Ann. Henri Poincaré 9 (2008), 347–372

1424-0637/020347-26, *published online* April 10, 2008
DOI 10.1007/s00023-008-0358-8

Annales Henri Poincaré

A Positive Mass Theorem on Asymptotically Hyperbolic Manifolds with Corners along a Hypersurface

Vincent Bonini and Jie Qing

Abstract. In this paper we take an approach similar to that in [13] to establish a positive mass theorem for spin asymptotically hyperbolic manifolds admitting corners along a hypersurface. The main analysis uses an integral representation of a solution to a perturbed eigenfunction equation to obtain an asymptotic expansion of the solution in the right order. This allows us to understand the change of the mass aspect of a conformal change of asymptotically hyperbolic metrics.

1. Introduction

In this paper we study the change of mass aspect for asymptotically hyperbolic manifolds under a conformal change of metric and establish a positive mass theorem for a class of asymptotically hyperbolic manifolds admitting corners along a hypersurface. This work follows an approach similar to that in [13]. The dimensions of all manifolds concerned in this paper are greater than 2. Positive mass theorems for asymptotically hyperbolic manifolds have been studied in many works, notably in [3,6,14,21]. A Riemannian manifold (M, g) with corners along a hypersurface Σ is a manifold that is separated by an embedded hypersurface $\Sigma \subset M$ such that each individual part is a smooth Riemannian manifold and the metric g is continuous across the hypersurface Σ. An asymptotically hyperbolic manifold with corners along a hypersurface is a Riemannian manifold with corners along a hypersurface with one part compact and the other part asymptotically hyperbolic. The issue at hand is to investigate the validity of a positive mass theorem for asymptotically hyperbolic manifolds with corners along a hypersurface if each part satisfies the

The first named author supported by MSRI Postdoctoral Fellowship.
The second named author supported partially by NSF grant DMS 0402294 .

scalar curvature condition. A good motivation given in [13] to initiate the study of such question is to use the Ricatti equation

$$R = R_\Sigma - \left(|A|^2 + H^2\right) - 2\frac{\partial H}{\partial n}\,, \tag{1.1}$$

which allows one to consider the scalar curvature in distributional sense across the hypersurface. It also turns out to relate to a notion of quasi-local mass in relativity (cf. [4, 13, 19, 20]. It is desirable to have a non-negative quantity associated with a compact domain Ω of an asymptotically hyperbolic manifold M, which is zero if and only if Ω can be isometrically embedded into the hyperbolic space and converges to the total mass when Ω exhausts M. Analogous to the suggestion for the asymptotically flat setting in [4], a natural candidate for such a quantity is given by taking the infimum of the total mass over the class of all asymptotically hyperbolic manifolds in which Ω can be isometrically embedded and to which positive mass theorem can apply. For more details readers are referred to [4, 13, 19, 20].

In case of an asymptotically hyperbolic manifold with corners along a hypersurface we will call the compact part the inside and the non-compact part the outside. We will denote the mean curvature of the hypersurface with respect to the inside metric in the outgoing direction by H_- and the mean curvature of the hypersurface with respective to the outside metric in the direction inward to the outside by H_+. Our main theorem is as follows:

Theorem 1.1. *Suppose that $(M^n,\ g)$ is a spin asymptotically hyperbolic manifold of dimension $n \geq 3$ with corners along a hypersurface. And suppose that the scalar curvature of both the inside and outside metrics are greater than or equal to $-n(n-1)$ and that*

$$H_-(x) \geq H_+(x)$$

for each x on the hypersurface. Then, if in a coordinate system at the infinity,

$$g = \sinh^{-2}\rho\left(d\rho^2 + g_0 + \frac{\rho^n}{n}h + O(\rho^{n+1})\right),$$

then

$$\int_{S^{n-1}} Tr_{g_0}h(x)dvol_{g_0}(x) \geq \left|\int_{S^{n-1}} x\,Tr_{g_0}h(x)dvol_{g_0}(x)\right|\,. \tag{1.2}$$

In [21] the vanishing of the mass is proved to imply the asymptotically hyperbolic manifold is isometric to the hyperbolic space. However, we did not find it is a straightforward consequence to have the same conclusion in our context nor did Miao in [13] in the context of asymptotically flat manifolds. We will give an affirmative answer to this question in a forthcoming paper. We would like to point out though it is easy to see that the scalar curvature should be the constant as the hyperbolic space.

We adopt an approach from [13] to smooth the corners, then conformally deform the metric so that the scalar curvature is greater than or equal to $-n(n-1)$ and then apply the positive mass theorem in [21]. Instead of solving an equation which is a perturbation of Laplace equation as in [13, 18] for asymptotically flat case, we realize, with our experience in [5, 16], that we should consider an equation which is a perturbation of the eigenfunction equation

$$-\Delta v + nv = 0 \tag{1.3}$$

on an asymptotically hyperbolic manifold, where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

on R^n in our notation in this paper. We also learned that in fact in each case the operator is simply the linearization of the Yamabe equation at the constant scalar curvature one. One of the consequences of this consideration gives hope that v decays in the right order to allow us to estimate the change of mass aspect after a conformal change of metric while another is the following key observation.

Lemma 1.2. *Suppose that $(M^n,\ g)$ is a Riemannian manifold and v is a positive smooth solution to the linear equation*

$$-\Delta v + nv - \frac{n-2}{4(n-1)}\big(R + n(n-1)\big)^{-} v = \frac{n-2}{4(n-1)}\big(R + n(n-1)\big)^{-} . \tag{1.4}$$

Then the scalar curvature of the metric $g_v = (1+v)^{\frac{4}{n-2}} g$ satisfies

$$R_{g_v} \geq -n(n-1) . \tag{1.5}$$

To find a solution v to (1.4) we use the analysis of weighted function spaces and uniformly degenerate elliptic equations, which are well developed in, for example, [1, 2, 8–12]. The positivity of the solution v to (1.4) follows from a clever use of a generalized maximum principle in [15]. We have noticed that the existence of the expansion of the solution v was studied in [2, 12]. But we need the explicit formula to estimate the change of mass aspects here. We followed the approach taken in [18] which used an integral representation to obtain an asymptotic expansion. To obtain an integral representation we used an explicit formula for the fundamental solution to the eigenfunction equation in the hyperbolic space

$$G_H(x,y) = \frac{c_n}{\sinh^{n-2} d_H(x,y) \cosh^2 d_H(x,y)} \theta\big(\cosh d_H(x,y)\big) , \tag{1.6}$$

where $d_H(x,y)$ is the hyperbolic distance between x and y in hyperbolic space H^n,

$$c_n = \frac{1}{(n-2)\mathrm{vol}(S^{n-1})} ,$$

$$\theta(s) = \frac{1}{\theta_0}\left(1 + \sum_{i=2}^{\infty} \prod_{j=2}^{i} \left(1 - \frac{n}{2j+n-1}\right) s^{-2i+2}\right) \tag{1.7}$$

and

$$\theta_0 = 1 + \sum_{i=2}^{\infty} \prod_{j=2}^{i} \left(1 - \frac{n}{2j+n-1}\right) . \tag{1.8}$$

For more detailed account on the above generalized eigenfunctions please see [2,12]. Thus

Lemma 1.3. *Suppose that* $(M^n,\ g)$ *is an asymptotically hyperbolic manifold,* M_c *is a compact set in* M *and* r_0 *is a large number. Let*

$$x = \psi(p) : M \setminus M_c \to R^n \setminus B_{r_0}(0)\,,$$

be a coordinate at the infinity in which

$$g = \sinh^{-2}\rho\left(d\rho^2 + g_0 + \frac{\rho^n}{n}h + O(\rho^{n+1})\right),$$

where $\sinh\rho = |x|^{-1}$. *Suppose that* $v \in C^{2,\alpha}_\delta(M)$ *with* $\delta > 0$ *solves the equation*

$$-\Delta v + nv + fv = w\,,$$

with

$$f \in C^{0,\alpha}_\kappa(M) \quad \text{and} \quad w \in C^{2,\alpha}_\eta(M)\,,$$

for some $\kappa > 2$ *and* $\eta > n+1$. *Then*

$$v(x) = A\left(\frac{x}{|x|}\right)|x|^{-n} + O\left(|x|^{-(n+1)}\right) \tag{1.9}$$

for some function A *on* S^{n-1}.

Note that the function $A(\frac{x}{|x|})$ in the above lemma in our proof will be given as a sum of several integrals which later allow us to estimate the size of change of the mass aspects, please see Lemma 6.5 in this note.

The paper is organized as follows: Section 2 is devoted to establishing an isomorphism theorem for a class of uniformly degenerate operators based on work in [10]. In Section 3 we introduce a linear equation whose solution gives a conformal factor for a metric with the scalar curvature greater than or equal to $-n(n-1)$. In Section 4 we derive an explicit formula for the fundamental solutions to the eigenfunction equation on hyperbolic space H^n. In Section 5 we use the standard fundamental solution to construct an approximate fundamental solution on an asymptotically hyperbolic manifold. This gives us an integral representation of a solution to the eigenfunction equation and the desired asymptotic expansion. In Section 6 we prove our main theorem by calculating the mass aspect of the deformed metric and applying the positive mass theorem in [21].

2. Analytic preliminaries

In this section we discuss some preliminaries of the analysis on weakly asymptotically hyperbolic manifolds. Let $\bar{M}^n$ be a smooth compact n-dimensional manifold with boundary ∂M and M^n be its interior. A nonnegative smooth function ρ on $\bar{M}$ is said to be a defining function for ∂M if

$$\begin{aligned} \rho > 0 \quad &\text{in} \quad M \\ \rho = 0 \quad &\text{on} \quad \partial M \end{aligned}$$

and $d\rho$ never vanishes on ∂M. For any non-negative integer m and any $0 \le \beta < 1$, a smooth Riemannian metric g on M is then said to be conformally compact of class $C^{m,\beta}$ if for any defining function ρ for ∂M, the conformal metric $\bar{g} = \rho^2 g$ extends as a $C^{m,\beta}$ metric on $\bar{M}$. The metric $\bar{g}$ restricted to $T(\partial M)$ induces a metric $\hat{g} := \bar{g}|_{T(\partial M)}$ on ∂M which rescales upon change in defining function and therefore defines a conformal structure $[\hat{g}]$ on ∂M called the conformal infinity of (M, g).

When $m + \beta \ge 2$, a straightforward computation as in [11] shows that the sectional curvatures of g approach $-|d\rho|^2_{\bar{g}}$ at ∂M. As in [5], we define weakly asymptotically hyperbolic manifolds as follows:

Definition 2.1. A connected complete Riemannian manifold $(M^n,\ g)$ is said to be weakly asymptotically hyperbolic of class $C^{m,\beta}$ if g is conformally compact of class $C^{m,\beta}$ with $m + \beta \ge 2$ and $|d\rho|^2_{\bar{g}} = 1$ on ∂M for a defining function ρ.

We will use the definitions of weighted function spaces from the papers of Lee [9, 10] (see also [1, 8]. Let $(M^n,\ g)$ be a weakly asymptotically hyperbolic manifold and let ρ be a defining function. The weighted Hölder spaces are defined, for $\delta \in R$,

$$C^{k,\alpha}_\delta(M) := \rho^\delta C^{k,\alpha}(M) = \left\{\rho^\delta u : u \in C^{k,\alpha}(M)\right\} \tag{2.1}$$

with the norm

$$\|u\|_{C^{k,\alpha}_\delta(M)} := \|\rho^{-\delta} u\|_{C^{k,\alpha}(M)}\,.$$

The weighted Sobolev spaces are defined, for $\delta \in R$,

$$W^{k,p}_\delta(M) := \rho^\delta W^{k,p}(M) = \left\{\rho^\delta u : u \in W^{k,p}(M)\right\} \tag{2.2}$$

with the norm

$$\|u\|_{W^{k,p}_\delta} := \|\rho^{-\delta} u\|_{W^{k,p}(M)}\,.$$

We recall the following weighted Sobolev embedding theorem from [10].

Lemma (Sobolev embedding). *Let (M^n, g) be weakly asymptotically hyperbolic manifold of class $C^{m,\beta}$ and $U \subset M$ an open subset. For $1 < p, q < \infty$, $0 < \alpha < 1$, $\delta \in R$, $1 \le k \le m$, and $k + \alpha \le m + \beta$, the inclusions*

$$W^{k,q}_\delta(U) \hookrightarrow W^{j,p}_\delta(U) \quad \textit{for} \quad k - \frac{n}{q} \ge j - \frac{n}{p} \tag{2.3}$$

and

$$W^{k,p}_\delta(U) \hookrightarrow C^{j,\alpha}_\delta(U) \quad \textit{for} \quad k - \frac{n}{p} \ge j + \alpha \tag{2.4}$$

are continuous.

The readers are referred to [10] (see also [1,8,9] for a more complete discussion of properties of the weighted Hölder and Sobolev spaces on weakly asymptotically hyperbolic manifolds. Our goal in this section is to derive an isomorphism result from [8, 10], particularly Theorem C in [10], for the operator $-\Delta + n + f$. We first state a simpler version of Theorem C in [10].

Lemma 2.2. *Suppose that $(M^n,\ g)$ is a weakly asymptotically hyperbolic manifold of class $C^{m,\beta}$. Let $k+1+\alpha \le m+\beta$ and $f \in C^{0,\alpha}_\gamma$ for some $\gamma > 0$. Then*

$$-\Delta + n + f : C^{2,\alpha}_\delta(M) \to C^{0,\alpha}_\delta(M)$$

is a zero index Fredholm operator whenever $\delta \in (0,n)$. The possible kernel is the L^2-kernel of $-\Delta + n + f$.

Then we derive an isomorphism result by asking that $-\Delta + n + f$ is a perturbation of $-\Delta + n$ with the negative part of f small in integral sense. We will denote

$$f = f^+ - f^-$$

where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

Proposition 2.3. *Suppose that $(M^n,\ g)$ is a weakly asymptotically hyperbolic manifold of class $C^{m,\beta}$. Let $4 \le m+\beta$ and $f \in C^{0,\alpha}_\gamma$ for some $\gamma > 0$. Then there is a positive number ϵ_0 such that, if*

$$\left(\int_M |f^-|^{\frac{n}{2}}\, dvol\right)^{\frac{2}{n}} \le \epsilon_0\,, \tag{2.5}$$

then

$$-\Delta + n + f : C^{2,\alpha}_\delta(M) \to C^{0,\alpha}_\delta(M) \tag{2.6}$$

is an isomorphism when $\delta \in (0,n)$.

Proof. Suppose that v is a function in the L^2-kernel of the operator $-\Delta + n + f$. Due to some standard weighted L^2 estimates (cf. Lemma 4.8 in [10], for instance) we know that $v \in W^{2,2}(M)$ and solves the equation

$$-\Delta v + nv + fv = 0\,. \tag{2.7}$$

Let ρ be a geodesic defining function for the weakly asymptotically hyperbolic manifold $(M^n,\ g)$. For $\epsilon > 0$ let

$$M_\epsilon = \left\{p \in M : 0 < \rho(p) < \epsilon\right\}.$$

Multiplying (1) by v and integrating by parts over $M \backslash M_\epsilon$ we see

$$\begin{aligned} 0 &= \int_{M\backslash M_\epsilon} -v\Delta v + fv^2 + nv^2 \\ &= \int_{M\backslash M_\epsilon} (|\nabla v|^2 + nv^2) + \int_{M\backslash M_\epsilon} fv^2 + \int_{\{\rho=\epsilon\}} v\frac{\partial v}{\partial \vec{n}}\, d\sigma\,. \end{aligned}$$

Now $v \in W^{2,2}(M)$ so for a fixed small number $\epsilon_1 > 0$

$$\int_0^{\epsilon_1} \int_{\rho=s} |v||\nabla v| d\sigma \frac{ds}{s} = \int_{M\backslash M_{\epsilon_1}} |v||\nabla v| < \infty\,.$$

Therefore, there is a sequence of $\epsilon_i \to 0$ such that

$$\int_{\rho=\epsilon_i} |v||\nabla v| d\sigma \to 0\,,$$

which implies

$$\int_M \left(|\nabla v|^2 + nv^2\right) = -\int_M f v^2 .$$

Then, by Hölder inequality,

$$\int_M \left(|\nabla v|^2 + nv^2\right) \leq \int_M f^- v^2 \leq \left(\int_M (f^-)^{\frac{n}{2}}\right)^{\frac{2}{n}} \left(\int_M v^{\frac{2n}{n-2}}\right)^{1-\frac{2}{n}} .$$

Next we apply the Sobolev embedding theorem and obtain

$$\int_M \left(|\nabla v|^2 + nv^2\right) \leq C \left(\int_M (f^-)^{\frac{n}{2}}\right)^{\frac{2}{n}} \int_M \left(|\nabla v|^2 + v^2\right) , \tag{2.8}$$

where C here is the Sobolev constant, which is independent of v. Thus, for

$$\epsilon_0 = \frac{1}{2C} ,$$

we may conclude that $v = 0$. So the proposition follows from Lemma 2.2. □

3. Conformal deformations

In this section we discuss the conformal deformation of the scalar curvature on an asymptotically hyperbolic manifold $(M^n,\ g)$. This idea comes from the work in [18] where the analogous situation was treated in the context of asymptotically flat manifolds.

Lemma 3.1. *Suppose that v is a positive solution to the following equation*

$$-\Delta v + nv - \frac{n-2}{4(n-1)}\big(R + n(n-1)\big)^- v = \frac{n-2}{4(n-1)}\big(R + n(n-1)\big)^- \tag{3.1}$$

on a manifold $(M^n,\ g)$. Then

$$R\left[(1+v)^{\frac{4}{n-2}} g\right] \geq -n(n-1) .$$

Proof. Let $u = 1 + v$. Then

$$\begin{aligned} -\Delta u + \frac{n-2}{4(n-1)} Ru &= -\Delta v + \frac{n-2}{4(n-1)}\big(R + n(n-1)\big)u - \frac{n(n-2)}{4} u \\ &\geq -\Delta v + nv - \frac{n-2}{4(n-1)}\big(R + n(n-1)\big)^- v \\ &\quad - \frac{n-2}{4(n-1)}\big(R + n(n-1)\big)^- \\ &\quad - nv - \frac{n(n-2)}{4}(1+v) \\ &= -\frac{(n-2)}{4(n-1)} n(n-1) \frac{1 + \frac{4}{n-2}\frac{v}{1+v}}{(1+v)^{\frac{4}{n-2}}} u^{\frac{n+2}{n-2}} . \end{aligned}$$

Hence to prove the lemma is to show that

$$1 + \frac{4}{n-2} - \frac{4}{n-2}\frac{1}{1+v} \leq (1+v)^{\frac{4}{n-2}} . \tag{3.2}$$

We differentiate the two sides with respect to v and compare

$$\frac{4}{n-2}(1+v)^{-2} < \frac{4}{n-2}(1+v)^{\frac{4}{n-2}-1} .$$

Therefore, by the fact that the two sides are the same when $v = 0$, the lemma follows. □

The rest of this section is devoted to solving for a positive solution to the equation

$$(-\Delta + n + f)v = h \tag{3.3}$$

on an asymptotically hyperbolic manifold $(M^n,\ g)$ with the function f suitably small in an integral sense. By the isomorphism proposition in the previous section we know, for $\delta \in (0, n)$ and each $h \in C^{0,\alpha}_\delta(M)$, there is a unique solution $v \in C^{2,\alpha}_\delta(M)$ to the equation (3.3). Hence what really need to do is to show that $v > 0$ in M. For simplicity we will denote

$$f = -\frac{n-2}{4(n-1)}\big(R + n(n-1)\big)^- \leq 0 .$$

Proposition 3.2. *Suppose that $(M^n,\ g)$ is a weakly asymptotically hyperbolic manifold of class $C^{m,\beta}$ with $m + \beta \geq 4$. Let ϵ_0 be the small positive number in Proposition 2.3 in the previous section and $\alpha \in (0, 1)$. Suppose that $f \in C^{0,\alpha}_\delta(M)$ for some $\delta \in (0, n)$ and that*

$$\left(\int_M |f|^{\frac{n}{2}}\right)^{\frac{2}{n}} \leq \epsilon_0 . \tag{3.4}$$

Then there is a positive solution $v \in C^{2,\alpha}_\delta(M)$ to the equation

$$-\Delta v + nv + fv = -f . \tag{3.5}$$

Proof. We first prove that v has to be nonnegative in M. Assume otherwise that v is negative somewhere in M so that

$$v_- = \min\big\{v(p) : p \in M\big\} < 0 .$$

Let us consider instead the function $u = v + v_0$ for a small positive number $v_0 < \min\{1, -\frac{v_-}{2}\}$. Then

$$-\Delta u + nu + fu = -f(1 - v_0) + nv_0 > 0$$

in M and $\min\{u(p) : p \in M\} < 0$. Since $v \in C^{2,\alpha}_\delta(M)$ for $\delta > 0$, for a geodesic defining function ρ, we may assume that

$$u > 0 \quad \text{on} \quad \partial(M \setminus M_\tau) = \big\{p \in M : \rho(p) = \tau\big\}$$

provided that $\tau > 0$ is sufficiently small. Now we are going to apply the generalized maximum principle in Section 2.5 in [15] to the function u on the manifold $M \setminus M_\tau$. According the generalized maximum principle what we need is to verify that the first eigenvalue of the operator $-\Delta + n + f$ on the domain $M \setminus M_{\tau'}$ for some $\tau' < \tau$ with Dirichlet boundary condition is positive. Therefore, for any $\phi \in C_c^\infty(M \setminus M_{\tau'})$, we consider the ratio

$$\frac{\int_M (|\nabla\phi|^2 + n\phi^2 + f\phi^2)}{\int_M \phi^2}$$
$$\geq \frac{1}{\int_M \phi^2}\left(\int_M (|\nabla\phi|^2 + \phi^2) - C\left(\int_M |f|^{\frac{n}{2}}\right)^{\frac{2}{n}}\left(\int_M (|\nabla\phi|^2 + \phi^2)\right)\right) \geq \frac{1}{2}\,.$$

Thus the first eigenvalue of the operator $-\Delta + n + f$ on the domain $M \setminus M_{\tau'}$ with the Dirichlet boundary condition is always positive. We may apply Theorem 10 in Section 2.5 of the book [15] to the function u/ϕ, where ϕ is the positive first eigenfunction over $M \setminus M_{\tau'}$, to obtain a contradiction. Therefore v is nonnegative in M. To show that v is in fact positive in M, for each $\tau > 0$, we apply the Hopf strong maximum principle to the function v/ϕ on the domain $M \setminus M_\tau$, where ϕ is the positive first eigenfunction over $M \setminus M_{\tau'}$ for any $0 < \tau' < \tau$. Thus the proof is complete. □

4. The fundamental solutions on the hyperbolic space

The materials in this section are well known and readers are refered to [1, 2, 10, 12] for more detailed account on the references. But for the convenience of the readers we will present a construction briefly. Let us first recall the definition of the hyperbolic space as a hyperboloid in the Minkowski space-time. The Minkowski space-time is R^{n+1} equipped with the Minkowski metric $-dt^2 + |dx|^2$ for $(t, x) \in R^{n+1}$. The upper hyperboloid is the submanifold

$$H^n = \left\{(t, x) \in R^{n+1} : -t^2 + |x|^2 = -1,\ t > 0\right\}. \tag{4.1}$$

Hence

$$(H^n,\ g_H) = \left(R^n,\ \frac{(d|x|)^2}{1 + |x|^2} + |x|^2 g_{S^{n-1}}\right), \tag{4.2}$$

where $g_{S^{n-1}}$ is the standard metric on the unit round $(n-1)$-sphere. We want to find the solution to the equation

$$-\Delta_{H^n} G_0(x) + nG_0(x) = \delta_0(x)\,, \tag{4.3}$$

which defines the Green's function in x centered at the origin of the differential operator $-\Delta + n$ on hyperbolic space H^n. We first compute, for $r = |x|$,

$$(-\Delta_{H^n} + n)r^{-n+2}t^{-k} = -(k-2)(k+n-1)r^{-n+2}t^{-k} + k(k+1)r^{-n+2}t^{-k-2}\,.$$

We then observe inductively that, for even number k

$$(-\Delta_{H^n}+n)\left(r^{-n+2}\left(t^{-2}+\frac{2\cdot 3}{2(n+3)}t^{-4}+\cdots+\frac{2\cdot 3\cdot 4\cdot 5\cdots(k-1)}{(2(n+3)\cdots(k-2)(k+n-1)}t^{-k}\right)\right) = \frac{2\cdot 3\cdot 4\cdot 5\cdots k\cdot(k+1)}{(2(n+3)\cdots(k-2)(k+n-1)}r^{-n+2}t^{-k-2}\,.$$

Therefore we consider the function

$$\tilde{\theta}(t)=\left(1+\sum_{i=2}^{\infty}\prod_{j=2}^{i}\left(1-\frac{n}{2j+n-1}\right)\frac{1}{t^{2i-2}}\right). \tag{4.4}$$

Notice that the infinite series $\tilde{\theta}$ is obviously convergent when $t>1$. In fact, when $t=1$, taking the logarithm of the general term we see

$$\log\prod_{j=2}^{i}\left(1-\frac{n}{2j+n-1}\right)\le -\frac{n}{2}\log\left(i+\frac{n-1}{2}\right)+c(n)$$

for some dimensional constant $c(n)$. Thus the infinite series

$$\tilde{\theta}(1)=1+\sum_{i=2}^{\infty}\prod_{j=2}^{i}\left(1-\frac{n}{2j+n-1}\right) \tag{4.5}$$

converges for all $n\ge 3$. We set

$$\theta(t)=\frac{\tilde{\theta}(t)}{\tilde{\theta}(1)} \tag{4.6}$$

and easily conclude that

Lemma 4.1. *Let*

$$G_0(x)=\frac{\theta(t)}{(n-2)vol(S^{n-1})}\frac{1}{r^{n-2}t^2}\,. \tag{4.7}$$

Then

$$-\Delta_{H^n}G_0(x)+nG_0(x)=\delta_0(x)$$

on hyperbolic space H^n.

To write the fundamental solution at any point in the hyperbolic space we want to express hyperbolic translation in the hyperboloid model of hyperbolic space H^n. Recall that the changes of coordinates between the ball model and hyperboloid model of the hyperbolic space are

$$x=\frac{2}{1-|\bar{x}|^2}\bar{x}\,,\quad t=\frac{1+|\bar{x}|^2}{1-|\bar{x}|^2}\,,$$

and

$$\bar{x}=\frac{1}{1+t}x\,.$$

Also recall that hyperbolic translation by $\bar{b}$ in the ball model is given in [17] by

$$\tau_{\bar{b}}(\bar{x}) = \frac{1-|\bar{b}|^2}{|\bar{x}|^2|\bar{b}|^2+2\bar{x}\cdot\bar{b}+1}\bar{x} + \frac{|\bar{x}|^2+2\bar{x}\cdot\bar{b}+1}{|\bar{x}|^2|\bar{b}|^2+2\bar{x}\cdot\bar{b}+1}\bar{b}\,, \tag{4.8}$$

where $t_x = \sqrt{1+|x|^2}$ and $t_b = \sqrt{1+|b|^2}$. Therefore we have

$$T_b(x) = x + t_x b + \frac{x\cdot b}{1+t_b} b \tag{4.9}$$

with

$$|T_b(x)| = \sinh d_H(x,\ -b)\,. \tag{4.10}$$

One key fact here is that

$$\cosh d_H(x,\ b) = t_x t_b - x\cdot b\,. \tag{4.11}$$

Thus

$$G_H(x,y) = G_y(x) = G_0\big(T_{-y}(x)\big)\,. \tag{4.12}$$

and explicitly

$$G_H(x,y) = \frac{c_n}{\sinh^{n-2} d_H(x,y)\cosh^2 d_H(x,y)}\theta\big(\cosh d_H(x,y)\big) \tag{4.13}$$

where

$$c_n = \frac{1}{(n-2)\mathrm{vol}(S^{n-1})}\,. \tag{4.14}$$

5. Asymptotic behavior

So far, for a weakly asymptotically hyperbolic manifold $(M^n,\ g)$ with

$$\big(R+n(n-1)\big)^- \in C^{0,\alpha}_\delta \quad \text{and} \quad \int_M \Big(\big(R+n(n-1)\big)^-\Big)^{\frac{n}{2}} \le \epsilon_0^{\frac{n}{2}}\,,$$

we have obtained a conformal deformation $g_v = (1+v)^{\frac{4}{n-2}} g$ such that

$$R[g_v] \ge -n(n-1)$$

and

$$0 < v \in C^{2,\alpha}_\delta(M)\,,$$

provided that $\delta \in (0,n)$. Unfortunately the decay rate of v just misses the decay rate on which the mass aspect of an asymptotically hyperbolic manifold is defined. We will use the Green's function we constructed in the pervious section to obtain an expansion at the infinity of the solution v to the equation

$$-\Delta v + nv - \frac{n-2}{4(n-1)}\big(R+n(n-1)\big)^- v = \frac{n-2}{4(n-1)}\big(R+n(n-1)\big)^-\,.$$

We follow the idea used in [18] to write an integral representation of the solution v with the help of the approximate Green's function $G_H(x,y)$ on the asymptotically hyperbolic manifold M. Let us start with a definition of asymptotically hyperbolic manifolds, which should be compared with the definition of weakly asymptotically hyperbolic manifolds given in Section 2. Since we will adopt the definition of mass aspect and mass for asymptotically hyperbolic manifolds from the work [21] we use his definition for asymptotically hyperbolic manifolds.

Definition 5.1. $(M^n,\ g)$ is said to be an asymptotically hyperbolic manifold if $(M^n,\ g)$ is a weakly asymptotically hyperbolic manifold with the standard round sphere $(S^{n-1},\ [g_0])$ as its conformal infinity, and, for a geodesic defining function ρ, in the conformally compact coordinates at the infinity,

$$g = \sinh^{-2}\rho\left(d\rho^2 + g_0 + \frac{1}{n}\rho^n h + O(\rho^{n+1})\right), \tag{5.1}$$

where h is symmetric two tensors on S^{n-1} at each point.

In the light of the above definition, we set up a conformally compact coordinate at the infinity associated with a defining function ρ as follows. Let

$$\psi : M \setminus M_c \to R^n \setminus B_{r_0}(0)\,,$$

for some compact subset $M_c \subset M$, such that

$$g_H = \frac{(d|x|)^2}{1+|x|^2} + |x|^2 g_0 = \sinh^{-2}\rho(d\rho^2 + g_0) \tag{5.2}$$

for $|x| > r_0$ and $\sinh\rho = \frac{1}{|x|}$.

We construct an approximate Green's function of an asymptotically hyperbolic manifold $(M^n,\ g)$. At each point $y \in R^n \setminus B_{r_0}(0)$, we consider the hyperbolic space H^n in the coordinate so that

$$g_H(x) = \frac{1}{1+r_y^2(x)}dr^2 + r_y^2(x)g_0 = (\tilde{g}_H)_{ij}(x)dx_i dx_j\,,$$

where

$$r_y(x) = \sqrt{A_{ij}(y)x_i x_j}\,. \tag{5.3}$$

This coordinate can be made into the standard coordinate by the linear transformation $B : R^n \to R^n$ such that $B^2 = A$. More importantly we need to ask

$$(\tilde{g}_H)_{ij}(y) = g_{ij}(y)\,. \tag{5.4}$$

A simple calculation yields

$$(\tilde{g}_H)_{ij}(x) = A_{ij} - \frac{A_{ik}x_k A_{jl}x_l}{1 + A_{kl}x_k x_l}\,. \tag{5.5}$$

Hence

$$A_{ij}(y) = g_{ij}(y) + \frac{g_{ik}(y)y_k g_{jl}(y)y_l}{1 - g_{kl}(y)y_k y_l}\,. \tag{5.6}$$

Therefore, since

$$g_{ij}(x) = \delta_{ij} - \frac{x_i x_j}{1+|x|^2} + O\big(|x|^{-n}\big)\tilde{h}_{ij}(x) \tag{5.7}$$

and

$$\tilde{h}_{ij}(x)x_j = 0\,, \tag{5.8}$$

we have

$$A_{ij}(y) = g_{ij}(y) + \frac{y_i y_j}{1+|y|^2} = \delta_{ij} + O\big(|y|^{-n}\big)\tilde{h}_{ij}(y)\,. \tag{5.9}$$

Let $\tilde{d}_H(x,y)$ be the hyperbolic distance function in the metric $(\tilde{g}_H)_{ij}(x)dx_i dx_j$ and let

$$G_y(x) = \frac{1}{(n-2)\mathrm{vol}(S^{n-1})} \frac{\theta\big(\cosh \tilde{d}_H(x,y)\big)}{\sinh^{n-2}\tilde{d}_H(x,y)\cosh^2\tilde{d}_H(x,y)}\,. \tag{5.10}$$

In the geodesic ball $B_1(y)$ in the metric g we calculate

$$g_{ij}(x) = (\tilde{g}_H)_{ij}(x) + \tilde{d}_H(x,y)O\big(|y|^{-n}\big) \tag{5.11}$$

and

$$\begin{aligned}\Delta_g &= \frac{1}{\sqrt{\det g}}\partial_j\left(\sqrt{\det g}\, g^{ij}\partial_j\right) \\ &= \Delta_H + \tilde{d}_H(x,y)O\big(|y|^{-n}\big)\Delta_H + (\tilde{g}_H)^{ij}\partial_i\big(O(|y|^{-n})\tilde{d}_H(x,y)\big)\partial_j\,.\end{aligned} \tag{5.12}$$

Thus, for any $x \in B_1(y)$ and $x \neq y$,

$$\Psi_y(x) = -\Delta_g G_y(x) + nG_y(x) = O\big(|y|^{-n}\big)O\left(\tilde{d}_H(x,y)^{-n+1}\right), \tag{5.13}$$

as $|x-y| \to 0$ and $|y| \to \infty$. On the other hand, outside the geodesic ball $B_1(y)$, we simply need

$$g_{ij}(x) = (\tilde{g}_H)_{ij}(x) + O\big(|x|^{-n}\big)\tilde{h}_{ij}(x) + O\big(|y|^{-n}\big)\xi_{ij}(x,y)\,,$$

as $|x| \to \infty$ and $|y| \to \infty$, which follows from some calculations, where

$$\xi_{ij}(x,y) = \tilde{h}_{ij}(y) - \frac{\tilde{h}_{ik}(y)x_k x_j + \tilde{h}_{jk}x_k x_i}{1+|x|^2} + \frac{x_i x_j}{1+|x|^2}\frac{\tilde{h}_{kl}x_k x_l}{1+|x|^2}\,.$$

Therefore

$$\xi_{ij}x_j = \frac{\tilde{h}_{ij}x_j}{1+|x|^2} - \frac{x_i}{1+|x|^2}\frac{\tilde{h}_{kl}x_k x_l}{1+|x|^2}$$

and

$$(\tilde{g}_H)^{ij} = \delta_{ij} + x_i x_j + O\big(|y|^{-n}\big)\xi_{ij}\,.$$

This implies

$$g^{ij} = (\tilde{g}_H)^{ij} + O\big(|y|^{-n}\big)\xi_{ij} + O\big(|x|^{-n}\big)\tilde{h}_{ij} + \text{ higher order terms}\,.$$

Here we use the facts that

$$(\delta_{ik} + x_i x_k)\xi_{kl}(\delta_{lj} + x_l x_j) = \xi_{ij}$$

and

$$(\delta_{ik} + x_i x_k)\tilde{h}_{kl}(\delta_{lj} + x_l x_j) = \tilde{h}_{ij}\,.$$

Therefore, outside the geodesic ball $B_1(y)$,

$$\begin{aligned}\Delta_g = \Delta_{\tilde{g}_H} &+ \Big(O\big(|y|^{-n}\big) + O\big(|x|^{-n}\big)\Big)\Delta_{\tilde{g}_H} \\ &+ (\tilde{g}_H)^{ij}\partial_i\Big(O(|y|^{-n}) + O(|x|^{-n})\Big)\partial_j\,.\end{aligned}$$

One last calculation we need is an estimate for $\Psi_y(x)$ outside the geodesic ball $B_1(y)$. We compute

$$\partial_i \tilde{d}_H(x,y) = \frac{1}{\sinh \tilde{d}_H(x,y)} \left(\frac{A_{ik}(y)x_k}{t_x} t_y - A_{ik}(y)y_k \right),$$

$$G'(s) = c_n \left(-\frac{(n-2)\theta(\cosh s)}{\sinh^{n-1} s \cosh s} - \frac{2\theta(\cosh s)}{\sinh^{n-3} s \cosh^3 s} + \frac{\theta'(\cosh s)}{\sinh^{n-3} s \cosh^2 s} \right),$$

and

$$\cosh^n \tilde{d}_H(y,x) G'\big(\tilde{d}_H(y,x)\big) \to -nc_n\,,$$

as $\tilde{d}_H(y,x) \to \infty$. Thus, outside the geodesic ball $B_1(y)$,

$$\begin{aligned} \Psi_y(x) &= -\Delta_g G_y(x) + nG_y(x) \\ &= \Big(O\big(|x|^{-n}\big) + O\big(|y|^{-n}\big) \Big) O\left(\frac{1}{\cosh^n \tilde{d}_H(x,y)} \right). \end{aligned} \tag{5.14}$$

Lemma 5.2. *Suppose that $(M^n,\ g)$ is an asymptotically hyperbolic manifold. Then*

$$-\Delta G_y(x) + nG_y(x) = \delta_y(x) + \Psi_y(x) \tag{5.15}$$

where $\Psi_y(x)$ satisfies the estimates (5.13) *and* (5.14).

As a consequence we have the following integral representation.

Proposition 5.3. *Suppose that $(M^n,\ g)$ is an asymptotically hyperbolic manifold and that*

$$\psi : M \setminus M_c \to R^n \setminus B_{r_0}(0)$$

is a conformally compact coordinate associated with a defining function ρ in which

$$g = \sinh^{-2}\rho \left(d\rho^2 + g_0 + \frac{\rho^n}{n} h + O\big(\rho^{n+1}\big) \right).$$

Suppose that $v \in C^{2,\alpha}_\delta(M)$ solves the equation

$$-\Delta v + nv + fv = w \in C^{0,\alpha}_\delta(M)\,,$$

where $f \in C^{0,\alpha}_\delta(M)$ and $\delta \in (0,n)$. Then, for each $x \in R^n \setminus B_{r_0}(0)$,

$$\begin{aligned} v(x) = &- \int_{R^n \setminus B_{r_0}(0)} v(y)\Psi_x(y) dvol_g(y) \\ &+ \int_{R^n \setminus B_{r_0}(0)} \big(w(y) - f(y)v(y)\big) G_x(y) dvol_g(y) \\ &- \int_{\partial B_{r_0}(0)} \frac{\partial G_x}{\partial n}(y) v(y) d\sigma_g(y) \\ &+ \int_{\partial B_{r_0}(0)} \frac{\partial v}{\partial n}(y) G_x(y) d\sigma_g(y)\,. \end{aligned} \tag{5.16}$$

Proof. We use the density property (cf. [10]) of the the space $C^{2,\alpha}_\delta(M)$ to have a sequence of functions $v_n \in C^\infty_c(M)$ such that

$$v_n \to v \quad \text{in} \quad C^{2,\alpha}_\delta(M)\,.$$

Then from (5.16) we have, for v_n,

$$\begin{aligned} v_n(x) = &-\int_{R^n\setminus B_{r_0}(0)} v_n(y)\Psi_x(y)dvol_g(y) \\ &+\int_{R^n\setminus B_{r_0}(0)} (-\Delta v_n + nv_n)G_x(y)dvol_g(y) \\ &-\int_{\partial B_{r_0}(0)} \frac{\partial G_x}{\partial n}(y)v_n(y)d\sigma_g(y) \\ &+\int_{\partial B_{r_0}(0)} \frac{\partial v_n}{\partial n}(y)G_x(y)d\sigma_g(y)\,. \end{aligned} \tag{5.17}$$

Hence, by taking the limit, we obtain (5.16) for v. □

Now we are ready to state and prove our main result of this section.

Theorem 5.4. *Suppose that $(M^n,\ g)$ is an asymptotically hyperbolic manifold and that*

$$\psi : M\setminus M_c \to R^n \setminus B_{r_0}(0)$$

is a conformally compact coordinate associated with a defining function ρ in which

$$g = \sinh^{-2}\rho\left(d\rho^2 + g_0 + \frac{\rho^n}{n}h + O(\rho^{n+1})\right).$$

Suppose that $v \in C^{2,\alpha}_\delta(M)$ with $\delta > 0$ solves the equation

$$-\Delta v + nv + fv = w$$

with

$$f \in C^{0,\alpha}_\kappa(M) \quad \textit{and} \quad w \in C^{2,\alpha}_\eta(M)$$

for some $\kappa > 2$ and $\eta > n+1$. Then, for each $x \in R^n \setminus B_{r_0}(0)$,

$$v(x) = A\left(\frac{x}{|x|}\right)|x|^{-n} + O\left(|x|^{-(n+1)}\right). \tag{5.18}$$

Remark 5.5. We would like to point out that the expansion (5.18) is a simple consequence of the work in [2, 12]. But we need some explicit expression of the coefficient A in (5.18) to prove Theorem 6.3 and Lemma 6.5 in the following section, which we did not find that it is easier to extract it from [2, 12] than to obtain it in the way presented here. The explicit expression of A will be obtained in the course of the following proof of Theorem 5.5 based on the integral representation of the solution v in (5.16).

Proof of Theorem 5.4. We are going to study the asymptotic behavior of $v(x)$ term by term in (5.16). We treat the easy ones first. First we consider

$$|x|^n \int_{\partial B_{r_0}(0)} \frac{\partial v}{\partial n}(y) G_x(y) d\sigma(y)\,,$$

as $|x| \to \infty$ and $y \in \partial B_{r_0}(0)$. Now

$$|x|^n G_x(y) = \frac{|x|^n}{\cosh^n \tilde{d}_H(y,x)} \frac{c_n \cosh^{n-2} \tilde{d}_H(y,x)}{\sinh^{n-2} \tilde{d}_H(y,x)} \theta\big(\cosh \tilde{d}_H(y,x)\big)\,,$$

where

$$\cosh \tilde{d}_H(y,x) = t_x t_y - A_{ij}(x) x^i\ y^j$$

and

$$A_{ij}(x) = \delta_{ij} + O\big(|x|^{-n}\big)\,.$$

Hence

$$|x|^n G_x(y) = \frac{1}{\left(\frac{t_x}{|x|} t_y - A_{ij}(x) \frac{x^i}{|x|}\ y^j\right)^n} \frac{c_n \cosh^{n-2} \tilde{d}_H(y,x)}{\sinh^{n-2} \tilde{d}_H(y,x)} \theta\big(\cosh \tilde{d}_H(y,x)\big) \in C^1(M)$$

and

$$\lim_{\lambda \to \infty} \lambda^n G_{\lambda \frac{x}{|x|}}(y) = c_n \left(t_y - \frac{x}{|x|} \cdot y\right)^{-n}\,.$$

Therefore

$$|x|^n \int_{\partial B_{r_0}(0)} \frac{\partial v}{\partial n}(y) G_x(y) d\sigma(y) \in C^1(M)$$

and

$$\begin{aligned} A_1\left(\frac{x}{|x|}\right) &= \lim_{\lambda \to \infty} \lambda^n \int_{\partial B_{r_0}(0)} \frac{\partial v}{\partial n}(y) G_{\lambda \frac{x}{|x|}}(y) d\sigma(y) \\ &= c_n \int_{\partial B_{r_0}(0)} \frac{\partial v}{\partial n}(y) \left(t_y - \frac{x}{|x|} \cdot y\right)^{-n} d\sigma(y)\,. \end{aligned} \tag{5.19}$$

Next we consider

$$|x|^n \int_{\partial B_{r_0}(0)} \frac{\partial G_x}{\partial n}(y) v(y) d\sigma(y)\,,$$

as $|x| \to \infty$ and $y \in \partial B_{r_0}(0)$. We compute

$$\begin{aligned} |x|^n \frac{\partial G_x}{\partial n}(y) &= |x|^n \rho(y) c_n G'\big(\tilde{d}_H(y,x)\big) \frac{\partial \tilde{d}_H(y,x)}{\partial r} \\ &= |x|^n \rho(y) G' \frac{t_x \frac{g_{ij} y^i\ y^j}{|y| t_y} - g_{ij} x^i\ \frac{y^j}{|y|}}{\sinh \tilde{d}_H(y,x)}\,, \end{aligned}$$

where

$$G'(s) = c_n \left(-\frac{(n-2)\theta(\cosh s)}{\sinh^{n-1} s \cosh s} - \frac{2\theta(\cosh s)}{\sinh^{n-3} s \cosh^3 s} + \frac{\theta'(\cosh s)}{\sinh^{n-3} s \cosh^2 s}\right)$$

and

$$\cosh^n \tilde{d}_H(y,x) G'\big(\tilde{d}_H(y,x)\big) \to -nc_n$$

as $\tilde{d}_H(y,x) \to \infty$. Therefore

$$\begin{aligned} A_2\left(\frac{x}{|x|}\right) &= \lim_{\lambda\to\infty} \lambda^n \int_{\partial B_{r_0}(0)} \frac{\partial G_{\lambda\frac{x}{|x|}}}{\partial n}(y) v(y) d\sigma(y) \\ &= -nc_n \int_{\partial B_{r_0}(0)} v(y) \left(t_y - \frac{x}{|x|}\cdot y\right)^{-n} \frac{\frac{|y|}{t_y} - \frac{x\cdot y}{|x||y|}}{t_y - \frac{x}{|x|}\cdot y} d\sigma(y)\,. \end{aligned} \tag{5.20}$$

For the term

$$|x|^n \int_{R^n\setminus B_{r_0}(0)} (h - fv) G_x(y) dvol_g(y)\,,$$

we know, for any given $y \in R^n \setminus B_{r_0}(0)$,

$$\lim_{\lambda\to\infty} \lambda^n G_{\lambda\frac{x}{|x|}}(y) = c_n \left(t_y - \frac{x}{|x|}\cdot y\right)^{-n}.$$

We observe that

$$t_y - \frac{x}{|x|}\cdot y = t_y - |y|\cos\phi \geq (1-\cos\phi)|y|\,,$$

where ϕ is the angle between x and y. Fixing a direction $\frac{x}{|x|}$, we easily see that for any $\epsilon_0 > 0$

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^n \int_{\{y\in R^n\setminus B_{r_0}(0):\ \cos\phi\leq 1-\epsilon_0\}} (h - fv) G_{\lambda\frac{x}{|x|}}(y) dvol_g(y) \\ &\qquad = \int_{\{y\in R^n\setminus B_{r_0}(0):\ \cos\phi\leq 1-\epsilon_0\}} (h - fv) \left(t_y - \frac{x}{|x|}\cdot y\right)^{-n} dvol_g(y)\,. \end{aligned}$$

On the other hand, when $\cos\phi > 1 - \epsilon_0$, it suffices to verify the claim

$$\int_{r_0}^{\infty} \int_{\{\cos\phi > 1-\epsilon_0\}} (h - fv)(t_y - r\cos\phi)^{-n} \frac{r^{n-1}}{t_y} d\sigma_0 dr < \infty\,. \tag{5.21}$$

Here we need to use the fact that $\eta > n$. We simply notice that

$$t_y - |y|\cos\phi = \frac{1 + \sin^2\phi |y|^2}{t_y + |y|\cos\phi}\,.$$

Hence

$$\begin{aligned} \int_{\{\cos\phi>1-\epsilon_0\}} \big(t_y - |y|\cos\phi\big)^{-n} d\sigma &\lessapprox \int_0^{\epsilon_0} \int_{S^{n-2}} \big(t_y - |y|\cos\phi\big)^{-n} \phi^{n-2} d\sigma d\phi \\ &\lessapprox \int_0^{\epsilon_1} \int_{S^{n-2}} \big(t_y - |y|\cos\phi\big)^{-n} \phi^{n-2} d\sigma d\phi \\ &\quad + \int_{\epsilon_1}^{\epsilon_0} \int_{S^{n-2}} \big(t_y - |y|\cos\phi\big)^{-n} \phi^{n-2} d\sigma d\phi \\ &\lessapprox |y|^n \epsilon_1^{n-1} + |y|^{-n} \epsilon_1^{-n-1} \lessapprox |y| \end{aligned}$$

for $\epsilon_1 = |y|^{-1} < \epsilon_0$. Therefore

$$\int_{\{\cos\phi > 1-\epsilon_0\}} (h - fv)\big(t_y - r\cos\phi\big)^{-n} \frac{r^{n-1}}{t_y} d\sigma_0 = O\big(r^{-\iota+n-1}\big)\,,$$

where $\iota = \min\{\eta, n + \frac{1}{2}\delta\} > n$, which implies our claim (5.21). Thus

$$\begin{aligned} A_0\left(\frac{x}{|x|}\right) &= \lim_{\lambda\to\infty} \lambda^n \int_{R^n\setminus B_{r_0}(0)} (h - fv) G_{\lambda\frac{x}{|x|}}(y) dvol_g(y) \\ &= \int_{R^n\setminus B_{r_0}(0)} (h - fv) \left(t_y - \frac{x}{|x|}\cdot y\right)^{-n} dvol_g(y)\,. \end{aligned} \tag{5.22}$$

A similar argument yields the next order when we have $\kappa > 2$ and $\eta > n+1$. For the last term

$$|x|^n \int_{R^n\setminus B_{r_0}(0)} v(y)\Psi_x(y) dvol_g(y)\,,$$

we need to use the estimates about the correction term $\Psi_x(y)$ in (5.13) and (5.14). We first look at

$$\begin{aligned} |x|^n \int_{B_1(x)} v(y)\Psi_x(y) dvol_g(y) &\lessapprox |x|^n \int_0^1 \int_{S^{n-1}} v(y)\Psi_x(y) \sinh^{n-1} r d\sigma dr \\ &\lessapprox |x|^n \int_0^1 |y|^{-n+\epsilon} |x|^{-n} r^{-n+1} r^{n-1} dr \end{aligned}$$

for any small positive number ϵ. Clearly

$$\lim_{|x|\to\infty} |x|^n \int_{B_1(x)} v(y)\Psi_x(y) dvol_g(y) = 0 \tag{5.23}$$

since $|y| \geq c|x|$ for $y \in B_1(x)$ and $|x| \to \infty$. Next we look at

$$|x|^n \int_{(R^n\setminus B_{r_0}(0))\setminus B_1(x)} v(y)\Psi_x(y) dvol_g(y)\,.$$

In the light of (5.14) and (5.23), using the argument we used to treat last term to obtain (5.21) and (5.22), we have

$$\begin{aligned} A_{-1}\left(\frac{x}{|x|}\right) &= \lim_{\lambda\to\infty} \lambda^n \int_{R^n\setminus B_{r_0}(0)} v(y)\Psi_{\lambda\frac{x}{|x|}}(y) dvol_g(y) \\ &= \lim_{\lambda\to\infty} \lambda^n \int_{(R^n\setminus B_{r_0}(0))\setminus B_1(x)} v(y)\Psi_{\lambda\frac{x}{|x|}}(y) dvol_g(y)\,. \end{aligned} \tag{5.24}$$

We have thus proven the theorem with

$$A\left(\frac{x}{|x|}\right) = A_{-1}\left(\frac{x}{|x|}\right) + A_0\left(\frac{x}{|x|}\right) + A_1\left(\frac{x}{|x|}\right) + A_2\left(\frac{x}{|x|}\right). \tag{5.25}$$

$\square$

6. Proof of the main theorem

In this section we prove the main theorem. We first recall a positive mass theorem for asymptotically hyperbolic manifolds from [21]. Readers are referred to [6] for more elaborated and complete discussions of positive mass theorems for asymptotically hyperbolic manifolds. Recall that, on an asymptotically hyperbolic manifold $(M^n,\ g)$ as defined in Definition 5.1, we have a coordinate at the infinity such that

$$g = \sinh^{-2}\rho\left(d\rho^2 + g_0 + \frac{\rho^n}{n}h + O(\rho^{n+1})\right). \tag{6.1}$$

In [21] it was proven that

Theorem 6.1 (Xiaodong Wang). *Suppose that $(M^n,\ g)$ is a spin asymptotically hyperbolic manifold and that $R_g \geq -n(n-1)$. Then*

$$\int_{S^{n-1}} Tr_{g_0}h(x)dvol_{g_0}(x) \geq \left|\int_{S^{n-1}} Tr_{g_0}h(x)xdvol_{g_0}(x)\right|. \tag{6.2}$$

Moreover the equality holds if and only if $(M^n,\ g)$ is isometric to the standard hyperbolic space H^n.

We adopt the idea from [13] to deal with asymptotically hyperbolic manifolds with corners along a hypersurface.

Definition 6.2. A Riemannian manifold $(M^n,\ g)$ is said to have corners along a hypersurface Σ if there is a smooth embedded hypersurface $\Sigma \subset M$ such that $M \setminus \Sigma = M_- \bigcup M_+$ and the inside $(M_-,\ g_-) = (M_-,\ g)$ is a smooth compact Riemannian manifold with a boundary Σ and the outside $(M_+,\ g_+) = (M_+,\ g)$ is a smooth Riemannian manifold with a boundary Σ. Moreover g_- and g_+ agree on the boundary Σ, that is, g continuous across the hypersurface $\Sigma \subset M$.

We will consider the outward mean curvature H_- of the hypersurface Σ in $(M_-,\ g_-)$ and the inward mean curvature H_+ of the hypersurface Σ in $(M_+,\ g_+)$. Near the hypersurface Σ we may use Gauss coordinates, that is, for some $\nu_0 > 0$, a point p within distance ν_0 from the hypersurface Σ is labeled by a point x on the hypersurface Σ and the signed distance $d = \text{dist}(p, \Sigma)$ to the hypersurface Σ. We now recall the smoothing operation given in Proposition 3.1 in [13] to have C^2 metrics on M approximating g.

Proposition 6.3 (Pengzi Miao). *Suppose that $(M,\ g)$ is a manifold with corners along a hypersurface Σ. Then there is a family of C^2 metrics g_ν, for $\nu \in (0, \nu_0)$, on M such that g_ν uniformly converges to g on M and $g_\nu = g$ outside $\Sigma \times (-\frac{1}{2}\nu, \frac{1}{2}\nu)$. Furthermore, the scalar curvature R_ν of the metric g_ν satisfies*

$$\begin{cases} R_\nu(p) = O(1)in & when \quad d \in \left(\frac{\nu^2}{100}, \frac{\nu}{2}\right] \\ R_\nu(p) = O(1) + 2(H_- - H_+)\left(\frac{100}{\nu^2}\phi\left(\frac{100}{\nu^2}\right)\right) & when \quad d \leq \frac{\nu^2}{100}, \end{cases} \tag{6.3}$$

where $O(1)$ stands for terms bounded independent of ν and $\phi(t) \in C_c^\infty(-1,1)$ is a standard mollifier.

Our next goal is to conformally deform the metric g_ν so that the scalar curvature is greater than or equal to $-n(n-1)$ so that the positive mass theorem in [21] applies. The reason that g_ν admits such conformal deformation relies on the fact that

$$\int_M \left[\frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^-\right]^{\frac{n}{2}} dvol_{g_\nu} \leq \epsilon_0^{\frac{n}{2}}$$

whenever ν is sufficiently small and $H_- - H_+ \geq 0$. Thus we are ready to state and prove our main theorem.

Theorem 6.4. *Suppose that $(M,\ g)$ is a spin Riemannian manifold with corners along a hypersurface Σ and that the outside is an asymptotically hyperbolic manifold and the inside is compact. Suppose that the scalar curvature of both the inside and outside metrics are greater than or equal to $-n(n-1)$ and that*

$$H_-(x) \geq H_+(x)$$

for each x on the hypersurface. Then, if in a coordinate system at the infinity,

$$g = \sinh^{-2}\rho\left(d\rho^2 + g_0 + \frac{\rho^n}{n}h + O(\rho^{n+1})\right),$$

then

$$\int_{S^{n-1}} Tr_{g_0}h(x)dvol_{g_0}(x) \geq \left|\int_{S^{n-1}} Tr_{g_0}h(x)x dvol_{g_0}(x)\right|.$$

Proof. We first use the smoothing operation given in [13] as stated in the above proposition. For each small $\nu < \nu_0$, we then solve the equation

$$-\Delta_{g_\nu}v + nv + f_\nu v = -f_\nu \tag{6.4}$$

on M for

$$f_\nu = -\frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^-.$$

According to Proposition 6.2 above

$$\int_M f_\nu^{\frac{n}{2}} dvol_{g_\nu} \leq C(g)\nu,$$

where $C(g)$ depends only on the metric g. For sufficiently small ν we apply Proposition 3.2 in Section 3 to obtain a positive solution v_ν to the above equation (6.4). Then we consider the new metric

$$\tilde{g}_\nu = (1+v_\nu)^{\frac{4}{n-2}}g_\nu.$$

In the light of Lemma 3.1 in Section 3 we know that the scalar curvature $\tilde{R}_\nu$ of the new metric $\tilde{g}_\nu$ is greater than or equal to $-n(n-1)$. To finish the proof we need to establish the following two lemmas. □

Lemma 6.5. *Suppose that $(M^n,\ g)$ is an asymptotically hyperbolic manifold and in a coordinate at the infinity associated with a geodesic defining function r*

$$g = \sinh^{-2}\rho\left(d\rho^2 + g_0 + \frac{\rho^n}{n}h + O(\rho^{n+1})\right),$$

where

$$r = \frac{\cosh\rho - 1}{\sinh\rho}.$$

And suppose that

$$v = A\left(\frac{x}{|x|}\right)\rho^n + O(\rho^{n+1})$$

is a positive function on M. Then there is a geodesic defining function $\tilde r$ for $\tilde g = (1+v)^{\frac{4}{n-2}} g$ such that

$$\tilde g = \sinh^{-2}\tilde\rho\left(d\tilde\rho^2 + g_0 + \frac{\tilde\rho^n}{n}\tilde h + O(\tilde\rho^{n+1})\right),$$

where

$$\tilde r = \frac{\cosh\tilde\rho - 1}{\sinh\tilde\rho}$$

and

$$\tilde h = \frac{4(n+1)}{n-2}A\left(\frac{x}{|x|}\right)g_0 + h\,. \tag{6.5}$$

Proof. First we recall that the geodesic defining function of the metric g is a defining function s such that

$$|ds|_{s^2 g} = 1$$

near the infinity. We refer the readers to Lemma 2.1 in [7] for the existence and uniqueness of the geodesic defining function associated with each boundary metric in the conformal infinity. We start with a geodesic defining function r for g. Then for each $\theta \in S^{n-1}$, let

$$\tilde r = e^w r \quad \text{and} \quad w(\theta, 0) = 0\,.$$

By the definition, w satisfies

$$2\frac{\partial w}{\partial r} + r|dw|^2_{r^2 g} = \frac{1}{r}\left((1+v)^{\frac{4}{n-2}} - 1\right) = \frac{4}{n-2}Ar^{n-1} + O(r^n)\,. \tag{6.6}$$

By an inductive argument we obtain

$$\frac{\partial^k w}{\partial r^k}(\theta, 0) = 0$$

for $k \le n-1$ and

$$\frac{\partial^n w}{\partial r^n}(\theta, 0) = (n-1)!\frac{2}{n-2}A(\theta)\,. \tag{6.7}$$

Hence

$$w(\theta, r) = \frac{2}{n(n-2)}A(\theta)r^n + O(r^{n+1})\,.$$

This gives

$$\tilde r(\theta, r) = r + \frac{2}{n(n-2)}A(\theta)r^{n+1} + O(r^{n+2})\,. \tag{6.8}$$

By the construction of the coordinate associated with a geodesic defining function, we need to compare the integral curves of the vector field $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \tilde{r}}$. We know

$$d\tilde{r} = \left(1 + \frac{2(n+1)}{n(n-2)} Ar^n\right) dr + \frac{2}{n(n-2)} r^{n+1} \frac{\partial A}{\partial \theta_i} d\theta_i + O(r^{n+1}),$$

which implies

$$\begin{aligned}\frac{\partial}{\partial \tilde{r}} &= (1+v)^{-\frac{4}{n-2}} \left(1 + \frac{2(n+1)}{n(n-2)} Ar^n\right) \frac{\partial}{\partial r} \\ &\quad + (1+v)^{-\frac{4}{n-2}} \left(r^{n+1} \frac{2}{n(n-2)} \frac{\partial A}{\partial \theta_j} + O\big(r^{n+2}\big)\right) g_r^{ij} \frac{\partial}{\partial \theta_i} \\ &= \frac{\partial}{\partial r} - \frac{2(n-1)}{n(n-2)} Ar^n \frac{\partial}{\partial r} + O\big(r^{n+1}\big).\end{aligned} \tag{6.9}$$

Therefore

$$\tilde{\theta}(\theta, r) = \theta + O\big(r^{n+1}\big). \tag{6.10}$$

Thus

$$\sinh^2 \tilde{\rho}\, \tilde{g} \left(\frac{\partial}{\partial \tilde{\theta}_i}, \frac{\partial}{\partial \tilde{\theta}_j}\right) = \sinh^2 \tilde{\rho}\, (1+v)^{\frac{4}{n-2}} g \left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right) + O\big(r^{n+1}\big).$$

In the light of the fact that

$$\begin{aligned}\frac{\sinh \tilde{\rho}}{1 + \cosh \tilde{\rho}} = \tilde{r} &= r\left(1 + \frac{2}{n(n-2)} Ar^n + O(r^{n+1})\right) \\ &= \frac{\sinh \rho}{1 + \cosh \rho}\left(1 + \frac{2}{n(n-2)} Ar^n + O(r^{n+1})\right)\end{aligned}$$

we have

$$\sinh^2 \tilde{\rho} = \sinh^2 \rho \left(\frac{1 + \cosh \tilde{\rho}}{1 + \cosh \rho}\right)^2 \left(1 + \frac{4}{n(n-2)} Ar^n + O\big(r^{n+1}\big)\right),$$

where

$$\begin{aligned}\frac{1 + \cosh \tilde{\rho}}{1 + \cosh \rho} &= 1 + \frac{\cosh \tilde{\rho} - \cosh \rho}{1 + \cosh \rho} \\ &= 1 + O(r)(\tilde{\rho} - \rho) \\ &= 1 + O(r)\big(\tanh^{-1} \tilde{r} - \tanh^{-1} r\big) \\ &= 1 + O\big(r^{n+1}\big).\end{aligned}$$

Finally, we arrive at

$$g_0 + \frac{\tilde{\rho}^n}{n} \tilde{h} + O(\tilde{\rho}^{n+1}) = g_0 + \left(\frac{4(n+1)}{n-2} \rho^n A(\theta) g_0 + \frac{\rho^n}{n} h(\theta)\right) + O(\rho^{n+1}), \tag{6.11}$$

which gives

$$\tilde{h} = \frac{4(n+1)}{n-2} A\left(\frac{x}{|x|}\right) g_0 + h \tag{6.12}$$

So the calculation is completed. □

The next lemma is an estimate of the perturbation of mass aspect $\frac{4(n+1)}{n-2} A_\nu(\frac{x}{|x|})g_0$ in terms of the small number ν as $\nu \to 0$ when $v = v_\nu$.

Lemma 6.6. *Suppose that $(M,\ g)$ is a complete Riemannian manifold with corners along a hypersurface and that the outside is an asymptotically hyperbolic manifold. Suppose that the scalar curvature of both the inside and outside metrics are greater than or equal to $-n(n-1)$ and that*

$$H_-(x) \geq H_+(x)$$

for each x on the hypersurface. Let g_ν be constructed as in Proposition 6.2. Then there is a unique positive solution $v_\nu \in C^{2,\alpha}_\delta(M)$ to the equation

$$-\Delta_{g_\nu} v + nv - \frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^- v = \frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^-\,,$$

when ν is sufficiently small. Moreover, in a coordinate at the infinity associated with a geodesic defining function r,

$$v_\nu = A_\nu\left(\frac{x}{|x|}\right) r^n + O(r^{n+1})$$

and

$$\left| A_\nu\left(\frac{x}{|x|}\right)\right| \leq C\nu^{\frac{1}{n+1}}\,, \tag{6.13}$$

where C is independent of ν.

Proof. By Proposition 6.2 we have

$$\frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^- \leq C$$

with compact support inside $\partial\Omega \times [-\frac{\nu}{2}, \frac{\nu}{2}]$, where C is independent of ν. Hence

$$\int_M \left(\frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^-\right)^{\frac{n}{2}} dvol_{g_\nu} \leq C\nu\,.$$

Therefore, by Proposition 3.2 and Theorem 5.5, there is exists the unique positive solution to the equation

$$-\Delta_{g_\nu} v + nv - \frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^- v = \frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^-\,,$$

when ν is sufficiently small and in a coordinate at the infinity associated with a geodesic defining function r,

$$v_\nu = A_\nu\left(\frac{x}{|x|}\right) r^n + O(r^{n+1})\,,$$

where $A(\frac{x}{|x|})$ is given in (5.25).

First of all, since

$$\left\|\frac{n-2}{4(n-1)}\big(R_\nu + n(n-1)\big)^-\right\|_{W^{0,n+1}_\gamma(M)} \leq C\nu^{\frac{1}{n+1}} \tag{6.14}$$

for any γ, we know by an isomorphism theorem similar to Proposition 2.3 (cf. Theorem C in [10]), that

$$\|v_\nu\|_{W_\gamma^{2,n+1}(M)} \leq C\nu^{\frac{1}{n+1}}$$

for any $\gamma < \frac{n+1}{2}$. Then by the Sobolev embedding theorem ([10]) we have

$$\|v_\nu\|_{C_\gamma^{1,\alpha}(M)} \leq C\nu^{\frac{1}{n+1}} \tag{6.15}$$

for some $\alpha \in (0,1)$.

Next we estimate $A(\frac{x}{|x|})$ term by term. We treat the easy terms first. For the term

$$A_0\left(\frac{x}{|x|}\right) = c_n \int_{R^n \setminus B_{r_0}(0)} \big(R_\nu + n(n-1)\big)^-(1+v)\left(\sqrt{1+|y|^2} - \frac{x}{|x|}\cdot y\right)^{-n} dvol_{g_\nu}(y)\,,$$

we simply ask r_0 is large enough so the support of $(R_\nu + n(n-1))^-$ is outside of $R^n \setminus B_{r_0}(0)$. Therefore, we may choose r_0 so that

$$A_0\left(\frac{x}{|x|}\right) = 0\,. \tag{6.16}$$

For the term

$$A_1\left(\frac{x}{|x|}\right) = c_n \int_{\partial B_{r_0}(0)} \frac{\partial v_\nu}{\partial n}\left(\sqrt{1+|y|^2} - \frac{x}{|x|}\cdot y\right)^{-n} d\sigma_{g_\nu}(y)\,,$$

we easily see that

$$A_1\left(\frac{x}{|x|}\right) \leq C\nu^{\frac{1}{n+1}}\,. \tag{6.17}$$

Similarly, for the term

$$A_2\left(\frac{x}{|x|}\right) = -nc_n \int_{\partial B_{r_0}(0)} v(y)\left(\sqrt{1+|y|^2} - \frac{x}{|x|}\cdot y\right)^{-n} \frac{\frac{|y|}{\sqrt{1+|y|^2}} - \frac{x}{|x|}\cdot\frac{y}{|y|}}{\sqrt{1+|y|^2} - \frac{x}{|x|}\cdot y} d\sigma_{g_\nu}(y)\,,$$

we easily derive from (6.15) that

$$\left|A_2\left(\frac{x}{|x|}\right)\right| \leq C\nu^{\frac{1}{n+1}}\,. \tag{6.18}$$

The last term is

$$A_{-1}\left(\frac{x}{|x|}\right) = \lim_{\lambda\to\infty} \lambda^n \int_{R^n \setminus B_{r_0}(0)} v_\nu(y)\Psi_{\lambda\frac{x}{|x|}}(y) dvol_g(y)\,.$$

Due to (6.15) and the estimate (5.14) we know

$$\begin{aligned} |A_{-1}| &\leq C\nu^{\frac{1}{n+1}} \lim_{\lambda\to\infty} \lambda^n \int |y|^{-\frac{n}{2}} O(|y|^{-n}) O\left(\frac{1}{\cosh^n d_H(\lambda\frac{x}{|x|}, y)}\right) dvol_g(y) \\ &\leq C\nu^{\frac{1}{n+1}}\,, \end{aligned} \tag{6.19}$$

where in the last step we use the same argument we used to establish (5.21) and (5.22) to deal with the term $(\sqrt{1+|y|^2} - \frac{x}{|x|}\cdot y)^{-n}$, which is only big when $\frac{y}{|y|}$ is very close to $\frac{x}{|x|}$. Thus we have proved that

$$\left| A_\nu \left(\frac{x}{|x|} \right) \right| \leq C\nu^{\frac{1}{n+1}} \tag{6.20}$$

for some C independent of ν. □

Proof of Theorem 6.3. To finish the proof of Theorem 6.3 we simply notice that for each ν sufficiently small, by Lemma 3.1 in Section 3, we may apply the positive mass theorem in [21] to the metric $(1+v_\nu)^{\frac{4}{n-2}} g_\nu$ and obtain that

$$\int_{S^{n-1}} \mathrm{Tr}_{g_0} \tilde{h} dvol_{g_0}(x) \geq \left| \int_{S^{n-1}} \mathrm{Tr}_{g_0} \tilde{h} x dvol_{g_0}(x) \right|$$

where

$$\tilde{h} = \frac{4(n+1)}{n-2} A_\nu \left(\frac{x}{|x|} \right) g_0 + h\,.$$

Here we note that the mass aspect of g_ν is the same as the mass aspect of g since g_ν is the same as g outside a compact set. Therefore, as $\nu \to 0$, we have

$$\int_{S^{n-1}} \mathrm{Tr}_{g_0} h dvol_{g_0}(x) \geq \left| \int_{S^{n-1}} \mathrm{Tr}_{g_0} h x dvol_{g_0}(x) \right|\,.$$

So the proof is finished. □

Acknowledgements

We would like to thank the referee for a very thorough reading of our manuscript and many constructive suggestions.

References

[1] L. Andersson, *Elliptic systems on manifolds with asymptotically negative curvature*, Indiana Math. Journal **42** (1993), no. 4, 1359–1388.

[2] L. Andersson and P. Chrusciel, *Solutions of the constraint equations in general relativity satisfying hyperboloidal boundary conditions*, Dissertationes Math. (Rozprawy Mat.) **355** (1996), 100.

[3] L. Andersson and M. Dahl, *Scalar curvature rigidity for asymptotically hyperbolic manifolds*, Ann. Global Anal. and Geo. **16** (1998), 1–27.

[4] R. Bartnik, *New definition of quasi-local mass*, Phys. Rev. Lett. **62** (1989), no. 20, 2346–2348.

[5] V. Bonini, P. Miao and J. Qing, *Ricci curvature rigidity for weakly asymptotically hyperbolic manifolds*, Comm. Anal. Geom. **14** (2006), no. 3, 603–612. ArXiv: hep-th/0310378.

[6] P. T. Chruściel and M. Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific J. Math. **212** (2003), no. 2, 231–264. ArXiv: math.DG/0110035.

[7] C. R. Graham, *Volume and area renormalizations for conformally compact Einstein metrics*, The Proceedings of the 19th Winter School "Geometry and Physics" (Srnì, 1999). Rend. Circ. Mat. Palermo (2) Suppl. No. 63 (2000), 31–42.

[8] C. R. Graham and J. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math., **87** (1991), no. 2, 186–225.

[9] J. Lee, *The spectrum of an asymptotically hyperbolic Einstein manifold*, Comm. Anal. Geom. **3** (1995), no. 1–2, 253–271.

[10] J. Lee, *Fredholm operators and Einstein metrics on conformally compact manifolds*, Mem. Amer. Math. Soc. **183** (2006), no. 864, vi+83 pp. ArXiv: math.DG/0105046.

[11] R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Differential Geom. **28** (1988), no. 2, 309–339.

[12] R. Mazzeo and R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), no. 2, 260–310.

[13] P. Miao, *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys., **6** (2002), no. 6, 1163–1182.

[14] M. Min-Oo, *Scalar curvature rigidity of asymptotically hyperbolic spin manifolds*, Math. Ann. **285** (1989), 527539.

[15] M. Protter and H. Weinberger, *The Maximum Principles in Differential Equations*, Springer, New York, 1984.

[16] J. Qing, *On the rigidity for conformally compact Einstein manifolds*, International Mathematics Research Notices **21** (2003), 1141–1153, ArXiv: math.DG/0305084.

[17] J. Ratcliff, *Foundations of Hyperbolic Manifolds*, GTM 149, Springer Verlag, 1994.

[18] R. Schoen and S. T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.

[19] Y. Shi and L. Tam, *Positive mass theorem and the boundary behaviors of a compact manifolds with nonnegative scalar curvature*, J. Differential Geom. **62** (2002), no. 1, 79–125. ArXiv:math.DG/0301047.

[20] Y. Shi and L. Tam, *Boundary behaviors and scalar curvature of compact manifolds*, ArXiv: math.DG/0611253.

[21] X. Wang, *The mass of asymptotically hyperbolic manifolds*, J. Diff. Geo. **57** (2001), no. 2, 273–299.

Vincent Bonini
Mathematical Science Research Institute
Berkeley, CA 94720, USA
e-mail: `vbonini@msri.org`

Jie Qing
Department of Mathematics, UC
Santa Cruz, CA 95064, USA
e-mail: `qing@math.ucsc.edu`

Communicated by Sergiu Klainerman.

Submitted: April 6, 2007.

Accepted: September 24, 2007.