\rho \int_{\mathbb{R}^d} (1-e^{-f(x)})dx\right\}.$$

The obtained limit is the distribution of a Poisson point process with uniform intensity ρdx on $\mathbb{R}^d$. But at temperature $T = 0$, all the particles are at 0 level of kinetic energy. Hence this limit $\Pi_{\rho,dx}$ is the distribution of the configurations of particles, with density ρ, at 0 level of energy and temperature 0. This has been already established (differently) by Goldin et al. [8]. Now remember that once the 0 state of kinetic energy is reached, the particles don't move anymore, hence the law of their configurations should not vary when the temperature goes down. But obviously $\nu_{\sqrt{2(\rho-\rho_c)}}$ is different from $\Pi_{(\rho-\rho_c),dx}$. Consequently the answer to the above question is negative. This implies that the presence of particles in the Bose–Einstein condensation state has an influence on the configurations of the still moving particles.

Moreover, (2.9) can be rewritten as

$$\zeta = \Pi_{\frac{1}{2}\eta^2} \ * \ \nu_{\sqrt{\rho-\rho_c}} \ * \ \Pi_{\frac{1}{2}\eta^2} \ * \ \nu_{\sqrt{\rho-\rho_c}}$$

which leads to

$$\zeta = \Pi_{\psi} \tag{2.10}$$

with $(\psi_x, x \in E) \overset{(\text{law})}{=} (\frac{1}{2}(\eta_x + \sqrt{\rho-\rho_c})^2 + \frac{1}{2}(\tilde{\eta}_x + \sqrt{\rho-\rho_c})^2, x \in E)$ and η and $\tilde{\eta}$ two independent centered Gaussian processes with covariance $(K(x,y), x, y \in \mathbb{R}^d)$. Under this writing it appears that ζ is the distribution of a Cox process. Similarly (2.9) leads to

$$\zeta = \Pi_{\frac{1}{2}(\eta+\sqrt{2(\rho-\rho_c)})^2} \ * \ \Pi_{\frac{1}{2}\eta^2}\,. \tag{2.11}$$

Under this last form, one can provide a physical interpretation in terms of fields (instead of particles). We thank Yvan Castin from Laboratoire Kaestler–Brossel for the following explanation. The Bosonic field $(\phi(x), x \in \mathbb{R}^d)$ satisfies $\phi(x) = \phi_0 + \phi_e(x)$, where ϕ_0 is a (spatially) uniform field corresponding to the condensated particles and $(\phi_e(x), x \in \mathbb{R}^d)$ is the field corresponding to the excited particles. This last field ϕ_e is a complex Gaussian field: $\phi_e = \frac{1}{\sqrt{2}}(\eta + i\tilde{\eta})$. Besides ϕ_0 is taken to be the constant $\sqrt{\rho-\rho_c}$. The real component of ϕ_e can interfere with ϕ_0 and provides the part $\Pi_{\frac{1}{2}(\eta+\sqrt{2(\rho-\rho_c)})^2}$ of ζ, while the imaginary component of ϕ_e does not interfere with ϕ_0 and its contribution to ζ is the same as for the gas without condensation $\Pi_{\frac{1}{2}\eta^2}$.

To prove Theorem 2.3 we will use the following characterization of the infinite divisible random measure. According to Theorem 11.2 (Chap. 11, p. 79) in Kallenberg's book [11], a random measure with distribution ζ is infinitely divisible iff for almost every x,w.r.t. $\mathbb{E}(\zeta)$, there exists a random measure with distribution μ_x

on Q such that

$$\zeta^x = \zeta * \mu_x \tag{2.12}$$

where $(\zeta^x, x \in E)$ denotes the Palm measures of ζ.

In the special case of a couple (α, K) satisfying condition (B), we hence obtain the existence of μ_x such that

$$\mu^x_{\alpha,K} = \mu_{\alpha,K} * \mu_x \tag{2.13}$$

which is precisely the factorization obtained by Shirai and Takahashi. Note that it is really an immediate consequence of the infinite divisibility of $\mu_{\alpha,K}$.

We are going to make use of Kallenberg's Theorem (2.12) to characterize the infinitely divisible Π_ψ.

Lemma 2.5. *Let Π_ψ be the distribution of a Cox process directed by a positive process $(\psi_x, x \in E)$ with respect to λ. Assume that $\mathbb{E}(\psi_x)$ is a locally bounded function of x. For each b in E such that $\mathbb{E}(\psi_b) > 0$, denote by $(\psi^{(b)}(x), x \in E)$ the process ψ under $\mathbb{E}[\frac{\psi(b)}{\mathbb{E}(\psi(b))};\ .]$. The Palm measure at b of Π_ψ, denoted by Π^b_ψ, admits the following factorization for $\mathbb{E}(\psi_x)\lambda(dx)$ almost every b*

$$\Pi^b_\psi = \Pi_{\psi^{(b)}} \ * \ \delta_b$$

where δ_b is the Dirac point mass at b.

The distribution Π_ψ is infinitely divisible iff for almost every b w.r.t. $\mathbb{E}(\psi_x)\lambda(dx)$, there exists a random measure with distribution μ_b such that

$$\Pi_{\psi^{(b)}} \ * \ \delta_b = \Pi_\psi \ * \ \mu_b \, .$$

Proof of Lemma 2.5. For every nonnegative function f on E, we have:

$$\int_Q \Pi_\psi(d\xi) e^{-\langle \xi, f\rangle} = \mathbb{E}\left[\exp\left(-\int_E (1 - e^{-f(x)})\psi(x)\lambda(dx)\right)\right]. \tag{2.14}$$

Call X the Cox process with distribution Π_ψ, then X admits a first moment measure M on $\mathcal{B}(E)$ defined by $\mathbb{E}(X(A)) = M(A) = \mathbb{E}(\int_A \psi(x)\lambda(dx))$, for every $A \in \mathcal{B}(E)$. Let $(\Pi^x_\psi, x \in E)$ be the family of Palm measures of Π_ψ, we define $\tilde{\Pi}^x_\psi$ by $\Pi^x_\psi = \tilde{\Pi}^x_\psi \ * \ \delta_x$. This means that $(\tilde{\Pi}^x_\psi, x \in E)$ satisfies

$$\int_Q \Pi_\psi(d\xi) \int_E \xi(dx) u(\xi, x) = \int_E M(dx) \int_Q \tilde{\Pi}^x_\psi(d\xi) u(\xi + \delta_x, x)\, .$$

As a consequence of this disintegration formula, we have for any f and g nonnegative functions on E with support in a compact set A

$$-\frac{d}{dt}\int_Q \Pi_\psi(d\xi) e^{-\langle \xi, f+tg\rangle}|_{t=0} = \int_E g(x) \int_Q \tilde{\Pi}^x_\psi(d\xi) e^{-\langle \xi+\delta_x, f\rangle} M(dx)$$

which thanks to (2.14) leads to

$$\int_E g(x)e^{-f(x)}\lambda(dx)\mathbb{E}\left[\psi(x)\exp\Big(-\int_R(1-e^{-f(y)})\psi(y)\lambda(dy)\Big)\right]$$
$$= \int_E g(x)\int_Q \tilde{\Pi}^x_\psi(d\xi)e^{-\langle\xi,f\rangle}e^{-f(x)}\mathbb{E}\big(\psi(x)\big)\lambda(dx)\,.$$

Consequently $\mathbb{E}[\psi(x)]\lambda(dx)$ a.e.

$$\int_Q \tilde{\Pi}^x_\psi(d\xi)e^{-\langle\xi,f\rangle} = \mathbb{E}\left[\frac{\psi(x)}{\mathbb{E}(\psi(x))}\exp\Big(-\int_R(1-e^{-f(y)})\psi(y)\lambda(dy)\Big)\right],$$

and

$$\tilde{\Pi}^x_\psi = \Pi_{\psi(x)}\,.$$

Lemma 2.5 now follows from (2.12), the infinite divisibility of Π_ψ, and the definition of $\tilde{\Pi}^x_\psi$. □

Proof of Theorem 2.3. Let N be a standard Gaussian variable independent of η. Then $\eta+N$ is a centered Gaussian process with covariance $K+1$ and we may take $\psi=\eta+N$.

(ii) ⇒ (iii) Assume that $\Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon}$ is infinitely divisible for every $\epsilon>0$.

Denote by $\Pi^x, x\in E\cup\{a\}$ the Palm measures of $\Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon}$. According to Lemma 2.5 there exists $\mathbb{E}(\eta_x+N)^2\lambda_\epsilon(dx)$ almost every x, a random measure with distribution μ_x on Q such that

$$\Pi^x = \Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon} * \mu_x\,.$$

Since $\lambda_\epsilon(\{a\})=\epsilon>0$, we have

$$\Pi^a = \Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon} * \mu_a\,.$$

We also have

$$\Pi^a = \tilde{\Pi}^a * \delta_a$$

where $\tilde{\Pi}^a$ is the law of a Cox process with intensity $\frac{1}{2}(\eta+N)^2$ under $\mathbb{E}(N^2,\ .\ )$ with respect to λ_ϵ. Consequently, we obtain

$$\tilde{\Pi}^a * \delta_a = \Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon} * \mu_a\,. \tag{2.15}$$

For a fixed positive constant $r>0$, the finite-dimensional Laplace transforms of the process $\frac{1}{2}(\eta+r)^2$ are given by

$$\mathbb{E}\left(\exp\left\{-\frac{1}{2}\sum_{i=1}^n \alpha_i(\eta_{x_i}+r)^2\right\}\right) = |I+\alpha K|^{-1/2}\exp\left\{-\frac{1}{2}r^2 1^t(I+\alpha K)^{-1}\alpha 1\right\}$$

for every $x_1, x_2, \dots, x_n$ in $E \cup \{a\}$ where 1 is the n-dimensional column vector of 1's and 1^t is its transpose. Consequently for every nonnegative function f on $E \cup \{a\}$

$$\int_Q \Pi_{\frac{1}{2}(\eta+r)^2,\lambda_\epsilon}(d\xi)e^{-\langle\xi,f\rangle} = I\!E\left[\exp\left\{-\frac{1}{2}\int_{E\cup\{a\}}(1-e^{-f(y)})\eta_y^2\lambda_\epsilon(dy)\right\}\right]\exp\left\{-\frac{1}{2}r^2F(f,K)\right\} \quad (2.16)$$

with $F(f,K) = (\sqrt{1-e^{-f}}, (I+K_\phi)^{-1}\sqrt{1-e^{-f}})$, where the inner product is with respect to λ_ϵ. Note that

$$F(f,\eta) = \int_E \sqrt{1-e^{-f(x)}}\left((I+K_\phi)^{-1}\sqrt{1-e^{-f}}\right)(x)\lambda(dx) + \epsilon(1-e^{-f(a)})\,. \quad (2.17)$$

We obtain

$$\int_Q \Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon}(d\xi)e^{-\langle\xi,f\rangle} = I\!E\left[\exp\left\{-\frac{1}{2}\int_{E\cup\{a\}}(1-e^{-f(y)})\eta_y^2\lambda_\epsilon(dy)\right\}\right] I\!E\left(\exp\left\{-\frac{1}{2}N^2F(f,K)\right\}\right) \quad (2.18)$$

and similarly

$$\int_Q \tilde{\Pi}^a(d\xi)e^{-\langle\xi,f\rangle} = I\!E\left[\exp\left\{-\frac{1}{2}\int_{E\cup\{a\}}(1-e^{-f(y)})\eta_y^2\lambda_\epsilon(dy)\right\}\right] I\!E\left[N^2\exp\left\{-\frac{1}{2}N^2F(f,K)\right\}\right]. \quad (2.19)$$

Now, making use of (2.18) and (2.19), equation (2.15) gives

$$\int_Q \mu_a(d\xi)e^{-\langle\xi,f\rangle} = I\!E\left[N^2\exp\left\{-\frac{1}{2}N^2F(f,K)\right\}\right] I\!E\left(\exp\left\{-\frac{1}{2}N^2F(f,K)\right\}\right)^{-1} \mathrm{e}^{-f(a)}$$

which thanks to elementary computations on the standard Gaussian law leads to

$$\int_Q \mu_a(d\xi)e^{-\langle\xi,f\rangle} = \big(1+F(f,K)\big)^{-1}\,\mathrm{e}^{-f(a)} = I\!E[\mathrm{e}^{-F(f,K)T}]\,\mathrm{e}^{-f(a)}$$

where T is an exponential variable with parameter 1 independent of η. Multiplying then each member of the above equation by $\int_Q \Pi_{\frac{1}{2}\eta^2}(d\xi)e^{\langle\xi,f\rangle}$, we obtain

$$\int_Q \mu_a(d\xi)e^{-\langle\xi,f\rangle}\int_Q \Pi_{\frac{1}{2}\eta^2}(d\xi)e^{-\langle\xi,f\rangle} = \int_Q \Pi_{\frac{1}{2}\eta^2}(d\xi)e^{-\langle\xi,f\rangle}\,I\!E[\mathrm{e}^{-F(f,K)T}]\,\mathrm{e}^{-f(a)}$$

which implies

$$\mu_a \ * \ \Pi_{\frac{1}{2}\eta^2} = \Pi_{\frac{1}{2}(\eta+\sqrt{2T})^2,\lambda_\epsilon} \ * \ \delta_a \,.$$

Now note that $\Pi_{\frac{1}{2}\eta^2}$ does not charge configurations including the site a, hence there exists a distribution $\tilde{\mu}_a$ such that $\mu_a = \tilde{\mu}_a * \delta_a$ and we finally obtain

$$\tilde{\mu}_a \ * \ \Pi_{\frac{1}{2}\eta^2} = \Pi_{\frac{1}{2}(\eta+\sqrt{2T})^2,\lambda_\epsilon} \,.$$

Denote by X_a, Y_{η^2} and $Y_{(\eta+\sqrt{2T})^2}$ the random measures corresponding respectively to the distributions $\tilde{\mu}_a$, $\Pi_{\frac{1}{2}\eta^2}$ and $\Pi_{\frac{1}{2}(\eta+\sqrt{2T})^2,\lambda_\epsilon}$. We then have

$$X_a \ + \ Y_{\eta^2} \overset{(\text{law})}{=} Y_{(\eta+\sqrt{2T})^2} \,.$$

In particular, since $Y_{\eta^2}(\{a\}) = 0$, we have $X_a(\{a\}) \overset{(\text{law})}{=} Y_{(\eta+\sqrt{2T})^2}(\{a\})$. It follows that

$$\big(X_a \ + \ Y_{\eta^2}, X_a(\{a\})\big) \overset{(\text{law})}{=} \big(Y_{(\eta+\sqrt{2T})^2}, Y_{(\eta+\sqrt{2T})^2}(\{a\})\big) \,.$$

Now, thanks to (2.17), we know that $Y_{(\eta+\sqrt{2T})^2}(\{a\}) = N_{\epsilon T}$ where $(N_t, t \geq 0)$ is a Poisson process independent of $(Y_{(\eta+\sqrt{2T})^2}{}_{|E}, T)$. Similarly $X_a(\{a\}) = N'_{\epsilon T_a}$ where T_a is an exponential variable with parameter 1, and $(N'_t, t \geq 0)$ is a Poisson process independent of $((X_a + Y_{\eta^2})_{|E}, T_a)$. Moreover, since $X_a(\{a\})$ is independent of Y_{η^2}, we may take T_a independent of Y_{η^2}. Hence

$$\big((X_a \ + \ Y_{\eta^2})_{|E}, N'_{\epsilon T_a}\big) \overset{(\text{law})}{=} (Y_{(\eta+\sqrt{2T})^2}{}_{|E}, N_{\epsilon T})$$

which implies that for every nonnegative measurable function f with compact support on E and every $\lambda > 0$

$$\mathbb{E}\big[e^{-\langle (X_a \ + \ Y_{\eta^2})_{|E}, f\rangle} \ e^{-\lambda N'_{\epsilon T_a}}\big] = \mathbb{E}\big[e^{-\langle Y_{(\eta+\sqrt{2T})^2}{}_{|E}, f\rangle} \ e^{-\lambda N_{\epsilon T}}\big]$$

which leads to

$$\mathbb{E}\big[e^{-\langle (X_a \ + \ Y_{\eta^2})_{|E}, f\rangle} \ e^{-(1-e^{-\lambda})\epsilon T_a}\big] = \mathbb{E}\big[e^{-\langle Y_{(\eta+\sqrt{2T})^2}{}_{|E}, f\rangle} \ e^{-(1-e^{-\lambda})\epsilon T}\big] \,.$$

Since the above is true for every $\epsilon > 0$, we obtain

$$(X_{a|E} \ + \ Y_{\eta^2}, T_a) \overset{(\text{law})}{=} (Y_{(\eta+\sqrt{2T})^2}{}_{|E}, T)$$

which leads to

$$(X_{a|E} \ |T_a = r) \ + \ Y_{\eta^2} \overset{(\text{law})}{=} Y_{(\eta+\sqrt{2r})^2}{}_{|E}$$

for almost every $r > 0$. In terms of distribution, this means that there exists a random measure on E with distribution ν_r satisfying

$$\nu_r \ * \ \Pi_{\frac{1}{2}\eta^2} = \Pi_{\frac{1}{2}(\eta+r)^2,\lambda} \,.$$

With the above equation and (2.16) for λ instead of λ_ϵ, we obtain

$$\int_Q \nu_r(d\xi) e^{-\langle \xi, f\rangle} = \exp\left\{-\frac{1}{2} r^2 \left(\sqrt{1-e^{-f}}, (I + K_\phi)^{-1} \sqrt{1-e^{-f}}\right)\right\}$$

where the inner product is with respect to the measure λ. We use now the result contained in Exercise 5.1 p. 33 Chap. 3 in Kallenberg's book [11], to check that for any sequence $(r_n, n \in \mathbb{N})$ of rational numbers converging to a given r, the sequence (ν_{r_n}) converges to a limit distribution satisfying both (2.5) and (2.6) for the measure λ. Hence (iii) is established for every $r > 0$. Since for every real r, $(\eta + r)^2 \overset{(\text{law})}{=} (\eta - r)^2$, (iii) is obtained for every real r.

(iii) $\Rightarrow$ (i) By assumption $\Pi_{\frac{1}{2}\eta^2,\lambda}$ is infinitely divisible. Since, $\nu_r = (\nu_{r/\sqrt{n}})^{*n}$, ν_r is infinitely divisible. Hence $\Pi_{\frac{1}{2}(\eta+r)^2,\lambda}$ is infinitely divisible for every r in $\mathbb{R}$.

(i) $\Rightarrow$ (v) Assume that $\Pi_{\frac{1}{2}(\eta+r)^2,\lambda}$ is infinitely divisible for every constant r. We have

$$\begin{aligned}
&\int_Q \Pi_{\frac{1}{2}(\eta+r)^2,\lambda_\epsilon}(d\xi) e^{-\langle \xi, f\rangle} \\
&\quad = \mathbb{E}\left[\exp\left\{\frac{1}{2}\int_{E\cup\{a\}} (1 - e^{-f(y)})(\eta_y + r)^2 \lambda_\epsilon(dy)\right\}\right] \\
&\quad = \mathbb{E}\left[\exp\left\{\frac{1}{2}\int_E (1 - e^{-f(y)})(\eta_y + r)^2 \lambda(dy)\right\}\right] \ \exp\left\{-\frac{1}{2}(1 - e^{-f(a)})\epsilon r^2\right\}
\end{aligned}$$

hence

$$\Pi_{\frac{1}{2}(\eta+r)^2,\lambda_\epsilon} = \Pi_{\frac{1}{2}(\eta+r)^2,\lambda} \ * \ \Pi_{\frac{1}{2}r^2,\epsilon\delta_a}\,.$$

As the convolution of two infinitely divisible distributions, $\Pi_{(\eta+r)^2,\lambda_\epsilon}$ is infinitely divisible too for every $\epsilon > 0$.

(v) $\Rightarrow$ (iv) We keep the notation of the proof of "(ii) $\Rightarrow$ (iii)".

$$\begin{aligned}
&\int_Q \Pi_{\frac{1}{2}(\eta+r)^2,\lambda_\epsilon}(d\xi) e^{-\langle \xi, f\rangle} \\
&\qquad = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\int_E (1 - e^{-f(y)})\eta_y^2 \lambda(dy)\right\}\right] \ \exp\left\{-\frac{1}{2}r^2 F(f,K)\right\}.
\end{aligned}$$

For every integer n and every r, $[\int_Q \Pi_{\frac{1}{2}(\eta+r)^2,\lambda_\epsilon}(d\xi) e^{-\langle \xi, f\rangle}]^{1/n}$ is still a Laplace transform of a random measure on Q. This is true in particular for $r\sqrt{n}$, but

$$\begin{aligned}
&\left[\int_Q \Pi_{\frac{1}{2}(\eta+r\sqrt{n})^2,\lambda_\epsilon}(d\xi) e^{-\langle \xi, f\rangle}\right]^{1/n} \\
&\qquad = \mathbb{E}\left[\exp\left\{-\frac{1}{2}\int_E (1 - e^{-f(y)})\eta_y^2 \lambda(dy)\right\}\right]^{1/n} \ \exp\left\{-\frac{1}{2}r^2 F(f,K)\right\}.
\end{aligned}$$

Letting n tend to ∞ we obtain, using Kallenberg's result (Exercise 5.1 p. 33 Chap. 3 in [11]) that there exists a limiting distribution with Laplace transform $\exp\{-\frac{1}{2}r^2 F(f,K)\}$.

(iv) $\Rightarrow$ (ii) We start from $\Pi_{\frac{1}{2}(\eta+r)^2,\lambda_\epsilon} = \Pi_{\frac{1}{2}\eta^2,\lambda_\epsilon} \ * \ \nu_r$.

Integrating the above equation with respect to $I\!P(N \in dr)$ (recall that N is a standard Gaussian random variable) we obtain

$$\Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon} = \Pi_{\frac{1}{2}\eta^2,\lambda_\epsilon} \ * \ \nu_N$$

where ν_N denotes the distribution satisfying for every positive function f on $E \cup \{a\}$

$$\int_Q \nu_N(d\xi)e^{-\langle \xi, f\rangle} = \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}r^2 F(f,K)\right\} I\!P(N \in dr)\,,$$

where $F(f,K)$ is defined by (2.16). Since N^2 is an infinitely divisible variable, for every integer n, there exists an i.i.d. sequence $(Z_1, Z_2, \dots, Z_n)$ of positive variables such that $N^2 \overset{(\text{law})}{=} Z_1 + Z_2 + \cdots + Z_n$. Hence we have

$$\begin{aligned}\int_Q \nu_N(d\xi)e^{-\langle \xi, f\rangle} &= I\!E\left[\exp\left\{-\frac{1}{2}N^2 F(f,K)\right\}\right] \\ &= I\!E\left[\exp\left\{-\frac{1}{2}Z_1 F(f,K)\right\}\right]^n \\ &= \left[\int_Q \nu_{\sqrt{Z_1}}(d\xi)e^{-\langle \xi, f\rangle}\right]^n .\end{aligned}$$

It follows that ν_N is infinitely divisible and thus, so is $\Pi_{\frac{1}{2}(\eta+N)^2,\lambda_\epsilon}$ (ie $\Pi_{\frac{1}{2}\psi^2,\lambda_\epsilon}$). □

3. A super-Isomorphism Theorem

The characterization of Gaussian processes $(\eta_x, x \in E)$ such that $(\eta_x^2, x \in E)$ is infinitely divisible is an old question that has been first raised up by Paul Lévy [12]. Several answers have been given since (see [5] for an extended bibliography of the subject). For example, in Remark 2.2.1 we have used Bapat's criteria [1] for centered Gaussian processes. In [4], we have characterized the Gaussian processes $(\eta_x, x \in E)$ such that $(\frac{1}{2}(\eta_x + r)^2, x \in E)$ is infinitely divisible for every real constant r: a Gaussian process has such a property iff its covariance is equal to the Green function of a recurrent Markov process X killed at the first hitting time of a given value a. The Markov process X is independent of η. This condition translates into the following factorization result. For every real r

$$\left(\frac{1}{2}(\eta_x + r)^2, x \in E\right) \overset{(\text{law})}{=} \left(\frac{1}{2}\eta_x^2 + L^x_{\tau_r}, x \in E\right) \tag{3.1}$$

where $(L^x_t, x \in E, t \geq 0)$ is the local time process of X and $\tau_r = \inf\{s \geq 0 : L^a_s > \frac{1}{2}r^2\}$. This identity is a so-called "Isomorphism Theorem" which has been established in [6]. It is a variant of the first Isomorphism Theorem established by Dynkin [3]. who drew inspiration from Symanzik [17].

With the notation of Theorem 2.3, (3.1) is equivalent to

$$\Pi_{\frac{1}{2}(\eta+r)^2,\lambda} = \Pi_{\frac{1}{2}\eta^2,\lambda} \ * \ \Pi_{L_{\tau_r},\lambda}$$

for every λ element of Q.

One can hence formulate the exact difference between the infinite divisibility of $(\eta+r)^2$ for every r and the infinite divisibility of $\Pi_{\frac{1}{2}(\eta+r)^2,\lambda}$ for every r: *$(\eta+r)^2$ is infinitely divisible for every r iff for every λ of Q, $\Pi_{\frac{1}{2}(\eta+r)^2,\lambda}$ is infinitely divisible for every r and ν_r is a Cox process with respect to λ .*

The identity generated by the infinite divisibility property of $\Pi_{\frac{1}{2}(\eta+r)^2,\lambda}$ is an extension of the above Isomorphism Theorem to point processes. It can hence be considered as a "super"- Isomorphism Theorem. We are going now to establish a lemma that will enlarge even more the point of view on the Isomorphism Theorems.

Lemma 3.1. *Let $(\psi_x, x \in E)$ be a positive process. For every a such that $\mathbb{E}(\psi_a) > 0$, denote by $(\psi_x^{(a)}, x \in E)$ the process having the law of $(\psi_x, x \in E)$ under the probability $\frac{1}{\mathbb{E}(\psi_a)}\mathbb{E}(\psi_a, .)$. Then, ψ is infinitely divisible if and only if for every a such that $\mathbb{E}(\psi_a) > 0$, there exists a process $(l_x^{(a)}, x \in E)$ independent of ψ such that*

$$\psi^{(a)} \overset{(\text{law})}{=} \psi + \ l^{(a)} . \tag{3.2}$$

Consider a centered Gaussian process η with a covariance equal to the Green function of transient Markov process. Then according Dynkin's Isomorphism Theorem [3], (3.2) holds for $\psi = \eta^2$. Hence Dynkin's Isomorphism Theorem can be seen as a characterization of the infinite divisibility property of η^2. Lemma 3.1 connects every infinitely divisible positive process $(\psi_x, x \in E)$ to a family of processes $((l_x^{(a)}, x \in E), a \in E)$. The identity (3.4) below relates the different $l^{(a)}$ as a varies. Similarly to Dynkin's Isomorphism Theorem or to (3.1), Lemma 3.1 relates path properties of ψ to path properties of $l^{(a)}$. For example, one immediately obtains that the continuity of ψ implies the continuity of $(l_x^{(a)}, x \in E)$ for every a. Lemma 3.1 can hence be seen as an "Isomorphism Theorem".

More generally, we can regard Lemma 2.5 as a super-Isomorphism Theorem characterizing the infinite divisibility of a given Cox process. When the considered Cox process has an infinitely divisible intensity ψ, the corresponding super Isomorphism Theorem is just the "super" identity existing above (3.2).

Proof of Lemma 3.1. If ψ is infinitely divisible then for every $x = (x_1, x_2, \dots, x_n) \in E^n$, there exists ν_x a Levy measure on $\mathbb{R}^n$ such that $\int_{\mathbb{R}_+^n} 1 \wedge |y| \nu_x(dy) < \infty$ and for every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ in $\mathbb{R}_+^n$

$$\mathbb{E}(e^{-\sum_{i=1}^n \alpha_i \psi_{x_i}}) = \exp\left\{-\int_{\mathbb{R}_+^n} (1 - e^{-(\alpha,y)})\nu_x(dy)\right\} \tag{3.3}$$

where $(\alpha, y) = \sum_{i=1}^n \alpha_i y_i$.

We hence have

$$\mathbb{E}\left(\frac{\psi(x_1)}{\mathbb{E}(\psi(x_1))} e^{-\sum_{i=1}^n \alpha_i \psi_{x_i}}\right) = \mathbb{E}(e^{-\sum_{i=1}^n \alpha_i \psi_{x_i}}) \int_{\mathbb{R}_+^n} \frac{y_1}{\mathbb{E}(\psi(x_1))} e^{-(\alpha,y)} \nu_x(dy)$$

from which it follows that there exists a process $l^{(x_1)}$ independent of ψ such that $\psi^{(x_1)} = \psi + l^{(x_1)}$.

Conversely, assume that for every a there exists a process $l^{(a)}$ satisfying (3.2). By computing the law of ψ under$\mathbb{E}[\psi_a\psi_b,\ .]$, applying the above formula twice, we see that for every couple (a,b) of E, we must have

$$c_a\mathbb{E}\big[l_b^{(a)}F(l_x^{(a)},x\in E)\big]=c_b\mathbb{E}\big[l_a^{(b)}F(l_x^{(b)},x\in E)\big] \tag{3.4}$$

where $c_x=\mathbb{E}(\psi_x)$ for every x in E. To lighten the writing, we set $x_1=a$. We also have

$$\frac{\frac{\partial}{\partial\alpha_1}\mathbb{E}(e^{-\sum_{i=1}^n\alpha_i\psi_{x_i}})}{\mathbb{E}(e^{-\sum_{i=1}^n\alpha_i\psi_{x_i}})}=-\mathbb{E}(e^{-\sum_{i=1}^n\alpha_i l_{x_i}^{(a)}})\mathbb{E}(\psi_{x_1})$$

and hence

$$\begin{aligned}&\mathbb{E}(e^{-\sum_{i=1}^n\alpha_i\psi_{x_i}})\\&=\mathbb{E}(e^{-\sum_{i=1}^n\alpha_i\psi_{x_i}})|_{\alpha_1=0}\exp\left\{-\mathbb{E}(\psi_{x_1})\mathbb{E}\left[\frac{1-e^{-\alpha_1 l_{x_1}^{(a)}}}{l_{x_1}^{(a)}}e^{-\sum_{i=2}^n\alpha_i l_{x_i}^{(a)}}\right]\right\}.\end{aligned} \tag{3.5}$$

We now use (3.5) and an induction argument to end our proof. For $n=1$ it follows immediately from (3.5) that $\mathbb{E}(e^{-\alpha_1\psi_{x_1}})=e^{-\int_0^\infty(1-e^{-\alpha_1y_1})\nu_x(dy_1)}$ where $\nu_x(dy_1)=\frac{\mathbb{E}(\psi_{x_1})}{y_1}\mathbb{P}(l_{x_1}^{(a)}\in dy_1)$.

Assume now that the law of $(\psi_{x_1},\psi_{x_2},\dots,\psi_{x_{n-1}})$ is given by

$$\mathbb{E}(e^{-\sum_{i=1}^{n-1}\alpha_i\psi_{x_i}})=\exp\left\{-\int_{[0,\infty)^{n-1}}(1-e^{-\sum_{i=1}^{n-1}\alpha_iy_i})\nu_x(dy)\right\},$$

with $\nu_x(dy)=\frac{c_{x_1}}{y_1}\mathbb{P}(l_x^{(x_1)}\in dy)$.

By (3.4) $\nu_x(dy)=\int_{\mathbb{R}+}\frac{c_{x_n}}{y_n}\mathbb{P}(l_x^{(x_n)}\in dy,l_{x_n}^{(x_n)}\in dy_n)$, for every x_n distinct from $x_1,x_2,\dots,x_{n-1}$.

Using this in (3.5) we obtain

$$\begin{aligned}&\mathbb{E}(e^{-\sum_{i=1}^n\alpha_i\psi_{x_i}})\\&=\exp\left\{-\int_{[0,\infty)^{n-1}}(1-e^{-\sum_{i=2}^n\alpha_iy_i})\nu_x(dy)\right\}\\&\qquad\exp\left\{-\int_{[0,\infty)^n}(e^{-\sum_{i=2}^n\alpha_iy_i}-e^{-\sum_{i=1}^n\alpha_iy_i})\frac{c_{x_1}}{y_1}\mathbb{P}(l_x^{(x_1)}\in dy_1dy_2\dots dy_n)\right\}\\&=\exp\left\{-\int_{[0,\infty)^n}(1-e^{-\sum_{i=1}^n\alpha_iy_i})\frac{c_{x_1}}{y_1}\mathbb{P}(l_x^{(x_1)}\in dy_1dy_2\dots dy_n)\right\}.\end{aligned}$$

□

We mention that Theorem 2.3 is easily extendable from squared Gaussian processes to permanental processes which are real valued processes characterized by the fact that any joint moment of such a process is equal to a permanent. These processes have been properly defined by Vere-Jones [20].

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