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Infrared Problem and Spatially Local Observables in Electrodynamics

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Abstract. An algebra previously proposed as an asymptotic field structure in electrodynamics is considered in respect of localization properties of fields. Fields are 'spatially local' – localized in regions resulting as unions of two intersecting (solid) lightcones: a future- and a past-lightcone. This localization remains in concord with the usual idealizations connected with the scattering theory. Fields thus localized naturally include infrared characteristics normally placed at spacelike infinity and form a structure respecting Gauss' law. When applied to the description of the radiation of an external classical current the model is free of 'infrared catastrophe'.

1. Introduction

The standard perturbative formulation of quantum electrodynamics, effective as it is in predicting a large scope of experimental results, leaves many fundamental questions concerning the structure of the theory unanswered. Among them is the one that will concern us here, the so called "infrared problem" of electromagnetic interaction (see [1–4] for discussion).

In short, the problem consists of difficulties in the theoretical characterization of charged particles (and charged states) and the construction of a rigourous, and at the same time having compelling quantum-mechanical interpretation, scattering theory in electrodynamics. Its origin is the masslessness of the photon and, consequently, the long-range character of the electromagnetic interaction. In standard perturbational calculations it manifests itself in the appearance of (IR-) divergencies, and at intermediate stages is cured by the removal of the source of trouble: either the photon is given a small mass, or the interaction is switched off in remote regions of spacetime. However, the eventual removal of the regularizing parameter is impossible unless previously an "averaging over unobservable low energy photons" has been carried out. This may be an effective calculational tool, but can hardly be regarded as a real solution of the problem.

Rigourous results on the issue are either incomplete or not widely recognized as physically satisfactory, or both. The superselection structure (related to the long-range behaviour) of prospective quantum electrodynamics based on local observables has been investigated within the general algebraic approach [5], but a consistent construction of a model following the assumptions has not been achieved so far, nor has a charged particle been given a clear-cut, widely accepted characterization, despite some tentative propositions in the direction [3, 6]. At a less mathematically elevated level there have been attempts at incorporating long-range aspects of electromagnetic interaction into the dynamics of asymptotic fields [7, 8]. The results, as assessed by Steinmann in his book on "perturbative axiomatic" electrodynamics, are "too complicated to be of much practical use either for the calculation of cross sections or for providing insights into the underlying structures of the theory" ([4], p. 219), which is an opinion we share. However, whether the approach proposed by Steinmann himself offers a more convincing alternative is open to debate: his analysis involves a complicated transformation from the calculational (indefinite-metric) representation to the physical one, and eventually relies on a somewhat arbitrary redefinition of the cross-section for charged particles.

Summing up, the infrared problem is still open to further study, which is a task we take up here. We further develop the approach to the infrared problem initiated in our earlier papers [9] and papers cited therein; see also [10]); this proposition may be regarded as an attempt at an algebraic formulation of an asymptotic dynamics respecting the long-range nature of electromagnetic field and Gauss' law. The term "asymptotic" is meant here in the sense used in the scattering theory, but the algebra is rather postulated then derived from a complete theory. In [9] the algebra was formulated in terms of independent asymptotic variables (the electromagnetic field part using the null infinity variables similar to those used by other authors before, as discussed in [9]; we do not use the conformal compactification technique of Penrose). Here we extend our discussion by including the issues of spacetime localization of fields and we formulate anew the interpretational issues involved. At the center of our discussion of localization stands an idea of spatial locality, which we postulate and proceed to explain.

We first recall that one of the standard paradigms of quantum field theory is spacetime locality of fundamental observables. This is made most explicit in the algebraic approach based on a net of algebras attached to bounded regions in spacetime [3]; other physical observables may be approximated by those strictly local. Strictly nonlocal effects, such as global charges, are only to be found in the characteristics of the *states* on the algebra of local observables; locally the global charge is irrelevant, one can always place a compensating charge "behind the Moon".

This picture is of course fruitful and seems to be well-founded in the physical practice. However, physics is full of idealizations contradicting, in strict sense, results or practice of experiment, but theoretically helpful. One of such idealizations at the base of the scattering theory (not only within quantum field theory) is the notion of a causally ("in" or "out") asymptotic quantity (asymptotic current, velocity, etc.). Strictly speaking, such a quantity is beyond any physical experiment, which always takes a finite time-span. Nevertheless, it is fruitful and helpful to accept that if we wait long enough then the stabilization of the result of a measurement gives us information on a quantity extending unchanged (in appropriate sense) to infinite past or future.

If we agree on this interpretation we can wonder why not include in the fundamental structure of the theory observables with localization extending to infinite past or future. We explore in this paper consequences of this suggestion. We do not see the need for including in the defining structure of the model localization regions of infinite spatial extension – the observables are "spatially local".

We give an outline of our ideas, ignoring most mathematical subtleties, in Section 2. Section 3 contains the construction of various test functions spaces needed for a rigourous construction of our algebraic model of asymptotic fields, which is discussed in Section 4. Section 5 explains the expected role of the model in the scattering theory and illustrates the idea on a simple example of radiation by a classical current. Appendices contain some technical material. The Lorentz product is denoted by $x \cdot y$ and has signature $(+,-,-,-)$. The spacetime integration element is denoted dx etc.

2. Main ideas

To place our analysis in context we review well-known structures appearing in the standard quantization of free electromagnetic fields. This quantization is best described with the use of a symplectic structure on the space of test fields. Let first our space be the space of free fields satisfying Maxwell equations in the whole spacetime, having compact support on each Cauchy surface. The symplectic form is then supplied by

$$\{F_1, F_2\}_C = \frac{1}{4\pi} \int_\Sigma \big(F_1{}^{ab} A_{2b} - F_2{}^{ab} A_{1b} \big)(x)\, d\sigma_a(x)\,, \tag{2.1}$$

where integration extends over a Cauchy surface Σ and $d\sigma_a$ is the dual integration form on Σ. This symplectic form is hypersurface- and gauge-independent. In particular, if a spacelike hyperplane $t = \text{const}$ in a given Minkowski frame is chosen, and the potentials are in radiation gauge, then the form becomes

$$\big\{\vec{E}_1, \vec{A}_1; \vec{E}_2, \vec{A}_2\big\}_R = \frac{1}{4\pi} \int \big[\vec{E}_1 \cdot \vec{A}_2 - \vec{E}_2 \cdot \vec{A}_1\big](t, \vec{x})\, d^3x\,. \tag{2.2}$$

In this representation the electromagnetic field is represented by a pair of 3-divergence-free fields $\vec{E}$, $\vec{A}$ supplying its initial data, which has the advantage of uniqueness. However, this representation is inconvenient for the discussion of spacetime localization, so another transform of (2.1) is needed.

If the potentials satisfy Lorenz condition, then one easily shows by using Stokes' theorem that (2.1) may be written as

$$\frac{1}{4\pi} \int_\Sigma \big(\nabla^a A_1^b\, A_{2b} - \nabla^a A_2^b\, A_{1b} \big)(x)\, d\sigma_a(x)\,. \tag{2.3}$$

Let now the potential A_1 be represented as the radiation (i.e., as usual, the retarded minus the advanced) potential of a conserved, smooth, compactly supported current J_1, that is

$$A_1^a(x) = 4\pi \int D(x-y) J_1^a(y)\, dy\,, \tag{2.4}$$

where $D(x) = \frac{1}{2\pi}\operatorname{sgn}(x^0)\delta(x^2) = -D(-x)$ is the Pauli–Jordan function. Using this in (2.3) and noting that A_2, being a solution of the wave equation, may be expressed in terms of initial data as

$$A_2^b(y) = \int_\Sigma \big[D(y-x)\nabla^a A_2^b(x) + \nabla^a D(y-x) A_2^b(x)\big]\, d\sigma_a(x)\,, \tag{2.5}$$

we find that (2.1) may be written as

$$\int J_1 \cdot A_2(y)\, dy\,. \tag{2.6}$$

This again is gauge-independent, so the Lorenz condition on A_2 may be dropped. Thus we find, by antisymmetry, that for smooth, compactly supported currents the symplectic structure (2.1) has another representation

$$\{J_1, J_2\} = \frac{1}{2}\int \big[J_1 \cdot A_2 - J_2 \cdot A_1\big](y)\, dy\,, \tag{2.7}$$

where A_i are any potentials of the fields F_i produced as radiation fields by the currents J_i. If A_i are chosen as in (2.4) then for local currents J_i this becomes

$$4\pi \int J_1^b(x) D(x-y) J_{2b}(y)\, dx\, dy\,. \tag{2.8}$$

The symplectic space is now formed by equivalence classes of smooth, compactly supported conserved currents producing the same electromagnetic radiation field.

Finally, one notes, that for each such current J^a there is a compactly supported 2-vector φ^{ab} such that $J^a = 2\nabla_b \varphi^{ab}$ (which is the consequence of the Poincaré lemma for the 3-form dual to J^a). With the use of Gauss' theorem expression (2.6) is rewritten as

$$\int \varphi_1^{ab} F_{2ab}(x)\, dx \tag{2.9}$$

and the symplectic structure acquires still another form

$$\{\varphi_1, \varphi_2\}_\varphi = \frac{1}{2}\int \big[\varphi_1^{ab} F_{2ab} - \varphi_2^{ab} F_{1ab}\big](x)\, dx\,. \tag{2.10}$$

The symplectic space is now formed by equivalence classes of smooth, compactly supported 2-vectors φ producing the same electromagnetic radiation field.

The quantization of the electromagnetic field consists, heuristically, in the replacement of one of the variables in the symplectic form by a "quantum variable" and imposition of the commutation rule:

$$\big[\{F_1, F\}_C, \{F_2, F\}_C\big] = i\{F_1, F_2\}_C\,, \tag{2.11}$$

where F symbolizes the quantum field and F_i are test fields. Any of the discussed above forms of the symplectic structure may be used instead. For the spacetime localization of variables forms (2.7) or (2.10) are suited. We choose the former, as the test fields have very clear physical interpretation in that case: they are conserved electromagnetic currents used to probe the quantum field. The tested field will be denoted $A(J_1)$, and according to (2.7) the intuitive content of this symbol is

$$A(J_1) = \frac{1}{2}\int \big[J_1^a(x)A_a(x) - J^a(x)A_{1\,a}(x)\big]\,dx\,, \tag{2.12}$$

with $J_a(x)$ related to $A_a(x)$ as in (2.4). For local J_1 this may be put in the form given by (2.6):

$$A(J_1) = \int J_1^a(x)A_a(x)\,dx\,, \quad J_1 \ \text{local}\,. \tag{2.13}$$

If this intuition is to be confirmed, the spacetime localization of $A(J_1)$ should be confined to the support of J_1. The quantization condition becomes

$$\big[A(J_1), A(J_2)\big] = i\{J_1, J_2\}\,, \tag{2.14}$$

which is consistent with the assumed localization and relativistic causality. All this is, of course, standard, possibly except for the use of currents J_1 instead of 2-vectors φ_1 as test fields, and consequently $A(J_1)$ instead of $F(\varphi_1) = \int \varphi_1^{ab}(x)F_{ab}(x)\,dx$ as smeared fields.

We want to remind the reader another point concerning the commutation relations (2.14). The symplectic product on the rhs of this equation may be interpreted as $A_2(J_1)$. If one introduces Weyl operators

$$W(J_1) = \exp\big[-iA(J_1)\big]\,, \tag{2.15}$$

then (2.14) may be written as

$$A(J_1)W(J_2) = W(J_2)\big[A(J_1) + A_2(J_1)\big]\,. \tag{2.16}$$

This has a clear interpretation: $W(J_2)$, when acting on a vector state, produces the radiation field due to the current J_2. It is important to note, that the coefficient in the exponent defining $W(J_2)$ is fixed by this relation. The commutation relations may be rewritten as a Weyl algebra:

$$W(J_1)W(J_2) = \exp\left[-\frac{i}{2}\{J_1, J_2\}\right]W(J_1 + J_2)\,. \tag{2.17}$$

The vacuum Fock representation of the commutation relations (2.14) is usually discussed in textbooks as *the* theory of free electromagnetic fields. However, if so, this theory is a rather poor one: it is not wide enough to include infrared-singular fields – with the long-range tail of the type produced in scattering of charged particles. Several kinds of response to this criticism are usually offered. One can argue that all physics is done locally, so even neglecting remote contributions one can approximate every physical situation. This is the point of view adopted in the standard perturbational electrodynamics; it goes together with the pragmatical approach to the charged particle, as mentioned above, and is fundamentally not convincing. If one accepts the real existence of the difficulty one can still retain the scope of algebra (2.14), but explore all "physically reasonable" representations; the problem then is an overabundance of those: every distribution of electromagnetic flux at spatial infinity labels a different representation. We think that it is legitimate to wonder whether, indeed, all those labels are superselected with respect to *all* physically admissible observables. Here is the place, logically, of attempts to introduce some "variables at (spatial) infinity" into electrodynamics, as those of [11] or [12] (see also [10] for an account of Staruszkiewicz's model). These attempts go in a sense against the paradigm of locality. It may be argued that all quantities should be obtainable as limits of local ones, so one should not introduce "by hand" variables escaping such limiting process. We only partly subscribe to this view, inasmuch as arbitrariness is concerned. However, below we want to argue that the interpretation of (2.14) described above naturally leads to the extension of the algebra of commutation relations. But first, as we want to treat some aspects of the interaction of the electromagnetic field with charged matter – for definiteness: electrons and positrons, to fix notation we briefly summarize the quantization of the Dirac field.

For the space of free Dirac fields with compact support on Cauchy surfaces there is an invariant scalar product

$$\langle \psi_1, \psi_2 \rangle_C = \int_\Sigma \overline{\psi_1} \gamma^a \psi_2(x)\, d\sigma_a(x)\,, \tag{2.18}$$

which makes the space into a pre-Hilbert space. Let the field ψ_1 be obtained as

$$\psi_1(x) = \frac{1}{i} \int S(x-y)\chi_1(y)\, dy\,, \tag{2.19}$$

where χ_1 is a smooth, compactly supported 4-spinor field, and $S(x)$ is the standard Green function of the free Dirac field:

$$\begin{aligned} S(x) &= (i\gamma \cdot \partial + m) D(m, x)\,, \\ D(m,x) &= \frac{i}{(2\pi)^3} \int \operatorname{sgn} p^0 \delta(p^2 - m^2) e^{-ip\cdot x} dp\,. \end{aligned} \tag{2.20}$$

Putting (2.19) into (2.18) and using the initial data problem solution for ψ_2:

$$\psi_2(y) = \frac{1}{i} \int_\Sigma S(y-x)\gamma^a \psi_2(x)\, d\sigma_a(x) \tag{2.21}$$

one finds that (2.18) may be written as

$$\int \overline{\chi_1(y)} \psi_2(y)\, dy\,. \tag{2.22}$$

If all free Dirac fields are represented as in (2.19) then our test fields space consists of equivalence classes of smooth compactly supported 4-spinor fields producing the same free Dirac field, with the pre-Hilbert structure given by the product

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{i} \int \overline{\chi_1(x)} S(x-y) \chi_2(y)\, dx\, dy\,. \tag{2.23}$$

The quantized Dirac field is now a quantum variable $\psi(\chi_1)$ depending anti-linearly on χ_1 and satisfying relations

$$\big[\psi(\chi_1), \psi(\chi_2)\big]_+ = 0\,, \quad \big[\psi(\chi_1), \psi(\chi_2)^*\big]_+ = \langle \chi_1, \chi_2 \rangle\,. \tag{2.24}$$

In standard perturbational electrodynamics of interacting fields one starts with the approximation of completely decoupled electromagnetic and Dirac fields. However, one has to remember, that electrodynamics is a constrained theory, in which Gauss' law should hold. Therefore the electromagnetic field of this approximation is qualitatively different from the total field, and the contradiction between locality and Gauss' constraint pervades much of more conscious approaches to the problem, with some complicated and physically not quite uncontroversial transformations "from local to charged sectors" appearing at late stages. We are of the opinion, that one should remove contradiction at the starting point, even at the expense of locality. Our aim is thus the replacement of the uncoupled algebra of the electromagnetic and Dirac fields by some modified asymptotic algebra, taking correctly into account the infrared aspects of the electromagnetic field and Gauss' law. In the asymptotic, "in" or "out" region, the interaction is weak, but the long-range structure should survive.

We return first to the electromagnetic part of the algebra. Recall that $A(J_1)$ is the field tested by a conserved current J_1. It is true that an idealization assuming currents extending to spatial infinity does not seem to be justified. However, other currents of noncompact support are commonplace in physics. More than that, no charged current can be truncated to vanish at late or early times, and this offers an example of a physical quantity, which cannot be approximated by quantities supported locally in spacetime. Currents normally carried by scattered matter may have compact support in spacelike directions, but for timelike directions the typical asymptotic falloff is

$$J(\lambda x) \sim \lambda^{-3} J_{\rm as}(x)\,, \quad x^2 > 0\,, \quad \lambda \to \infty\,, \tag{2.25}$$

which defines $J_{\rm as}$ as a homogeneous of degree -3 asymptote of J, supported inside the future and past lightcones. Moreover, in all physical scattering situations there is

$$x \wedge J_{\rm as}(x) = 0\,, \tag{2.26}$$

which reflects the fact that $J_{\rm as}$ is only due to asymptotically free matter carrying electric charge (no magnetic charges).

Accordingly, we shall admit as the space of test fields a class of conserved currents of the type carried by charged matter, moving freely at early and late times, but having compact support in spacelike directions. As we shall see, this is sufficient to result in the appearance of infrared degrees of freedom in the algebra. This should not come as a surprise if one recalls classical analogues: an infrared singular field, however low-energetic, induces a finite phase change of the wave function of a quantum particle, and also causes an adiabatic shift of the trajectory of a classical charged particle [13,14]. In both cases the size of the effect depends on the infrared characteristic of the electromagnetic field, which shows that physically typical currents are able to "test" those aspects of the field.

We supplement, however, this extension of the test space with the following restriction. The test currents space will slightly differ for the "in" and "out" cases. In the former only the past asymptotics, and in the latter the future asymptotics admitted by (2.25) may be different from zero. This seems rational from the point of view of the regions in which those fields are "tested". On the other hand, as we shall see, each of these classes of currents produces the same class of radiation fields (and the same as that obtained without these restrictions on asymptotics), so in each case the role of $W(J_1)$ is the same. We note that the restrictions automatically imply that the test currents are globally charge-free (but may carry nonzero charges in different asymptotic directions).

With the test space of currents thus extended, we shall have to be more cautious with the symplectic form (2.7). As we shall see, this form is not completely gauge-independent any more on the enlarged space. Therefore from now on we put for A_i in this form radiation potentials obtained according to (2.4). The integrand in (2.7) will be thus specified, and absolutely integrable. On the other hand, the integrand in the double integral of (2.8) will not be absolutely integrable in general, so this form will not be used. After this specification the symplectic form will become unambiguous for charge-free test currents.

We now add charged particles. We assume that the fields interact only weakly, but we want to construct for this situation a closed algebra. The only remnant of the interaction which we take into account is the fact that free charged particles carry their Coulomb fields. Thus the quantum variables will now be interpreted as:

$$\begin{aligned} &\psi(\chi_1) \quad &&\text{– free charged field carrying its Coulomb field,} \\ &A(J_1) \quad &&\text{– total electromagnetic field.} \end{aligned} \tag{2.27}$$

For A and ψ separately we retain previous commutation relations, but the above interpretation implies that these variables should not be assumed to commute with each other, one should expect a relation of the intuitive form $A\,\psi = \psi\;[A+$ Coulomb field carried by $\psi]$. Recall once more that $A(J_1)$ is loosely $\{J_1, J\}$. Moreover, with the use of Fourier-transformed fields

$$\widehat{\chi_1}(p) = \frac{1}{(2\pi)^2}\int \chi_1(x) e^{ip\cdot x} dx \tag{2.28}$$

we have $\psi(\chi_1) = \int \overline{\widehat{\chi_1}}(p)\widehat{\psi}(p)\,dp$, with $\widehat{\psi}(p)$ describing a particle with charge $-e$ moving with the momentum p. Therefore we postulate the relation

$$A(J_1)\widehat{\psi}(p) = \widehat{\psi}(p)\big[A(J_1) - \{J_1, J_{p/m}\}\big]\,, \tag{2.29}$$

where J_v is the current connected with the particle with charge e moving freely with four-velocity v; remember that $\widehat{\psi}$ is supported on the mass hyperboloid. This current is non-radiating, but on the other hand it differs from test currents of the "in" and "out" space by having non-vanishing both asymptotes.

A priori, one has a potential difficulty in the relation (2.29): a particle with fixed momentum is completely delocalized, so there is an ambiguity in the current $J_{p/m}$. It turns out, however, that taking for J_v the current of a point particle moving along any straight line parallel to v one obtains the same value of $\{J_1, J_v\}$, depending only on the long-range tail of the potential produced by J_1. We rewrite (2.29) in the form

$$W(J_1)\psi(\chi_1) = \psi(S_{J_1}\chi_1)W(J_1)\,, \tag{2.30}$$

where

$$\widehat{S_{J_1}\chi_1}(p) = e^{-i\{J_1, J_{p/m}\}}\,\widehat{\chi_1}(p) \quad \text{for} \quad p^2 = m^2\,. \tag{2.31}$$

There are two important points to be made. Recall that the test spinors for the Dirac field were assumed smooth and compactly supported. This turns out to be inconsistent with the above relation – if χ_1 is compactly supported, then $S_{J_1}\chi_1$ is not. Therefore, similarly as in the electromagnetic case, we have to extend the test function space. We shall find that it is possible to choose these fields as compactly supported in spacelike directions and decaying polynomially in the timelike directions; the degree of the decay may be chosen arbitrarily high without changing the element $\psi(\chi_1)$.

The second point concerns gauge invariance. Consider element $W(J_1)$ with J_1 producing pure gauge potential. As stated above, the symplectic form is unambiguously defined for the currents in one of the test classes ("in" or "out"), so this element commutes with electromagnetic field. However, it does not commute with the Dirac field and (2.30) gives in that case

$$W(J_1)\psi(\chi_1) = e^{i\Lambda_1 e}\psi(\chi_1)W(J_1)\,, \quad A_1 \text{ pure gauge}\,, \tag{2.32}$$

where the scalar Λ_1 is determined by the infrared characteristic of the Lorentz potential A_1 produced by J_1. Thus for such J_1 the element $W(J_1)$ should be interpreted as $\exp[-i\Lambda_1 Q]$, with Q – the total charge observable. The quantization of charge in units of e means that $\Lambda_1 e$ should be interpreted as a phase variable, which we shall take into account below.

The elements $W(J_1)$ and $\psi(\chi_1)$ satisfying relations (2.17), (2.24) and (2.30) form our algebra. In the next section we give precise meaning to test fields of these generating elements. Section 4 gives then a precise formulation of the algebra.

3. Test functions spaces

The geometry of the spacetime is given by the affine Minkowski space $\mathcal{M}$. If a reference point O is chosen, then each point P in $\mathcal{M}$ is represented by a vector x in the associated Minkowski vector space M according to $P = O + x$. We mostly keep O fixed and use this representation, but we also remember to control the independence of structures from O. If a Minkowski basis $(e_0, \dots, e_3)$ in M is chosen, then we denote $x = x^i e_i$. We also then use the standard multi-index notation $x^\alpha = (x^0)^{\alpha_0} \dots (x^3)^{\alpha_3}$, $|\alpha| = \alpha_0 + \dots + \alpha_3$, $D^\beta = \partial_0^{\beta_0} \dots \partial_3^{\beta_3}$, where $\partial_i = \partial/\partial x^i$. We associate with the chosen Minkowski basis a Euclidean metric with unit matrix in that basis, and denote $|x|$ the norm of x in that metric. For a tensor or spinor $\mathcal{C}^\infty(\mathcal{M})$ field ϕ we introduce for each $\kappa \geq 0$ and $l = 0, 1, \ldots$ a seminorm

$$\|\phi\|_{\kappa,\,l} = \sup \big(1 + |x|\big)^\kappa |D^\beta \phi_j(x)|\,, \tag{3.1}$$

where supremum is taken over $x \in M$, all β such that $|\beta| = l$, and j running over component indexes of the field in the chosen basis. For fixed l the seminorms form an increasing net over $\kappa \geq 0$. For fixed κ and l seminorms $\|.\|_{\kappa,l}$ and $\|.\|'_{\kappa,l}$ associated with two reference systems $(O, (e_i))$ and $(O', (e'_i))$ are equivalent. For fixed O this follows from the equivalence of norms $|x|$ and $|x|'$ and from the linearity of components transformations, while for the translation $O' = O + a$ from the estimate $(1 + |x - a|)^\kappa \leq \mathrm{const}(\kappa, a)(1 + |x|)^\kappa$. If one denotes $\phi_a(x) = \phi(x - a)$ then it follows

$$\|\phi_a\|_{\kappa,\,l} \leq \mathrm{const}\, \|\phi\|_{\kappa,\,l}\,. \tag{3.2}$$

Seminorms (3.1) are used in this section to construct the spaces $\mathcal{J}_{\mathrm{as}}$ and $\mathcal{K}^\Box$ which will supply test functions for elements $W(J)$ and $\psi(\chi)$ respectively. We also equip the space $\mathcal{J}_{\mathrm{as}}$ with the natural topology, although it will not be used in this paper. Space $\mathcal{K}^\Box$, being a subspace of the Hilbert space of the scalar product (2.23), inherits its topology. The reference for inductive limit spaces are the books [15] and [16].

3.1. Spaces $\boldsymbol{\mathcal{S}_{\kappa+}}$

Consider the space $\mathcal{C}^\infty(\mathcal{M})$ of fields of a given geometric type – not to burden notation this type will be kept implicit. For each $\kappa > 0$ we define the subspace

$$\mathcal{S}_\kappa = \big\{\phi \in \mathcal{C}^\infty \mid \|\phi\|_{\kappa+l,\,l} < \infty,\, l = 0, 1, \ldots\big\}\,. \tag{3.3}$$

With the topology determined by the family of seminorms which define them, these spaces are locally convex, Fréchet spaces, independent of the choice of a reference system $(O, (e_i))$. We denote the topology of $\mathcal{S}_\kappa$ by $\mathfrak{T}_\kappa$. For each κ the net of spaces $\mathcal{S}_{\kappa+\epsilon}$, $\epsilon \in (0, 1)$, is decreasing, so the union

$$\mathcal{S}_{\kappa+} = \bigcup_{0<\epsilon<1} \mathcal{S}_{\kappa+\epsilon} \tag{3.4}$$

is a vector space, a subspace of $\mathcal{S}_\kappa$. For $\epsilon > \epsilon'$ the natural embedding $\mathcal{S}_{\kappa+\epsilon} \mapsto \mathcal{S}_{\kappa+\epsilon'}$ is continuous. Denote by $\mathfrak{T}_{\kappa+}$ the strongest locally convex topology on $\mathcal{S}_{\kappa+}$ in which all natural embeddings $\mathcal{S}_{\kappa+\epsilon} \mapsto \mathcal{S}_{\kappa+}$ are continuous. It is easy to see that this topology is stronger than the topology induced on $\mathcal{S}_{\kappa+}$ by the topology $\mathfrak{T}_\kappa$. Therefore $\mathfrak{T}_{\kappa+}$ is Hausdorff and we can conclude that the space $(\mathcal{S}_{\kappa+}, \mathfrak{T}_{\kappa+})$ is the topological inductive limit of the spaces $(\mathcal{S}_{\kappa+\epsilon}, \mathfrak{T}_{\kappa+\epsilon})$. The relations between the spaces and topologies are summarized by

$$\mathcal{S}_{\kappa+\epsilon} \subset \mathcal{S}_{\kappa+} \subset \mathcal{S}_\kappa\,, \quad \mathfrak{T}_{\kappa+\epsilon} > \mathfrak{T}_{\kappa+} > \mathfrak{T}_\kappa\,, \tag{3.5}$$

where the sign $>$ means that the topology to the left is stronger than the topology induced by the one to the right.

3.2. Spaces $\mathcal{S}^{\kappa}_{\kappa+}$

Consider the homogeneity operator $H = x\cdot\partial$ and denote $H_\kappa = H + \kappa\,\mathrm{id}$ for $\kappa > 0$. If $H_\kappa f(x) = 0$ then f is homogeneous of degree $-\kappa$, so H_κ is injective on $\mathcal{C}^\infty$. One has $|D^\beta H_\kappa\phi(x)| \leq \sum_{i=0}^{3}|x^i||\partial_i D^\beta\phi(x)| + (\kappa+|\beta|)|D^\beta\phi(x)|$, so for $\phi\in\mathcal{S}_{\kappa+\epsilon}$ there is

$$\|H_\kappa\phi\|_{\kappa+\epsilon+l,\,l} \leq \mathrm{const}\,\big(\|\phi\|_{\kappa+\epsilon+l+1,\,l+1} + \|\phi\|_{\kappa+\epsilon+l,\,l}\big)\,. \tag{3.6}$$

Conversely, let now $\psi\in\mathcal{S}_{\kappa+\epsilon}$ and define

$$\phi(x) = \int_0^1 u^{\kappa-1}\psi(ux)du\,. \tag{3.7}$$

Using the bounds on $D^\beta\psi(x)$ one finds that differentiation may be pulled under the integral sign

$$D^\beta\phi(x) = \int_0^1 u^{\kappa+|\beta|-1}[D^\beta\psi](ux)\,du\,, \tag{3.8}$$

and then it is easily seen that $H_\kappa\phi = \psi$. The estimation of the integral gives

$$|D^\beta\phi(x)| \leq \mathrm{const}\|\psi\|_{\kappa+\epsilon+|\beta|,|\beta|}\big(1+|x|\big)^{-\kappa-|\beta|}\,, \tag{3.9}$$

so we find

$$\|\phi\|_{\kappa+l,\,l} \leq \mathrm{const}\,\|H_\kappa\phi\|_{\kappa+\epsilon+l,\,l}\,. \tag{3.10}$$

Furthermore, define

$$\phi_{\mathrm{as}}(x) = \int_0^\infty u^{\kappa-1}\psi(ux)du\,, \tag{3.11}$$

which is homogeneous of degree $-\kappa$ and $\mathcal{C}^\infty$ outside $x = 0$. Then

$$D^\beta[\phi-\phi_{\mathrm{as}}](x) = -\int_1^\infty u^{\kappa+|\beta|-1}[D^\beta\psi](ux)\,du\,. \tag{3.12}$$

Estimating the integral one finds that for $|x|\geq 1$ we have

$$|x|^{\kappa+\epsilon+|\beta|}\big|D^\beta[\phi-\phi_{\mathrm{as}}](x)\big| \leq \mathrm{const}\|H_\kappa\phi\|_{\kappa+\epsilon+|\beta|,\,|\beta|}\,. \tag{3.13}$$

It follows that

$$\phi_{\mathrm{as}}(x) = \lim_{R\to\infty} R^\kappa\phi(Rx)\,, \tag{3.14}$$

and this asymptote is independent of the choice of the central point O. With the use of the estimates (3.10) and (3.13) one finds that in the special case of vanishing asymptote we have a bound stronger than (3.10):

$$\|\phi\|_{\kappa+\epsilon+l,\,l} \leq \mathrm{const}\,\|H_\kappa\phi\|_{\kappa+\epsilon+l,\,l} \quad \text{iff} \quad \phi_{\mathrm{as}} = 0\,. \tag{3.15}$$

The estimates (3.6) and (3.10) imply that

$$H_\kappa\mathcal{S}_{\kappa+\epsilon} \subset \mathcal{S}_{\kappa+\epsilon} \subset H_\kappa\mathcal{S}_\kappa\,, \tag{3.16}$$

so H_κ^{-1} maps $\mathcal{S}_{\kappa+\epsilon}$ bijectively onto $\mathcal{S}^\kappa_{\kappa+\epsilon} = H_\kappa^{-1}(\mathcal{S}_{\kappa+\epsilon})$, and $\mathcal{S}_{\kappa+}$ onto

$$\mathcal{S}^\kappa_{\kappa+} = \bigcup_{0<\epsilon<1}\mathcal{S}^\kappa_{\kappa+\epsilon}\,. \tag{3.17}$$

We also use H_κ^{-1} to transfer the topological inductive limit structure from $\mathcal{S}_{\kappa+}$ to $\mathcal{S}^\kappa_{\kappa+}$. Thus the topology $\mathfrak{T}^\kappa_{\kappa+}$ of $\mathcal{S}^\kappa_{\kappa+}$ is the inductive limit of topologies $\mathfrak{T}^\kappa_{\kappa+\epsilon}$ of $\mathcal{S}^\kappa_{\kappa+\epsilon}$, which are determined by the seminorms $\|\phi\|^\kappa_{\kappa+\epsilon+l,\,l} = \|H_\kappa\phi\|_{\kappa+\epsilon+l,\,l}$, $l = 0, 1, \ldots$. From (3.16) and the relations between seminorms (3.6), (3.10) and (3.15) we have

$$\mathcal{S}_{\kappa+} \subset \mathcal{S}^\kappa_{\kappa+} \subset \mathcal{S}_\kappa\,, \quad \mathfrak{T}_{\kappa+} \sim \mathfrak{T}^\kappa_{\kappa+} > \mathfrak{T}_\kappa\,; \tag{3.18}$$

here the sign $\sim$ means that the topology to the left is equal to the topology induced by the one to the right. Space $\mathcal{S}_{\kappa+}$ consists of all those functions ϕ in $\mathcal{S}^\kappa_{\kappa+}$ for which $\phi_{\rm as} = 0$, and forms a closed subspace of $\mathcal{S}^\kappa_{\kappa+}$ (if $\phi_\lambda \to \phi$ in $\mathcal{S}^\kappa_{\kappa+}$, then this limit is also achieved in the topology $\mathfrak{T}_\kappa$, which implies that $\phi_{\rm as} = 0$ if $\phi_{\lambda{\rm as}} = 0$).

The topological inductive limit structure summarized in (3.17) and (3.18) is independent of the choice of reference system $(O, (e_i))$. This is immediate for the change of basis. Also, the independence of $\mathcal{S}^\kappa_{\kappa+\epsilon}$ and $\mathcal{S}^\kappa_{\kappa+}$ as sets from the choice of O follows immediately from the independence of $\mathcal{S}_{\kappa+\epsilon}$ from that choice. It remains to be shown that $\mathfrak{T}^\kappa_{\kappa+\epsilon}$ is translationally invariant. This follows from the estimates

$$\|H_\kappa\phi_a\|_{\kappa+\epsilon+l,\,l} \leq \mathrm{const}\, \|H_\kappa\phi\|_{\kappa+\epsilon+l,\,l} + \mathrm{const}\, \|H_\kappa\phi\|_{\kappa+\epsilon+l+1,\,l+1}\,, \tag{3.19}$$

which are obtained by writing $H_\kappa\phi_a(x) = [H_\kappa\phi]_a(x) + a\cdot\partial\phi_a(x)$ and using (3.2) and (3.10).

3.3. Spaces s.l.($\mathcal{S}^\kappa_{\kappa+}$)

Let $\mathcal{O}$ be any open set in $\mathcal{M}$ with the closure $\overline{\mathcal{O}}$. The set of fields in the space $\mathcal{S}^\kappa_{\kappa+}$ with support in $\overline{\mathcal{O}}$ forms a closed subspace equipped with the induced topology; we denote it $\mathcal{S}^\kappa_{\kappa+}(\overline{\mathcal{O}})$. Let $V_\pm$ be the open future (past) lightcones in M. We shall use notation $\mathcal{C}_\pm$ for any of the open sets $\mathcal{C}_\pm = P_\pm + V_\pm$, $P_\pm \in \mathcal{M}$, and we shall also write $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ for any $\mathcal{C}_\pm$ such that $\mathcal{C}_+ \cap \mathcal{C}_- \neq \emptyset$. The family $\mathcal{S}^\kappa_{\kappa+}(\overline{\mathcal{C}})$ with the induced topologies $\mathfrak{T}^\kappa_{\kappa+}(\overline{\mathcal{C}})$ forms an increasing net of locally convex spaces. It is now easy to see that the sum (s.l. stands for "spatially local")

$$\mathrm{s.l.}(\mathcal{S}^\kappa_{\kappa+}) = \bigcup_{\mathcal{C}} \mathcal{S}^\kappa_{\kappa+}(\overline{\mathcal{C}}) \subset \mathcal{S}^\kappa_{\kappa+} \tag{3.20}$$

forms a strict inductive limit and the limit topology s.l.$(\mathfrak{T}^\kappa_{\kappa+})$ satisfies

$$\mathfrak{T}^\kappa_{\kappa+}(\overline{\mathcal{C}}) \sim \mathrm{s.l.}(\mathfrak{T}^\kappa_{\kappa+}) > \mathfrak{T}^\kappa_{\kappa+}\,. \tag{3.21}$$

For each ϕ in this space there is: $\operatorname{supp}\phi_{\rm as} \subseteq \overline{V_+ \cup V_-}$.

Finally, we note that the derivative ∂_i maps continuously $\mathcal{S}_\kappa \mapsto \mathcal{S}_{\kappa+1}$ and $\mathcal{S}^\kappa_{\kappa+\epsilon} \mapsto \mathcal{S}^{\kappa+1}_{\kappa+1+\epsilon}$, while multiplication by x^i maps continuously $\mathcal{S}_{\kappa+1} \mapsto \mathcal{S}_\kappa$ and $\mathcal{S}^{\kappa+1}_{\kappa+1+\epsilon} \mapsto \mathcal{S}^\kappa_{\kappa+\epsilon}$. Then similar continuous connections also take place between pairs of spaces of the type $\mathcal{S}_{\kappa+}$, $\mathcal{S}^\kappa_{\kappa+}$, $\mathcal{S}^\kappa_{\kappa+}(\overline{\mathcal{C}})$ and s.l.$(\mathcal{S}^\kappa_{\kappa+})$.

3.4. Spaces $\mathcal{J}_{\rm in}$ and $\mathcal{J}_{\rm out}$

Choose now $\kappa = 3$ and consider the space s.l.$(\mathcal{S}^3_{3+})$ of vector fields. We denote by $\mathcal{J}_{\rm in}$ ($\mathcal{J}_{\rm out}$) the subspace of fields J which satisfy the following additional conditions:

$$\partial\cdot J = 0\,, \quad x\wedge J_{\rm as} = 0\,, \quad \operatorname{supp} J_{\rm as} \subseteq \overline{V_-}\ (\text{resp. } \overline{V_+})\,, \tag{3.22}$$

(compare (2.25) and (2.26)). Let J_λ be any net in $\mathcal{J}_{\rm in}$ ($\mathcal{J}_{\rm out}$), J an element of s.l.$(\mathcal{S}^3_{3+})$, and $J_\lambda \to J$. The mapping $J \mapsto \partial \cdot J$ is continuous (between suitable spaces – see the end of the last subsection) so the first condition is conserved under the limit and $\partial \cdot J = 0$. As s.l.$(\mathfrak{T}^\kappa_{\kappa+})$ is stronger than the topology induced by $\mathfrak{T}_\kappa$ it is easy to see that the support properties of $(J_\lambda)_{\rm as}$ are conserved under the limit, so J satisfies the third condition. Finally, using the continuity of the mapping $J \mapsto x \wedge J$ and again the conservation of support properties one finds that J satisfies the second condition. Thus $\mathcal{J}_{\rm in}$ and $\mathcal{J}_{\rm out}$ are closed subspaces of s.l.$(\mathcal{S}^3_{3+})$. We shall write $\mathcal{J}_{\rm as}$ for $\mathcal{J}_{\rm in}$ or $\mathcal{J}_{\rm out}$, and we shall also set $J_{\rm in} = J_{\rm as}$ in $\mathcal{J}_{\rm in}$, and $J_{\rm out} = J_{\rm as}$ in $\mathcal{J}_{\rm out}$. We denote by $\mathcal{J}_{\rm as}(\overline{\mathcal{C}})$ the subspace of $\mathcal{J}_{\rm as}$ consisting of currents supported in $\overline{\mathcal{C}}$.

Let $s \in \mathbb{R}$ and l be a future-pointing lightlike vector and choose a region $\mathcal{C}$. It is shown by a straightforward calculation that for $\kappa > 2$

$$\int_{\mathcal{C}} \frac{\delta(s - x\cdot l)}{\big(|x|+1\big)^{\kappa}} dx \leq \frac{\mathrm{const}(\mathcal{C})}{\big(|s|+1\big)^{\kappa-3}}\,, \tag{3.23}$$

where $\delta(.)$ is the Dirac delta function. If vectors l are scaled to $l^0 = 1$ then the bounding constant in the above relation is l-independent. Therefore for $J \in \mathcal{J}_{\rm as}$ the integral

$$V(s,l) = \int J(x)\delta(s - x\cdot l)\, dx \tag{3.24}$$

is absolutely convergent, the function $V(s,l)$ is homogeneous of degree -1: $V(\mu s, \mu l) = \mu^{-1} V(s,l)$ for $\mu > 0$, and if l's are scaled to $l^0 = 1$ then it is bounded. We denote $L_{ab} = l_a\partial/\partial l^b - l_b \partial/\partial l^a$, $X_{ab} = x_a \partial/\partial x^b - x_b\partial/\partial x^a$ and observe that

$$(L_{ab} + X_{ab})\delta(s - x\cdot l) = 0\,, \quad (s\partial_s + x\cdot\partial + 1)\delta(s - x\cdot l) = 0\,. \tag{3.25}$$

Using these identities we find that $V(s,l)$ is infinitely differentiable (outside the vertex of the cone; operators L_{ab} incorporate all intrinsic derivatives in the cone), and we have (dot denotes the derivative ∂/∂_s)

$$sL_{a_1b_1}\dots L_{a_nb_n}\dot{V}_c(s,l) = \int \delta(s - x\cdot l) X_{a_1b_1}\dots X_{a_nb_n} H_3 J_c(x)\, dx\,. \tag{3.26}$$

Let J be in $\mathcal{J}_{\rm as} \cap \mathcal{S}^3_{3+\epsilon}(\overline{\mathcal{C}})$. Then estimating the integrand by

$$|X_{a_1b_1}\dots X_{a_nb_n} H_3 J_c(x)| \leq \mathrm{const} \sum_{l=0}^{n} \|H_3 J\|_{3+\epsilon+l,\,l} (1+|x|)^{-3-\epsilon} \tag{3.27}$$

and using (3.23) we obtain

$$\big|L_{a_1b_1}\dots L_{a_nb_n}\dot{V}(s,l)\big| \leq \mathrm{const}(\mathcal{C}) \sum_{l=0}^{n} \|J\|^3_{3+\epsilon+l,\,l}\big(1+|s|\big)^{-1-\epsilon} \tag{3.28}$$

for l's scaled to $l^0 = 1$. The limits $V(\pm\infty, l)$ are determined by $J_{\rm as}$ and one finds

$$\begin{aligned} &V(-\infty, l) = \int J_{\rm in}(x)\delta(x\cdot l + 1)\, dx\,, \quad V(+\infty, l) = 0 \quad \text{for } J \in \mathcal{J}_{\rm in}\,,\\ &V(-\infty, l) = 0\,, \quad V(+\infty, l) = \int J_{\rm out}(x)\delta(x\cdot l - 1)\, dx \quad \text{for } J \in \mathcal{J}_{\rm out}\,. \end{aligned} \tag{3.29}$$

In addition the following identities are satisfied

$$L_{[ab}V_{c]}(\pm\infty, l) = 0\,, \quad l\cdot V(s, l) = 0\,. \tag{3.30}$$

The first of them is the consequence of the second condition in (3.22). To prove the second one we observe that $\partial_s(s\, l\cdot \dot V(s, l)) = \int \delta(s - x\cdot l) H_4 \partial\cdot J(x)\, dx = 0$ by (3.26) and conservation of J, and then the result follows by (3.29). For currents with non-vanishing both future and past asymptotes $J_{\rm as}$ the rhs of the second equation in (3.30) is the total charge.

The radiation potential produced by the current $J \in \mathcal{J}_{\rm as}$ is completely determined by $\dot V(s, l)$ according to the formula [9]

$$A(x) = -\frac{1}{2\pi}\int \dot V(x\cdot l, l)\, d^2 l\,, \tag{3.31}$$

which follows from the representation $D(x) = -(1/8\pi^2)\int \delta'(x\cdot l)\, d^2 l$. Here $d^2 l$ is the invariant measure on the set of null directions: we remind the reader that if $f(l)$ is homogeneous of degree -2 then the integral

$$\int f(l)\, d^2 l = \int f(1, \vec l)\, d\Omega(\vec l)\,, \tag{3.32}$$

where $d\Omega(\vec l)$ is the solid angle measure in the direction of the unit 3-vector $\vec l$, is independent of the choice of Minkowski basis, and satisfies

$$\int L_{ab} f(l)\, d^2 l = 0\,. \tag{3.33}$$

Using (3.31) to express A_1 and A_2 in the symplectic form (2.7) we find that the integrand in that form is absolutely integrable and one obtains

$$\{J_1, J_2\} = \frac{1}{4\pi}\int \big(\dot V_1\cdot V_2 - \dot V_2\cdot V_1\big)(s, l)\, ds\, d^2 l\,, \tag{3.34}$$

so $\mathcal{J}_{\rm as}$ becomes a symplectic space. Moreover, using (3.28) and the fact that one of the asymptotic limits $V(\pm\infty, l)$ vanishes, one easily obtains the estimate

$$\big|\{J_1, J_2\}\big| \leq {\rm const}(\mathcal{C})\, \|J_1\|^3_{3+\epsilon, 0}\, \|J_2\|^3_{3+\epsilon, 0}\,. \tag{3.35}$$

Using properties of V_i it is easy to find the kernel of the symplectic form:

$${\rm Ker}\{.\,,.\} = \{J \in \mathcal{J}_{\rm as} \mid l\wedge V = 0\}\,. \tag{3.36}$$

If J_v is a current of a point particle carrying charge e and moving freely along any world-line parallel to the four-velocity $v = p/m$ then it is easy to find that

$$\{J, J_{p/m}\} = \frac{e}{4\pi}\int \frac{p\cdot \Delta V(l)}{p\cdot l} d^2 l\,, \tag{3.37}$$

where

$$\Delta V(l) = V(+\infty, l) - V(-\infty, l) = \mp V(\mp\infty, l) \tag{3.38}$$

in the "in" and "out" case respectively. By the substitution $p \to w = p + iq$ the rhs of (3.37) may be analytically extended to the domain $\{w \mid p^2 > 0 \text{ or } q^2 > 0\}$; we denote this extension $F_J(w)$. It is shown in Appendix A that there exist continuous limit functions on M: $F_{J\pm}(p) = \lim_{\lambda\searrow 0} F_J(p \pm \lambda iq)$, where q is a timelike future-pointing vector, with $F_{J\pm}(p) = F_J(p)$ for $p^2 > 0$.

The electromagnetic potential represented by (3.31) is infrared singular (has a spacelike tail of the decay rate of the Coulomb field) if $\Delta V(l) \neq 0$, and this function characterizes this singularity completely. We observe that our form of the symplectic structure remains well-defined for those fields. Also, note that (3.37) vanishes for infrared-regular fields, so $W(J)$ for such currents commutes with charged particle field (see (2.30), (2.31) and below).

3.5. Space $\mathcal{K}^\square$

For the space $\mathcal{C}^\infty(\mathcal{M})$ of fields of given geometric type we define subspaces

$$\begin{aligned} \mathcal{S}^\square_\kappa = \{\phi \in \mathcal{C}^\infty \mid \sup_x \big(1 + |x|\big)^{\kappa+n} |\square^n D^\beta \phi_j(x)| < \infty\,, \\ \forall n = 0, 1, \ldots\,,\ \forall\beta\,,\ \forall j\} \end{aligned} \tag{3.39}$$

(independent of the choice of reference system). We also introduce subspaces $\mathcal{S}^\square_\kappa(\overline{\mathcal{C}})$ of functions with support in $\overline{\mathcal{C}}$ and the algebraic inductive limit space

$$\text{s.l.}(\mathcal{S}^\square_\kappa) = \bigcup_{\mathcal{C}} \mathcal{S}^\square_\kappa(\overline{\mathcal{C}})\,. \tag{3.40}$$

Consider the space of 4-spinor fields of the type $\mathcal{K}^\square = \text{s.l.}(\mathcal{S}^\square_5)$; we shall also write $\mathcal{K}^\square(\overline{\mathcal{C}}) = \mathcal{S}^\square_5(\overline{\mathcal{C}})$. The Fourier transforms of fields in that space (after fixing the origin) are among continuous functions vanishing faster then polynomially at infinity. Two test fields χ_1 and χ_2 are in one class producing the same Dirac field according to (2.19) if, and only if, the restrictions of $\widehat{\chi_1}$ and $\widehat{\chi_2}$ to the hyperboloid $p^2 = m^2$ are equal. For $\chi \in \mathcal{K}^\square$ we shall denote by $[\chi]$ the class of fields producing the same Dirac field as χ, and by $[\mathcal{K}^\square]$ the quotient space of these classes; also, $[\mathcal{K}^\square(\overline{\mathcal{C}})]$ will denote the set of classes $[\chi]$ with $\chi \in \mathcal{K}^\square(\overline{\mathcal{C}})$.

Let $\chi \in \mathcal{S}^\square_5(\overline{\mathcal{C}})$ and $k > 5$. Then we have

$$\begin{aligned} \chi &= \chi_k + (m^2 + \square)\chi'_k\,, \\ \chi_k &= \left(\frac{-\square}{m^2}\right)^{k-5} \chi \in \mathcal{S}^\square_k(\overline{\mathcal{C}})\,, \\ \chi'_k &= \frac{1}{m^2}\left[1 + \left(\frac{-\square}{m^2}\right) + \cdots + \left(\frac{-\square}{m^2}\right)^{k-6}\right]\chi \in \mathcal{S}^\square_5(\overline{\mathcal{C}})\,. \end{aligned} \tag{3.41}$$

Therefore for each $\chi \in \mathcal{K}^\square$ the class $[\chi]$ contains for each $k \geq 5$ a field $\chi_k \in \mathcal{S}^\square_k$ with the same support properties as χ.

For each $J \in \mathcal{J}_{\rm as}$ we define a linear operator S_J in $[\mathcal{K}^\Box]$ as follows. For $\chi \in \mathcal{S}^\Box_5(\overline{\mathcal{C}})$ with $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ we find any $\chi_k \in [\chi] \cap \mathcal{S}^\Box_k(\overline{\mathcal{C}})$ with $k \geq 10$ and split $\chi_k = \chi_{k+} + \chi_{k-}$, $\chi_{k\pm} \in \mathcal{S}^\Box_k(\overline{\mathcal{C}_\pm})$, which is possible, as shown in Appendix B. We put

$$S_J[\chi] = [\chi'] \,, \quad \text{where} \quad \widehat{\chi'}(p) = e^{-iF_{J+}(p)}\widehat{\chi_{k+}}(p) + e^{-iF_{J-}(p)}\widehat{\chi_{k-}}(p) \,, \tag{3.42}$$

and the functions $F_{J\pm}$ were defined at the end of the last subsection. The results of Appendix B guarantee that this is a correct definition, i.e., the function χ' is in $\mathcal{S}^\Box_5(\overline{\mathcal{C}})$ and the class $[\chi']$ is independent of the choice of χ_k and its split into $\chi_{k\pm}$. Moreover, it is easy to convince oneself that

$$S_{J_1}\, S_{J_2} = S_{J_1+J_2} \,, \quad S_0 = \mathrm{id} \,, \quad S_J\big[\mathcal{K}^\Box(\overline{\mathcal{C}})\big] = \big[\mathcal{K}^\Box(\overline{\mathcal{C}})\big] \,, \tag{3.43}$$

so S_J is bijective for each J and the mapping $S_\bullet : J \mapsto S_J$ is a homomorphism of the additive group of currents into the group of automorphisms of $[\mathcal{K}^\Box]$ and of each of the spaces $[\mathcal{K}^\Box(\overline{\mathcal{C}})]$. The kernel of this homomorphism is the subgroup of currents given by

$$\mathrm{Ker}\, S_\bullet = \big\{ J \in \mathcal{J}_{\rm as} \mid \{J, J_v\} = 2k\pi, \quad k = 0, \pm 1, \dots \big\} \,. \tag{3.44}$$

The mappings are unitary with respect to the scalar product (2.23):

$$\text{if} \quad [\chi'_i] = S_J[\chi_i] \quad \text{then} \quad \langle \chi'_1, \chi'_2 \rangle = \langle \chi_1, \chi_2 \rangle \,. \tag{3.45}$$

4. Asymptotic algebras of fields

The structures of the last section allow now for a rigourous formulation of our ideas in the algebraic form.

4.1. *-algebras $\mathcal{B}_{\rm as}$

We define the *-algebras of fields. For each $J \in \mathcal{J}_{\rm as}$ we assume an element of the algebra $W_{\rm as}(J)$, and for each $\chi \in \mathcal{K}^\Box$ an element $\psi_{\rm as}(\chi)$. We also assume a unit element E and impose the algebraic relations

$$\begin{gathered} W_{\rm as}(J)^* = W_{\rm as}(-J) \,, \quad W_{\rm as}(0) = E \,, \\ W_{\rm as}(J_1) W_{\rm as}(J_2) = \exp\left[-\frac{i}{2}\{J_1, J_2\}\right] W_{\rm as}(J_1 + J_2) \,, \\ \big[\psi_{\rm as}(\chi_1), \psi_{\rm as}(\chi_2)\big]_+ = 0 \,, \quad \big[\psi_{\rm as}(\chi_1), \psi_{\rm as}(\chi_2)^*\big]_+ = \langle \chi_1, \chi_2 \rangle \,, \\ W_{\rm as}(J)\psi_{\rm as}(\chi) = \psi_{\rm as}(\chi') W_{\rm as}(J) \,, \quad \text{where} \quad [\chi'] = S_J[\chi] \,. \end{gathered} \tag{4.1}$$

We want to identify those elements $W_{\rm as}(J)$ which generate the same relations. The following is a subgroup of the additive group of currents $\mathcal{J}_{\rm as}$:

$$\begin{aligned} \mathcal{J}^0_{\rm as} &= \mathrm{Ker}\{.\,,.\} \cap \mathrm{Ker}\, S_\bullet \\ &= \left\{ J \in \mathcal{J}_{\rm as} \mid V(s,l) = l\alpha(s,l),\ (e/4\pi)\!\int\! \Delta\alpha(l)\, d^2l = 2k\pi,\ k = 0, \pm 1, \dots \right\} , \end{aligned} \tag{4.2}$$

where $\Delta\alpha(l) = \alpha(+\infty, l) - \alpha(-\infty, l) = \mp\alpha(\mp\infty, l)$ in the "in" and "out" case respectively. We shall denote by $[\mathcal{J}_{\rm as}]$ the quotient group $\mathcal{J}_{\rm as}/\mathcal{J}^0_{\rm as}$, and set $W_{\rm as}(J_1) = W_{\rm as}(J_2)$ if $[J_2] = [J_1]$. Similarly we identify $\psi_{\rm as}(\chi_1) = \psi_{\rm as}(\chi_2)$ if $[\chi_1] = [\chi_2]$. After these identifications we shall call the *-algebra generated by the relations (4.1) the field *-algebra $\mathcal{B}_{\rm as}$. For each $\mathcal{C}$ the elements $W_{\rm as}(J)$ and $\psi_{\rm as}(\chi)$ with $J \in \mathcal{J}_{\rm as}(\overline{\mathcal{C}})$ and $\chi \in \mathcal{K}^\square(\overline{\mathcal{C}})$ generate a subalgebra, denoted $\mathcal{B}_{\rm as}(\overline{\mathcal{C}})$, and we have $\mathcal{B}_{\rm as} = \cup_{\mathcal{C}}\mathcal{B}_{\rm as}(\overline{\mathcal{C}})$.

This construction can also be characterized as follows. The relations (4.1) generate a *-algebra. The elements $A\big(W_{\rm as}(J) - E\big)$, $J \in [0]$, and $A\psi_{\rm as}(\chi)$, $\chi \in [0]$, where A goes over all elements of the algebra, generate a two-sided ideal of the algebra. The quotient of the algebra through this ideal is the *-algebra $\mathcal{B}_{\rm as}$, as=in or out.

The elements $\psi_{\rm as}(\chi)$ generate a subalgebra $\mathcal{B}^+_{\rm as}$ of the CAR type, and elements $W_{\rm as}(J)$ – a subalgebra $\mathcal{B}^-_{\rm as}$ of the CCR type. Each element of $\mathcal{B}_{\rm as}$ may be brought to the form $\sum_{i=1}^k C_i W_{\rm as}(J_i)$, where $C_i \in \mathcal{B}^+_{\rm as}$ and with currents J_i such that $[J_i] \neq [J_j]$ for $i \neq j$. The last relation in (4.1) may be used to define a group of automorphisms of $\mathcal{B}^+_{\rm as}$:

$$\begin{aligned} &\beta_J(C) = W_{\rm as}(J) C W_{\rm as}(-J)\,, \quad C \in \mathcal{B}^+_{\rm as}\,,\\ &\beta_{J_1}\beta_{J_2} = \beta_{J_1+J_2}\,, \quad \beta_0 = \mathrm{id}\,. \end{aligned} \tag{4.3}$$

The universal covering group $\mathcal{P}$ of the Poincaré group has a representation in the automorphism group of algebra $\mathcal{B}_{\rm as}$. After choosing the origin in $\mathcal{M}$ each element in $\mathcal{P}$ is represented by (a, A), $a \in M$, $A \in SL(2,\mathbb{C})$, and the respective automorphism $\alpha_{a,A}$ is given by the standard formulas

$$\begin{aligned} \alpha_{a,A}\big(W_{\rm as}(J)\big) &= W_{\rm as}(T_{a,A}J)\,, \quad [T_{a,A}J](x) = \Lambda(A) J\big(\Lambda(A)^{-1}(x-a)\big)\,,\\ \alpha_{a,A}\big(\psi_{\rm as}(\chi)\big) &= \psi_{\rm as}(R_{a,A}\chi)\,, \quad [R_{a,A}\chi](x) = S(A)\chi\big(\Lambda(A)^{-1}(x-a)\big)\,, \end{aligned} \tag{4.4}$$

where $\Lambda(A)$ and $S(A)$ are elements of the vector and 4-spinor representations of $SL(2,\mathbb{C})$ respectively.

In the above construction of the algebras $\mathcal{B}_{\rm as}$ we identified elements labelled by test functions falling into a common equivalence class. The net effect of these identifications is that the elements $\psi(\chi)$ could be labelled by the restriction of the Fourier transform of χ to the hyperboloid $p^2 = m^2$, and the elements $W(J)$ – by the corresponding classes of V's up to addition of an element of $\mathcal{J}^0_{\rm as}$. If formulated in this way, the algebra $\mathcal{B}_{\rm out}$ is a subalgebra of the algebra $\mathcal{B}$ of [9], Eqs. (3.40–43). The use of the present test spaces adds to the elements of the algebra the spacetime localization properties. In the following (sub)sections we use results of [9], although with some modifications indicated below. The change from $\mathcal{B}_{\rm out}$ to $\mathcal{B}_{\rm in}$ is almost trivial in all what follows and we shall not separate these two cases.

To be precise, there is one minor difference in the use of V's in [9] and the present paper. Let us concentrate on the "out" case. Here we have $V(-\infty, l) = 0$ in that case, while in [9] we used variable $V^{\rm out}(s, l)$ for which $V^{\rm out}(+\infty, l) = 0$. The relation between the two variables is $V^{\rm out}(s, l) = V(s, l) - V(+\infty, l)$. It is easy to check that the value of the symplectic form remains unchanged under this transformation, and also $\Delta V^{\mathrm{out}}(l) = \Delta V(l)$, so S_J is unaffected. The convention of V is more naturally connected with the outgoing test current, while the convention of V^{out} is connected with the description of the future null asymptotics of the radiation potential (3.31): for a lightlike future-pointing vector k one has

$$\lim_{R\to\infty} RA(x+Rk) = V^{\mathrm{out}}(x\cdot k, k)\,. \tag{4.5}$$

Similar connections, with past interchanged with future, take place in the "in" case.

4.2. Selection of representations

We consider *-representations π of the algebra $\mathcal{B}_{\mathrm{as}}$ by bounded operators in a Hilbert space $\mathcal{H}$ (all representations considered will be *-representations and we suppress this qualification in the sequel). We are interested only in regular representations – those for which all one-parameter groups $\lambda \mapsto \pi\big(W_{\mathrm{as}}(\lambda J)\big)$ are strongly continuous; the Weyl exponentiation is a technical device, which should be invertible on the level of representations having physical significance. Moreover, we further restrict attention to the translationally covariant representations with positive energy. This means that there exists a representation of the translation group by unitary operators $U(a)$ in $\mathcal{H}$ which implement the automorphisms $\alpha_a \equiv \alpha_{a,\mathbf{1}}$, i.e., $\pi(\alpha_a(B)) = U(a)\pi(B)U^*(a)$ for each $B\in\mathcal{B}_{\mathrm{as}}$, and whose spectrum is contained in $\overline{V_+}$.

Let π_F be the standard positive energy Fock representation of $\mathcal{B}^+_{\mathrm{as}}$ on the Hilbert space $\mathcal{H}_F$, with the Fock vacuum vector denoted Ω_F, and π_r be a regular, translationally covariant positive energy representation of $\mathcal{B}^-_{\mathrm{as}}$ on $\mathcal{H}_r$. Define the following operators $\pi(A)$ on the space $\mathcal{H} = \mathcal{H}_F\otimes\mathcal{H}_r$ by

$$\begin{gathered}\pi(C) = \pi_F(C)\otimes \mathrm{id}_r\,,\quad C\in\mathcal{B}^+_{\mathrm{as}}\,,\\ \pi\big(W_{\mathrm{as}}(J)\big)[\pi_F(B)\Omega_F\otimes\varphi] = \pi_F(\beta_J B)\Omega_F\otimes\pi_r\big(W_{\mathrm{as}}(J)\big)\varphi\,,\quad B\in\mathcal{B}^+_{\mathrm{as}}\,.\end{gathered} \tag{4.6}$$

Then π extends to a regular, translationally covariant positive energy representation of $\mathcal{B}_{\mathrm{as}}$. Conversely, if π is a representation of $\mathcal{B}_{\mathrm{as}}$ with these properties then up to a unitary equivalence it has the form given by (4.6).

The theorem formulated in the last paragraph results from a slight modification of the Theorem 4.4 of [9], taken over here by the remarks of the two last paragraphs of the preceding subsection. The modification is twofold. First, the theorem is formulated in terms of a C^*-algebra $\mathcal{F}$ generated by $\mathcal{B}$, but in the proofs of this theorem and its lemmas one can replace $\mathcal{F}$ by $\mathcal{B}$. Second, we need to replace $\mathcal{B}$ by $\mathcal{B}_{\mathrm{as}}$. This however poses no problem, one only has to use the fact that our present test space $\mathcal{K}^\square$ is dense in the Hilbert space of the scalar product (2.23).

We now restrict the choice of representations still further. We demand that the elements $\pi(W_{\mathrm{as}}(J))$ are gauge-invariant when acting on the subspace $\Omega_F\otimes\mathcal{H}_r$. The physical motivation for that seems plausible enough – the electromagnetic field alone should be gauge-invariant. Let $[[\mathcal{J}_{\mathrm{as}}]]$ be the quotient space $\mathcal{J}_{\mathrm{as}}/\operatorname{Ker}\{.\,,.\}$ with elements denoted by $[[J]]$. The assumed gauge-invariance means that $\pi_r(W_{\mathrm{as}}(J_1)) = \pi_r(W_{\mathrm{as}}(J_2))$ if $[[J_1]] = [[J_2]]$. This is equivalent to the assumption that π_r is a representation of the algebra $[[\mathcal{B}^-_{\rm as}]]$ (in which such elements $W_{\rm as}(J_1)$ and $W_{\rm as}(J_2)$ have been identified).

4.3. C^* field algebras $\mathcal{F}_{\rm as}$

We can now equip the algebra $\mathcal{B}_{\rm as}$ with a C^*-norm defined by $\|A\| = \|\pi(A)\|$, where π is any representation in the class defined by the above assumptions. This norm is independent of the choice of the particular representation π. The uniqueness follows in three steps. First, the representation π_F extends uniquely to the Fock representation of the full (unique) CAR Dirac fields algebra – this is because fields $[\chi]$, $\chi \in \mathcal{K}^\square$, are dense in the Hilbert space defined by the product (2.23) (as, in particular, compactly supported χ's are). Second, as the symplectic form is nondegenerate on $[[\mathcal{J}_{\rm as}]]$, the algebra $[[\mathcal{B}^-_{\rm as}]]$ generates the unique C^* Weyl algebra. Each representation π_r in the assumed class extends to a representation of this Weyl algebra. Third, these extended representations π_F and π_r are faithful, so the operator norm on $\pi_F(\mathcal{B}^+_{\rm as}) \otimes_{\rm alg} \pi_r(\mathcal{B}^-_{\rm as})$ is independent of the choice of π_r. This is sufficient to conclude the claimed uniqueness.

We shall call the C^*-algebra generated by $\mathcal{B}_{\rm as}$ equipped with the norm introduced above the C^* asymptotic field algebra, $\mathcal{F}_{\rm as} = \overline{\mathcal{B}_{\rm as}}^{\|\cdot\|}$. Each representation π of $\mathcal{B}_{\rm as}$ in the assumed class extends to a faithful representation of $\mathcal{F}_{\rm as}$. We also introduce algebras $\mathcal{F}_{\rm as}(\overline{\mathcal{C}}) = \overline{\mathcal{B}_{\rm as}(\overline{\mathcal{C}})}^{\|\cdot\|}$.

We note that the present construction of the algebras $\mathcal{F}_{\rm as}$ is more restrictive than the one leading from $\mathcal{B}$ to $\mathcal{F}$ in [9]. There the construction of $\mathcal{F}$ was based on all possible Hilbert space representations of $\mathcal{B}$. However, now I think that this is both more involved and unjustified. Some of the elements of this larger algebra could be brought to zero in representations having physical interpretation.

4.4. Examples of the representations π_r

Let ρ be a real smooth function on $\mathcal{M}$ of compact support, such that $\int \rho(x)dx = 1$. For each $J \in \mathcal{J}_{\rm as}(\overline{\mathcal{C}})$ we denote

$$J_\rho = J - \rho * J_{\rm as}\,, \quad (\rho * J_{\rm as})(x) = \int \rho(x-y) J_{\rm as}(y)\, dy\,. \tag{4.7}$$

Modifying slightly the steps of Appendix C one finds that $\rho * J_{\rm as}, J_\rho \in \mathcal{J}_{\rm as}(\overline{\mathcal{C}})$. The asymptote of $\rho * J_{\rm as}$ is equal to $J_{\rm as}$, therefore the asymptote of J_ρ vanishes, $J_{\rho\rm as} = 0$. Moreover, taking into account the support property of $J_{\rm as}$ one finds

$$\begin{aligned} V(s,l) - V_\rho(s,l) &= \int \delta(s - x\cdot l)(\rho * J_{\rm as})(x)dx \\ &= \int \rho(z)\Big\{ \int \delta(s - z\cdot l - y\cdot l) J_{\rm as}(y)\, dy \Big\}\, dz \\ &= \int \theta\big(\mp(s - z\cdot l)\big)\rho(z)dz \int \delta(y\cdot l \pm 1) J_{\rm as}(y) dy\,, \end{aligned}$$

where the upper and lower signs refer to the “in” and “out” case respectively and the last step results from rescaling $s - z\cdot l$ in the inner integral to ∓ 1 respectively.

If we introduce

$$H(s,l) = \int \operatorname{sgn}(s - z\cdot l)\rho(z)dz\,, \quad \text{with} \quad \lim_{s\to\pm\infty} H(s,l) = \pm 1\,, \tag{4.8}$$

then

$$V(s,l) = V_\rho(s,l) + \frac{1}{2}\big[1 \mp H(s,l)\big]\, V(\mp\infty, l)\,. \tag{4.9}$$

Using this split in (3.34) it is easy to show that

$$\{J_1, J_2\} = \{J_{1\rho}, J_{2\rho}\} + \{J_1, \rho * J_{2\mathrm{as}}\} - \{J_2, \rho * J_{1\mathrm{as}}\}\,. \tag{4.10}$$

Currents $J_{i\rho}$ produce infrared-regular fields and the first term on the rhs coincides with the standard symplectic form for such fields, and expressed in terms of V's reads

$$\{J_{1\rho}, J_{2\rho}\} = \frac{1}{4\pi}\int \big(\dot V_{1\rho}\cdot V_{2\rho} - \dot V_{2\rho}\cdot V_{1\rho}\big)(s,l)\, ds\, d^2 l\,, \tag{4.11}$$

The rest of the rhs of (4.10) is another symplectic form which we now transform. By a straightforward calculation we have

$$\{J_1, \rho * J_{2\mathrm{as}}\} = \int \left(\frac{1}{4\pi}\int \dot V_1 H(s,l) ds\right)\cdot \Delta V_2(l)\, d^2 l\,, \tag{4.12}$$

with $\Delta V(l)$ defined in (3.38). We recall from [9] that for vector fields $f(l)$ homogeneous of degree -1 and orthogonal to l the scalar product $(f,g)_0 = -\int f(l)\cdot g(l)\, d^2 l$ defines a Hilbert space $\mathcal{H}_0$ of equivalence classes, and that the fields satisfying in addition $L\wedge f = 0$ (cf. Eq. (3.30)) span its subspace $\mathcal{H}_{IR}$. (All fields differing by fields of the form $l\alpha(l)$ fall into one class. To simplify notation we suppress the square brackets which were used in [9] to distinguish a class $[f]$ from the field f.) Following the notation of [9] we denote $p(\dot V) = \frac{1}{2\pi}\Delta V$. Also, we set $h = \pi \dot H$, and again following [9] write $r_h(\dot V)$ for the orthogonal projection in $\mathcal{H}_0$ onto $\mathcal{H}_{IR}$ of $\frac{1}{2}\int \dot V H(s,l) ds$. Now we can write

$$\{J_1, \rho * J_{2\mathrm{as}}\} - \{J_2, \rho * J_{1\mathrm{as}}\} = \big\{p(\dot V_1)\oplus r_h(\dot V_1), p(\dot V_2)\oplus r_h(\dot V_2)\big\}_{IR}\,, \tag{4.13}$$

where

$$\{g_1\oplus k_1, g_2\oplus k_2\}_{IR} = (g_1, k_2)_{\mathcal{H}_{IR}} - (g_2, k_1)_{\mathcal{H}_{IR}} \tag{4.14}$$

is a nondegenerate symplectic form on the space $\mathcal{H}_{IR}\oplus\mathcal{H}_{IR}$.

The additive split of the symplectic form into the parts (4.11) and (4.13) suggests the following construction. Let $\hat W(V)$ generate the standard Weyl algebra over the space of infrared-regular fields with the symplectic form (4.11), and $w(g\oplus k)$ generate the Weyl algebra over the symplectic space given by (4.14). Choose representations π_{reg} and π_{sing} of these two algebras. Then the formula

$$\pi_r\big(W(J)\big) = \pi_{\mathrm{reg}}\big(\hat W(V_\rho)\big)\otimes \pi_{\mathrm{sing}}\Big(w\big(p(\dot V)\oplus r_h(\dot V)\big)\Big) \tag{4.15}$$

defines a representation of the algebra $\mathcal{B}^-_{\mathrm{as}}$. If π_{reg} and π_{sing} are cyclic, determined by the GNS construction from the states ω_{reg} and ω_{sing} respectively, then π_r is acyclic representation determined by the state $\omega_r(W(J)) =$

$\omega_{\mathrm{reg}}(\hat{W}(V_\rho))\omega_{\mathrm{sing}}(w(p(\dot{V}) \oplus r_h(\dot{V})))$. In particular, we take for ω_{reg} the standard vacuum state, $\omega_{\mathrm{reg}}(\hat{W}(V_\rho)) = \exp[-\frac{1}{2}F(\dot{V}_\rho, \dot{V}_\rho)]$, where

$$\begin{aligned} F(\dot{V}_1, \dot{V}_2) &= -\int_{\omega \geq 0} \overline{\widetilde{\dot{V}}_1(\omega, l)} \cdot \widetilde{\dot{V}}_2(\omega, l)\, \frac{d\omega}{\omega}\, d^2 l \\ &= \frac{1}{(2\pi)^2} \int \log(s - \tau - i0)\, \dot{V}_1(s, l) \cdot \dot{V}_2(\tau, l)\, ds\, d\tau\, d^2 l\,, \end{aligned} \tag{4.16}$$

$$\widetilde{\dot{V}}(\omega, l) = \frac{1}{2\pi} \int e^{i\omega s} \dot{V}(s, l)\, ds\,. \tag{4.17}$$

Let B be a positive, trace-class operator in $\mathcal{H}_{IR}$ such that $B^{1/2}\mathcal{H}_{IR}$ contains the subspace C^∞_{IR} of all smooth fields in $\mathcal{H}_{IR}$. Then the formula

$$\omega_{\mathrm{sing}}\big(w(g \oplus k)\big) = \exp\left[-\frac{1}{4}s(g \oplus k, g \oplus k)\right], \tag{4.18}$$

$$s(g_1 \oplus k_1, g_2 \oplus k_2) = \frac{1}{2}\big(B^{-1/2}g_1, B^{-1/2}g_2\big)_{\mathcal{H}_{IR}} + 2\big(B^{1/2}k_1, B^{1/2}k_2\big)_{\mathcal{H}_{IR}}, \tag{4.19}$$

defines a quasi-free state. The resulting representation π_r satisfies all our selection conditions. If, in addition, $\overline{B^{-1/2}\mathcal{C}^\infty_{IR}}^{\mathcal{H}_{IR}} = \mathcal{H}_{IR}$, then π_r is irreducible [9].

Representations thus obtained seem to depend on the choice of ρ. However, this dependence is spurious: it was shown in [9] that (with fixed B) they are all unitarily equivalent. In fact, one can construct them in a version which does not need this auxiliary function. States ω_r with different ρ's are then realized by different vector states in the representation space.

The spectrum of energy-momentum covers in each representation from the given class the whole future lightcone, and is purely continuous [9]. Thus there is no vacuum state, but the energy content can be arbitrarily close to zero. In fact, one finds that in our formulation the translational invariance is in contradiction with the regularity of infrared-singular Weyl operators. However, for infrared-regular test fields the states of the form discussed above can approximate (weakly) the vacuum. Indeed, for these test currents J there is $J_{\mathrm{as}} = 0$, so $J_\rho = J$ and the state ω_{reg} is the vacuum. Next, we have to consider the infrared part ω_{sing}. As $p(\dot{V}) = 0$, we have $\omega_{\mathrm{sing}}(\omega(0 \oplus r_h(\dot{V}))) = \exp[-\|B^{1/2}r_h(\dot{V})\|^2_{IR}/2]$. Take now a family of states with $\rho_\lambda(x) = \rho(x + \lambda t)$, where t is any timelike vector. It is then easy to show that for $\lambda \to \pm\infty$ the corresponding H_λ tends point-wise to ∓ 1, and then $r_{h_\lambda}(\dot{V})$ tends in norm to zero. As the operator B is bounded, the singular part tends to 1, which ends the proof.

5. Scattering

In standard treatments of scattering the ingoing and outgoing fields are different representations of one asymptotic field algebra $\mathcal{F}_{\mathrm{as}}$ (say, of the free scalar field) unitarily connected by the scattering operator: $\pi_{\mathrm{out}}(A) = S^*\pi_{\mathrm{in}}(A)S$, $A \in \mathcal{F}_{\mathrm{as}}$. The "in" and "out" fields are (expected to be) obtainable from the actual field variables of the theory by some limiting process. We have to explain in what way we expect this picture should be extended to accommodate two asymptotic algebras $\mathcal{F}_{\rm in}$ and $\mathcal{F}_{\rm out}$.

5.1. Canonical isomorphism $\gamma_\iota : \mathcal{F}_{\mathbf{in}} \mapsto \mathcal{F}_{\mathbf{out}}$

We still conjecture that the asymptotic fields are outcomes of some limiting process. The gap between $\mathcal{F}_{\rm in}$ and $\mathcal{F}_{\rm out}$ will be bridged by showing that there exists a canonical isomorphism $\gamma_\iota : \mathcal{F}_{\rm in} \mapsto \mathcal{F}_{\rm out}$. This automorphism will be interpreted to describe the "no interaction" situation, in the sense that the formula $\pi_{\rm out}(\gamma_\iota A) = \pi_{\rm in}(A)$, $A \in \mathcal{F}_{\rm in}$, gives the "out" field in terms of the "in" field in that case. In general case we expect the relation

$$\pi_{\rm out}(\gamma_\iota A) = S^* \pi_{\rm in}(A) S\,, \quad A \in \mathcal{F}_{\rm in}\,, \tag{5.1}$$

with S playing the role of the scattering operator.

To construct the automorphism γ_ι we first define the group isomorphism $\iota : [\mathcal{J}_{\rm in}] \mapsto [\mathcal{J}_{\rm out}]$. To this end we first define relation $\mathcal{R} \subset [\mathcal{J}_{\rm in}] \times [\mathcal{J}_{\rm out}]$ as follows: $([J_1], [J_2]) \in \mathcal{R}$ iff the current $J_1 - J_2$ radiates no electromagnetic field and $(e/4\pi) \int [\Delta V_1(l) - \Delta V_2(l)]\, d^2 l = 2k\pi$. Recalling the definition of $[\mathcal{J}_{\rm as}]$ as given after (4.2) it is easy to see that this is an unambiguous definition independent of the choice of J_i in the respective classes, and that the relation is one to one. Moreover, if the pairs $([J_1], [J_2])$ and $([J_1'], [J_2'])$ satisfy the relation, then also does the pair $([J_1] + [J_1'], [J_2] + [J_2'])$. We show in Appendix C that for each J_1 there exists J_2 such that $([J_1], [J_2])$ satisfy the relation (and conversely, for each J_2 there is a respective J_1). With this result we can set $\iota[J_1] = [J_2]$ iff $([J_1], [J_2]) \in \mathcal{R}$, and conclude that this defines a group isomorphism. With the results of Appendix C it is also easily shown that ι is a symplectic mapping, and also that $\{J_1, J_v\} = \{J_2, J_v\}$ for $[J_2] = \iota[J_1]$.

We now set

$$\gamma_\iota\big(\psi_{\rm in}(\chi)\big) = \psi_{\rm out}(\chi)\,, \quad \gamma_\iota\big(W_{\rm in}(J_1)\big) = W_{\rm out}(J_2)\,, \quad [J_2] = \iota[J_1]\,. \tag{5.2}$$

If $\pi_{\rm in}$ is the representation of $\mathcal{B}_{\rm in}$ in the assumed class (see (4.6) and the accompanying discussion) then $\pi_{\rm out}$ defined by $\pi_{\rm out}(\psi_{\rm out}(\chi)) = \pi_{\rm in}(\psi_{\rm in}(\chi))$, $\pi_{\rm out}(W_{\rm out}(J_2)) = \pi_{\rm in}(W_{\rm in}(J_1))$, $[J_2] = \iota[J_1]$, defines a representation of $\mathcal{B}_{\rm out}$ in the same class. Therefore γ_ι extends to a topological isomorphism $\gamma_\iota : \mathcal{F}_{\rm in} \mapsto \mathcal{F}_{\rm out}$.

5.2. Radiation by external current

How to construct a scattering theory in the given language in the general case of full quantum theory is an open question. Here we consider only electromagnetic field scattered by classical external current.

Let $J_{\rm ext}$ be a current satisfying the conditions of spaces $\mathcal{J}_{\rm as}$, except that its asymptote has both an incoming and outgoing parts (has support in $\overline{V_+ \cup V_-}$); this is a classical conserved current typical of charged matter. We denote $V_{\rm ext}(s,l) = \int J_{\rm ext}(x)\delta(s - x\cdot l)dx$. This current produces the Lorenz radiation potential $A_{\rm ext}$ in accordance with (2.4). We need test currents producing the same radiation as

J_{ext} but belonging to $\mathcal{J}_{\text{in}}$ or $\mathcal{J}_{\text{out}}$. Let $J_{\text{ext,in}}$ and $J_{\text{ext,out}}$ denote the incoming and outgoing asymptote of J_{ext} respectively. We denote further

$$\begin{aligned} J^+_{\text{ext,out}}(x) &= J_{\text{ext,out}}(x) + J_{\text{ext,out}}(-x)\,, \qquad & J_{\text{ext},1} &= J_{\text{ext}} - \rho * J^+_{\text{ext,out}}\,,\\ J^+_{\text{ext,in}}(x) &= J_{\text{ext,in}}(x) + J_{\text{ext,in}}(-x)\,, & J_{\text{ext},2} &= J_{\text{ext}} - \rho * J^+_{\text{ext,in}}\,. \end{aligned} \tag{5.3}$$

Then proceeding as in Appendix C one shows that $J_{\text{ext},1} \in \mathcal{J}_{\text{in}}$, $J_{\text{ext},2} \in \mathcal{J}_{\text{out}}$ and they both produce radiation potential A_{ext}; the classes $[J_{\text{ext},1}]$ and $[J_{\text{ext},2}]$ are independent of the choice of ρ and $[J_{\text{ext},2}] = \iota[J_{\text{ext},1}]$. Denoting $V_{\text{ext},i}(s,l) = \int J_{\text{ext},i}\delta(s - x\cdot l)dx$ $(i = 1,2)$ we have $V_{\text{ext},1}(s,l) = V_{\text{ext}}(s,l) - V_{\text{ext}}(+\infty,l)$, $V_{\text{ext},2}(s,l) = V_{\text{ext}}(s,l) - V_{\text{ext}}(-\infty,l)$.

The process of scattering of electromagnetic field by the external classical current should have the effect of adding the radiation field of the current. Therefore, remembering the interpretation of $W(J)$ we expect

$$\pi_{\text{out}}\big(W_{\text{out}}(J_2)\big) = \pi_{\text{in}}\big(W_{\text{in}}(J_1)\big)e^{-i\{J_2,J_{\text{ext},2}\}} = \pi_{\text{in}}\big(W_{\text{in}}(J_1)\big)e^{-i\{J_1,J_{\text{ext},1}\}}\,, \tag{5.4}$$

where $[J_2] = \iota[J_1]$. Using the commutation relations it is easy to find that the operator

$$S = \pi_{\text{in}}\big(W_{\text{in}}(J_{\text{ext},1})\big) = \pi_{\text{out}}\big(W_{\text{out}}(J_{\text{ext},2})\big) \tag{5.5}$$

satisfies the condition (5.1) and turns to identity when there is no radiation. There is no "infrared catastrophe" difficulty in this formulation.

Formula (5.5) cannot be directly applied when J_{ext} is a current of a point charge,

$$J_{\text{point}}(x) = e\int \dot z(\tau)\delta^4\big(x - z(\tau)\big)\,d\tau\,. \tag{5.6}$$

However, if π_{in} (and π_{out}) is of the form discussed in Section 4.4 it can be extended to this case. One easily finds that

$$\begin{aligned} V_{\text{point}}(s,l) &= e\frac{\dot z\big(\tau(s,l)\big)}{\dot z\big(\tau(s,l)\big)\cdot l}\,,\\ V_{\text{point}\rho}(s,l) &= \int \dot H(s-u,l)V_{\text{point}}(u,l)\,du\,, \end{aligned} \tag{5.7}$$

where $\tau(s,l)$ is the solution of $z(\tau)\cdot l = s$. For a smooth trajectory $z(\tau)$ with sufficiently fast achieved asymptotic velocities the element $\hat W(V_{\text{point}\rho})$ is well defined and the formula (5.5) may be extended with the use of the rhs of (4.15).

6. Summary

This paper introduces the notion of spatially local observables and fields in electrodynamics; we have developed the test functions machinery needed for that. We have shown that the enlarged algebra of fundamental asymptotic fields naturally includes the infrared degrees of freedom, which care for the implementation of Gauss' law at the algebraic level. The algebra proved to be a reformulation of the algebra postulated before by the author, but now we gained control over the spacetime localization of fields.

The perturbational construction of the standard electrodynamics is based on the uncoupled free fields algebra. We expect that the scattering theory based in similar way on the algebra discussed in this article could throw new light on the infrared and charge problems of the standard theory. Here, as a first step towards this task, we have shown how to apply our formalism to the simple case of the radiation produced by external classical current. The description is free from "infrared catastrophe".

Appendices

Appendix A. Functions $F_\pm(p)$

Let $V_a(l)$ be a smooth vector function on the future lightcone, homogeneous of degree -1 and such that $l \cdot V(l) = 0$. If t is any unit timelike future-pointing vector and $V(l)$ is continued to a neighbourhood of the cone with the preservation of its properties then $L_{ab}[t^a V^b(l)/t \cdot l] = \partial \cdot V(l)$. Therefore the rhs is independent of the extension in the assumed class and

$$\int \partial \cdot V(l)\, d^2 l = 0\,. \tag{A.1}$$

Let now $F(w)$ be defined for $w = p + iq$, $p, q \in M$, $q^2 > 0$, by

$$F(w) = \int \frac{w \cdot V(l)}{w \cdot l}\, d^2 l\,. \tag{A.2}$$

This is a homogeneous of degree 0, analytical function on its domain. Choosing any unit timelike future-pointing vector t and using property (A.1) we rewrite F as

$$F(w) = -\int \partial \cdot V(l) \log \frac{w \cdot l}{t \cdot l}\, d^2 l + \int \frac{t \cdot V(l)}{t \cdot l}\, d^2 l\,. \tag{A.3}$$

This is used to show that $F(w)$ is bounded on its domain. Due to homogeneity it is sufficient to consider two cases: $|q| \leq |p| = 1/2$, and $|p| \leq |q| = 1/2$. In the first case the first integral is bounded by const $\int [|\log(p^0 - \vec{p} \cdot \vec{l})^2| + 2\pi] d\Omega(\vec{l}) \leq$ const; the second case is treated similarly.

It is easy to see that there exist limit functions on $M \setminus \{0\}$

$$\begin{aligned} F_\pm(p) &= \lim_{\lambda \searrow 0} F(p \pm i\lambda t) \\ &= -\int \partial \cdot V(l) \left[\log \frac{|p \cdot l|}{t \cdot l} \mp i\frac{\pi}{2} \operatorname{sgn}(p \cdot l)\right] d^2 l \\ &\quad + \int \frac{t \cdot V(l)}{t \cdot l}\, d^2 l = F_\mp(-p)\,, \end{aligned} \tag{A.4}$$

which are independent of the choice of t (specified as above), homogeneous and equal to $F(p)$ for $p \in V_\pm$. We show below that these limits are achieved uniformly on each set separated from zero, $F_\pm$ are continuous outside $p = 0$ and $\mathcal{C}^\infty$ outside $p^2 = 0$, and for each multiindex α functions $(p^2)^{|\alpha|} D^\alpha F_\pm(p)$ have continuous extensions to M.

Proof. If $g(l)$ is any smooth function on the cone, homogeneous of degree -2 and such that $\int g(l)\, d^2l = 0$, then for $q^2 > 0$ we have

$$\begin{aligned}\frac{\partial}{\partial w^a} \int g(l) \log \frac{w \cdot l}{t \cdot l}\, d^2l = \frac{w^b}{w^2} \bigg[\int L_{ba} g(l) \log \frac{w \cdot l}{t \cdot l}\, d^2l \\ + \int g(l) \frac{l_a t_b - l_b t_a}{t \cdot l} d^2l \bigg] .\end{aligned} \tag{A.5}$$

This is shown by transferring L_{ba} on the rhs by parts and observing that

$$[L_{ba} + W_{ba}] \log(w \cdot l / t \cdot l) = (l_a t_b - l_b t_a)/t \cdot l \,,$$

where $W_{ba} = w_b\, \partial/\partial w^a - w_a\, \partial/\partial w^b$. Applying (A.5) inductively to (A.3) one finds that for $q^2 > 0$

$$(w^2)^{|\alpha|} D^\alpha F(w) = \sum_i Q_i(w) \int h_i(l) \log \frac{w \cdot l}{t \cdot l}\, d^2l + Q(w) \,,$$

where Q_i and Q are polynomials, homogeneous of degree $|\alpha|$, and h_i are smooth functions, homogeneous of degree -2 and such that $\int h_i(l) d^2l = 0$. To end the proof it is now sufficient to show that uniform limits (A.4) exist on each set separated from zero for each function of the form $G(w) = \int h(l) \log(w \cdot l / t \cdot l)\, d^2l$, with $\int h(l) d^2l = 0$. We have $(\lambda > 0)$

$$\log \frac{(p \pm i\lambda t) \cdot l}{t \cdot l} = \frac{1}{2} \log \left[\left(\frac{p \cdot l}{t \cdot l} \right)^2 + \lambda^2 \right] \mp i \arctan \left(\frac{p \cdot l}{\lambda\, t \cdot l} \right) \pm i \frac{\pi}{2} \,,$$

but the last term falls out of the integral. Denote

$$G_\pm(p) = \int h(l) \left[\log \frac{|p \cdot l|}{t \cdot l} \mp i \frac{\pi}{2} \operatorname{sgn}(p \cdot l) \right] d^2l \,.$$

If we denote $k = p/\lambda$ then

$$\begin{aligned}\big| G(p \pm i\lambda t) - G_\pm(p) \big| &= \big| G(k \pm it) - G_\pm(k) \big| \\ &\leq \text{const} \int \bigg\{ \log \left[1 + \left(\frac{t \cdot l}{k \cdot l} \right)^2 \right] \\ &\quad + \left[\pi - 2 \arctan \frac{|k \cdot l|}{t \cdot l} \right] \bigg\} d^2l \\ &= \frac{\text{const}}{|\vec{k}|} \int_{|k^0| - |\vec{k}|}^{|k^0| + |\vec{k}|} \left\{ \log \left[1 + \frac{1}{u^2} \right] + \pi - 2 \arctan |u| \right\} du \\ &\leq \text{const} \left\{ \log \left[1 + \frac{1}{|k^0| + |\vec{k}|} \right] + \frac{\log \big[1 + |k^0| + |\vec{k}| \big]}{|k^0| + |\vec{k}|} \right\} .\end{aligned}$$

where the spherical coordinates have been used. To get this result we first estimate the integrand by const $\log(1 + 1/|u|)$ and then consider the cases (a) $|k^0| \geq 2|\vec{k}|$ and (b) $|k^0| \leq 2|\vec{k}|$ separately. In case (a) the integral is bounded by const $\times \log[1 + 1/(|k^0| - |\vec{k}|)]$, which is sufficient as here $|k^0| - |\vec{k}| \geq (|k^0| + |\vec{k}|)/3$. In case (b) we extend the integration limits to $\pm(|k^0| + |\vec{k}|)$, calculate the integral explicitly and observe that here $|\vec{k}| \geq (|k^0| + |\vec{k}|)/3$, which yields the final result.

For $|p^0| + |\vec{p}| \geq r \geq \lambda$ we now have

$$|G(p \pm i\lambda t) - G_\pm(p)| \leq \text{const}\frac{1 + \log[1 + r/\lambda]}{r/\lambda} \to 0 \quad \text{for} \quad \lambda \to 0\,,$$

which ends the proof. $\square$

Appendix B. Transformation E^F

Let F and $F_\pm$ be defined as in Appendix A and let $\chi \in \mathcal{S}_k^\Box(\overline{\mathcal{C}_\pm})$ for a given $k \geq 10$. Define linear mappings $\chi \mapsto E_\pm^F \chi$ by

$$\widehat{E_\pm^F \chi}(p) = \exp\big[-iF_\pm(p)\big]\hat{\chi}(p)\,. \tag{B.1}$$

We show here that there is $E_\pm^F \chi \in \mathcal{S}_{k-5}^\Box(\overline{\mathcal{C}_\pm})$. Furthermore, if $\chi \in \mathcal{S}_k^\Box(\overline{\mathcal{C}})$ and $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ then it is possible to separate $\chi = \chi_{1+} + \chi_{1-}$, $\chi_{1\pm} \in \mathcal{S}_k^\Box(\overline{\mathcal{C}_\pm})$, and define $E_1^F \chi = E_+^F \chi_{1+} + E_-^F \chi_{1-} \in \mathcal{S}_{k-5}^\Box(\overline{\mathcal{C}})$; subscript 1 indicates the choice of (in general non-unique) separation. If another separation is indexed by 2 then $\widehat{E_1^F \chi}(p) = \widehat{E_2^F \chi}(p)$ for $p^2 > 0$.

Proof. If $\chi \in \mathcal{S}_k^\Box$ $(k \geq 5)$ then one shows by induction with respect to $|\alpha|$ that $\hat{\chi}(p)$ is $\mathcal{C}^\infty$ outside $p^2 = 0$ and for $|\alpha| \leq k - 5 + n$ the functions $(p^2)^n D^\alpha \hat{\chi}(p)$ have continuous extensions to M, vanishing faster than polynomially at infinity. Using this fact and the properties of $F_\pm$ shown in Appendix A it is easy to show that the same remains valid when $\hat{\chi}(p)$ is replaced by $\hat{\chi}'(p) = \exp[-iF_\pm(p)]\hat{\chi}(p)$. It follows that $\sup_x |x^\alpha \Box^n D^\beta \chi'(x)| < \infty$ whenever $|\alpha| \leq (k-5) + n$. This is sufficient to conclude that $E_\pm^F \chi \in \mathcal{S}_{k-5}^\Box$.

Let $\chi \in \mathcal{S}_k^\Box(\overline{\mathcal{C}_\pm})$, $k \geq 10$, so $E_\pm^F \chi \in \mathcal{S}_{k-5}^\Box$. We can choose the reference point at the vertex of the cone, and then the support of χ is in $\overline{V_\pm}$. Then the Fourier transform $\hat{\chi}(w)$ exists as an analytical function of the complex variable for $w \in M + iV_\pm$, satisfies the bound $|\hat{\chi}(p + iq)| < \text{const}(1 + |p|)^{-n}$ for each n and gives $\lim_{\lambda \searrow 0} \hat{\chi}(p + i\lambda q) = \hat{\chi}(p)$. Using the properties of $F(w)$ one finds that also $\exp[-iF(w)]\hat{\chi}(w)$ is analytical in the same domain, satisfies similar bounds and gives in the limit $\lim_{\lambda \searrow 0} \exp[-iF(p + i\lambda q)]\hat{\chi}(p + i\lambda q) = \exp[-iF_\pm(p)]\chi(p)$. By the theorem connecting cone-like support properties of a distribution with the analyticity properties of its transform (see [17], Thm.IX.16) this is sufficient to conclude that $E_\pm^F \chi \in \mathcal{S}_{k-5}^\Box(\overline{\mathcal{C}_\pm})$.

Let now $\chi \in \mathcal{S}_k^\Box(\overline{\mathcal{C}})$, $\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+$. As $\mathcal{C}_- \cap \mathcal{C}_+ \neq \emptyset$ one can choose the reference system such that $\mathcal{C}_\pm = \mp Re_0 + V_\pm$, $R > 0$ (if the origin is identified with the zero vector). Let f be a real smooth function of a real variable, such that $f(s) = -1$ for $s < -1/3$, $f(s) = 1$ for $s > 1/3$. We define functions $\rho_\pm$ on $\mathcal{M}$ as follows:

$$\rho_\pm(x) = \begin{cases} \frac{1}{2}\Big[1 \pm f\Big(\frac{Rx^0}{R^2-|\vec{x}|^2}\Big)\Big] & \text{for} \quad x \in \mathcal{C}_- \cap \mathcal{C}_+ \\ 1 & \text{on the rest of} \quad \mathcal{C}_\pm \\ 0 & \text{outside} \quad \mathcal{C}_\pm\,. \end{cases} \tag{B.2}$$

It is easy to show that $\rho_\pm \in \mathcal{C}^\infty(\mathcal{C})$ and the functions $\chi_\pm = \rho_\pm \chi$ are in $\mathcal{S}_k^\square(\overline{\mathcal{C}_\pm})$ respectively (use the fact that χ and its derivatives vanish at the boundary of $\mathcal{C}$ faster than any power of the Euclidean distance from that boundary). As the sum $\rho_+ + \rho_-$ is the characteristic function of the set $\mathcal{C}$ we have $\chi = \chi_+ + \chi_-$ and this separation satisfies conditions stated at the beginning. If $\chi = \chi_{i+} + \chi_{i-}$, $i = 1, 2$, are any two separations satisfying these conditions, then $\widehat{E_2^F\chi}(p) - \widehat{E_1^F\chi}(p) = (\exp[-iF_+(p)] - \exp[-iF_-(p)])(\hat{\chi}_{2+}(p) - \hat{\chi}_{1+}(p))$, which vanishes for $p^2 > 0$. □

Appendix C. Non-radiating currents

Let $J_1 \in \mathcal{J}_{\rm in}(\overline{\mathcal{C}})$, $\mathcal{C} = \mathcal{C}_- \cup \mathcal{C}_+$, with the asymptote $J_{1\rm as}$. Choose a real smooth function ρ on $\mathcal{M}$ with support in $\mathcal{C}_- \cap \mathcal{C}_+$, such that $\int \rho(y)dy = 1$, and define

$$J_2 = J_1 - \rho * J_{1\rm as}^+\,, \quad (\rho * J_{1\rm as}^+)(x) = \int \rho(x-y) J_{1\rm as}^+(y)\,dy\,,$$
$$J_{1\rm as}^+(y) = J_{1\rm as}(y) + J_{1\rm as}(-y)\,. \tag{C.1}$$

Then we have $J_2 \in \mathcal{J}_{\rm out}(\overline{\mathcal{C}})$, $J_2 - J_1$ produces no radiation potential and $J_{2\rm as}(x) = -J_{1\rm as}(-x)$. In terms of V's we have $V_2(s,l) = V_1(s,l) - V_1(-\infty,l)$ and $\Delta V_2(l) = V_2(+\infty,l) = -V_1(-\infty,l) = \Delta V_1(l)$.

Proof. It is easy to show that the asymptote $J_{1\rm as}(x)$ is divergence-free outside $x=0$ (as a limit of a conserved current). But it is also a distribution and by smearing it with a test function one also shows that $J_{1\rm as}$ is a conserved distributional current. Then it is immediate that $\rho * J_{1\rm as}^+$ is a smooth, conserved current. The support properties of ρ and J_1 imply that the support of $\rho * J_{1\rm as}^+$, and consequently also the support of J_2, are contained in $\overline{\mathcal{C}}$.

Let $\mathcal{C}_- \cap \mathcal{C}_+$ be contained in $|x| \leq R$. Then for $|x| \geq 2R$ we have

$$\big|D^\beta H_3(\rho * J_{1\rm as}^+)(x)\big| = \left|\int \rho(z)\, z \cdot \partial D^\beta J_{1\rm as}^+(x-z)dz\right| \leq \text{const}\,|x|^{-|\beta|-4}\,.$$

This guarantees that $\rho * J_{1\rm as}^+, J_2 \in \mathcal{S}_{3+}^3(\overline{\mathcal{C}})$ together with J_1. The asymptote of $\rho * J_{1\rm as}^+$ is easily found: $\lim_{\lambda\to\infty} \lambda^3(\rho * J_{1\rm as}^+)(\lambda x) = J_{1\rm as}^+(x)$ (use the assumption on the integral of ρ). Thus $J_{2\rm as}(x) = -J_{1\rm as}(-x)$ and $J_2 \in \mathcal{J}_{\rm out}(\overline{\mathcal{C}})$.

Finally, we calculate

$$V_1(s,l) - V_2(s,l) = \int \delta(s - x\cdot l)(\rho * J^+_{1\mathrm{as}})(x)dx$$
$$= \lim_{\lambda\searrow 0} \int_{|s-z\cdot l|\geq\lambda} \rho(z)\left\{\int \delta(s - z\cdot l - y\cdot l)J^+_{1\mathrm{as}}(y)dy\right\}dz\,.$$

But now using the fact that $|\xi|^3 J^+_{1\mathrm{as}}(\xi y) = J^+_{1\mathrm{as}}(y)$ for all $\xi \neq 0$ one can scale $s - z\cdot l$ to -1, and then one finds

$$V_1(s,l) - V_2(s,l) = \int \delta(y\cdot l + 1)J^+_{1\mathrm{as}}(y)dy = \int \delta(y\cdot l + 1)J_{1\mathrm{as}}(y)dy = V_1(-\infty, l)\,,$$

which ends the proof. □

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