Ann. Henri Poincaré 8 (2007), 1279–1301

1424-0637/071279-23, *published online* October 24, 2007
DOI 10.1007/s00023-007-0335-7

Annales Henri Poincaré

The Rotation Number for the Generalized Kronig–Penney Hamiltonians

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Abstract. We discuss the one-dimensional Schrödinger operator with generalized point interaction on a lattice. We give a characterization of the band edges of its spectrum by the rotation number.

1. Introduction and main result

In this paper we study the one-dimensional Schrödinger operators with generalized point interactions on a lattice. Our purpose is to give a characterization of the band edges of its spectrum by the rotation number. To describe our main theorem, we introduce notations. We fix $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. Let $0 = \kappa_1 < \kappa_2 < \cdots < \kappa_{n+1} = 2\pi$ be a partition of the interval $(0, 2\pi)$. We put $\Gamma_j = \{\kappa_j\} + 2\pi\mathbb{Z}$ for $j = 1, 2, \dots, n$, and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$. We now introduce the special linear group

$$SL_2(\mathbb{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1 \right\}.$$

We further introduce the Sobolev space

$$H^2(I) = \left\{ y(x) \in L^2(I) \middle| \quad y'(x), y''(x) \in L^2(I) \right\},$$

where the symbol $'$ stands for the differentiation with respect to x, and $I \subset \mathbb{R}$ is an open set. For $\{\theta_j\}_{j=1}^n \subset \mathbb{R}$ and $\{A_j\}_{j=1}^n \subset SL_2(\mathbb{R})$, we define the one-dimensional Schrödinger operator $H = H(\theta_1, \theta_2, \dots, \theta_n, A_1, A_2, \dots, A_n)$ in $L^2(\mathbb{R})$ as follows.

$$(Hy)(x) = -y''(x)\,, \quad x \in \mathbb{R} \setminus \Gamma\,, \tag{1.1}$$

$$\mathrm{Dom}(H) = \left\{ y \in H^2(\mathbb{R} \setminus \Gamma) \middle| \begin{array}{c} \left(\begin{array}{c} y(x+0) \\ y'(x+0) \end{array} \right) = e^{i\theta_j} A_j \left(\begin{array}{c} y(x-0) \\ y'(x-0) \end{array} \right) \\ \text{for} \quad x \in \Gamma_j, \quad j = 1, 2, \dots, n \end{array} \right\}. \tag{1.2}$$

The operator H is self-adjoint. The proof of this fact is similar to [14, Proposition 2.1].

Now we describe the band structure of the spectrum of H. Since H has 2π-periodic point interactions, we can make use of a direct integral decomposition for H (see [16, Section XIII.16]). For $\mu \in \mathbb{R}$, we define the Hilbert space

$$\mathcal{H}_\mu = \left\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}) \middle| \quad u(x+2\pi) = e^{i\mu} u(x) \quad \text{a.e.} \quad x \in \mathbb{R} \right\}$$

equipped with the inner product

$$\langle u, v \rangle_{\mathcal{H}_\mu} = \int_0^{2\pi} u(x)\overline{v(x)}dx\,, \quad u, v \in \mathcal{H}_\mu\,.$$

We define a fiber operator $H_\mu = H_\mu(\theta_1, \ldots, \theta_n, A_1, \ldots, A_n)$ in $\mathcal{H}_\mu$ as

$$(H_\mu y)(x) = -y''(x)\,, \quad x \in \mathbb{R} \setminus \Gamma\,,$$

$$\mathrm{Dom}(H_\mu) = \left\{ y \in \mathcal{H}_\mu \,\middle|\, \begin{array}{l} y \in H^2\big((0,2\pi) \setminus \{\kappa_2, \ldots, \kappa_n\}\big)\,, \\ \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j\,, \quad j = 1, 2, \ldots, n \end{array} \right\}.$$

On the other hand, we define a unitary operator

$$\mathcal{U} : L^2(\mathbb{R}) \to \int_0^{2\pi} \oplus \mathcal{H}_\mu d\mu$$

as

$$(\mathcal{U}u)(x,\mu) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} e^{il\mu} u(x - 2l\pi)\,.$$

Then we have the direct integral representation of H:

$$\mathcal{U} H \mathcal{U}^{-1} = \int_0^{2\pi} \oplus H_\mu d\mu\,.$$

Let $\lambda_j(\mu)$ stand for the jth eigenvalue of H_μ counted with multiplicity for $j \in \mathbb{N}$. For each $j = 1, 2, \ldots, n$, we write

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}.$$

Since $A_j \in SL_2(\mathbb{R})$, we have $a_j + d_j \neq 0$ if $b_j = 0$. Let

$$S = \{1 \le k \le n | \quad b_k \neq 0\}\,,$$
$$T = \{1 \le k \le n | \quad b_k = 0\}\,,$$

and

$$\xi = \prod_{j \in S} b_j \prod_{k \in T} (a_k + d_k)\,,$$

where $\Pi_{j \in S} b_j := 1$ if $S = \emptyset$, while $\Pi_{k \in T}(a_k + d_k) := 1$ if $T = \emptyset$. We note that $\xi \neq 0$.

The fundamental properties of $\sigma(H)$, the spectrum of H, are summarized in the following way.

Proposition 1.1. (a) *The function* $\lambda_j(\cdot)$ *is continuous on* $[0, 2\pi]$.

(b) *It holds that* $\lambda_j(\mu) = \lambda_j(-\mu + 2\theta_1 + \cdots + 2\theta_n)$.

(c) *If* $\mu - (\theta_1 + \cdots + \theta_n) \notin \pi\mathbb{Z}$, *then every eigenvalue of* H_μ *is simple.*

(d) *The spectrum of* $H(\theta_1, \ldots, \theta_n, A_1, \ldots, A_n)$ *is given by*

$$\begin{aligned}
&\sigma\big(H(\theta_1, \ldots, \theta_n, A_1, \ldots, A_n)\big) \\
&\quad = \bigcup_{\mu \in [\theta_1 + \cdots + \theta_n, \theta_1 + \cdots + \theta_n + \pi]} \sigma\big(H_\mu(\theta_1, \ldots, \theta_n, A_1, \ldots, A_n)\big) \\
&\quad = \bigcup_{j=1}^{\infty} \lambda_j([\theta_1 + \cdots + \theta_n, \theta_1 + \cdots + \theta_n + \pi]) \\
&\quad = \bigcup_{j=1}^{\infty} \bigcup_{\mu \in [\theta_1 + \cdots + \theta_n, \theta_1 + \cdots + \theta_n + \pi]} \{\lambda_j(\mu)\}
\end{aligned}$$

(e) *The set* $\sigma(H(\theta_1, \ldots, \theta_n, A_1, \ldots, A_n))$ *is independent of* $\{\theta_j\}_{j=1}^n$.

(f) *If* $\xi > 0$ *and* $\theta_1 = \theta_2 = \cdots = \theta_n = 0$, *then the function* $\lambda_j(\cdot)$ *is strictly monotone increasing* (*respectively, decreasing*) *function on* $[0, \pi]$ *for odd* (*respectively, even*) j.

(g) *If* $\xi < 0$ *and* $\theta_1 = \theta_2 = \cdots = \theta_n = 0$, *then the function* $\lambda_j(\cdot)$ *is strictly monotone increasing* (*respectively, decreasing*) *function on* $[0, \pi]$ *for even* (*respectively, odd*) j.

We show this proposition in Section 2. Here we define the spectral gaps of H. Thanks to Proposition 1.1 (e), we may suppose that

$$\theta_1 = \theta_2 = \cdots = \theta_n = 0\,,$$

which does not cause any loss of generality. We define

$$G_j = \begin{cases} \big(\lambda_j(\pi), \lambda_{j+1}(\pi)\big) & \text{for } j \text{ odd}\,, \\ \big(\lambda_j(0), \lambda_{j+1}(0)\big) & \text{for } j \text{ even} \end{cases}$$

in the case where $\xi > 0$, while we put

$$G_j = \begin{cases} \big(\lambda_j(\pi), \lambda_{j+1}(\pi)\big) & \text{for } j \text{ even}\,, \\ \big(\lambda_j(0), \lambda_{j+1}(0)\big) & \text{for } j \text{ odd} \end{cases}$$

if $\xi < 0$. The open interval G_j is called the jth gap of the spectrum of H. Furthermore, we put $B_j = \lambda_j([0, \pi])$. Then, we refer to the closed interval B_j as the jth band of the spectrum of H.

Next we introduce the rotation number. For this purpose, we consider the Schrödinger equation

$$-y''(x, \lambda) = \lambda y(x, \lambda)\,, \quad x \in \mathbb{R} \setminus \Gamma\,, \tag{1.3}$$

$$\begin{pmatrix} y(x+0, \lambda) \\ y'(x+0, \lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0, \lambda) \\ y'(x-0, \lambda) \end{pmatrix}, \quad x \in \Gamma_j\,, \quad j = 1, 2, \ldots, n\,, \tag{1.4}$$

where λ is a real parameter. We define the Prüfer transform of a nontrivial solution $y(x,\lambda)$ to (1.3) and (1.4) as follows. Let (r,ω) be the polar coordinates of (y,y'):

$$y = r\sin\omega\,, \quad y' = r\cos\omega\,.$$

Then we call the function $\omega = \omega(x,\lambda)$ the Prüfer transform of $y(x,\lambda)$. The boundary condition (1.4) reduces to

$$r(x+0,\lambda)\sin\omega(x+0,\lambda) = r(x-0,\lambda)\big(a_j\sin\omega(x-0,\lambda) + b_j\cos\omega(x-0,\lambda)\big)\,,$$
$$r(x+0,\lambda)\cos\omega(x+0,\lambda) = r(x-0,\lambda)\big(c_j\sin\omega(x-0,\lambda) + d_j\cos\omega(x-0,\lambda)\big)$$

for $x \in \Gamma_j$ and $j = 1,2,\dots,n$. Since y is a nontrivial solution of (1.3) and (1.4), we have $r(x+0,\lambda) > 0$ and $r(x-0,\lambda) > 0$ for $x \in \Gamma$. Therefore $\omega(x,\lambda)$ satisfies the equation

$$\omega'(x,\lambda) = \cos^2\omega(x,\lambda) + \lambda\sin^2\omega(x,\lambda)\,, \quad x \in \mathbb{R}\setminus\Gamma \tag{1.5}$$

as well as the boundary conditions

$$\begin{aligned}\sin\omega(x+0,\lambda)\big(c_j\sin\omega(x-0,\lambda) + d_j\cos\omega(x-0,\lambda)\big)&\\ = \cos\omega(x+0,\lambda)\big(a_j\sin\omega(x-0,\lambda)&\\ + b_j\cos\omega(x-0,\lambda)\big)\,,&\end{aligned} \tag{1.6}$$

$$\operatorname{sgn}\big(\sin\omega(x+0,\lambda)\big) = \operatorname{sgn}\big(a_j\sin\omega(x-0,\lambda) + b_j\cos\omega(x-0,\lambda)\big)\,, \tag{1.7}$$

$$\operatorname{sgn}\big(\cos\omega(x+0,\lambda)\big) = \operatorname{sgn}\big(c_j\sin\omega(x-0,\lambda) + d_j\cos\omega(x-0,\lambda)\big) \tag{1.8}$$

for $x \in \Gamma_j$ and $j = 1,2,\dots,n$, where

$$\operatorname{sgn}\, x = \begin{cases} 1 & \text{if} \quad x > 0\,,\\ 0 & \text{if} \quad x = 0\,,\\ -1 & \text{if} \quad x < 0\,. \end{cases}$$

To determine the principal value of $\omega(x+0,\lambda)$ by the boundary condition (1.6), (1.7), and (1.8), we must select a branch of $\omega(x+0,\lambda)$ for $x \in \Gamma$. We choose the branch of $\omega(x+0,\lambda)$ as

$$-\pi \le \omega(x+0,\lambda) - \omega(x-0,\lambda) < \pi \quad \text{for} \quad x \in \Gamma\,. \tag{1.9}$$

Thanks to this selection, $\omega(x+0,\lambda)$ is uniquely determined. We pick $\omega_0 \in \mathbb{R}$. Let $\omega = \omega(x,\lambda,\omega_0)$ be the solution of (1.5)–(1.9) subject to the initial condition

$$\omega(+0,\lambda) = \omega_0\,. \tag{1.10}$$

We define the rotation number of (1.5)–(1.9) as

$$\rho(\lambda) = \lim_{n\to\infty}\frac{\omega(2n\pi+0,\lambda,\omega_0) - \omega_0}{2n\pi}\,. \tag{1.11}$$

By a similar way to the proof of [7, Theorem 2.1], it follows that the limit exists and is independent of the initial value ω_0. Furthermore, the function $\rho(\lambda)$ is non-decreasing on $\mathbb{R}$.

For a finite set A, we denote by $\sharp A$ the number of the elements of A. We put

$$\begin{aligned} B_j &= [\alpha_j, \beta_j] \quad \text{for} \quad j \in \mathbb{N}\,, \\ l &= \sharp\big\{1 \leq j \leq n \big| \quad (b_j < 0) \quad \text{or} \quad (b_j = 0, \quad d_j < 0)\big\}\,. \end{aligned}$$

The main result is the following theorem, which describes a relationship between the rotation number and the band edges of $\sigma(H)$.

Theorem 1.2. *For $m \in \mathbb{N}$, we have*

$$\alpha_m = \max\left\{\lambda \in \mathbb{R} \,\middle|\, \quad \rho(\lambda) = \frac{m-1}{2} - \frac{l}{2}\right\}, \tag{1.12}$$

$$\beta_m = \min\left\{\lambda \in \mathbb{R} \,\middle|\, \quad \rho(\lambda) = \frac{m}{2} - \frac{l}{2}\right\}. \tag{1.13}$$

We stress that the results (1.12) and (1.13) critically depend on the choice of the branch of $\omega(x+0,\lambda)$ for $x \in \Gamma$, which we will demonstrate in Section 4.

Our study is motivated by the works [4,9,13,14]. The rotation number has a close relation to the spectrum of the Schrödinger operators with almost periodic potentials. Such a relationship was first established by Johnson and Moser [9,13]. In order to recall that, we introduce the almost periodic Schrödinger operators $L = -d^2/dx^2 + q(x)$, where q is an almost periodic function with a frequency module $\mathcal{M}$. They proved that the rotation number $\alpha(\lambda)$ of L exists and defines a continuous function in $\{\lambda \in \mathbb{C} | \quad \mathrm{Im}\ \lambda \geq 0\}$. Furthermore, $\alpha(\lambda)$ is constant in an open interval I in a spectral gap and $2\alpha(\lambda) \in \mathcal{M}$ for $\lambda \in I$. As the special case where q is periodic of period ℓ, they labeled the jth band $\tilde{B}_j$ of $\sigma(L)$ as

$$\tilde{B}_j = \overline{\left\{\lambda \in \mathbb{R} \,\middle|\, \quad j-1 < \frac{\ell\alpha(\lambda)}{\pi} < j\right\}} \tag{1.14}$$

for $j \in \mathbb{N}$. Besides, they showed that

$$\lim_{x\to\infty} \frac{N(x,\lambda)}{x} = \frac{\alpha(\lambda)}{\pi}$$

if $N(x,\lambda)$ is the number of the zeroes in $[0,x]$ of a nontrivial solution to $(L\varphi)(x) = \lambda\varphi(x)$. This limit is called the density of states in physics. These facts are summarized in [13] and proved in detail in [9]. The rotation number is useful in the spectral analysis of the Hill equations; Gan and Zhang [4] utilized (1.14) in the investigation of the resonance pockets of the Hill equations with two-step potentials.

The one-dimensional Schrödinger operators with periodic singular potentials have been studied by numerous authors; we refer to [2,5,6,8,12,14,17,18] and [1] for a thorough review. Such an operator is the Hamiltonian for an electron in a one-dimensional crystal and plays an important role in solid state physics (see, e.g., [11]). In 1931, Kronig and Penney [12] introduced and discussed the Hamiltonian which is formally expressed as

$$L_1 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta(x - 2\pi l) \quad \text{in} \quad L^2(\mathbb{R})\,,$$

where $\delta(x)$ is the Dirac delta function at the origin and $\beta \in \mathbb{R}\backslash\{0\}$. The Dirac delta function is the most typical point interaction. This operator is nowadays called the Kronig–Penney Hamiltonian. In our notations the operators L_1 is expressed as $L_1 = H(0, M_1)$, where

$$M_1 = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}.$$

They drew the band function of L_1. Afterward, a new class of point interactions was inspired by Gesztesy, Holden, and Kirsch (see [5,6]). They studied the operator $L_2 := H(0, M_2)$ with

$$M_2 = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

We mention that this operator has the formal expression

$$L_2 = -\frac{d^2}{dx^2} + \beta \sum_{l=-\infty}^{\infty} \delta'(x - 2\pi l) \quad \text{in} \quad L^2(\mathbb{R})\,.$$

Furthermore, the δ'-interaction was generalized by Šeba [17] (see also [2] and [1, Section K.1.4]). He proved that the domain of any self-adjoint extension of $(-d^2/dx^2)|_{C_0^\infty(\mathbb{R}\backslash\{0\})}$ in $L^2(\mathbb{R})$ of coupled type is expressed as

$$\left\{ y \in H^2\big(\mathbb{R} \setminus \{0\}\big) \,\middle|\, \begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = cA \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix} \right\}$$

with $A \in SL_2(\mathbb{R})$, $c \in \mathbb{C}$, and $|c| = 1$. On the other hand, Hughes [8] gave the Floquet–Bloch decomposition of the Schrödinger operator with generalized point interactions of the form $L_3 = H(\theta_1, A_1)$ with $A_1 \in SL_2(\mathbb{R})$. These backgrounds attract our interest to the generalized Kronig–Penney Hamiltonians $H = H(\theta_1, \theta_2, \ldots, \theta_n, A_1, A_2, \ldots, A_n)$ and their spectra.

In our previous work [14] we determined the indices of the absent spectral gaps of H in the case where

$$n = 2 \quad \text{and} \quad A_1, A_2 \in SO(2) \setminus \{E, -E\}\,, \tag{1.15}$$

E being the 2×2 unit matrix. For that purpose we established (1.12) and (1.13) in the case (1.15); see [14, Theorem 4.3]. We emphasize that the proof of (1.12) and (1.13) for general A_j is subtler than that for (1.15). Indeed, in the latter case the boundary conditions (1.6)–(1.8) reduce to a rather simpler form

$$\omega(x + 0, \lambda) - \omega(x - 0, \lambda) = \gamma_j \quad \text{for} \quad x \in \Gamma_j\,,$$

where

$$A_j = \begin{pmatrix} \cos\gamma_j & -\sin\gamma_j \\ \sin\gamma_j & \cos\gamma_j \end{pmatrix}.$$

In a subsequent paper [15], we utilize Theorem 1.2 to determine the indices of the absent spectral gaps in various classes of the generalized Kronig–Penney Hamiltonians.

We organize this paper as follows. In Section 2, we show Proposition 1.1. Section 3 is devoted to the proof of Theorem 1.2. In Section 4, we demonstrate how the choice (1.9) affects the results in Theorem 1.2. In Section 5, we perform the perturbation theory for the generalized Kronig–Penney Hamiltonians; we consider the operator $\tilde{H} = \tilde{H}(\theta_1, \ldots, \theta_2, A_1, \ldots, A_n)$ in $L^2(\mathbb{R})$ defined as

$$\tilde{H} = H + q\,,$$

where $q \in L^\infty(\mathbb{R};\mathbb{R})$ is a 2π-periodic function. We describe a relationship between the rotation number and the band edges of $\sigma(\tilde{H})$.

2. Proof of Proposition 1.1

We consider the equations

$$-y''(x,\lambda) = \lambda y(x,\lambda)\,, \quad x \in \mathbb{R}\setminus\Gamma\,, \tag{2.1}$$

$$\begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix}, \quad x \in \Gamma_j\,, \quad j = 1,2,\ldots,n\,, \tag{2.2}$$

where λ is a real parameter. These equations have two solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ which are uniquely determined by the initial conditions

$$y_1(+0,\lambda) = 1\,, \quad y_1'(+0,\lambda) = 0\,,$$

and

$$y_2(+0,\lambda) = 0\,, \quad y_2'(+0,\lambda) = 1\,,$$

respectively. We introduce the discriminant $D(\lambda)$ of (2.1) and (2.2):

$$D(\lambda) = y_1(2\pi+0,\lambda) + y_2'(2\pi+0,\lambda)\,. \tag{2.3}$$

The matrix

$$M(\lambda) := \begin{pmatrix} y_1(2\pi+0,\lambda) & y_2(2\pi+0,\lambda) \\ y_1'(2\pi+0,\lambda) & y_2'(2\pi+0,\lambda) \end{pmatrix} \tag{2.4}$$

is called the monodromy matrix of (2.1) and (2.2). The characteristic equation of $M(\lambda)$ is

$$t^2 - D(\lambda)t + e^{2i(\theta_1+\theta_2+\cdots+\theta_n)} = 0\,, \tag{2.5}$$

since $\det M(\lambda) = e^{2i(\theta_1+\theta_2+\cdots+\theta_n)}$. It follows that λ is an eigenvalue of H_μ if and only if $e^{i\mu}$ is a root of (2.5). Thus, the sequence $\{\lambda_j(\mu)\}_{j=1}^{\infty}$ provides all the zeros of $D(\lambda) - (e^{i\mu} + e^{2i(\theta_1+\theta_2+\cdots+\theta_n)-i\mu})$. We put

$$A_{n+1} = \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = A_1 \tag{2.6}$$

and $\theta_{n+1} = \theta_1$. The solutions y_1 and y_2 satisfy the recurrence relations

$$y_k(\kappa_j + 0, \lambda) = \alpha_j(\lambda) y_k(\kappa_{j-1} + 0, \lambda) + \beta_j(\lambda) y_k'(\kappa_{j-1} + 0, \lambda)\,, \tag{2.7}$$

$$y_k'(\kappa_j + 0, \lambda) = \gamma_j(\lambda) y_k(\kappa_{j-1} + 0, \lambda) + \delta_j(\lambda) y_k'(\kappa_{j-1} + 0, \lambda) \tag{2.8}$$

for $k = 1, 2$, where

$$\alpha_j(\lambda) = e^{i\theta_j}\left(a_j \cos\tau_j\sqrt{\lambda} - b_j\sqrt{\lambda}\sin\tau_j\sqrt{\lambda}\right), \tag{2.9}$$

$$\beta_j(\lambda) = e^{i\theta_j}\left(\frac{a_j}{\sqrt{\lambda}}\sin\tau_j\sqrt{\lambda} + b_j\cos\tau_j\sqrt{\lambda}\right), \tag{2.10}$$

$$\gamma_j(\lambda) = e^{i\theta_j}\left(c_j \cos\tau_j\sqrt{\lambda} - d_j\sqrt{\lambda}\sin\tau_j\sqrt{\lambda}\right), \tag{2.11}$$

$$\delta_j(\lambda) = e^{i\theta_j}\left(\frac{c_j}{\sqrt{\lambda}}\sin\tau_j\sqrt{\lambda} + d_j\cos\tau_j\sqrt{\lambda}\right), \tag{2.12}$$

and

$$\tau_j = \kappa_j - \kappa_{j-1} > 0$$

for $j = 2, \ldots, n+1$. In the next lemma, we investigate the asymptotic behavior of $y_1(2\pi + 0, \lambda)$ and $y_2'(2\pi + 0, \lambda)$ as $\lambda \to -\infty$. Let

$$\Lambda_1(n) = \{2 \le l \le n+1 |\quad b_l = 0\}\,, \tag{2.13}$$

$$\Lambda_2(n) = \{2 \le l \le n+1 |\quad b_l \neq 0\}\,, \tag{2.14}$$

$$k(n) = \sharp\Lambda_1(n)\,, \tag{2.15}$$

and

$$\mu(n) = \begin{cases} \left(\prod_{j\in\Lambda_2(n-1)} b_j\right)\left(\prod_{i\in\Lambda_1(n-1)}(a_i + d_i)\right) & \text{if} \quad n \in \mathbb{N}\setminus\{1\}\,, \\ 1 & \text{if} \quad n = 1\,. \end{cases} \tag{2.16}$$

We have $\mu(n) \neq 0$ for $n \in \mathbb{N}$.

Lemma 2.1. *We have*

$$\frac{y_1(\kappa_{n+1} + 0, \lambda)}{\prod_{j=2}^{n+1} e^{i\theta_j}\cosh\tau_j\sqrt{-\lambda}} = \begin{cases} a_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)} + \mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)-1}\right) & \text{if} \quad b_{n+1} = 0\,, \\ b_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)} + \mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)-1}\right) & \text{if} \quad b_{n+1} \neq 0\,, \end{cases} \tag{2.17}$$

$$\frac{y_1'(\kappa_{n+1}+0,\lambda)}{\prod_{j=2}^{n+1} e^{i\theta_j}\cosh\tau_j\sqrt{-\lambda}} = \begin{cases} d_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)+1} + \mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)}\right) & \text{if} \quad b_{n+1}=0\,, \\ d_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)} \\ +\left\{c_{n+1}\mu(n)+d_{n+1}\mathcal{O}(1)\right\}\left(\sqrt{-\lambda}\right)^{n-k(n)-1} \\ +\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)-2}\right) \quad \text{if} \quad b_{n+1}\neq 0\,, \end{cases} \tag{2.18}$$

$$\frac{y_2(\kappa_{n+1}+0,\lambda)}{\prod_{j=2}^{n+1} e^{i\theta_j}\cosh\tau_j\sqrt{-\lambda}} = \begin{cases} a_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)-1} + \mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)-2}\right) & \text{if} \quad b_{n+1}=0\,, \\ b_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)-1} + \mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)-2}\right) & \text{if} \quad b_{n+1}\neq 0\,, \end{cases} \tag{2.19}$$

$$\frac{y_2'(\kappa_{n+1}+0,\lambda)}{\prod_{j=2}^{n+1} e^{i\theta_j}\cosh\tau_j\sqrt{-\lambda}} = \begin{cases} d_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)} + \mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)-1}\right) & \text{if} \quad b_{n+1}=0\,, \\ d_{n+1}\mu(n)\left(\sqrt{-\lambda}\right)^{n-k(n)-1} \\ +\left\{c_{n+1}\mu(n)+d_{n+1}\mathcal{O}(1)\right\}\left(\sqrt{-\lambda}\right)^{n-k(n)-2} \\ +\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{n-k(n)-3}\right) \quad \text{if} \quad b_{n+1}\neq 0 \end{cases} \tag{2.20}$$

as $\lambda \to -\infty$.

Proof. First of all, we simultaneously prove (2.17) and (2.18) by induction. It follows by (2.7) and (2.8) that

$$y_1(\kappa_2+0,\lambda) = e^{i\theta_2}\cosh\tau_2\sqrt{-\lambda}\left(a_2+b_2\sqrt{-\lambda}\tanh\tau_2\sqrt{-\lambda}\right),$$
$$y_1'(\kappa_2+0,\lambda) = e^{i\theta_2}\cosh\tau_2\sqrt{-\lambda}\left(c_2+d_2\sqrt{-\lambda}\tanh\tau_2\sqrt{-\lambda}\right)$$

for $\lambda<0$. This implies that (2.17) and (2.18) hold when $n=1$. We pick an integer $m\geq 2$, arbitrarily. We suppose that (2.17) and (2.18) are valid if $n=m-1$.

We consider the case where $b_m=0$. It follows by definition that

$$\mu(m) = (a_m+d_m)\mu(m-1)\,.$$

Then it follows by (2.7) and (2.8) that

$$\begin{aligned} &\frac{y_1(\kappa_{m+1}+0,\lambda)}{\prod_{j=2}^{m+1} e^{i\theta_j}\cosh\tau_j\sqrt{-\lambda}} \\ &= \left(a_{m+1}+b_{m+1}\sqrt{-\lambda}\tanh\tau_{m+1}\sqrt{-\lambda}\right) \\ &\quad\times\left\{a_m\mu(m-1)\left(\sqrt{-\lambda}\right)^{m-k(m-1)-1} + \mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m-1)-2}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{a_{m+1}}{\sqrt{-\lambda}}\tanh\tau_{m+1}\sqrt{-\lambda}+b_{m+1}\right)\\
&\times\left\{d_m\mu(m-1)\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)-1}+\mathcal{O}\left(\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)-2}\right)\right\}\\
&=\begin{cases}
a_{m+1}(a_m+d_m)\mu(m-1)\left(\sqrt{-\lambda}\right)^{m-k(m-1)-1}\\
+\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m-1)-2}\right) \quad \text{if} \quad b_{m+1}=0\,,\\
b_{m+1}(a_m+d_m)\mu(m-1)\left(\sqrt{-\lambda}\right)^{m-k(m-1)}\\
+\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m-1)-1}\right) \quad \text{if} \quad b_{m+1}\neq 0\,,
\end{cases}\\
&=\begin{cases}
a_{m+1}\mu(m)\left(\sqrt{-\lambda}\right)^{m-k(m)}+\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m)-1}\right) \quad \text{if} \quad b_{m+1}=0\,,\\
b_{m+1}\mu(m)\left(\sqrt{-\lambda}\right)^{m-k(m)}+\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m)-1}\right) \quad \text{if} \quad b_{m+1}\neq 0\,,
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{y_1'(\kappa_{m+1}+0,\lambda)}{\prod_{j=2}^{m+1}e^{i\theta_j}\cosh\tau_j\sqrt{-\lambda}}\\
&=\Big(c_{m+1}+d_{m+1}\sqrt{-\lambda}\tanh\tau_{m+1}\sqrt{-\lambda}\Big)\\
&\quad\times\left\{a_m\mu(m-1)\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)-1}+\mathcal{O}\left(\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)-2}\right)\right\}\\
&\quad+\left(\frac{c_{m+1}}{\sqrt{-\lambda}}\tanh\tau_{m+1}\sqrt{-\lambda}+d_{m+1}\right)\\
&\quad\times\left\{d_m\mu(m-1)\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)}+\mathcal{O}\Big(\left(\sqrt{-\lambda}\right)^{m-k(m-1)-1}\Big)\right\}\\
&=\begin{cases}
d_{m+1}(a_m+d_m)\mu(m-1)\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)}+\mathcal{O}\left(\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)-1}\right)\\
\text{if} \quad b_{m+1}=0\,,\\
d_{m+1}(a_m+d_m)\mu(m-1)\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)}\\
+\Big\{c_{m+1}(a_m+d_m)\mu(m-1)+d_{m+1}\mathcal{O}(1)\Big\}\Big(\sqrt{-\lambda}\Big)^{m-1-k(m-1)}\\
+\mathcal{O}\left(\Big(\sqrt{-\lambda}\Big)^{m-k(m-1)-2}\right) \qquad \text{if} \quad b_{m+1}\neq 0\,,
\end{cases}\\
&=\begin{cases}
d_{m+1}\mu(m)\left(\sqrt{-\lambda}\right)^{m-k(m)+1}+\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m)}\right) \quad \text{if} \quad b_{m+1}=0\,,\\
d_{m+1}\mu(m)\left(\sqrt{-\lambda}\right)^{m-k(m)}\\
+\Big\{c_{m+1}\mu(m)+d_{m+1}\mathcal{O}(1)\Big\}\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m)-1}\right)\\
+\mathcal{O}\left(\left(\sqrt{-\lambda}\right)^{m-k(m)-2}\right) \quad \text{if} \quad b_{m+1}\neq 0
\end{cases}
\end{aligned}
$$

as $\lambda\to-\infty$. In a similar way we get the claim for $b_m\neq 0$. Therefore, we have (2.17) and (2.18) for all $n\in\mathbb{N}$.

Similarly, we obtain (2.19) and (2.20). □

We write $f(\lambda) \sim g(\lambda)$ as $\lambda \to -\infty$ if and only if $f(\lambda)/g(\lambda) \to 1$ as $\lambda \to -\infty$. Using Lemma 2.1 and (2.3), we have

$$D(\lambda) \sim \left(\prod_{j=2}^{n+1} e^{i\theta_j} \cosh \tau_j \sqrt{-\lambda} \right) \mu(n+1) \Big(\sqrt{-\lambda} \Big)^{n-k(n)} \tag{2.21}$$

as $\lambda \to -\infty$. Now we have prepared to show Proposition 1.1.

Proof of Proposition 1.1. We have only to prove the statements (e), (f), and (g), since the demonstrations of (a), (b), (c), and (d) are similar to those of [10, Theorem XIII.89 (a), (b), and (c)] and of [10, Theorem XIII.90 (a)].

We now prove (e). By the definition of H_μ and that of $\mathcal{H}_\mu$, we claim that

$$\sigma\big(H_\mu(\theta_1, \dots, \theta_n, A_1, \dots, A_n)\big) = \sigma\big(H_{(\mu-(\theta_1+\dots+\theta_n))}(0, \dots, 0, A_1, \dots, A_n)\big) .$$

This combined with (b) and (d) implies the claim (e).

Next we show (f) and (g). Let $\theta_1 = \cdots = \theta_n = 0$. It follows from (2.17) and (2.6) that

$$\lim_{\lambda \to -\infty} D(\lambda) = \begin{cases} +\infty & \text{if} \quad \mu(n+1) > 0\,, \\ -\infty & \text{if} \quad \mu(n+1) < 0\,. \end{cases}$$

Therefore we arrive at the conclusions (f) and (g) in a similar way to [12, Proposition 1, (d) and (e)]. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We recall (2.6). Let

$$\begin{aligned} q_j &= \sharp\big\{2 \le k \le j | \quad (b_k < 0) \quad \text{or} \quad (b_k = 0, \quad d_k < 0)\big\}\,, \\ q_1 &= 0\,, \end{aligned}$$

and

$$\eta_j = \begin{cases} \operatorname{Arctan}(b_j/d_j) - q_{j-1}\pi & \text{if} \quad b_j > 0\,, \quad d_j > 0\,, \\ \operatorname{Arctan}(b_j/d_j) + \pi - q_{j-1}\pi\,, & \text{if} \quad b_j > 0\,, \quad d_j < 0\,, \\ \pi/2 - q_{j-1}\pi & \text{if} \quad b_j > 0\,, \quad d_j = 0\,, \\ \operatorname{Arctan}(b_j/d_j) - \pi - q_{j-1}\pi\,, & \text{if} \quad b_j < 0\,, \quad d_j < 0\,, \\ \operatorname{Arctan}(b_j/d_j) - q_{j-1}\pi & \text{if} \quad b_j < 0\,, \quad d_j > 0\,, \\ -\pi/2 - q_{j-1}\pi & \text{if} \quad b_j < 0\,, \quad d_j = 0\,, \\ -q_{j-1}\pi & \text{if} \quad b_j = 0\,, \quad d_j > 0\,, \\ -\pi - q_{j-1}\pi & \text{if} \quad b_j = 0\,, \quad d_j < 0 \end{cases}$$

for $2 \le j \le n+1$, where $\operatorname{Arctan}(x) \in (-\pi/2, \pi/2)$ for $x \in \mathbb{R}$. Since

$$q_j = \begin{cases} q_{j-1} + 1 & \text{if} \quad (b_j < 0) \quad \text{or} \quad (b_j = 0, \quad d_j < 0)\,, \\ q_{j-1} & \text{otherwise}\,, \end{cases}$$

we have

$$\eta_j \in [-q_j\pi, -q_j\pi + \pi)\,. \tag{3.1}$$

We pick a $\gamma \in (0,\pi)$ such that

$$\eta_j < -q_j\pi + \gamma \quad \text{for} \quad j = 2,3,\dots,n+1\,.$$

Then we have the following lemma.

Lemma 3.1. *Let $0 \le \omega_0 < \pi$. Then, there exists $\lambda_0 \in \mathbb{R}$ such that*

$$-\pi(q_j + pq_{n+1}) \le \omega(\kappa_j + 2\pi p + 0, \lambda, \omega_0) \le -\pi(q_j + pq_{n+1}) + \gamma$$

for any $p \in \mathbb{N}\cup\{0\}$, $j = 2,3,\dots,n+1$, $\lambda \le \lambda_0$, and $\omega_0 \in [0,\gamma]$.

Proof. To prove this lemma, we recall a fundamental fact on the Prüfer transform from [2, Chapter 8, Theorem 2.1]. Let $c < d$. For $\beta \in [0,\pi)$, let $\theta = \theta(x,\lambda,c,\beta)$ be the solution to the initial value problem

$$\frac{d}{dx}\theta = \cos^2\theta + \lambda\sin^2\theta \quad \text{on} \quad \mathbb{R}\,, \tag{3.2}$$

$$\theta|_{x=c} = \beta\,. \tag{3.3}$$

Then, it holds that

$$\lim_{\lambda\to-\infty}\theta(d,\lambda,c,\beta) = 0\,. \tag{3.4}$$

Moreover, the function $\theta(d,\cdot,c,\beta)$ is strictly monotone increasing on $\mathbb{R}$.

We fix $\omega_0 \in [0,\gamma]$. By (3.4), we have

$$\lim_{\lambda\to-\infty}\omega(\kappa_2 - 0,\lambda,\omega_0) = 0\,. \tag{3.5}$$

By induction on $j = 2,3,\dots,n+1$, we shall show the following statements.

$$\text{The limit } \beta_j := \lim_{\lambda\to-\infty}\omega(\kappa_j - 0,\lambda,\omega_0) \in \mathbb{R} \text{ exists, and we have } \beta_j = -q_{j-1}\pi. \tag{3.6}$$

$$\text{The function } \omega(\kappa_j - 0,\cdot,\omega_0) \text{ is strictly monotone increasing on } \mathbb{R}. \tag{3.7}$$

In the first instance, it follows by (3.5) that (3.6) and (3.7) are valid for $j = 2$. We pick $m \in \{2,3,\dots,n+1\}$, arbitrarily. Suppose that (3.6) and (3.7) hold for $j = m$. We show that the limit

$$\alpha_m := \lim_{\lambda\to-\infty}\omega(\kappa_m + 0,\lambda,\omega_0)$$

exists.

We consider the case where $d_m \ne 0$. By (1.6), we have

$$\tan\omega(\kappa_m + 0,\lambda,\omega_0) = \frac{a_m\tan\omega(\kappa_m - 0,\lambda,\omega_0) + b_m}{c_m\tan\omega(\kappa_m - 0,\lambda,\omega_0) + d_m}\,. \tag{3.8}$$

Since $\beta_m = -q_{m-1}\pi$ by assumption, it follows that

$$\lim_{\lambda\to-\infty}\tan\omega(\kappa_m + 0,\lambda,\omega_0) = b_m/d_m\,.$$

Since the function $\tan x$ is continuous and strictly monotone increasing in a neighborhood of the point b_m/d_m, it follows that the limit α_m exists.

Next we discuss the case where $d_m = 0$. Suppose that $d_m = 0$. Then, it follows by (1.6) that

$$\begin{aligned} &c_m \sin\omega(\kappa_m + 0, \lambda, \omega_0) \sin\omega(\kappa_m - 0, \lambda, \omega_0) \\ &\quad = \cos\omega(\kappa_m + 0, \lambda, \omega_0)\big(a_m \sin\omega(\kappa_m - 0, \lambda, \omega_0) + b_m \cos\omega(\kappa_m - 0, \lambda, \omega_0)\big) . \end{aligned}$$

Since $\beta_m = -q_{m-1}\pi$, we have

$$|c_m \sin\omega(\kappa_m + 0, \lambda, \omega_0) \sin\omega(\kappa_m - 0, \lambda, \omega_0)| \le |c_m \sin\omega(\kappa_m - 0, \lambda, \omega_0)| \to 0$$

as $\lambda \to -\infty$. On the other hand, it follows by $a_m d_m - b_m c_m = 1$ and $d_m = 0$ that

$$a_m \sin\omega(\kappa_m - 0, \lambda, \omega_0) + b_m \cos\omega(\kappa_m - 0, \lambda, \omega_0) \to b_m(-1)^{q_{m+1}} \ne 0$$

as $\lambda \to -\infty$. Thereby we conclude that $\lim_{\lambda\to-\infty} \cos\omega(\kappa_m + 0, \lambda, \omega_0) = 0$. So $\lim_{\lambda\to-\infty} \omega(\kappa_m + 0, \lambda, \omega_0) \in \mathbb{R}$ exists. Therefore α_m exists in every case.

Next we show

$$\alpha_m = \eta_m . \tag{3.9}$$

By (1.6) and $\beta_m = -q_{m-1}\pi$ it follows that

$$d_m \sin\alpha_m = b_m \cos\alpha_m . \tag{3.10}$$

First, we consider the case where $b_m = 0$ and $d_m > 0$. Suppose that $b_m = 0$ and $d_m > 0$. Then we have $\sin\alpha_m = 0$. Note that

$$c_m \sin\omega(\kappa_m - 0, \lambda, \omega_0) + d_m \cos\omega(\kappa_m - 0, \lambda, \omega_0) \to d_m(-1)^{q_{m-1}} \ne 0 .$$

Since the function $\operatorname{sgn}(\cdot)$ is continuous on $\mathbb{R} \setminus \{0\}$, it follows by (1.8) that

$$\begin{aligned} &\lim_{\lambda\to-\infty} \operatorname{sgn}\big(\cos\omega(\kappa_m + 0, \lambda, \omega_0)\big) \\ &\qquad = \lim_{\lambda\to-\infty} \operatorname{sgn}\big(c_m \sin\omega(\kappa_m - 0, \lambda, \omega_0) + d_m \cos\omega(\kappa_m - 0, \lambda, \omega_0)\big) \\ &\qquad = \operatorname{sgn}\big(d_m(-1)^{q_{m-1}}\big) \\ &\qquad = (-1)^{q_{m-1}} . \end{aligned} \tag{3.11}$$

Because $\sin\alpha_m = 0$, we have $\lim_{\lambda\to-\infty} \cos\omega(\kappa_m + 0, \lambda, \omega_0) = \cos\alpha_m = \pm 1$. Since the function $\operatorname{sgn}(\cdot)$ is continuous in a neighborhood of ± 1, we have

$$\lim_{\lambda\to-\infty} \operatorname{sgn}\big(\cos\omega(\kappa_m + 0, \lambda, \omega_0)\big) = \operatorname{sgn}\left(\lim_{\lambda\to-\infty} \cos\omega(\kappa_m + 0, \lambda, \omega_0)\right) . \tag{3.12}$$

By (3.11) and (3.12), it follows that

$$\operatorname{sgn}(\cos\alpha_m) = (-1)^{q_{m-1}} .$$

This combined with $\sin\alpha_m = 0$, $\beta_m = -q_{m-1}\pi$, and (1.9) means

$$\alpha_m = -q_{m-1}\pi .$$

Namely we have $\alpha_m = \eta_m$ if $b_m = 0$ and $d_m > 0$.

Next, we deal with the case where $b_m > 0$ and $d_m > 0$. Suppose that $b_m > 0$ and $d_m > 0$. We notice that

$$a_m \sin\omega(\kappa_m - 0, \lambda, \omega_0) + b_m \cos\omega(\kappa_m - 0, \lambda, \omega_0) \to b_m(-1)^{q_{m-1}} \neq 0$$

and

$$c_m \sin\omega(\kappa_m - 0, \lambda, \omega_0) + d_m \cos\omega(\kappa_m - 0, \lambda, \omega_0) \to d_m(-1)^{q_{m-1}} \neq 0 .$$

Since the function $\mathrm{sgn}(\cdot)$ is continuous on $\mathbb{R}\setminus\{0\}$, it follows by (1.7) and (1.8) that

$$\begin{aligned}\lim_{\lambda\to-\infty} \mathrm{sgn}\big(\sin\omega(\kappa_m + 0, \lambda, \omega_0)\big) &= \lim_{\lambda\to-\infty} \mathrm{sgn}\big(\cos\omega(\kappa_m + 0, \lambda, \omega_0)\big) \\ &= (-1)^{q_{m-1}} . \end{aligned} \tag{3.13}$$

The equation (3.10) implies $\sin\alpha_m \neq 0$ and $\cos\alpha_m \neq 0$. The function $\mathrm{sgn}(\cdot)$ is also continuous in a neighborhood of $\sin\alpha_m$ and in that of $\cos\alpha_m$. Hence, (3.13) reduces to

$$\mathrm{sgn}(\sin\alpha_m) = (-1)^{q_{m-1}} = \mathrm{sgn}(\cos\alpha_m) .$$

Combining this with (3.10) and (1.9), we have $\alpha_m = -q_{m-1}\pi + \arctan(b_m/d_m)$. Therefore we obtain $\alpha_m = \eta_m$ if $b_m > 0$ and $d_m > 0$. In a similarly we have $\alpha_m = \eta_m$ in the other cases.

Next, we show that $\omega(\kappa_m + 0, \cdot, \omega_0)$ is strictly monotone increasing on $\mathbb{R}$. Let $f(x) = (a_m x + b_m)/(c_m x + d_m)$. The function $f(x)$ is strictly monotone increasing on $\mathbb{R}$ in the case where $c_m = 0$, since $a_m d_m - b_m c_m = 1$. In the case where $c_m \neq 0$, we see that $\lim_{x\to-d_m/c_m\pm0} f(x) = \mp\infty$ and that the function $f(x)$ is strictly monotone increasing on $(-\infty, -d_m/c_m)$ and on $(-d_m/c_m, \infty)$. Combining these fact with (3.8) and the monotonicity of $\omega(\kappa_m - 0, \cdot, \omega_0)$, we infer that $\omega(\kappa_m + 0, \cdot, \omega_0)$ is strictly monotone increasing.

Since $\omega(\kappa_{m+1} - 0, \lambda, \omega_0) = \theta(\kappa_{m+1}, \lambda, \kappa_m, \omega(\kappa_m + 0, \lambda, \omega_0))$, (3.7) is valid for $j = m + 1$. Using the monotonicity of $\omega(\kappa_m + 0, \cdot, \omega_0)$ and (3.1), we conclude that there exists $\lambda_m \in \mathbb{R}$ such that

$$-q_m\pi \leq \omega(\kappa_m + 0, \lambda, \omega_0) \leq -q_m\pi + \gamma$$

for $\lambda \leq \lambda_m$. By the comparison theorem [3, Chapter 8] and this inequality, we have

$$\theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi) \leq \omega(\kappa_{m+1} - 0, \lambda, \omega_0) < \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi + \gamma)$$

for $\lambda \leq \lambda_m$. Since (3.2) is π-periodic, we derive

$$\lim_{\lambda\to-\infty} \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi) = \lim_{\lambda\to-\infty} \theta(\kappa_{m+1}, \lambda, \kappa_m, -q_m\pi + \gamma) = -q_m\pi ,$$

so that

$$\beta_{m+1} = -q_m\pi .$$

Therefore we have proved (3.6) and (3.7) for $j = m + 1$. Therefore, (3.6) and (3.7) are valid for $j = 2, 3, \ldots, n + 1$.

Put $\lambda_0 = \min_{2\leq j\leq n+1} \lambda_j$. We have

$$-\pi q_j \leq \omega(\kappa_j + 0, \lambda, \omega_0) < -\pi q_j + \gamma \tag{3.14}$$

for $j = 2, 3, \ldots, n + 1$ and $\lambda \leq \lambda_0$.

Next we show the uniformity of the estimate (3.14) with respect to $\omega_0 \in [0, \gamma]$. According to the comparison theorem and $\omega_0 \in [0, \gamma]$, we have

$$\omega(\kappa_j + 0, \lambda, 0) \leq \omega(\kappa_j + 0, \lambda, \omega_0) \leq \omega(\kappa_j + 0, \lambda, \gamma)\,.$$

Therefore the estimate (3.14) is uniform with respect to $\omega_0 \in [0, \gamma]$.

Since (1.5)–(1.8) is 2π-periodic with respect to x, we have the desired assertion from (3.14). □

Lemma 3.2. *Put*

$$l = \sharp\{1 \leq j \leq n| \quad (b_j < 0) \quad \text{or} \quad (b_j = 0, \quad d_j < 0)\}\,.$$

Then, we have

$$\lim_{\lambda \to -\infty} \rho(\lambda) = -\frac{l}{2}\,.$$

Proof. We notice $q_{n+1} = l$. By Lemma 2.1 we have

$$-\pi pl \leq \omega(2\pi p + 0, \lambda, \omega_0) \leq -\pi pl + \gamma \tag{3.15}$$

for $0 \leq \omega_0 \leq \gamma$, $\lambda \leq \lambda_0$, and $p \in \mathbb{N}$. Therefore we arrive at our goal by dividing (3.15) by $2p\pi$. □

Proof of Theorem 1.2. Combining Lemma 3.2 with the proof of [4, Proposition 2.1], we get the assertions. □

4. Remarks on Theorem 1.2

We demonstrate how the choice (1.9) of the principal value of $\omega(x + 0, \lambda)$ on Γ affects (1.12) and (1.13). In order to see that, we choose the principal value of $\omega(x + 0, \lambda)$ on Γ as

$$0 \leq \omega(x + 0, \lambda) - \omega(x - 0, \lambda) < 2\pi \quad \text{for} \quad x \in \Gamma_j \quad \text{and} \quad j = 1, 2, \ldots, n \tag{4.1}$$

instead of (1.9). Then we have the following theorem in a similar way to Theorem 1.2.

Theorem 4.1. *Put*

$$l = \sharp\{1 \leq j \leq n| \quad (b_j < 0) \quad \text{or} \quad (b_j = 0, \quad d_j < 0)\}\,.$$

Then we have

$$\alpha_m = \max\left\{\lambda \in \mathbb{R} \,\middle|\, \rho(\lambda) = \frac{m-1}{2} + \frac{l}{2}\right\},$$

$$\beta_m = \min\left\{\lambda \in \mathbb{R} \,\middle|\, \rho(\lambda) = \frac{m}{2} + \frac{l}{2}\right\}.$$

This is why our main result is largely affected by (1.9). Thereby, we must be careful about how the principal value is chosen in applications.

5. Perturbation theory for the generalized Kronig–Penney Hamiltonian

In this section we perturb the operator H by a bounded, periodic function and discuss its spectral properties. We introduce

$$L^\infty(\mathbb{R};\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \quad \|f\|_{L^\infty(\mathbb{R})} < \infty \right\},$$

where $\|f\|_{L^\infty(\mathbb{R})} = \text{ess.sup}_{x\in\mathbb{R}}|f(x)|$. Assume that $q \in L^\infty(\mathbb{R};\mathbb{R})$ and q is a 2π-periodic function. We recall (1.1) and (1.2). We define the operator

$$\tilde{H} = \tilde{H}(\theta_1, \dots, \theta_n, A_1, \dots, A_n) \quad \text{in} \quad L^2(\mathbb{R})$$

as

$$\tilde{H} = H + q\,,$$
$$\text{Dom}(\tilde{H}) = \text{Dom}(H)\,.$$

Since H is self-adjoint operator and $q \in L^\infty(\mathbb{R}, \mathbb{R})$, $\tilde{H}$ is also self-adjoint(see [10, Section V. Theorem 4.10]). We have the direct integral representation of $\tilde{H}$:

$$\mathcal{U}\tilde{H}\mathcal{U}^{-1} = \int_0^{2\pi} \oplus \tilde{H}_\mu d\mu\,,$$

where $\tilde{H}_\mu = H_\mu + q$. Let $\tilde{\lambda}_j(\mu)$ be the jth eigenvalue of $\tilde{H}_\mu$ counted with multiplicity for $j \in \mathbb{N}$. We have the following statements, which are analogous to Proposition 1.1.

Proposition 5.1. (a) *The function $\tilde{\lambda}_j(\cdot)$ is continuous on $[0, 2\pi]$.*

(b) *It holds that $\tilde{\lambda}_j(\mu) = \tilde{\lambda}_j(-\mu + 2\theta_1 + \cdots + 2\theta_n)$.*

(c) *If $\mu - (\theta_1 + \cdots + \theta_n) \notin \pi\mathbb{Z}$, then every eigenvalue of $\tilde{H}_\mu$ is simple.*

(d) *The spectrum of $\tilde{H}(\theta_1, \dots, \theta_n, A_1, \dots, A_n)$ is given by*

$$\begin{aligned}
&\sigma\big(\tilde{H}(\theta_1, \dots, \theta_n, A_1, \dots, A_n)\big) \\
&\quad = \bigcup_{\mu\in[\theta_1+\cdots+\theta_n, \theta_1+\cdots+\theta_n+\pi]} \sigma\big(\tilde{H}_\mu(\theta_1, \dots, \theta_n, A_1, \dots, A_n)\big) \\
&\quad = \bigcup_{j=1}^{\infty} \tilde{\lambda}_j([\theta_1 + \cdots + \theta_n, \theta_1 + \cdots + \theta_n + \pi]) \\
&\quad = \bigcup_{j=1}^{\infty} \bigcup_{\mu\in[\theta_1+\cdots+\theta_n, \theta_1+\cdots+\theta_n+\pi]} \{\tilde{\lambda}_j(\mu)\}\,.
\end{aligned}$$

(e) *The set $\sigma(\tilde{H}(\theta_1, \dots, \theta_n, A_1, \dots, A_n))$ is independent of $\{\theta_j\}_{j=1}^n$.*

(f) *If $\xi > 0$ and $\theta_1 = \theta_2 = \cdots = \theta_n = 0$, then the function $\tilde{\lambda}_j(\cdot)$ is strictly monotone increasing (respectively, decreasing) function on $[0, \pi]$ for odd (respectively, even) j.*

(g) *If* $\xi < 0$ *and* $\theta_1 = \theta_2 = \cdots = \theta_n = 0$*, then the function* $\tilde{\lambda}_j(\cdot)$ *is strictly monotone increasing* (*respectively, decreasing*) *function on* $[0,\pi]$ *for even* (*respectively, odd*) j.

The proof of (a)–(e) in this proposition is exactly same as that in Proposition 1.1. Since the proof of (f) and (g)in Proposition 5.1 is even subtler than that in Proposition 1.1, we give a rather detailed proof of Proposition 5.1 (f) and (g) for the sake of completeness. We consider the equations

$$-y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda)\,, \quad x \in \mathbb{R}\setminus\Gamma\,, \tag{5.1}$$

$$\begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix}, \qquad x \in \Gamma_j\,, \quad j = 1,2,\ldots,n\,, \tag{5.2}$$

where λ is a real parameter. These equations also have two solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ which are uniquely determined by the initial conditions

$$y_1(+0,\lambda) = 1\,, \quad y_1'(+0,\lambda) = 0\,,$$

and

$$y_2(+0,\lambda) = 0\,, \quad y_1'(+0,\lambda) = 1\,,$$

respectively. The differential equation (5.1) is equivalent to

$$-y''(x,\lambda) + \big(q(x) + \|q\|_{L^\infty(\mathbb{R})}\big)y(x,\lambda) = \big(\lambda + \|q\|_{L^\infty(\mathbb{R})}\big)y(x,\lambda)\,, \tag{5.3}$$

where $x \in \mathbb{R}\setminus\Gamma$. Let

$$\tilde{q}(x) = q(x) + \|q\|_{L^\infty(\mathbb{R})}\,, \tag{5.4}$$

and

$$\tilde{\lambda} = \lambda + \|q\|_{L^\infty(\mathbb{R})}\,.$$

Let $\lambda < -\|q\|_{L^\infty(\mathbb{R})}$ and $\eta = \sqrt{-\tilde{\lambda}}$. We have $\tilde{\lambda} < 0$. We define $u_m(k,j-1,x,\lambda)$ as

$$u_0(k,j-1,x,\lambda) = y_k(\kappa_{j-1}+0,\lambda)\cosh\big(\eta(x-\kappa_{j-1})\big) + y_k'(\kappa_{j-1}+0,\lambda)\frac{\sinh\big(\eta(x-\kappa_{j-1})\big)}{\eta}\,, \tag{5.5}$$

$$u_m(k,j-1,x,\lambda) = \frac{1}{\eta}\int_{\kappa_{j-1}}^{x} \sinh\big(\eta(x-\xi)\big)\tilde{q}(\xi)u_{m-1}(k,j-1,\xi,\lambda)d\xi \tag{5.6}$$

for $m \in \mathbb{N}$, $k = 1,2$, $j = 2,3,\ldots,n+1$, and $x \in (\kappa_{j-1},\kappa_j)$. By applying the Picard method of iteration to the differential equation (5.3), we have

$$y_k(x,\lambda) = \sum_{m=0}^{\infty} u_m(k,j-1,x,\lambda) \quad \text{for} \quad x \in (\kappa_{j-1},\kappa_j)\,, \tag{5.7}$$

$$y_k'(x,\lambda) = \sum_{m=1}^{\infty} u_m'(k,j-1,x,\lambda) \quad \text{for} \quad x \in (\kappa_{j-1},\kappa_j)\,. \tag{5.8}$$

Differentiating (5.6) with respect to x, we have

$$u'_m(k,j-1,x,\lambda) = \int_{\kappa_{j-1}}^{x} \cosh\big(\eta(x-\xi)\big)\tilde{q}(\xi)u_{m-1}(k,j-1,\xi,\lambda)d\xi \tag{5.9}$$

for $k = 1, 2$, $j = 2, 3, \ldots, n+1$, and $x \in (\kappa_{j-1}, \kappa_j)$.

For $m \in \mathbb{N}$, $k = 1, 2$, $j = 2, 3, \ldots, n+1$, and $x \in (\kappa_{j-1}, \kappa_j)$, we define $I_m(k, j-1, x, \lambda)$, $J_m(k, j-1, x, \lambda)$, $\tilde{I}_m(k, j-1, x, \lambda)$, and $\tilde{J}_m(k, j-1, x, \lambda)$ as

$$\begin{aligned} u_m(k,j-1,x,\lambda) &= \cosh\big(\eta(x-\kappa_{j-1})\big) \\ &\times \left\{ y_k(\kappa_{j-1}+0,\lambda) I_m(k,j-1,x,\lambda) + \frac{y'_k(\kappa_{j-1}+0,\lambda)}{\eta} J_m(k,j-1,x,\lambda) \right\}, \end{aligned} \tag{5.10}$$

$$\begin{aligned} u'_m(k,j-1,x,\lambda) &= \cosh\big(\eta(x-\kappa_{j-1})\big) \\ &\times \Big\{ y_k(\kappa_{j-1}+0,\lambda)\eta \tanh\big(\eta(x-\kappa_{j-1})\big) \tilde{I}_m(k,j-1,x,\lambda) \\ &+ y'_k(\kappa_{j-1}+0,\lambda)\tilde{J}_m(k,j-1,x,\lambda) \Big\}. \end{aligned} \tag{5.11}$$

Summarizing (5.7), (5.8), (5.10), and (5.11), we obtain

$$\begin{aligned} y_k(\kappa_j-0,\lambda) = \cosh\tau_j\eta & \\ \times \Big\{ & y_k(\kappa_{j-1}+0,\lambda) \sum_{m=0}^{\infty} I_m(k,j-1,\kappa_j-0,\lambda) \\ &+ \frac{y'_k(\kappa_{j-1}+0,\lambda)}{\eta} \sum_{m=0}^{\infty} J_m(k,j-1,\kappa_j-0,\lambda) \Big\}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} y'_k(\kappa_j-0,\lambda) = \cosh\tau_j\eta & \\ \times \Big\{ & y_k(\kappa_{j-1}+0,\lambda)\eta\,(\tanh\tau_j\eta) \sum_{n=0}^{\infty} \tilde{I}_m(k,j-1,\kappa_j-0,\lambda) \\ &+ y'_k(\kappa_{j-1}+0,\lambda) \sum_{m=0}^{\infty} \tilde{J}_m(k,j-1,\kappa_j-0,\lambda) \Big\}. \end{aligned} \tag{5.13}$$

By (5.2), (5.12), and (5.13), we obtain

$$\begin{aligned} y_k(\kappa_j+0,\lambda) &= \alpha_j(k,\lambda)y_k(\kappa_{j-1}+0,\lambda) + \beta_j(k,\lambda)y'_k(\kappa_{j-1}+0,\lambda), \\ y'_k(\kappa_j+0,\lambda) &= \gamma_j(k,\lambda)y_k(\kappa_{j-1}+0,\lambda) + \delta_j(k,\lambda)y'_k(\kappa_{j-1}+0,\lambda) \end{aligned}$$

for $k = 1, 2$, $j = 2, 3, \ldots, n+1$, and $\lambda < -\|q\|_{L^\infty(\mathbb{R})}$, where

$$\alpha_j(k,\lambda) = e^{i\theta_j}\Bigg(a_j \cosh(\tau_j\eta)\sum_{m=0}^{\infty} I_m(k, j-1, \kappa_j - 0, \lambda) + b_j\eta\sinh(\tau_j\eta)\sum_{m=0}^{\infty}\tilde{I}_m(k, j-1, \kappa_j - 0, \lambda)\Bigg), \tag{5.14}$$

$$\beta_j(k,\lambda) = e^{i\theta_j}\Bigg(\frac{a_j}{\eta} \cosh(\tau_j\eta)\sum_{m=0}^{\infty} J_m(k, j-1, \kappa_j - 0, \lambda) + b_j\cosh(\tau_j\eta)\sum_{m=0}^{\infty}\tilde{J}_m(k, j-1, \kappa_j - 0, \lambda)\Bigg), \tag{5.15}$$

$$\gamma_j(k,\lambda) = e^{i\theta_j}\Bigg(c_j \cosh(\tau_j\eta)\sum_{m=0}^{\infty} I_m(k, j-1, \kappa_j - 0, \lambda) + d_j\eta\sinh(\tau_j\eta)\sum_{m=0}^{\infty}\tilde{I}_m(k, j-1, \kappa_j - 0, \lambda)\Bigg), \tag{5.16}$$

$$\delta_j(k,\lambda) = e^{i\theta_j}\Bigg(\frac{c_j}{\eta} \cosh(\tau_j\eta)\sum_{m=0}^{\infty} J_m(k, j-1, \kappa_j - 0, \lambda) + d_j\cosh(\tau_j\eta)\sum_{m=0}^{\infty}\tilde{J}_m(k, j-1, \kappa_j - 0, \lambda)\Bigg). \tag{5.17}$$

Lemma 5.2. *For $k = 1, 2$, and $j = 2, 3, \ldots, n+1$, we have*

$$\sum_{m=0}^{\infty} I_m(k, j-1, \kappa_j - 0, \lambda) = 1 + \mathcal{O}\left((\sqrt{-\lambda})^{-1}\right), \tag{5.18}$$

$$\sum_{m=0}^{\infty} \tilde{I}_m(k, j-1, \kappa_j - 0, \lambda) = 1 + \mathcal{O}\left((\sqrt{-\lambda})^{-1}\right), \tag{5.19}$$

$$\sum_{m=0}^{\infty} J_m(k, j-1, \kappa_j - 0, \lambda) = 1 + \mathcal{O}\left((\sqrt{-\lambda})^{-1}\right), \tag{5.20}$$

$$\sum_{m=0}^{\infty} \tilde{J}_m(k, j-1, \kappa_j - 0, \lambda) = 1 + \mathcal{O}\left((\sqrt{-\lambda})^{-1}\right) \tag{5.21}$$

as $\lambda \to -\infty$.

Proof. We pick $k = 1, 2$, $j = 2, 3, \ldots, n+1$, and $\lambda < -\|q\|_{L^\infty(\mathbb{R})}$, arbitrarily. By (5.5), we notice

$$I_0(k, j-1, x, \lambda) = 1, \quad J_0(k, j-1, x, \lambda) = \tanh\left(\eta(x - \kappa_{j-1})\right). \tag{5.22}$$

We choose $m \in \mathbb{N}$, arbitrarily. Substituting (5.10) for (5.6), we obtain the recurrence relations

$$\begin{aligned} &I_m(k, j-1, x, \lambda) \\ &= \frac{1}{\eta} \int_{\kappa_{j-1}}^{x} \sinh\big(\eta(x-\xi)\big)\tilde{q}(\xi) \frac{\cosh\big(\eta(\xi-\kappa_{j-1})\big)}{\cosh\big(\eta(x-\kappa_{j-1})\big)} I_{m-1}(k, j-1, \xi, \lambda) d\xi\,, \qquad (5.23) \\ &J_m(k, j-1, x, \lambda) \\ &= \frac{1}{\eta} \int_{\kappa_{j-1}}^{x} \sinh\big(\eta(x-\xi)\big)\tilde{q}(\xi) \frac{\cosh\big(\eta(\xi-\kappa_{j-1})\big)}{\cosh\big(\eta(x-\kappa_{j-1})\big)} J_{m-1}(k, j-1, \xi, \lambda) d\xi\,. \qquad (5.24) \end{aligned}$$

On the other hand, it follows by differentiating (5.5) that

$$\tilde{I}_0(k, j-1, x, \lambda) = 1\,, \quad \tilde{J}_0(k, j-1, x, \lambda) = 1\,. \tag{5.25}$$

Substituting (5.11) for (5.9), we derive

$$\begin{aligned} &\tilde{I}_m(k, j-1, x, \lambda) \\ &= \frac{1}{\eta} \int_{\kappa_{j-1}}^{x} \cosh\big(\eta(x-\xi)\big)\tilde{q}(\xi) \frac{\cosh\big(\eta(\xi-\kappa_{j-1})\big)}{\sinh\big(\eta(x-\kappa_{j-1})\big)} I_{m-1}(k, j-1, \xi, \lambda) d\xi\,, \qquad (5.26) \\ &\tilde{J}_m(k, j-1, x, \lambda) \\ &= \frac{1}{\eta} \int_{\kappa_{j-1}}^{x} \cosh\big(\eta(x-\xi)\big)\tilde{q}(\xi) \frac{\sinh\big(\eta(\xi-\kappa_{j-1})\big)}{\cosh\big(\eta(x-\kappa_{j-1})\big)} J_{m-1}(k, j-1, \xi, \lambda) d\xi\,. \qquad (5.27) \end{aligned}$$

The integral kernel in (5.23) is estimated as

$$\begin{aligned} 0 &\le \frac{\sinh\big(\eta(x-\xi)\big)\cosh\big(\eta(\xi-\kappa_{j-1})\big)}{\cosh\big(\eta(x-\kappa_{j-1})\big)} \tilde{q}(\xi) \\ &= \left(\frac{1}{2} \tanh\big(\eta(x-\kappa_{j-1})\big) + \frac{\sinh(x+\kappa_{j-1}-2\xi)}{2\cosh\big(\eta(x-\kappa_{j-1})\big)} \right) \tilde{q}(\xi) \\ &\le \Big(\tanh\big(\eta(x-\kappa_{j-1})\big)\Big)\tilde{q}(\xi) \\ &\le \tilde{q}(\xi) \end{aligned} \tag{5.28}$$

for $\xi \in (\kappa_{j-1}, x)$, because it follows by (5.4) that

$$\tilde{q}(x) \ge 0\,, \qquad \text{a.e.} \quad x \in \mathbb{R}\,. \tag{5.29}$$

Thereby, we get

$$I_m(k, j-1, \kappa_j - 0, \lambda) \ge 0 \tag{5.30}$$

by induction. Furthermore, we show

$$I_m(k, j-1, x, \lambda) \le \frac{1}{m!} \left(\frac{\|\tilde{q}\|_{L^\infty(\mathbb{R})}(x-\kappa_{j-1})}{\eta} \right)^m \tag{5.31}$$

by induction. By (5.22), it follows that (5.31) holds when $m = 1$. We pick $l \geq 2$, arbitrarily. Suppose that (5.31) holds for $m = l - 1$. We have

$$\begin{aligned} I_l(k, j-1, x, \lambda) &\leq \frac{1}{\eta(l-1)!} \int_{\kappa_{j-1}}^{x} \tilde{q}(\xi) \left(\frac{\|\tilde{q}\|_{L^\infty(\mathbb{R})}(\xi - \kappa_{j-1})}{\eta} \right)^{l-1} d\xi \\ &\leq \frac{1}{(l-1)!} \left(\frac{\|\tilde{q}\|_{L^\infty(\mathbb{R})}}{\eta} \right)^{l} \int_{\kappa_{j-1}}^{x} (\xi - \kappa_{j-1})^{l-1} d\xi \\ &= \frac{1}{l!} \left(\frac{\|\tilde{q}\|_{L^\infty(\mathbb{R})}(x - \kappa_{j-1})}{\eta} \right)^{l} . \end{aligned}$$

Therefore we have (5.31) for all $m \in \mathbb{N}$. Now we prove (5.18). By (5.22), (5.30), and (5.31), we have

$$\begin{aligned} 0 &\leq \sum_{m=0}^{\infty} I_m(k, j-1, \kappa_j - 0, \lambda) - 1 \\ &\leq \sum_{m=1}^{\infty} \frac{1}{m!} \left(\frac{\|\tilde{q}\|_{L^\infty(\mathbb{R})} \tau_j}{\eta} \right)^{m} \\ &\leq \frac{\|\tilde{q}\|_{L^\infty(\mathbb{R})}}{\eta} \exp\left(\frac{\|\tilde{q}\|_{L^\infty(\mathbb{R})}}{\eta} \right) , \end{aligned}$$

and thus (5.18). Similarly, we get (5.19)–(5.21). □

We recall (2.13)–(2.16). By Lemma 5.2, we have the following lemma.

Lemma 5.3. *We have*

$$\frac{y_1(\kappa_{n+1} + 0, \lambda)}{\prod_{j=2}^{n+1} e^{i\theta_j} \cosh \tau_j \eta} = \begin{cases} a_{n+1}\mu(n)\eta^{n-k(n)} + \mathcal{O}\left(\eta^{n-k(n)-1}\right) & \text{if} \quad b_{n+1} = 0 \, , \\ b_{n+1}\mu(n)\eta^{n-k(n)} + \mathcal{O}\left(\eta^{n-k(n)-1}\right) & \text{if} \quad b_{n+1} \neq 0 \, , \end{cases} \tag{5.32}$$

$$\frac{y_2'(\kappa_{n+1} + 0, \lambda)}{\prod_{j=2}^{n+1} e^{i\theta_j} \cosh \tau_j \eta} = \begin{cases} d_{n+1}\mu(n)\eta^{n-k(n)} + \mathcal{O}\left(\eta^{n-k(n)-1}\right) \quad \text{if} \quad b_{n+1} = 0 \, , \\ d_{n+1}\mu(n)\eta^{n-k(n)-1} \\ +\left\{c_{n+1}\mu(n) + d_{n+1}\mathcal{O}(1)\right\}\eta^{n-k(n)-2} \\ +\mathcal{O}\left(\eta^{n-k(n)-3}\right) \quad \text{if} \quad b_{n+1} \neq 0 \end{cases} \tag{5.33}$$

as $\lambda \to -\infty$.

Thanks to Lemma 5.3, we have Proposition 5.1 (f) and (g).

Let $(\tilde{r}, \tilde{\omega})$ be the polar coordinates of $(\tilde{y}, \tilde{y}')$, where $\tilde{y}(x, \lambda)$ is a nontrivial solution to (5.1) and (5.2). Then $\tilde{\omega}(x, \lambda)$ satisfies the equations

$$\frac{d}{dx}\tilde{\omega}(x, \lambda) = \cos^2 \tilde{\omega}(x, \lambda) + \big(\lambda + q(x)\big) \sin^2 \tilde{\omega}(x, \lambda\,), \quad x \in \mathbb{R} \setminus \Gamma \tag{5.34}$$

as well as the boundary conditions

$$\begin{aligned}\sin\tilde{\omega}(x+0,\lambda)\big(c_j\sin\tilde{\omega}(x-0,\lambda)+d_j\cos\tilde{\omega}(x-0,\lambda)\big)\\ =\cos\tilde{\omega}(x+0,\lambda)\big(a_j\sin\tilde{\omega}(x-0,\lambda)+b_j\cos\tilde{\omega}(x-0,\lambda)\big),\end{aligned}\tag{5.35}$$

$$\operatorname{sgn}\big(\sin\tilde{\omega}(x+0,\lambda)\big)=\operatorname{sgn}\big(a_j\sin\tilde{\omega}(x-0,\lambda)+b_j\cos\tilde{\omega}(x-0,\lambda)\big),\tag{5.36}$$

$$\operatorname{sgn}\big(\cos\tilde{\omega}(x+0,\lambda)\big)=\operatorname{sgn}\big(c_j\sin\tilde{\omega}(x-0,\lambda)+d_j\cos\tilde{\omega}(x-0,\lambda)\big)\tag{5.37}$$

for all $x\in\Gamma_j$ and $j=1,2,\dots,n$. We choose the branch of $\tilde{\omega}(x+0,\lambda)$ as

$$-\pi\le\tilde{\omega}(x+0,\lambda)-\tilde{\omega}(x-0,\lambda)<\pi\quad\text{for}\quad x\in\Gamma\,.\tag{5.38}$$

Thanks to this selection, $\tilde{\omega}(x+0,\lambda)$ is uniquely determined. Let $\tilde{\omega}=\tilde{\omega}(x,\lambda,\tilde{\omega}_0)$ be the solution of (5.34)–(5.38) subject to the initial condition

$$\tilde{\omega}(+0,\lambda)=\tilde{\omega}_0\,.\tag{5.39}$$

We define the rotation number of (5.34)–(5.39) as

$$\tilde{\rho}(\lambda)=\lim_{n\to\infty}\frac{\tilde{\omega}(2n\pi+0,\lambda,\tilde{\omega}_0)-\tilde{\omega}_0}{2n\pi}\,.\tag{5.40}$$

In a similar way to Section 3, we have the following theorem, which is an extension of (1.14).

Theorem 5.4. *For $j\in\mathbb{N}$, let $\tilde{B}_j=[\tilde{\alpha}_j,\tilde{\beta}_j]$ be the jth band of the spectrum of $\tilde{H}$. Put*

$$l=\sharp\big\{1\le j\le n\big|\quad(b_j<0)\quad\text{or}\quad(b_j=0,\quad d_j<0)\big\}\,.$$

Then we have

$$\tilde{\alpha}_j=\max\left\{\lambda\in\mathbb{R}\;\middle|\;\tilde{\rho}(\lambda)=\frac{m-1}{2}-\frac{l}{2}\right\},$$

$$\tilde{\beta}_j=\min\left\{\lambda\in\mathbb{R}\;\middle|\;\tilde{\rho}(\lambda)=\frac{m}{2}-\frac{l}{2}\right\}$$

for $j\in\mathbb{N}$.

Acknowledgements

The author thanks the referees for useful comments which improved the manuscript. He also thanks Professor Kazushi Yoshitomi for helpful advices.

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Communicated by Claude-Alain Pillet.
Submitted: December 18, 2006.
Accepted: February 23, 2007.