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The Inverse Problem for Perturbed Harmonic Oscillator on the Half-Line with a Dirichlet Boundary Condition

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Dedicated to Vladimir Buslaev on the occasion of his 70th birthday

Abstract. We consider the perturbed harmonic oscillator $T_D\psi = -\psi'' + x^2\psi + q(x)\psi$, $\psi(0) = 0$, in $L^2(\mathbb{R}_+)$, where $q \in \mathbf{H}_+ = \{q', xq \in L^2(\mathbb{R}_+)\}$ is a real-valued potential. We prove that the mapping $q \mapsto$ spectral data $=$ $\{$eigenvalues of $T_D\} \oplus \{$norming constants$\}$ is one-to-one and onto. The complete characterization of the set of spectral data which corresponds to $q \in \mathbf{H}_+$ is given.

1. Introduction and main results

Consider the Schrödinger operator

$$\mathrm{H} = -\frac{\partial^2}{\partial \mathbf{x}^2} + |\mathbf{x}|^2 + q(|\mathbf{x}|) \quad \mathbf{x} \in \mathbb{R}^3\,, \tag{1.1}$$

acting in the space $L^2(\mathbb{R}^3)$. Let $x = |\mathbf{x}|$ and q be a real-valued bounded function. The operator H has pure point spectrum. Using the standard transformation $u(\mathbf{x}) \mapsto xu(x)$ and expansion in spherical harmonics, we obtain that H is unitary equivalent to a direct sum of the Schrödinger operators acting on $L^2(\mathbb{R}_+)$. The first operator from this sum is given by

$$T_D\psi = -\psi'' + x^2\psi + q(x)\psi\,, \quad \psi(0) = 0\,, \quad x \geq 0\,. \tag{1.2}$$

The second is $-\frac{d^2}{dx^2} + x^2 + \frac{2}{x^2} + q(x)$ etc. Below we consider the simplest case, i.e., the operator T_D. In our paper we assume that

$$q \in \mathbf{H}_+ = \left\{q \in L^2(\mathbb{R}_+) : q', xq \in L^2(\mathbb{R}_+)\right\}.$$

The similar class of potentials was used to solve the corresponding inverse problem on the real line [5]. Define the unperturbed operator $T_D^0\psi = -\psi'' + x^2\psi$, $\psi(0) = 0$.

The spectrum $\sigma(T_D)$ of T_D is the increasing sequence of simple eigenvalues $\sigma_n = \sigma_n^0 + o(1)$, where $\sigma_n^0 = 4n+3$, $n \ge 0$, are the eigenvalues of T_D^0. Note that $\sigma(T_D)$ does not determine q uniquely, see Theorem 1.3. Then what does the isospectral set

$$\mathrm{Iso}_D(q) = \big\{p \in \mathbf{H}_+ : \sigma_n(p) = \sigma_n(q) \text{ for all } n \ge 0\big\}$$

of all potentials p with the same Dirichlet spectrum as q look like?

The inverse problem consists of two parts:

1) to characterize the set of all sequences of real numbers which arise as the Dirichlet spectra of $q \in \mathbf{H}_+$.
2) to describe the set $\mathrm{Iso}_D(q)$.

We shall give the complete solution of these problems in Theorem 1.3. To describe the set $\mathrm{Iso}_D(q)$ we define the **norming constants** $\nu_n(q)$ by[1]

$$\nu_n(q) = \log \big\|\varphi\big(\cdot, \sigma_n(q), q\big)\big\|_+^{-2} = 2\log|\psi'_{n,D}(0,q)|, \quad n \ge 0, \tag{1.3}$$

where $\varphi(x) = \varphi(x, \lambda, q)$ is the solution of the equation

$$-\varphi'' + x^2\varphi + q(x)\varphi = \lambda\varphi, \quad \varphi(0) = 0, \quad \varphi'(0) = 1, \quad (\lambda, q) \in \mathbb{C} \times \mathbf{H}_+, \tag{1.4}$$

and $\psi_{n,D}$ is the n-th normalized (in $L^2(\mathbb{R}_+)$) eigenfunction of the operator T_D.

Remark. Let $\Psi_{n,0}(\mathbf{x}) = \Psi_{n,0}(|\mathbf{x}|)$ be the n-th normalized (in $L^2(\mathbb{R}^3)$) spherically-symmetric eigenfunction of the operator H given by (1.1). Then we obtain

$$\Psi_{n,0}(\mathbf{x}) = \frac{\psi_{n,D}(|\mathbf{x}|)}{2\sqrt{\pi}|\,\mathbf{x}|} \quad \text{and} \quad |\Psi_{n,0}(0)|^2 = \frac{e^{\nu_n(q)}}{4\pi}, \quad n \ge 0.$$

We describe papers about the inverse problem for the perturbed harmonic oscillator, which are relevant to our paper. McKean and Trubowitz [11] considered the problem of reconstruction on the real line. They gave an algorithm for the reconstruction of q from norming constants for the class of real infinitely differentiable potentials, vanishing rapidly at $\pm\infty$, for fixed eigenvalues $\lambda_n(q) = \lambda_n^0$ for all n and "norming constants" $\to 0$ rapidly as $n \to \infty$. Later on, Levitan [10] reproved some results of [11] without an exact definition of the class of potentials. Some uniqueness theorems were proved by Gesztesy, Simon [7] and Chelkak, Kargaev, Korotyaev [4]. Chelkak, Kargaev, Korotyaev [5] obtained the characterization and described the isospectral set for the case on the real line for $q \in \mathbf{H} = \{q \in L^2(\mathbb{R}) : q', xq \in L^2(\mathbb{R})\}$. For uniqueness theorems we need some asymptotics of fundamental solutions and eigenvalues at high energy. For characterization we need "sharp" asymptotics of these values. Usually it is not simple. Note that recently the asymptotics $\lambda_n(q)$ were determined for bounded potentials in [9]. Gesztesy and Simon [6] proved that the each $\mathrm{Iso}_D(q)$ is connected for various classes of potentials. Note that the inverse problem for harmonic oscillator on the half-line for the boundary conditions $\psi'(0) = b\psi(0)$, $b \in \mathbb{R}$ is solved in [3].

[1] Here and below we use the notations $\|\cdot\|_+ = \|\cdot\|_{L^2(\mathbb{R}_+)}$, $\langle\cdot,\cdot\rangle_+ = \langle\cdot,\cdot\rangle_{L^2(\mathbb{R}_+)}$.

Our approach is based on the methods from [5] and [13] (devoted to the inverse Dirichlet problem on $[0,1]$). The main point in the inverse problem for the perturbed harmonic oscillator T_D on $\mathbb{R}_+$ is the characterization of $\mathrm{Iso}_D(q)$. Note that, in contrast to the case of perturbed harmonic oscillator on the real-line [5], the characterization of $\{\nu_n\}_{n=0}^{\infty}$, i.e., the parameterization of isospectral manifolds, is given in terms of the standard weighted ℓ^2-space. Thus there is a big difference between the case of the real line and the case of the half-line.

The present paper continues the series of papers [4,5] devoted to the inverse spectral problem for the perturbed harmonic oscillator on the real line. Note that the set of spectra which correspond to potentials from $\mathbf{H}_+$ (see Section 3 for details) is similar to the space of spectral data in [5]. In particular, the range of the linear operator $f(z)\mapsto(1-z)^{-1/2}f(z)$ acting in some Hardy–Sobolev space in the unit disc plays an important role. As a byproduct of our analysis, we give the simple proof of the equivalence between two definitions of the space of spectral data (Theorem 4.2 in [5]), which was established in [5] using a more complicated techniques.

We recall some basic results from [5]. Consider the operator

$$T\psi = -\psi'' + x^2\psi + q(x)\psi\,,$$
$$q \in \mathbf{H}_{even} = \big\{q \in L^2(\mathbb{R}) : q', xq \in L^2(\mathbb{R});\ q(x) = q(-x),\ x \in \mathbb{R}\big\}\,,$$

acting in the space $L^2(\mathbb{R})$. The spectrum $\sigma(T)$ is an increasing sequence of simple eigenvalues given by

$$\lambda_n(q) = \lambda_n^0 + \mu_n(q)\,, \quad \text{where} \quad \lambda_n^0 = \lambda_n(0) = 2n+1\,, \quad n \ge 0\,, \qquad \text{and} \quad \mu_n(q) \to 0 \quad \text{as} \quad n \to \infty\,.$$

Define the real weighted ℓ^2-space

$$\ell_r^2 = \left\{ c = \{c_n\}_{n=0}^{\infty} :\ c_n \in \mathbb{R}\,,\ \|c\|_{\ell_r^2}^2 = \sum_{n\ge 0}(1+n)^{2r}|c_n|^2 < +\infty \right\}, \quad r \ge 0\,,$$

and the Hardy–Sobolev space of analytic functions in the unit disc $\mathbb{D}=\{z : |z|<1\}$:

$$H_r^2 = H_r^2(\mathbb{D}) = \left\{ f(z) \equiv \sum_{n\ge 0} f_n z^n,\ z\in\mathbb{D} :\ f_n \in \mathbb{R}\,,\ |f\|_{H_r^2} = \|\{f_n\}_{n=0}^{\infty}\|_{\ell_r^2} < +\infty \right\}, \quad r \ge 0\,.$$

Introduce the **space of spectral data** from [5]

$$\mathcal{H} = \left\{ h = \{h_n\}_{n=0}^{\infty} :\ \sum_{n\ge 0} h_n z^n \equiv \frac{f(z)}{\sqrt{1-z}}\,,\ f \in H_{\frac{3}{4}}^2 \right\}, \quad \|h\|_{\mathcal{H}} = \|f\|_{H_{3/4}^2}\,. \tag{1.5}$$

Theorem 1.1 ([5]). *The mapping* $q \to \{\lambda_n(q) - \lambda_n^0\}_{n=0}^\infty$ *is a real-analytic isomorphism*[2] *between the space of even potentials* $\mathbf{H}_{even}$ *and the following open convex subset*

$$\mathcal{S} = \left\{\{h_n\}_{n=0}^\infty \in \mathcal{H} : \lambda_0^0 + h_0 < \lambda_1^0 + h_1 < \lambda_2^0 + h_2 < \cdots \right\} \subset \mathcal{H}\,.$$

Remark. The inequalities in the definition of $\mathcal{S}$ correspond to the monotonicity of eigenvalues.

Recall the following sharp representation from [5]:

$$\lambda_n(q) = \lambda_n^0 + \frac{\int_{\mathbb{R}} q(t)dt}{\pi\sqrt{\lambda_n^0}} + \widetilde{\mu}_n(q)\,, \quad \left\{\widetilde{\mu}_n(q)\right\}_0^\infty \in \mathcal{H}_0\,, \quad q \in \mathbf{H}\,,$$

where the subspace $\mathcal{H}_0 \subset \mathcal{H}$ of codimension 1 is given by

$$\mathcal{H}_0 = \left\{ h \in \mathcal{H} : \sqrt{1-z}\sum_{n\ge 0} h_n z^n \Bigg|_{z=1} = f(1) = 0 \right\}. \tag{1.6}$$

Remark that Lemma 3.4 yields $\widetilde{\mu}_n(q) = O(n^{-3/4}\log^{1/2} n)$ as $n \to \infty$. Also, we need

Proposition 1.2 (Trace formula). *For each* $q \in \mathbf{H}_{even}$ *the following identity holds:*

$$q(0) = 2\sum_{n\ge 0}\left(\lambda_{2n}(q) - \lambda_{2n+1}(q) + 2\right), \tag{1.7}$$

where the sum converges absolutely.

We come to the inverse problem for the operator T_D on $\mathbb{R}_+$. For each $q \in \mathbf{H}_+$ we set $q(-x) = q(x)$, $x \ge 0$. This gives a natural isomorphism between $\mathbf{H}_+$ and $\mathbf{H}_{even}$. Then

$$\sigma_n(q) = \lambda_{2n+1}(q)\,, \quad n \ge 0\,.$$

Let

$$\mathcal{S}_D = \left\{\{h_n\}_{n=0}^\infty \in \mathcal{H} : \sigma_0^0 + h_0 < \sigma_1^0 + h_1 < \sigma_2^0 + h_2 < \cdots \right\}. \tag{1.8}$$

We formulate our main result.

Theorem 1.3.

(i) *The sequence* $\{\sigma_n(q) - \sigma_n^0\}_{n=0}^\infty$ *belongs to* $\mathcal{S}_D$ *for each potential* $q \in \mathbf{H}_+$.

(ii) *For each* $q \in \mathbf{H}_+$ *the sequence* $\{r_n(q)\}_{n=0}^\infty \in \ell^2_{3/4}$*, where* r_n *is given by*

$$\nu_n(q) = \nu_n^0 - \frac{q(0)}{2(2n+1)} + r_n(q)\,, \quad \text{and} \quad \nu_n^0 = \nu_n(0) = \log\left[\frac{4(2n+1)!}{\sqrt{\pi}\,2^{2n}[n!]^2}\right].$$

[2]By definition, the mapping of Hilbert spaces $F : H_1 \to H_2$ is a local real-analytic isomorphism iff for any $y \in H_1$ it has an analytic continuation $\widetilde{F}$ into some complex neighborhood $y \in U \subset H_{1\mathbb{C}}$ of y such that $\widetilde{F}$ is a bijection between U and some complex neighborhood $F(y) \in \widetilde{F}(U) \subset H_{2\mathbb{C}}$ of $F(y)$ and both $\widetilde{F}$, $\widetilde{F}^{-1}$ are analytic. The local isomorphism F is a (global) isomorphism iff it is a bijection.

(iii) *The mapping* $q \mapsto (\{\sigma_n(q) - \sigma_n^0\}_{n=0}^{\infty}, q(0), \{r_n(q)\}_{n=0}^{\infty})$ *is a real-analytic isomorphism between* $\mathbf{H}_+$ *and* $\mathcal{S}_D \times \mathbb{R} \times \ell^2_{3/4}$.

Remark. In particular, $(p(0), \{r_n(p)\}_{n=0}^{\infty}) \in \mathbb{R} \times \ell^2_{3/4}$ are "independent coordinates" in $\mathrm{Iso}_D(q)$.

The ingredients of the proof of Theorem 1.3 are:

i) Uniqueness Theorem. We adopt the proof from [13] and [4]. This proof requires only some estimates of the fundamental solutions.
ii) Analysis of the Fréchet derivative of the nonlinear spectral mapping $\{\text{potentials}\} \mapsto \{\text{spectral data}\}$ at the point $q = 0$. We emphasize that this linear operator is complicated (in particular, it is not the Fourier transform, as it was in [13]). Here we essentially use the technique of generating functions (from [5]), which are analytic in the unit disc.
iii) Asymptotic analysis of the difference between spectral data and its Fréchet derivatives at $q = 0$. Here the calculations and asymptotics from [5] play an important role.
iv) The proof that the spectral mapping is a surjection, i.e., the fact that each element of an appropriate Hilbert space can be obtained as spectral data of some potential $q \in \mathbf{H}_+$. Here we use the standard Darboux transform of second-order differential equations.

The plan of the paper. Section 2 is devoted to the basic asymptotics of the eigenvalues $\sigma_n(q)$ and the values $\log[(-1)^n \psi_+'(0, \sigma_n(q), q)]$. In Section 3 we introduce the space $\mathcal{H}$ and obtain its equivalent definition (Corollary 3.6). Furthermore, we consider a kind of linear approximation of our spectral data and prove Theorem 3.10 that is, in a sense, the linear analogue of the main Theorem 1.3. Section 4 is devoted to the asymptotics of the norming constants $\nu_n(q)$. Also, in this sect. we prove Proposition 1.2. In Section 5 we prove the main Theorem 1.3. All needed properties of fundamental solutions, gradients of spectral data and some technical lemmas are collected in Appendix.

2. Basic asymptotics

Let $\psi_+^0(x, \lambda) = D_{\frac{\lambda-1}{2}}(\sqrt{2}x)$ be the decreasing near $+\infty$ solution of the unperturbed equation

$$-\psi'' + x^2 \psi = \lambda \psi .$$

We use the standard notation $D_\mu(x)$ for the Weber functions (or the parabolic cylinder functions), see [1]. Note that for each $q \in \mathbf{H}_+$ the perturbed equation

$$-\psi'' + x^2 \psi + q(x)\psi = \lambda \psi$$

has the unique solution $\psi_+(x, \lambda, q)$ such that $\psi_+(x) = \psi_+^0(x)(1 + o(1))$ as $x \to +\infty$ (see (A.11)).

Lemma 2.1. *For each $q \in \mathbf{H}_+$ and $n \geq 0$ the following identities hold:*

$$\nu_n(q) = 2\log\left[-\frac{\psi'_+\big(0,\sigma_n(q),q\big)}{\dot\psi_+\big(0,\sigma_n(q),q\big)}\right], \tag{2.1}$$

$$\dot\psi_+\big(0,\sigma_n(q),q\big) = \frac{\psi^0_+\big(0,\sigma_n(q)\big)}{\sigma_n(q)-\sigma^0_n}\cdot\prod_{m:m\neq n}\frac{\sigma_n(q)-\sigma_m(q)}{\sigma_n(q)-\sigma^0_m}, \qquad \frac{\partial}{\partial\lambda}\psi_+ = \dot\psi_+\,. \tag{2.2}$$

Remark. It is important that the values $\dot\psi_+(0,\sigma_n(q),q)$ are uniquely determined by the spectrum $\sigma(T_D)$. In particular, $\dot\psi_+(0,\sigma_n(p),p) = \dot\psi_+(0,\sigma_n(q),q)$ for all $p \in \mathrm{Iso}_D(q)$ and $n \geq 0$.

Proof. The standard identity[3] $\psi_+^2 = \{\dot\psi_+\,,\psi_+\}'$ yields

$$\int_0^{+\infty}\psi_+^2(x)dx = \{\dot\psi_+\,,\psi_+\}\Big|_0^{+\infty} = -\dot\psi_+(0)\psi'_+(0)\,,$$

where we omit $\sigma_n(q)$ and q for short. Therefore,

$$e^{\nu_n(q)} = \big[\psi'_{n,D}(0,q)\big]^2 = \left[\frac{\psi'_+\big(0,\sigma_n(q),q\big)}{\|\psi_+\big(\cdot,\sigma_n(q),q\big)\|_+}\right]^2 = -\frac{\psi'_+\big(0,\sigma_n(q),q\big)}{\dot\psi_+\big(0,\sigma_n(q),q\big)}\,.$$

Using the Hadamard Factorization Theorem, we obtain

$$\psi_+(0,\sigma,q) = \psi^0_+(0,\sigma)\cdot\prod_{m\geq 0}\frac{\sigma-\sigma_m(q)}{\sigma-\sigma^0_m}\,, \quad \sigma\in\mathbb{C}\,.$$

The differentiation of $\psi_+(0,\sigma,q)$ gives (2.2). □

Let ψ^0_n be the normalized (in $L^2(\mathbb{R})$) eigenfunctions of the unperturbed harmonic oscillator on $\mathbb{R}$. Note that $\psi^0_{n,D}(\cdot) = \psi_{n,D}(\cdot,0) = \sqrt{2}\psi^0_{2n+1}(\cdot)$. It is well-known that

$$\psi^0_n(x) = \big(n!\sqrt{\pi}\big)^{-\frac12}D_n\big(\sqrt{2}x\big) = \big(2^n n!\sqrt{\pi}\big)^{-\frac12}H_n(x)e^{-\frac{x^2}{2}}\,, \quad n\geq 0\,,$$

where $H_n(x)$ are the Hermite polynomials. For each $n \geq 0$ we consider the second solution

$$\chi^0_n(x) = \left(\frac{n!\sqrt{\pi}}{2}\right)^{1/2}\begin{cases}(-1)^{\frac n2}\,\mathrm{Im}\,D_{-n-1}(i\sqrt{2}x)\,, & n \text{ is even}\,,\\ (-1)^{\frac{n-1}{2}}\,\mathrm{Re}\,D_{-n-1}(i\sqrt{2}x)\,, & n \text{ is odd}\,,\end{cases}$$

of the equation $-\psi''+x^2\psi = \lambda^0_n\psi$ which is uniquely defined by the conditions

$$\{\chi^0_n\,,\psi^0_n\} = 1\,, \quad (\psi^0_n\chi^0_n)(-x) = -(\psi^0_n\chi^0_n)(x)\,, \quad x\in\mathbb{R}\,.$$

Note that $(\psi^0_n\chi^0_n)(x) = (-1)^{n+1}x + O(x^2)$ as $x\to 0$, and $(\psi^0_n\chi^0_n)(x) = -x^{-1} + O(x^{-2})$ as $x\to\infty$, see [5]. Following [5], for $q \in \mathbf{H}_+$, we introduce

$$\hat q_n = \big\langle q,(\psi^0_n)^2\big\rangle_+\,, \quad \check q_n = \big\langle q,\psi^0_n\chi^0_n\big\rangle_+ \quad n\geq 0\,. \tag{2.3}$$

[3]Here and below we use the notations $\{f,g\} = fg' - f'g$, $u' = \frac{\partial}{\partial x}u$, $\dot u = \frac{\partial}{\partial\lambda}u$.

Also, we introduce the constants

$$\kappa_n = \psi_+^0(0,\lambda_n^0)\,, \quad \kappa_n' = (\psi_+^0)'(0,\lambda_n^0)\,, \quad \dot\kappa_n = \dot\psi_+^0(0,\lambda_n^0) \quad \text{and so on}\,.$$

Theorem 2.2. *For each $q \in \mathbf{H}_+$ the following asymptotics*[4] *hold:*

$$\sigma_n(q) = \sigma_n^0 + 2\hat{q}_{2n+1} + \ell^2_{\frac34+\delta}(n)\,, \tag{2.4}$$

$$\log\frac{\psi_+'(0,\sigma_n(q),q)}{\kappa_{2n+1}'} = \frac{\dot\kappa_{2n+1}'}{\kappa_{2n+1}'}\left(\sigma_n(q)-\sigma_n^0\right) - \check{q}_{2n+1} + \ell^2_{\frac34+\delta}(n)\,, \tag{2.5}$$

uniformly on bounded subsets of $\mathbf{H}_+$, for some absolute constant $\delta > 0$.

Remark.

i) Proposition 3.2 immediately yields $\{\hat{q}_n\}_{n=0}^\infty \in \mathcal{H}$. Using basic properties of the spaces $\mathcal{H}$, $\mathcal{H}_0$ (see Proposition 3.3 and Lemma 3.4), we obtain

$$\hat{q}_{2n+1} = \pi^{-1}\textstyle\int_{\mathbb{R}_+} q(t)dt\cdot(\sigma_n^0)^{-\frac12} + O\left(n^{-\frac34}\log^{\frac12} n\right).$$

ii) In the proof we use some technical results from [5], formulated in Appendix A.1–A.4.

Proof. Let $\mu = \sigma_n(q)-\sigma_n^0$ and $m = 2n+1$. Recall that $\sigma_n^0 = \lambda_m^0$ and $\psi_+(0,\lambda_m^0+\mu,q) = 0$. Lemma A.4 (i) yields $\mu = O(m^{-1/2})$. Due to Corollary A.3 and asymptotics (A.13), we have

$$0 = \frac{\psi_+(0,\lambda_m^0+\mu,q)}{\dot\kappa_m} = \frac{\psi_+^{(1)}(0,\lambda_m^0,q)+\dot\kappa_m\cdot\mu}{\dot\kappa_m} + O\left(m^{-1}\log m\right).$$

Hence, Lemma A.7 (ii) gives $\mu = 2\hat{q}_m + O(m^{-1}\log^2 m)$. Using the similar arguments, we deduce that

$$0 = \frac{1}{\dot\kappa_m}\left(\psi_+^{(1)}+\dot\psi_+^0\cdot\mu+\psi_+^{(2)}+\dot\psi_+^{(1)}\cdot 2\hat{q}_m+\frac{\ddot\psi_+^0}{2}\,(2\hat{q}_m)^2\right)(0,\lambda_m^0,q)+O\left(m^{-\frac32}\log^3 m\right).$$

Together with Lemmas A.7 (ii), A.6, this yields

$$\begin{aligned}\frac{\mu}{2} &= \hat{q}_m - \check{q}_m\hat{q}_m + \left(\frac{\dot\kappa_m'}{\kappa_m'}\,\hat{q}_m + \frac12\,\check{q}_m\right)\cdot 2\hat{q}_m - \frac{\ddot\kappa_m}{\kappa_m}\,(\hat{q}_m)^2 + \ell^2_{\frac34+\delta}(m)\\ &= \hat{q}_m + \left(\frac{2\dot\kappa_m'}{\kappa_m'} - \frac{\ddot\kappa_m}{\kappa_m}\right)\cdot(\hat{q}_m)^2 + \ell^2_{\frac34+\delta}(m) = \hat{q}_m + \ell^2_{\frac34+\delta}(m)\,.\end{aligned}$$

[4] Here and below $a_n = b_n + \ell_r^2(n)$ means that $\{a_n - b_n\}_{n=0}^\infty \in \ell_r^2$. We say that $a_n(q) = b_n(q) + \ell_r^2(n)$ holds true uniformly on some set iff norms $\|\{a_n(q)-b_n(q)\}_{n=0}^\infty\|_{\ell_r^2}$ are uniformly bounded on this set.

Furthermore, using Corollary A.3 and Lemma A.7 (ii), we obtain

$$\begin{aligned}
&\frac{\psi'_+\big(0,\sigma_n(q),q\big)}{\kappa'_m} \\
&= \frac{\Big(\psi^0_+ + \psi^{(1)}_+ + \psi^{(2)}_+\Big)(0,\lambda^0_m+\mu,q)}{\kappa'_m} + O(m^{-\frac{3}{2}}) \\
&= \frac{\Big(\psi^0_+ + \psi^{(1)}_+ + \dot\psi^0_+\cdot\mu + \psi^{(2)}_+ + \dot\psi^{(1)}_+\cdot 2\hat{q}_m + \frac12\ddot\psi^0_+\cdot(2\hat{q}_m)^2\Big)'(0,\lambda^0_m,q)}{\kappa'_m} \\
&\quad + O\big(m^{-\frac{3}{2}}\log^3 m\big) \\
&= 1 - \check{q}_m + \frac{\dot\kappa'_m}{\kappa'_m}\cdot\mu + \left(\frac12(\check{q}_m)^2 - \frac{\pi^2}{8}(\hat{q}_m)^2\right) + \left(-\frac{\dot\kappa'_m}{\kappa'_m}\check{q}_m + \frac{\pi^2}{8}\hat{q}_m\right)\cdot 2\hat{q}_m \\
&\quad + \frac{2\ddot\kappa'_m}{\kappa'_m}(\hat{q}_m)^2 + \ell^2_{\frac34+\delta}(m)\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\log\frac{\psi'_+\big(0,\sigma_n(q),q\big)}{\kappa'_{2n+1}} &= -\check{q}_m + \frac{\dot\kappa'_m}{\kappa'_m}\cdot\mu + 2\left(\frac{\ddot\kappa'_m}{\kappa'_m} - \frac{(\dot\kappa'_m)^2}{(\kappa'_m)^2} + \frac{\pi^2}{16}\right)\cdot(\hat{q}_m)^2 + \ell^2_{\frac34+\delta}(m) \\
&= -\check{q}_m + \frac{\dot\kappa'_m}{\kappa'_m}\cdot\mu + \ell^2_{\frac34+\delta}(m)\,,
\end{aligned}$$

where we have used Lemma A.6. □

3. Coefficients $\hat{q}_{2n+1}$, $\check{q}_{2n+1}$ and $\tilde{q}_n$

Let

$$\widetilde{\psi}^0_n(x) = 2^{1/4}\psi^0_n(\sqrt{2}x)\,, \quad n \ge 0\,.$$

Note that the mapping

$$q \mapsto \big\{\langle q, \widetilde{\psi}^0_n\rangle\big\}^\infty_{n=0}\,, \quad \mathbf{H} \to \ell^2_{1/2}\,. \tag{3.1}$$

is a linear isomorphism[5]. Moreover, since $\{\widetilde{\psi}^0_{2m}\}^\infty_{m=0}$ is the orthogonal basis of the space $\mathbf{H}_{even}$, it is the orthogonal basis of $\mathbf{H}_+$. On the contrary, $\{\widetilde{\psi}^0_{2m+1}\}^\infty_{m=0}$ is the orthogonal basis of the subspace

$$\mathbf{H}^0_+ = \{q \in \mathbf{H}_+ : q(0) = 0\} \subsetneq \mathbf{H}_+\,.$$

[5]We say that the linear operator is a linear isomorphism iff it is bounded and its inverse is bounded too.

Following [5], for each potential $q \in \mathbf{H}_+$ we define two (analytic in the unit disc $\mathbb{D}$) functions

$$\begin{aligned} (Fq)(z) &\equiv \frac{1}{(2\pi)^{1/4}} \sum_{k \geq 0} \sqrt{E_k}\, \langle q, \widetilde{\psi}^0_{2k} \rangle_+ \cdot z^k\,, \quad z \in \mathbb{D}\,, \\ (Gq)(z) &\equiv -\frac{(2\pi)^{1/4}}{2} \sum_{k \geq 0} \frac{\langle q, \widetilde{\psi}^0_{2k+1} \rangle_+}{\sqrt{(2k+1)E_k}}\, z^k\,, \quad z \in \mathbb{D}\,, \end{aligned} \tag{3.2}$$

where

$$E_k = \frac{(2k)!}{2^{2k}(k!)^2} \sim \pi^{-\frac{1}{2}} k^{-\frac{1}{2}} \quad \text{as} \quad k \to \infty\,. \tag{3.3}$$

Lemma 3.1.

(i) *The mapping* $q \mapsto Fq$ *is a linear isomorphism between* $\mathbf{H}_+$ *and* $H^2_{3/4}$.

(ii) *The mapping* $q \mapsto Gq$ *is a linear isomorphism between* $\mathbf{H}^0_+$ *and* $H^2_{3/4}$.

(iii) $(Gq)(\cdot) \in C(\mathbb{T} \setminus \{-1\})$ *and* $(Gq)(\cdot) \in L^1(\mathbb{T})$ *for each* $q \in \mathbf{H}_+$.

Proof. (i), (ii) Using (3.1) and (3.3), we deduce that the mappings

$$\begin{aligned} q &\mapsto \big\{ \langle q, \widetilde{\psi}^0_{2m} \rangle_+ \big\}_{m=0}^{\infty} \mapsto Fq\,, \qquad && \mathbf{H}_+ \to \ell^2_{1/2} \to H^2_{3/4} \\ q &\mapsto \big\{ \langle q, \widetilde{\psi}^0_{2m+1} \rangle_+ \big\}_{m=0}^{\infty} \mapsto Gq\,, \qquad && \mathbf{H}^0_+ \to \ell^2_{1/2} \to H^2_{3/4} \end{aligned}$$

are linear isomorphisms.

(iii) Let $q \in \mathbf{H}_+$. Since $\widetilde{\psi}^0_0(0) = 2^{1/4}\pi^{-1/4}$ (see (3.8)), we obtain

$$q(x) = 2^{-\frac{1}{4}} \pi^{\frac{1}{4}} q(0) \cdot \widetilde{\psi}^0_0(x) + q_0(x)\,, \quad x \geq 0\,,$$

for some $q_0 \in \mathbf{H}^0_+$. Due to (ii), we have $Gq_0 \in H^2_{3/4} \subset C(\mathbb{T}) \subset L^1(\mathbb{T})$. Furthermore, (3.9) yields

$$(G\widetilde{\psi}^0_0)(z) = -\frac{(2\pi)^{1/4}}{2\sqrt{2\pi}} \sum_{m \geq 0} \frac{(-1)^m z^m}{2m+1}\,.$$

Hence, $G\widetilde{\psi}^0_0 \in C(\mathbb{T} \setminus \{-1\})$, $G\widetilde{\psi}^0_0 \in L^1(\mathbb{T})$ and the same holds for Gq. □

Lemma 3.2 ([5]). *Let* $q \in \mathbf{H}_+$. *Then the following identities*[6] *hold:*

$$\sum_{n \geq 0} \hat{q}_n z^n \equiv \frac{(Fq)(z)}{\sqrt{1-z}}\,, \qquad \sum_{n \geq 0} \check{q}_n z^n \equiv P_+ \left[\frac{(Gq)(\zeta)}{\sqrt{1-\overline{\zeta}}} \right], \quad z \in \mathbb{D}\,, \tag{3.4}$$

$$(Fq)(1) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}_+} q(t)dt\,, \qquad (Fq)(-1) = 2^{-\frac{3}{2}} q(0)\,, \tag{3.5}$$

where the coefficients $\hat{q}_n$ *and* $\check{q}_n$, $n \geq 0$, *are defined by* (2.3).

[6] We write $f(z) \equiv g(z)$ iff the identity $f(z) = g(z)$ holds true for all $z \in \mathbb{D}$.

Remark.

i) Here and below we put $(P_+f)(z) \equiv \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)d\zeta}{\zeta-z}$ for any $f\in L^1(\mathbb{T})$ and $z\in\mathbb{D}$. In particular, the identity $(P_+\sum_{n=-k}^{k}c_n\zeta^n)(z)\equiv\sum_{n=0}^{k}c_nz^n$ holds true for any $c_n\in\mathbb{C}$.
ii) Definition (1.5) of the space $\mathcal{H}$ is directly motivated by asymptotics (2.4) and (3.4)

Proof. Identities (3.4) were proved in [5] (Propositions 1.2 and 2.9). Also, in [5] it was shown that

$$1=\sum_{k\geq 0}\widetilde{\psi}_{2k}^0(x)\int_{\mathbb{R}}\widetilde{\psi}_{2k}^0(t)dt=(2\pi)^{\frac14}\sum_{k\geq0}\sqrt{E_k}\,\widetilde{\psi}_{2k}^0(x)\,,$$

in the sense of distributions, which gives $(Fq)(1)=(2\pi)^{-1/2}\int_{\mathbb{R}_+}q(t)dt$. Furthermore,

$$\delta(x)=\sum_{k\geq0}\widetilde{\psi}_{2k}^0(x)\cdot\widetilde{\psi}_{2k}^0(0)=\sum_{k\geq0}\widetilde{\psi}_{2k}^0(x)\cdot\frac{2^{1/4}H_{2k}(0)}{\left(\sqrt{\pi}\,2^{2k}(2k)!\right)^{1/2}}$$
$$=\left(\frac{2}{\pi}\right)^{\frac14}\sum_{k\geq0}(-1)^k\sqrt{E_k}\,\widetilde{\psi}_{2k}^0(x)$$

in the sense of distributions. Together with (3.2), this implies $(Fq)(-1)=2^{-1/2}\cdot q(0)/2$. □

We need some results from [5] (see Lemmas 2.10, 2.11 [5]).

Proposition 3.3.

(i) *For each $\{h_n\}_{n=0}^{\infty}\in\mathcal{H}$ there exist unique $v\in\mathbb{R}$ and $\{h_n^{(0)}\}_{n=0}^{\infty}\in\mathcal{H}_0$ such that $h_n=v\cdot(\lambda_n^0)^{-1/2}+h_n^{(0)}$. The mapping*

$$h\mapsto\big(v,h^{(0)}\big)$$

is a linear isomorphism between $\mathcal{H}$ and $\mathbb{R}\times\mathcal{H}_0$. If $h=\{\hat{q}_n\}_{n=0}^{\infty}$, $q\in\mathbf{H}_+$, then

$$v=2^{\frac12}\pi^{-\frac12}\sqrt{1-z}\sum_{n=0}^{+\infty}h_nz^n\Big|_{z=1}=\pi^{-1}\int_{\mathbb{R}_+}q(t)dt\,.$$

(ii) *The set of finite sequences $\{(h_0,\ldots,h_k,0,0,\ldots),\ k\geq0,\ h_j\in\mathbb{R}\}$ is dense in $\mathcal{H}_0$.*
(iii) *The embeddings $\ell^2_{3/4}\subset\mathcal{H}_0\subset\ell^2_{1/4}$ are fulfilled.*
(iv) *If $\{h_n\}_{n=0}^{\infty}\in\mathcal{H}$, then $\{h_n-h_{n+1}\}_{n=0}^{\infty}\in\ell^2_{3/4}$.*

Remark. Since $\mathcal{H}_0\subset\ell^2_{1/4}$, the sequence of leading terms $\{(\lambda_n^0)^{-1/2}v\}_{n=0}^{\infty}$ doesn't belong to $\mathcal{H}_0$.

The next lemma gives the O-type estimate for sequences from $\mathcal{H}_0$.

Lemma 3.4. *Let $\{h_n\}_{n=0}^{\infty}\in\mathcal{H}_0$. Then $h_n=O(n^{-3/4}\log^{1/2}n)$ as $n\to\infty$.*

Proof. The proof is similar to the proof of Lemma 2.1 in [2]. Definition (1.6) of $\mathcal{H}_0$ yields

$$\sum_{n\geq 0} h_n z^n \equiv \frac{\sum_{k\geq 0} f_k z^k}{\sqrt{1-z}}\,, \quad \{f_k\}_{k=0}^{\infty} \in \ell^2_{\frac{3}{4}}\,, \quad \sum_{k\geq 0} f_k = f(1) = 0\,.$$

Recall that $(1-z)^{-1/2} \equiv \sum_{m\geq 0} E_m z^m$. Hence,

$$h_n = \sum_{k=0}^{n} E_{n-k} f_k = \sum_{k=1}^{n} (E_{n-k} - E_n) f_k - E_n \sum_{k=n+1}^{\infty} f_k\,.$$

It is easy to see that $E_n = O(n^{-1/2})$ and $E_{n-k} - E_n = O(kn^{-1}(n-k+1)^{-1/2})$. Therefore,

$$\left| E_n \sum_{k=n+1}^{\infty} f_k \right| \leq E_n \left(\sum_{k=n+1}^{+\infty} k^{-\frac{3}{2}} \right)^{\frac{1}{2}} \left(\sum_{k=n+1}^{+\infty} k^{\frac{3}{2}} |f_k|^2 \right)^{\frac{1}{2}} = O(n^{-\frac{3}{4}})\,,$$

$$\left| \sum_{k=1}^{n} (E_{n-k} - E_n) f_k \right| \leq \left(\sum_{k=1}^{n} O\left(\frac{k^{\frac{1}{2}}}{n^2(n-k+1)} \right) \right)^{1/2} \left(\sum_{k=1}^{n} k^{\frac{3}{2}} |f_k|^2 \right)^{1/2}$$
$$= O(n^{-\frac{3}{4}} \log^{\frac{1}{2}} n)\,,$$

where the estimate $\sum_{k=1}^{+\infty} k^{3/2} |f_k|^2 < +\infty$ has been used. □

Recall that Fq and Gq are defined by (3.2) and the system of functions $\{\widetilde{\psi}^0_{2k}\}_{k=0}^{\infty}$ is a basis of $\mathbf{H}_+$. Therefore, it is possible to rewrite Gq in terms of Fq. Note that this situation differs from the case $q \in \mathbf{H}$ (the perturbed harmonic oscillator on the whole real line [5]), where the functions Fq and Gq are "independent coordinates" in the space of potentials.

For $\zeta = e^{i\phi} \in \mathbb{T}, \phi \in (-\pi, \pi)$, $\zeta \neq -1$, we define $\sqrt{\zeta} = e^{\frac{i\phi}{2}}$. We have the identity

$$\frac{1}{\sqrt{\zeta}} = \frac{2}{\pi} \sum_{s\in\mathbb{Z}} \frac{(-1)^s}{2s+1} \zeta^s \quad \text{in} \quad L^2(\mathbb{T})\,. \tag{3.6}$$

Lemma 3.5. *For each $q \in \mathbf{H}_+$ the following identity holds:*

$$(Gq)(z) \equiv -\frac{\pi}{2} P_+ \left[\frac{(Fq)(\zeta)}{\sqrt{\zeta}} \right], \quad z \in \mathbb{D}\,. \tag{3.7}$$

Proof. We determine the coefficients of the function ψ^0_{2m+1} with respect to the basis $\{\psi^0_{2k}\}_{k=0}^{\infty}$. The standard identity $\{\psi^0_{2k}\,, \psi^0_{2m+1}\}' = (\lambda^0_{2k} - \lambda^0_{2m+1}) \psi^0_{2k} \psi^0_{2m+1}$ yields

$$(\psi^0_{2m+1}\,, \psi^0_{2k})_+ = \int_{\mathbb{R}_+} \psi^0_{2m+1}(x) \psi^0_{2k}(x) dx = \frac{\{\psi^0_{2k}\,, \psi^0_{2m+1}\}(0)}{\lambda^0_{2m+1} - \lambda^0_{2k}}$$
$$= \frac{\psi^0_{2k}(0)(\psi^0_{2m+1})'(0)}{2\big(2(m-k)+1\big)}\,.$$

Note that

$$\psi_{2k}^0(0)=\frac{H_{2k}(0)}{\big(\sqrt{\pi}\,2^{2k}(2k)!\big)^{1/2}}=\frac{(-1)^k2^k(2k-1)!!}{\big(\sqrt{\pi}\,2^{2k}(2k)!\big)^{1/2}}=\frac{(-1)^k}{\pi^{1/4}}\sqrt{E_k}\,, \qquad (3.8)$$

$$(\psi_{2m+1}^0)'(0)=\frac{H'_{2m+1}(0)}{\big(\sqrt{\pi}\,2^{2m+1}(2m+1)!\big)^{1/2}}=\frac{(-1)^m2^{m+1}(2m+1)!!}{\big(\sqrt{\pi}\,2^{2m+1}(2m+1)!\big)^{1/2}}$$

$$=\frac{(-1)^m\sqrt{2}}{\pi^{1/4}}\sqrt{(2m+1)E_m}\,.$$

Therefore,

$$(\widetilde{\psi}_{2m+1}^0\,,\widetilde{\psi}_{2k}^0)_+=(\psi_{2m+1}^0\,,\psi_{2k}^0)_+=\frac{1}{\sqrt{2\pi}}\cdot\frac{(-1)^{m-k}}{2(m-k)+1}\sqrt{(2m+1)E_mE_k}\,. \qquad (3.9)$$

Since $\|\widetilde{\psi}_{2k}^0\|_+^2=1/2$, we obtain

$$\frac{\widetilde{\psi}_{2m+1}^0}{\sqrt{(2m+1)E_m}}=\sqrt{\frac{2}{\pi}}\sum_{k=0}^{+\infty}\frac{(-1)^{m-k}}{2(m-k)+1}\cdot\sqrt{E_k}\,\widetilde{\psi}_{2k}^0\,.$$

This gives

$$P_+\left[\frac{(Fq)(\zeta)}{\sqrt{\zeta}}\right]\equiv\frac{2^{3/4}}{\pi^{5/4}}P_+\left[\sum_{l=-\infty}^{+\infty}\frac{(-1)^l}{2l+1}\zeta^l\cdot\sum_{k=0}^{+\infty}\sqrt{E_k}(q,\widetilde{\psi}_{2k}^0)_+\zeta^k\right]$$

$$\equiv\frac{2^{1/4}}{\pi^{3/4}}\sum_{m=0}^{+\infty}\frac{(q,\widetilde{\psi}_{2m+1}^0)_+}{\sqrt{(2m+1)E_m}}z^m\equiv-\frac{2}{\pi}(Gq)(z)\,,\quad z\in\mathbb{D}\,,$$

where definition (3.2) of the functions Fq and Gq has been used. □

We introduce the formal linear operator $\mathcal{A}$ by

$$(\mathcal{A}f)(z)\equiv P_+\left[\frac{f(\zeta)}{\sqrt{-\zeta}}\right]\equiv P_+\left[\sqrt{1-\overline{\zeta}}\cdot\frac{f(\zeta)}{\sqrt{1-\zeta}}\right],\quad f\in L^1(\mathbb{T})\,. \qquad (3.10)$$

Let

$$\overset{\circ}{H}{}^2_{3/4}=\big\{f\in H^2_{3/4}:f(1)=0\big\}\subset H^2_{3/4}\,.$$

Using Lemma 3.5 we shall obtain the simple proof of Theorem 4.2 from [5] about the equivalent definition of $\mathcal{H}_0$.

Corollary 3.6.

(i) *The operator* $\mathcal{A}:\overset{\circ}{H}{}^2_{3/4}\to H^2_{3/4}$ *and its inverse are bounded.*

(ii) *The following identity holds:*

$$\mathcal{H}_0=\left\{\{h_n\}_{n=0}^\infty:\sum_{n\geq0}h_nz^n\equiv P_+\left[\frac{g(\zeta)}{\sqrt{1-\overline{\zeta}}}\right],\ g\in H^2_{3/4}\right\}. \qquad (3.11)$$

The norms $\|h\|_{\mathcal{H}}$ *and* $\|g\|_{H^2_{3/4}}$ *are equivalent, i.e.,* $C_1\|g\|_{H^2_{3/4}}\leq\|h\|_{\mathcal{H}}\leq C_2\|g\|_{H^2_{3/4}}$ *for any* $g\in H^2_{3/4}$ *and some absolute constants* $C_1,C_2>0$.

Remark. This equivalence was proved in [5] using different and complicated arguments.

Proof. (i) Recall that the mapping $q \mapsto (Gq)(-z)$ is a linear isomorphism between $\mathbf{H}_+^0$ and $H^2_{3/4}$. Also, due to the identity $(Fq)(-1) = 2^{-3/2}q(0)$ (see Lemma 3.2), the mapping $q \mapsto (Fq)(-z)$ is a linear isomorphism between $\mathbf{H}_+^0$ and $\overset{\circ}{H}{}^2_{3/4}$. Therefore, the mapping

$$f(z) \equiv (Fq)(-z) \mapsto q \mapsto (Gq)(-z) \equiv -\frac{\pi}{2} P_+ \left[\frac{(Fq)(-\zeta)}{\sqrt{\zeta}} \right] \equiv \mathcal{A}f(z)$$

is a linear isomorphism between $\overset{\circ}{H}{}^2_{3/4}$, $\mathbf{H}_+^0$ and $H^2_{3/4}$ respectively.

(ii) If $g \in H^2_{3/4}$, then $g_0(z) \equiv g(z) - g(1) \in \overset{\circ}{H}{}^2_{3/4}$ and so $|g_0(\zeta)| \le C|\zeta - 1|^{1/4}$, $|\zeta| = 1$, for some constant $C > 0$. Hence, the following equivalence is valid:

$$c\sum_{n\ge 0} h_n z^n \equiv P_+ \left[\frac{g(\zeta)}{\sqrt{1-\overline{\zeta}}} \right] \quad \Leftrightarrow \quad \sum_{n\ge 0} h_n z^n \equiv g(1) + \frac{g_0(z)}{\sqrt{1-\overline{z}}} - P_- \left[\frac{g_0(\zeta)}{\sqrt{1-\overline{\zeta}}} \right]$$

$$\Leftrightarrow \quad P_+ \left[\sqrt{1-\overline{\zeta}} \sum_{n\ge 0} h_n \zeta^n \right] \equiv g(1) + g_0(z) \equiv g(z)\,,$$

where $P_- f \equiv f - P_+ f$ is the projector to the subspace of antianalytic functions in $\mathbb{D}$. Therefore, the equation

$$\frac{f(z)}{\sqrt{1-z}} \equiv \sum_{n\ge 0} h_n z^n \equiv P_+ \left[\frac{g(\zeta)}{\sqrt{1-\overline{\zeta}}} \right], \quad \text{where} \quad f \in \overset{\circ}{H}{}^2_{3/4}\,, \quad g \in H^2_{3/4}\,,$$

is equivalent to $g(z) \equiv (\mathcal{A}f)(z)$. Then, (3.11) follows from (i). □

Lemma 3.7. *For each $q \in \mathbf{H}_+$ the following identity holds:*

$$\sum_{n\ge 0} \check{q}_{2n+1} z^n \equiv \frac{\pi}{2} P_+ \left[\frac{1}{\sqrt{-\zeta}} \sum_{n\ge 0} \hat{q}_{2n} \zeta^n \right], \quad z \in \mathbb{D}\,. \tag{3.12}$$

Proof. Due to identities (3.2) and Lemma 3.5, we have

$$\sum_{n\ge 0} \check{q}_n z^n \equiv P_+ \left[\frac{(Gq)(\zeta)}{\sqrt{1-\overline{\zeta}}} \right] \equiv -\frac{\pi}{2} P_+ \left[\frac{(Fq)(\zeta)}{\sqrt{\zeta}\sqrt{1-\overline{\zeta}}} \right]$$

$$\equiv -\frac{\pi}{2} P_+ \left[\frac{\sqrt{1-\zeta}}{\sqrt{\zeta}\sqrt{1-\overline{\zeta}}} \sum_{n\ge 0} \hat{q}_n \zeta^n \right] \equiv -\frac{\pi}{2} P_+ \left[\frac{\sqrt{-\zeta}}{\sqrt{\zeta}} \sum_{n\ge 0} \hat{q}_n \zeta^n \right].$$

Therefore,

$$\sum_{n\ge 0} \check{q}_{2n+1} z^{2n+1} \equiv -\frac{\pi}{4} P_+ \left[\frac{\sqrt{-\zeta}}{\sqrt{\zeta}} \sum_{n\ge 0} \hat{q}_n \zeta^n - \frac{\sqrt{\zeta}}{\sqrt{-\zeta}} \sum_{n\ge 0} \hat{q}_n (-\zeta)^n \right]$$

$$\equiv \frac{\pi}{4} P_+ \left[\frac{\sqrt{\zeta}}{\sqrt{-\zeta}} \sum_{n\ge 0} \hat{q}_n \left(\zeta^n + (-\zeta)^n\right) \right] \equiv \frac{\pi}{2} P_+ \left[\frac{\sqrt{\zeta}}{\sqrt{-\zeta}} \sum_{n\ge 0} \hat{q}_{2n} \zeta^{2n} \right].$$

This yields

$$\sum_{n\ge 0} \check{q}_{2n+1} z^{2n} \equiv \frac{\pi}{2} P_+ \left[\frac{1}{\zeta} \cdot \frac{\sqrt{\zeta}}{\sqrt{-\zeta}} \sum_{n\ge 0} \hat{q}_{2n} \zeta^{2n} \right] \equiv \frac{\pi}{2} P_+ \left[\frac{1}{\sqrt{-\zeta^2}} \sum_{n\ge 0} \hat{q}_{2n} \zeta^{2n} \right],$$

since $\sqrt{-\zeta} \cdot \sqrt{\zeta} = \sqrt{-\zeta^2}$ for $\zeta \in \mathbb{T}$, $\zeta \neq \pm 1$. □

We consider linear terms $\{\hat{q}_{2n+1}\}_{n=0}^{\infty}$ and $\{\check{q}_{2n+1}\}_{n=0}^{\infty}$ in asymptotics (2.4), (2.5).

Proposition 3.8.

(i) *For each* $q \in \mathbf{H}_+$ *the following identity is fulfilled:*

$$\sum_{n\ge 0} \hat{q}_{2n+1} z^n \equiv \frac{(F_D q)(z)}{\sqrt{1-z}}, \quad z \in \mathbb{D}, \tag{3.13}$$

where

$$(F_D q)(z^2) \equiv \frac{1}{2z} \left((Fq)(z)\sqrt{1+z} - (Fq)(-z)\sqrt{1-z} \right).$$

(ii) *For each* $q \in \mathbf{H}_+$ *the following identity is fulfilled:*

$$\sum_{n\ge 0} \check{q}_{2n+1} z^n \equiv \frac{\pi}{2} P_+ \left[\frac{1}{\sqrt{-\zeta}} \left((G_D q)(\zeta) + \sum_{n\ge 0} \hat{q}_{2n+1} \zeta^n \right) \right], \quad z \in \mathbb{D}, \tag{3.14}$$

where

$$(G_D q)(z^2) \equiv \frac{1}{2z} \left((Fq)(-z)\sqrt{1+z} - (Fq)(z)\sqrt{1-z} \right).$$

(iii) *The mapping*

$$q \mapsto (F_D q\,; G_D q)$$

is a linear isomorphism between $\mathbf{H}_+$ *and* $H^2_{3/4} \times H^2_{3/4}$.

Proof. (i) Due to (3.2), we have

$$\sum_{n\ge 0} \hat{q}_{2n+1} z^{2n} \equiv \frac{1}{2z} \left(\frac{(Fq)(z)}{\sqrt{1-z}} - \frac{(Fq)(-z)}{\sqrt{1+z}} \right) \equiv \frac{(F_D q)(z^2)}{\sqrt{1-z^2}}.$$

This gives (3.13).

(ii) Recall that Lemma 3.7 yields

$$\sum_{n\geq 0} \check{q}_{2n+1} z^n \equiv \frac{\pi}{2} P_+ \left[\frac{1}{\sqrt{-\zeta}} \sum_{n\geq 0} \hat{q}_{2n} \zeta^n \right].$$

Using (3.2), we obtain

$$\sum_{n\geq 0} (\hat{q}_{2n} - \hat{q}_{2n+1}) z^{2n} \equiv \frac{1}{2}\left(\frac{(Fq)(z)}{\sqrt{1-z}} + \frac{(Fq)(-z)}{\sqrt{1+z}} \right) - \frac{(F_D q)(z^2)}{\sqrt{1-z^2}} \equiv (G_D q)(z^2).$$

This gives (3.14).

(iii) Recall that the mapping $q \mapsto Fq$ is a linear isomorphism between H_+ and $H^2_{3/4}$. Therefore, we need to prove that $Fq \mapsto (F_D q; G_D q)$ is a linear isomorphism between $H^2_{3/4}$ and $H^2_{3/4} \times H^2_{3/4}$. Due to definitions of F_D and G_D, the direct mapping is bounded. Since

$$(Fq)(z) \equiv (F_D q)(z^2)\sqrt{1+z} + (G_D q)(z^2)\sqrt{1-z},$$

the inverse mapping is bounded too. □

Definition 3.9. For $q \in \mathbf{H}_+$ define coefficients $\tilde{q}_n$, $n \geq 0$, by

$$\sum_{n\geq 0} \tilde{q}_n z^n \equiv \frac{\pi}{2} P_+ \left[\frac{1}{\sqrt{-\zeta}} \big((G_D q)(\zeta) - (G_D q)(1) \big) \right], \quad z \in \mathbb{D}.$$

Remark. Due to Proposition 3.8, we have $G_D q \in H^2_{3/4}$. Hence, $(G_D q)(\cdot) - (G_D q)(1) \in \mathring{H}^2_{3/4}$ and Corollary 3.6 gives $\{\tilde{q}_n\}_{n=0}^{\infty} \in \ell^2_{3/4}$.

Theorem 3.10.

(i) *For each* $q \in \mathbf{H}_+$ *the following identities hold:*

$$\check{q}_{2n+1} = \frac{q(0)}{4(2n+1)} + \tilde{q}_n + \sum_{m\geq 0} \frac{\hat{q}_{2m+1}}{2(n-m)+1}, \quad n \geq 0. \tag{3.15}$$

(ii) *The mapping*

$$q \mapsto \big(\{\hat{q}_{2n+1}\}_{n=0}^{\infty}; q(0); \{\tilde{q}_n\}_{n=0}^{\infty} \big)$$

is a linear isomorphism between $\mathbf{H}_+$ *and* $\mathcal{H} \times \mathbb{R} \times \ell^2_{3/4}$.

Proof. (i) Due to identity (3.14) and Definition 3.9, we have

$$\sum_{n\geq 0} \check{q}_{2n+1} z^n \equiv \sum_{n\geq 0} \tilde{q}_n z^n + \frac{\pi}{2} P_+ \left[\frac{1}{\sqrt{-\zeta}} \left((G_D q)(1) + \sum_{n\geq 0} \hat{q}_{2n+1} z^n \right) \right]. \tag{3.16}$$

Note that $(G_D q)(1) = 2^{-1/2}(Fq)(-1)$. Then, identity (3.5) yields $(G_D q)(1) = \frac{1}{4} q(0)$. Substituting the identity $1/\sqrt{-\zeta} = \frac{2}{\pi} \sum_{s\in\mathbb{Z}} \frac{\zeta^s}{2s+1}$ in $L^2(\mathbb{T})$ (see (3.6)) into (3.16), we obtain (3.15).

(ii) Due to Proposition 3.8 and identity (3.13), the mappings

$$\begin{array}{ccccc} q & \mapsto & (F_D q\,; G_D q) & \mapsto & \big(\{\hat{q}_{2n+1}\}_{n=0}^{\infty}\,; G_D q\big)\,, \\ \mathbf{H}_+ & \to & H^2_{3/4}\times H^2_{3/4} & \to & \mathcal{H}\times H^2_{3/4}\,, \end{array}$$

are linear isomorphisms. Using Corollary 3.6, we deduce that the mapping

$$G_D q \mapsto \big((G_D q)(1)\,; \{\tilde{q}_n\}_{n=0}^{\infty}\big)\,, \quad H^2_{3/4} \to \mathbb{R}\times \ell^2_{3/4}\,,$$

is a linear isomorphism too. The identity $(G_D q)(1) = \frac{1}{4}q(0)$ completes the proof. □

4. Asymptotics of $\nu_{\mathbf{n}}(\mathbf{q})$ and proof of Proposition 1.2

Lemma 4.1. *For each $q \in \mathbf{H}_+$ the following identity holds:*

$$\sum_{n\geq 0}\big(\sigma_n(q) - \sigma_n^0 - 2\hat{q}_{2n+1}\big) = 0\,. \tag{4.1}$$

where the series converges absolutely.

Proof. By asymptotics (2.4), the series $\sum_{n\geq 0}(\sigma_n(q) - \sigma_n^0 - 2\hat{q}_{2n+1})$ converges absolutely. Due to Lemma A.4[7], $\frac{\partial\sigma_n(q)}{\partial q(x)} = \psi^2_{n,D}(x,q)$, where $\psi_{n,D}$ is the n-th normalized (in $L^2(\mathbb{R}_+)$) eigenfunction of the operator T_D. Therefore,

$$\sigma_n(q) - \sigma_n^0 = \int_0^1 \frac{d}{ds}\,\sigma_n(sq)ds = \int_0^1 \big\langle \psi^2_{n,D}(x,sq), q(x)\big\rangle_+ ds\,.$$

Recall that $\psi^2_{n,D}(x,0) = 2(\psi^0_{2n+1})^2(x)$. Then,

$$\sigma_n(q) - \sigma_n^0 - 2\hat{q}_{2n+1} = \int_0^1 \big\langle \psi^2_{n,D}(x,sq) - \psi^2_{n,D}(x,0), q(x)\big\rangle_{L^2(\mathbb{R}_+\,,\,dx)}\, ds\,.$$

The standard perturbation theory (e.g., see [8]) yields

$$\frac{\partial\psi_{n,D}(x,q)}{\partial q(y)} = \sum_{m:m\neq n}\frac{\psi_{n,D}(y,q)\psi_{m,D}(y,q)}{\sigma_n(q)-\sigma_m(q)}\,\psi_{m,D}(x,q)\,.$$

Hence,

$$\begin{aligned} &\big\langle \psi^2_{n,D}(x,sq) - (\psi^0_{n,D})^2(x), q(x)\big\rangle_{L^2(\mathbb{R}_+\,,\,dx)} \\ &= \int_0^s \left\langle \frac{d}{dt}\,\psi^2_{n,D}(x,tq), q(x)\right\rangle_{L^2(\mathbb{R}_+\,,\,dx)} dt \end{aligned}$$

[7] Here and below $\partial\xi(q)/\partial q = \zeta(q)$ means that for any $v \in L^2$ the equation $(d_q\xi)(v) = \langle v, \overline{\zeta}\rangle_{L^2}$ holds true.

$$= \int_0^s \Big\langle 2\psi_{n,D}(x,tq)$$
$$\times \Big\langle \sum_{m:m\neq n} \frac{\psi_{n,D}(y,tq)\psi_{m,D}(y,tq)}{\sigma_n(q)-\sigma_m(q)}\,\psi_{m,D}(x,tq), q(y)\Big\rangle_{L^2(dy)} q(x)\Big\rangle_{L^2(dx)} dt\,.$$

This gives

$$\sum_{n\geq 0}\big(\sigma_n(q)-\sigma_n^0-2\hat{q}_{2n+1}\big) = 2\int_0^1 ds \int_0^s \sum_{n\geq 0}\sum_{m:m\neq n} \frac{\big\langle(\psi_{n,D}\psi_{m,D})(tq),q\big\rangle_+^2}{\sigma_n(tq)-\sigma_m(tq)}\,dt\,.$$

Let

$$S_k = \sum_{n=0}^{k}\sum_{m:m\neq n} \frac{\big\langle(\psi_{n,D}\psi_{m,D})(tq),q\big\rangle_+^2}{\sigma_n(tq)-\sigma_m(tq)} = \sum_{n=0}^{k}\sum_{m=k+1}^{+\infty} \frac{\big\langle(\psi_{n,D}\psi_{m,D})(tq),q\big\rangle_+^2}{\sigma_n(tq)-\sigma_m(tq)}\,.$$

Due to Lemma A.8, for some absolute constant $\varepsilon>0$ we have

$$\big\langle(\psi_{n,D}\psi_{m,D})(tq),q\big\rangle_+^2 = \begin{cases} O\big(n^{-\frac12}\, m^{-\frac12}\big) & \text{for all } n,m\geq 0\,,\\ O\big(n^{-\frac12-\frac{\varepsilon}{2}} m^{-\frac12}\big)\,, & \text{if } m\geq n+n^{\frac12+\varepsilon}\,.\end{cases}$$

Using the simple estimate $|\sigma_n(tq)-\sigma_m(tq)|^{-1} = O(|n-m|^{-1})$ and technical Lemma A.9, we obtain $S_k\to 0$ as $k\to\infty$, i.e., $\sum_{n\geq 0}(\sigma_n(q)-\sigma_n^0-2\hat{q}_{2n+1})=0$. □

Proof of Proposition 1.2. Repeating the proof of Lemma 4.1, we obtain

$$\sum_{n\geq 0}\big(\lambda_n(q)-\lambda_n^0-2\hat{q}_n\big)=0\,,$$

for $q\in \mathbf{H}_{even}$. Recall that $\lambda_{2n+1}(q)=\sigma_n(q)$ and $\lambda_{2n+1}^0=\sigma_n^0$. Using (4.1), we get

$$\sum_{n\geq 0}(-1)^n\big(\lambda_n(q)-\lambda_n^0-2\hat{q}_n\big) = \sum_{n\geq 0}\big(\lambda_n(q)-\lambda_n^0-2\hat{q}_n\big) - 2\sum_{n\geq 0}\big(\sigma_n(q)-\sigma_n^0-2\hat{q}_{2n+1}\big)=0\,.$$

Due to Lemma 3.2, Propsition 3.3 (iv) this yields

$$\sum_{n\geq 0}(-1)^n\big(\lambda_n(q)-\lambda_n^0\big) = 2\sum_{n\geq 0}(-1)^n\hat{q}_n = 2\cdot\frac{(Fq)(z)}{\sqrt{1-z}}\Big|_{z=-1} = \frac{q(0)}{2}\,.$$

Hence, $q(0)=2\sum_{n\geq 0}(-1)^n(\lambda_n(q)-\lambda_n^0)$, which gives (1.2). □

Recall that each sequence $\{\sigma_n(q)-\sigma_n^0\}, q\in\mathbf{H}_+$ belongs to the set $\mathcal{S}_D\subset\mathcal{H}$ given by (1.8).

Theorem 4.2.

(i) *Each function* $r_n(q) = \nu_n(q) - \nu_n^0 + \frac{q(0)}{2(2n+1)}, q \in \mathbf{H}_+$ *satisfies*

$$r_n(q) = -2\widetilde{q}_n + R_n(\mu) + \ell^2_{\frac{3}{4}+\delta}(n), \quad \mu = \{\mu_m\}_{m=0}^{\infty} = \big\{\sigma_m(q) - \sigma_0\big\}_{m=0}^{\infty} \qquad (4.2)$$

uniformly on bounded subsets of $\mathbf{H}_+$*,where* $\delta > 0$ *is some absolute constant and*

$$\begin{aligned} R_n(\mu) &= -2\log\left[\frac{\psi_+^0(0,\sigma_n^0+\mu_n)}{\dot\kappa_{2n+1}\cdot\mu_n}\prod_{m:m\neq n}\frac{(\sigma_n^0+\mu_n)-(\sigma_m^0+\mu_m)}{(\sigma_n^0+\mu_n)-\sigma_m^0}\right] \\ &\quad + \frac{2\dot\kappa'_{2n+1}}{\kappa'_{2n+1}}\cdot\mu_n - \sum_{m\geq 0}\frac{\mu_m}{2(n-m)+1}, \quad n \geq 0. \end{aligned} \qquad (4.3)$$

(ii) *For each* $\{\mu_m\}_{m=0}^{\infty} \in \mathcal{S}_D$ *the sequence* $\{R_n\}_{n=0}^{\infty}$ *belongs to the space* $\ell^2_{3/4}$. *Moreover, the mapping* $\mathcal{R} : \mathcal{S}_D \to \ell^2_{3/4}$ *given by* $\{\mu_m\}_{m=0}^{\infty} \mapsto \{R_n\}_{n=0}^{\infty}$, *is locally bounded.*

Remark. Note that $\widetilde{q}_n = \ell^2_{3/4}(n)$ due to Theorem 3.10 (ii). Therefore, $\{r_n(q)\}_{n=0}^{\infty} \in \ell^2_{3/4}$.

Proof. (i) Let $\sigma_n = \sigma_n(q)$ and $\mu_n = \sigma_n(q) - \sigma_n^0$, $n \geq 0$. Lemma 2.1 yields

$$\frac{\nu_n(q)-\nu_n^0}{2} = \log\left[\frac{\psi'_+(0,\sigma_n,q)}{\dot\psi_+(0,\sigma_n,q)}\cdot\frac{\dot\kappa_{2n+1}}{\kappa'_{2n+1}}\right].$$

Using Theorem 2.2 (ii) and Theorem 3.10 (i), we obtain

$$\log\frac{\psi'_+(0,\sigma_n,q)}{\kappa'_{2n+1}} = \frac{\dot\kappa'_{2n+1}}{\kappa'_{2n+1}}\mu_n - \frac{q(0)}{4(2n+1)} - \widetilde{q}_n - \sum_{m\geq 0}\frac{\hat{q}_{2m+1}}{2(n-m)+1} + \ell^2_{\frac{3}{4}+\delta}(n).$$

Furthermore, identity (2.2) gives

$$\log\frac{\dot\psi_+(0,\sigma_n,q)}{\dot\kappa_{2n+1}} = \log\left[\frac{\psi_+^0(0,\sigma_n)}{\dot\kappa_{2n+1}\cdot\mu_n}\prod_{m:m\neq n}\frac{\sigma_n-\sigma_m}{\sigma_n-\sigma_m^0}\right]$$

Hence,

$$r_n(q) = \nu_n(q) - \nu_n^0 + \frac{q(0)}{2(2n+1)} = -2\widetilde{q}_n + R_n(\mu) + \ell^2_{\frac{3}{4}+\delta}(n) + h_n,$$

where

$$h_n = \sum_{m\geq 0}\frac{\mu_m - 2\hat{q}_{2m+1}}{2(n-m)+1}, \quad n \geq 0.$$

In order to prove that $h_n = \ell^2_{3/4+\delta}(n)$, we note that identity (3.6) yields

$$h(z) \equiv \sum_{n\geq 0} h_n z^n \equiv \frac{\pi}{2}P_+\left[\frac{g(\zeta)}{\sqrt{-\zeta}}\right], \quad \text{where} \quad g(z) \equiv \sum_{m\geq 0}(\mu_m - 2\hat{q}_{2m+1})z^m$$

Due to asymptotics (2.4) and identity (4.1), we have $g \in H^2_{3/4+\delta}$ and $g(1) = 0$. Hence[8],

$$\frac{g(\zeta)}{\sqrt{-\zeta}} \in W^2_{\frac{3}{4}+\delta}(\mathbb{T}), \quad \text{and so} \quad P_+\left[\frac{g(\zeta)}{\sqrt{-\zeta}}\right] \in H^2_{\frac{3}{4}+\delta}.$$

Thus, $\sum_{n\geq 0} h_n z^n \in H^2_{3/4+\delta}$, i.e., $\{h_n\}_{n=0}^\infty \in \ell^2_{3/4+\delta}$.

(ii) Let $\{\mu_m\}_{m=0}^\infty \in \mathcal{S}_D$. We rewrite (4.3) in the form $R_n = R_n^{(1)} + R_n^{(2)} + R_n^{(3)}$, where

$$\begin{aligned} R_n^{(1)} &= -2\log\left[\frac{\psi_+^0(0,\sigma_n^0+\mu_n)}{\dot{\kappa}_{2n+1}\cdot\mu_n}\right] + \frac{2\dot{\kappa}'_{2n+1}}{\kappa'_{2n+1}}\cdot\mu_n\,, \\ R_n^{(2)} &= -\sum_{m:m\neq n}\left(2\log\left[1-\frac{\mu_m}{4(n-m)+\mu_n}\right] + \frac{\mu_m}{2(n-m)}\right), \\ R_n^{(3)} &= \frac{1}{2}\sum_{m:m\neq n}\frac{\mu_m}{n-m} - \sum_{m\geq 0}\frac{\mu_m}{2(n-m)+1}\,. \end{aligned}$$

In the following Lemmas 4.3–4.5 we will analyze these terms separately. Recall that Proposition 3.3 and Lemma 3.4 give $\mu_n = O(n^{-1/2})$ as $n\to\infty$ and

$$\mu_n = v\cdot(n+1)^{-\frac{1}{2}} + \ell^2_{1/4}(n)\,, \tag{4.4}$$

where $v \in \mathbb{R}$ is some constant.

Lemma 4.3. *The asymptotics* $R_n^{(1)} = \frac{\pi^2 v^2}{48}(n+1)^{-1} + \ell^2_{3/4}(n)$ *hold true.*

Proof. Due to $\psi_+^0(0,\sigma_n^0) = 0$, $\mu_n = O(n^{-1/2})$ and the estimates from Corollary A.3, we have

$$\frac{\psi_+^0(0,\lambda_n^0+\mu_n)}{\dot{\kappa}_{2n+1}\cdot\mu_n} = 1 + \frac{\ddot{\kappa}_{2n+1}}{2\dot{\kappa}_{2n+1}}\,\mu_n + \frac{\dddot{\kappa}_{2n+1}}{6\dot{\kappa}_{2n+1}}\,\mu_n^2 + O(n^{-\frac{3}{2}}\log^4 n)\,.$$

Therefore,

$$\begin{aligned} R_n^{(1)} &= \left(\frac{2\dot{\kappa}'_{2n+1}}{\kappa'_{2n+1}} - \frac{\ddot{\kappa}_{2n+1}}{\dot{\kappa}_{2n+1}}\right)\mu_n - 2\left(\frac{\dddot{\kappa}_{2n+1}}{6\dot{\kappa}_{2n+1}} - \frac{(\ddot{\kappa}_{2n+1})^2}{8(\dot{\kappa}_{2n+1})^2}\right)\mu_n^2 + O(n^{-\frac{3}{2}}\log^4 n) \\ &= \frac{\pi^2}{48}\,\mu_n^2 + O(n^{-\frac{3}{2}}\log^4 n) = \frac{\pi^2 v^2}{48}\,(n+1)^{-1} + \ell^2_{3/4}(n)\,, \end{aligned}$$

where we have used Lemma A.6 and (4.4). □

Lemma 4.4. *The asymptotics* $R_n^{(2)} = -\frac{\pi^2 v^2}{48}(n+1)^{-1} + \ell^2_{3/4}(n)$ *hold true.*

[8] Here and below $W^2_r(\mathbb{T})$ is the Sobolev space on the unit circle $\mathbb{T} = \{\zeta\in\mathbb{C} : |\zeta| = 1\}$.

Proof. For $m \neq n$ we have

$$
\begin{aligned}
2\log\left[1-\frac{\mu_m}{4(n-m)+\mu_n}\right]+\frac{\mu_m}{2(n-m)} &= \frac{\mu_m\mu_n}{2(n-m)\big(4(n-m)+\mu_n\big)}\\
&\quad -\frac{\mu_m^2}{\big(4(n-m)+\mu_n\big)^2}+O\left(\frac{m^{-3/2}}{(n-m)^3}\right)\\
&= \frac{\mu_m\mu_n}{8(n-m)^2}-\frac{\mu_m^2}{16(n-m)^2}\\
&\quad +O\left(\frac{m^{-3/2}+m^{-1/2}n^{-1}}{(n-m)^3}\right).
\end{aligned}
$$

Therefore,

$$
16R_n^{(2)} = -2\mu_n\cdot\sum_{m:m\neq n}\frac{\mu_m}{(n-m)^2} + \sum_{m:m\neq n}\frac{\mu_m^2}{(n-m)^2} + O(n^{-\frac{3}{2}}).
$$

Recall that $\mu_n = v(n+1)^{-1/2}+\ell^2_{1/4}(n)$ and $\mu_n^2 = v^2(n+1)^{-1}+\ell^2_{3/4}(n)$. Using simple technical Lemma A.10, we deduce that

$$
\begin{aligned}
16R_n^{(2)} = -2&\left(\frac{v}{(n+1)^{1/2}}+\ell^2_{\frac{1}{4}}(n)\right)\left(\frac{\pi^2 v}{3(n+1)^{1/2}}+\ell^2_{\frac{1}{4}}(n)\right)\\
&+\left(\frac{\pi^2v^2}{3(n+1)}+\ell^2_{\frac{3}{4}}(n)\right)+O(n^{-\frac{3}{2}}).
\end{aligned}
$$

This gives $48R_n^{(2)} = -\pi^2v^2(n+1)^{-1}+\ell^2_{3/4}(n)$. □

Lemma 4.5. *The asymptotics* $R_n^{(3)} = \ell^2_{3/4}(n)$ *hold true.*

Proof. Note that the following identities are fulfilled in $L^2(\mathbb{T})$:

$$
2\sum_{l=-\infty}^{+\infty}\frac{\zeta^l}{2l+1} = \frac{\pi}{\sqrt{-\zeta}}, \quad \sum_{l:l\neq 0}\frac{\zeta^l}{l} = -\log(-\zeta),
$$

where the branches of $\sqrt{-\zeta}$ and $\log(-\zeta)$, $\zeta\in\mathbb{T}\setminus\{1\}$ are such that $\sqrt{1}=1$ and $\log 1 = 0$. Then,

$$
\sum_{n\geq 0}R_n^{(3)}z^n \equiv -\frac{1}{2}P_+\left[\left(\frac{\pi}{\sqrt{-\zeta}}+\log(-\zeta)\right)\cdot\sum_{n\geq 0}\mu_n\zeta^n\right].
$$

Since $\{\mu_n\}_{n=0}^{\infty}\in\mathcal{H}$, we have

$$
\sum_{n\geq 0}\mu_n z^n \equiv \frac{F(z)}{\sqrt{1-z}}, \quad \text{where} \quad F\in H^2_{3/4}.
$$

Introduce the function

$$
\gamma(\zeta) = \frac{\pi/\sqrt{-\zeta}+\log(-\zeta)}{\sqrt{1-\zeta}}, \quad \zeta\in\mathbb{T}. \tag{4.5}
$$

It is clear that $\gamma \in C^\infty(\mathbb{T} \setminus \{1\})$. Note that $\gamma(\zeta) \to 0$ as $\zeta \to 1 \pm i0$. This yields $\gamma \in W^2_{3/4}(\mathbb{T})$, $\gamma F \in W^2_{3/4}(\mathbb{T})$ and $P_+[\gamma F] \in H^2_{3/4}$. The last statement is equivalent to $\{R_n^{(3)}\}_{n=0}^\infty \in \ell^2_{3/4}$. □

Lemmas 4.3–4.5 give $R_n = R_n^{(1)} + R_n^{(2)} + R_n^{(3)} = \ell^2_{3/4}(n)$. Note that all estimates are uniform on bounded subsets of $\mathcal{S}_D$. The proof of Theorem 4.2 is finished. □

5. Proof of Theorem 1.3

Introduce the mapping

$$\Phi : q \mapsto \Big(\big\{\mu_n(q)\big\}_{n=0}^\infty ; q(0) ; \big\{r_n(q)\big\}_{n=0}^\infty \Big) ,$$

where $\mu_n(q) = \sigma_n(q) - \sigma_n^0$. Due to Theorems 1.1, 4.2, we have

$$\Phi : \mathbf{H}_+ \to \mathcal{S}_D \times \mathbb{R} \times \ell^2_{\frac{3}{4}} .$$

Theorem 1.3 claims that Φ is a real-analytic isomorphism. The proof given below consists of five steps: Φ is injective (Section 5.1); Φ is real-analytic (Section 5.2); the Fréchet derivative $d_q\Phi$ is a Fredholm operator for each $q \in \mathbf{H}_+$ (Section 5.3); $d_q\Phi$ is invertible for each $q \in \mathbf{H}_+$, i.e., Φ is a local real-analytic isomorphism (Section 5.4); Φ is surjective (Section 5.5).

5.1. Uniqueness theorem

Let $(\{\mu_n(p)\}_{n=0}^\infty ; p(0) ; \{r_n(p)\}_{n=0}^\infty) = (\{\mu_n(q)\}_{n=0}^\infty ; q(0) ; \{r_n(q)\}_{n=0}^\infty)$ for some $p, q \in \mathbf{H}_+$. By definitions of μ_n and r_n, it is equivalent to

$$\sigma_n = \sigma_n(p) = \sigma_n(q) \quad \text{and} \quad \nu_n = \nu_n(p) = \nu_n(q) \quad \text{for all} \quad n \geq 0 .$$

Using Lemma 2.1, we obtain

$$\psi'_+(0, \sigma_n, p) = \psi'_+(0, \sigma_n, q) \quad \text{for all} \quad n \geq 0 .$$

The rest of the proof is standard (see also [4]). Recall that $\varphi(x, \lambda, q)$ is the solution of (1.4) such that $\varphi(0, \lambda, q) = 0$, $\varphi'(0, \lambda, q) = 1$. Introduce the functions

$$\begin{aligned}
f_1(\lambda; x, q, p) &= \frac{F_1(\lambda; x, q, p)}{\psi_+(0, \lambda, q)} , \\
F_1(\lambda; x, q, p) &= \psi_+(x, \lambda, p)\varphi'(x, \lambda, q) - \varphi(x, \lambda, p)\psi'_+(x, \lambda, q) , \\
f_2(\lambda; x, q, p) &= \frac{F_2(\lambda; x, q, p)}{\psi_+(0, \lambda, q)} , \\
F_2(\lambda; x, q, p) &= \psi_+(x, \lambda, p)\varphi(x, \lambda, q) - \varphi(x, \lambda, p)\psi_+(x, \lambda, q) .
\end{aligned}$$

Both f_1 and f_2 are entire with respect to λ for each $x \in \mathbb{R}_+$. Indeed, all roots σ_n, $n \geq 0$, of the denominator $\psi_+(0, \cdot, q)$ are simple and all these values are roots of the numerators F_1, F_2, since

$$\frac{\psi_+(x, \sigma_n, p)}{\varphi(x, \sigma_n, p)} = \psi'_+(0, \sigma_n, p) = \psi'_+(0, \sigma_n, q) = \frac{\psi_+(x, \sigma_n, q)}{\varphi(x, \sigma_n, q)}$$

for all $x \in \mathbb{R}_+$ and $n \geq 0$. Standard estimates (see Lemma A.2 and asymptotics (A.6)) of φ and ψ_+ give

$$f_1(\lambda; x, p, q) = 1 + O\left(|\lambda|^{-\frac{1}{2}}\right), \quad f_2(\lambda; x, p, q) = O\left(|\lambda|^{-\frac{1}{2}}\right), \quad |\lambda| = \lambda^0_{2k}, \quad k \to \infty.$$

Then, the maximum principle implies

$$f_1(\lambda; x, p, q) = 1, \quad f_2(\lambda; x, p, q) = 0, \quad \lambda \in \mathbb{C}.$$

This yields $\varphi(x, \lambda, p) = \varphi(x, \lambda, q)$ and $\psi_+(x, \lambda, p) = \psi_+(x, \lambda, q)$, i.e., $p = q$. □

5.2. Φ is a real-analytic mapping

Recall that for some $\delta > 0$ the following asymptotics are fulfilled (see (2.4) and (4.2)):

$$\mu_n(q) = 2\hat{q}_{2n+1} + \ell^2_{\frac{3}{4}+\delta}(n), \quad r_n(q) = -2\tilde{q}_n + R_n\left(\left\{\mu_m(q)\right\}_{m=0}^{\infty}\right) + \ell^2_{\frac{3}{4}+\delta}(n), \tag{5.1}$$

where

$$\mathcal{R} : \mathcal{S}_D \to \ell^2_{\frac{3}{4}}, \quad \mathcal{R} : \{\mu_m\}_{m=0}^{\infty} \mapsto \{R_n\}_{n=0}^{\infty},$$

is a locally bounded mapping given by (4.3). Let $\mathbf{H}_{+\mathbb{C}}$ be the complexification of $\mathbf{H}_+$. Due to Lemma A.4 (ii), for each $q \in \mathbf{H}_+$ all functions $\sigma_n(q)$ and $\psi'_+(0, \sigma_n(q), q)$, $n \geq 0$, have analytic continuations into some complex neighborhood of q. Moreover, due to Lemma 2.1, all functions $\nu_n(q)$ have analytic continuations into some complex neighborhood of q. Therefore, for each *real* potential $q \in \mathbf{H}_+$ all "coordinate functions" $\mu_n(q)$, $q(0)$, $r_n(q)$ of the mapping Φ have an analytic continuation into some small *complex* neighborhood Q of q.

Repeating the proof of (5.1) we obtain that these asymptotics hold true uniformly on bounded subsets of Q. Let

$$\Phi^{(0)} : q \mapsto \left(\{2\hat{q}_{2n+1}\}_{n=0}^{\infty}; q(0); \{-2\tilde{q}_n\}_{n=0}^{\infty}\right).$$

Due to Theorem 3.10 (i), $\Phi^{(0)}$ is a linear isomorphism between $\mathbf{H}_+$ and $\mathcal{H} \times \mathbb{R} \times \ell^2_{3/4}$. In particular, $\Phi^{(0)}$ is a real-analytic mapping. Consider the difference

$$\Phi - \Phi^{(0)} : q \mapsto \left(\left\{\mu_n(q) - 2\hat{q}_{2n+1}\right\}_{n=0}^{\infty}; 0; \left\{r_n(q) + 2\tilde{q}_n\right\}_{n=0}^{\infty}\right),$$
$$\Phi - \Phi^{(0)} : \mathbf{H}_+ \to \ell^2_{\frac{3}{4}+\delta} \times \mathbb{R} \times \ell^2_{\frac{3}{4}}.$$

All "coordinate functions" $\mu_n(q) - 2\hat{q}_{2n+1}$, $r_n(q) + 2\tilde{q}_n$ are analytic and $\Phi - \Phi^{(0)}$ is correctly defined and bounded in some small complex neighborhood of each real potential (since (5.1) holds true uniformly on bounded subsets). Then, $\Phi - \Phi^{(0)}$ is a real-analytic mapping from $\mathbf{H}_+$ into $\ell^2_{3/4+\delta}\times\mathbb{R}\times\ell^2_{3/4}$ and Φ is real-analytic too, since $\ell^2_{3/4+\delta}\subset\ell^2_{3/4}\subset\mathcal{H}$. $\square$

5.3. The Fréchet derivative $\mathbf{d_q\Phi}$ is a Fredholm operator for each $q\in\mathbf{H}_+$

In other words, we will prove that $d_q\Phi$ is the sum of invertible and compact operators. Let

$$\Phi^{(1)}:q\mapsto\Big(0\,;0\,;\mathcal{R}\Big(\big\{\mu_m(q)\big\}_{m=0}^{\infty}\Big)\Big)\quad\text{and}\quad\Phi^{(2)}=\Phi-\Phi^{(0)}-\Phi^{(1)}\,.$$

Using the same arguments as above, we obtain that $\mathcal{R}_D:\mathcal{S}_D\to\ell^2_{3/4}$ is a real-analytic mapping (since it is locally bounded in some small complex neighborhood of each real point $\mu\in\mathcal{S}_D$ and all "coordinate function" R_n are analytic). Then, $\Phi^{(1)}$ is real-analytic as a composition of real-analytic mappings. Theorems 2.2, 4.2 yield

$$\Phi^{(2)}:\mathbf{H}_+\to\ell^2_{\frac34+\delta}\times\mathbb{R}\times\ell^2_{\frac34+\delta}\,.$$

Repeating above arguments again, we obtain that $\Phi^{(2)}$ is a real-analytic mapping too.

Fix some $q\in\mathbf{H}_+$. The Fréchet derivatives $d_q\Phi$, $d_q\Phi^{(j)}$ of the analytic mappings Φ, $\Phi^{(j)}$ at the point q are bounded linear operators and

$$d_q\Phi=d_q\Phi^{(0)}+d_q\Phi^{(1)}+d_q\Phi^{(2)}=\big(\Phi^{(0)}+d_q\Phi^{(1)}\big)+d_q\Phi^{(2)}\,.$$

Note that the operator

$$d_q\Phi^{(2)}:\mathbf{H}_+\to\mathcal{H}\times\mathbb{R}\times\ell^2_{\frac34}$$

is compact since it maps $\mathbf{H}_+$ into $\ell^2_{3/4+\delta}\times\mathbb{R}\times\ell^2_{3/4+\delta}$ and the embedding $\ell^2_{3/4+\delta}\subset\ell^2_{3/4}$ is compact. In order to prove that $\Phi^{(0)}+d_q\Phi^{(1)}$ is invertible, we introduce two linear operators

$$A:\mathbf{H}_+\to\mathcal{H}\,,\ p\mapsto Ap=\{2\hat{p}_{2n+1}\}_{n=0}^{\infty}\,,$$
$$B:\mathbf{H}_+\to\ell^2_{\frac34}\,,\ p\mapsto Bp=\{-2\tilde{p}_n\}_{n=0}^{\infty}\,.$$

Recall that $\Phi^{(0)}p=(Ap\,;p(0)\,;Bp)$. The chain rule implies

$$\Big(d_q\Phi^{(1)}\Big)p=\big(0\,;0\,;(d_{\mu(q)}\mathcal{R})Ap\big)\,,$$

where $d_{\mu(q)}\mathcal{R}$ is the Fréchet derivative of the mapping $\mathcal{R}$ at the point $\mu(q)=\{\mu_m(q)\}_{m=0}^{\infty}\in\mathcal{S}_D$. Hence, $\Phi^{(0)}+d_q\Phi^{(1)}=C\Phi^{(0)}$, where both operators C and C^{-1} given by

$$C^{\pm1}:\mathcal{H}\times\mathbb{R}\times\ell^2_{3/4}\to\mathcal{H}\times\mathbb{R}\times\ell^2_{3/4}\,,$$
$$C:(h;t;r)\mapsto\big(h;t;r\pm(d_{\mu(q)}\mathcal{R}_D)h\big)$$

are bounded. Recall that $(\Phi_D^{(0)})^{-1}$ is bounded due to Theorem 3.10 (ii). Therefore, the operator $(\Phi^{(0)}+d_q\Phi^{(1)})^{-1}$ is bounded too. $\square$

5.4. Φ is a local real-analytic isomorphism

By Fredholm's Theory, in order to prove that $(d_q\Phi)^{-1}$ is bounded, it is sufficient to check that the range $\operatorname{Ran} d_q\Phi$ is dense:

$$\mathcal{H} \times \mathbb{R} \times \ell^2_{3/4} = \overline{\operatorname{Ran} d_q\Phi}\,. \tag{5.2}$$

Note that Lemma A.4 (ii) gives

$$\frac{\partial \sigma_n(q)}{\partial q(t)} = \psi^2_{n,D}(t,q)\,, \quad \frac{\partial \log\Big[(-1)^n \psi'_+\big(0,\sigma_n(q),q\big)\Big]}{\partial q(t)} = -(\psi_{n,D}\chi_{n,D})(t,q)\,, \tag{5.3}$$

where $\psi_{n,D}(\cdot,q)$ is the n-th normalized eigenfunction of T_D and $\chi_{n,D}(\cdot,q)$ is some special solution of (1.4) for $\lambda = \sigma_n(q)$ such that $\{\chi_{n,D}\,,\psi_{n,D}\} = 1$. In particular,

$$(\psi_{n,D}\chi_{n,D})(t,q) \sim t\,, \quad t \to 0\,,$$
$$(\psi_{n,D}\chi_{n,D})(t,q) \sim -t^{-1}, \quad t \to +\infty\,.$$

Due to Lemma A.5 (i), for each $q \in \mathbf{H}_+$ the following standard identities are fulfilled:

$$\begin{aligned} \big((\psi^2_{n,D})'(q), \psi^2_{m,D}(q)\big)_+ &= 0\,, \\ \big((\psi^2_{n,D})'(q), \psi_{m,D}\chi_{m,D})(q)\big)_+ &= \tfrac{1}{2}\,\delta_{mn}\,, \\ \big((\psi_{n,D}\chi_{n,D})'(q), (\psi_{m,D}\chi_{m,D})(q)\big)_+ &= 0\,, \quad n,m \ge 0\,. \end{aligned} \tag{5.4}$$

Note that $(\psi^2_{m,D})'(\cdot,q) \in \mathbf{H}_+$. Using (5.3), (5.4), we obtain

$$\left(\frac{\partial \mu_n(q)}{\partial q}\,, (\psi^2_{m,D})'(q)\right)_+ = 0$$

and

$$\left(\frac{\partial \log\big[(-1)^n\psi'_+(0,\sigma_n(q),q)\big]}{\partial q}\,, (\psi^2_{m,D})'(q)\right)_+ = \frac{\delta_{nm}}{2}$$

for all $n,m \ge 0$. Due to Lemma 2.1, this implies

$$\left(\frac{\partial \dot\psi_+(0,\sigma_n(q),q)}{\partial q}\,, (\psi^2_{m,D})'(q)\right)_+ = 0 \quad \text{and} \quad \left(\frac{\partial \nu_n(q)}{\partial q}\,, (\psi^2_{m,D})'(q)\right)_+ = \delta_{nm}\,.$$

The identity

$$\left(\frac{\partial q(0)}{\partial q}\,, (\psi^2_{m,D})'(q)\right)_+ = (\psi^2_{m,D})'(0) = 0$$

gives

$$\left(\frac{\partial r_n(q)}{\partial q}\,, (\psi^2_{m,D})'(q)\right)_+ = \left(\frac{\partial \nu_n(q)}{\partial q}\,, (\psi^2_{m,D})'(q)\right)_+ = \delta_{nm}\,.$$

Thus,

$$(d_q\Phi)\big((\psi^2_{m,D})'(q)\big) = (\mathbf{0}\,;0\,;\mathbf{e}_m)\,,$$

where $\mathbf{0} = (0,0,0,\dots)$, $\mathbf{e}_0 = (1,0,0,\dots)$, $\mathbf{e}_1 = (0,1,0,\dots)$ and so on. Therefore,

$$\big\{(\mathbf{0};0)\big\} \times \ell^2_{3/4} = \big\{(\mathbf{0};0;\mathbf{c}) : \mathbf{c} \in \ell^2_{3/4}\big\} \subset \overline{\operatorname{Ran} d_q\Phi}\,. \tag{5.5}$$

We come to the second component of $(d_q\Phi)\xi$, i.e., to the value $\xi(0)$. We consider the lowest eigenvalue $\lambda_0(q)$ of the operator T_N (with the same potential q and the Neumann boundary condition $\psi'(0) = 0$) and the function

$$\xi(t) = (\varphi\vartheta)'\big(t, \lambda_0(q), q\big)\,, \quad t \in \mathbb{R}\,,$$

where $\vartheta(t)$ is the solution of $-\psi'' + x^2\psi + q(x)\psi = \lambda\psi$ such that $\vartheta(0) = 1$ and $\vartheta'(0) = 1$. Note that $\xi(0) = 1$. Asymptotics (A.11), (A.12) give $\xi \in \mathbf{H}_+$ since $\vartheta(\cdot, \lambda_0(q), q)$ is proportional to $\psi_+(\cdot, \lambda_0(q), q)$. Moreover, using $\varphi(0, \lambda_0(q), q) = \psi_{n,D}(0, q) = 0$, we obtain

$$\left(\frac{\partial \mu_n(q)}{\partial q}\,, \xi\right)_+ = \big(\psi_{n,D}^2(q)\,, \xi\big)_+ = \frac{-\{\psi_{n,D}\,, \varphi\}\{\psi_{n,D}\,, \vartheta\}\big(0, \lambda_0(q), q\big)}{2\big(\sigma_n(q) - \lambda_0(q)\big)} = 0\,.$$

Hence,

$$(d_q\Phi)\xi = \big(\mathbf{0}\,; 1\,; (d_q r)\xi\big)\,, \quad \text{where} \quad (d_q r)\xi = \left\{\left(\frac{\partial r_n(q)}{\partial q}\,, \xi\right)_+\right\}_{n=0}^{\infty} \in \ell^2_{3/4}\,.$$

Together with (5.5) this implies

$$\{\mathbf{0}\} \times \mathbb{R} \times \ell^2_{3/4} \subset \overline{\operatorname{Ran} d_q\Phi}\,.$$

Furthermore, we consider the functions $-2(\psi_{m,D}\chi_{m,D})'(q) \in \mathbf{H}_+$ (see asymptotics (A.11), (A.12)). Identities (5.3), (5.4) and $(\psi_{m,D}\chi_{m,D})'(0, q) = 1$ give

$$(d_q\Phi)\big(-2(\psi_{m,D}\chi_{m,D})'(q)\big) = \Big(\mathbf{e}_m\,; -2\,; (d_q r)\big(-2(\psi_{m,D}\chi_{m,D})'(q)\big)\Big)\,.$$

Due to Proposition 3.3 (ii), the set of finite sequences is dense in $\mathcal{H}_0$. Therefore,

$$\mathcal{H}_0 \times \mathbb{R} \times \ell^2_{3/4} \subset \overline{\operatorname{Ran} d_q\Phi}\,. \tag{5.6}$$

In conclusion, we consider an arbitrary function $\zeta \in \mathbf{H}_+$ such that $\int_{\mathbb{R}_+} \zeta(t)dt \neq 0$. Proposition 3.3 (i) implies

$$(d_q\Phi)\zeta \notin \mathcal{H}_0 \times \mathbb{R} \times \ell^2_{3/4}\,.$$

Together with (5.6) this yields (5.2), since the codimension of $\mathcal{H}_0$ in $\mathcal{H}$ is equal to 1. □

5.5. Φ is surjective

Lemma 5.1. *Let $q \in \mathbf{H}_+$, $n \geq 0$ and $t \in \mathbb{R}$. Denote*

$$q_n^t(x) = q(x) - 2\frac{d^2}{dx^2}\log \eta_n^t(x, q)\,,$$
$$\eta_n^t(x, q) = 1 + (e^t - 1)\int_x^{+\infty} \psi_{n,D}^2(s, q)ds\,.$$

Then $q_n^t \in \mathbf{H}_+$ and

$$\sigma_m(q_n^t) = \sigma_m(q)\,,$$
$$\nu_m(q_n^t) = \nu_m(q) + t\delta_{mn}$$

for all $m \geq 0$. Moreover, $q_n^t(0) = q(0)$.

Remark. Therefore, $r_m(q_n^t) = r_m(q) + t\delta_{nm}$ for all $m \geq 0$.

Proof. This Lemma is similar to [5] Theorem 3.5 and can be proved by direct calculations using the so-called Darboux transform of second-order differential equation (see also [11], [13]) and Lemma 2.1. Note that $\eta_n^t(x,q) = e^t - (e^t - 1)\int_0^x \psi_{n,D}^2(s,q)ds = e^t + O(x^3)$, $x \downarrow 0$. This implies $q_n^t(0) = q(0)$. □

We consider an arbitrary spectral data $(h^*; u^*; c^*) \in \mathcal{S}_D \times \mathbb{R} \times \ell^2_{3/4}$. Due to Theorem 1.1 and Proposition 1.2, there exists a potential $q^* \in \mathbf{H}_+$ such that

$$\mu_n(q^*) = h_n^* \quad \text{for all} \quad n \geq 0 \quad \text{and} \quad q^*(0) = u^*.$$

This yields

$$\big(h^*; u^*; r(q^*)\big) \in \Phi(\mathbf{H}_+), \quad \text{where} \quad r(q^*) = \big(r_0(q^*), r_1(q^*), \dots\big).$$

Due to Proposition 3.3 (ii), for each $\varepsilon > 0$ there exist a finite sequence $t_\varepsilon = (t_0, \dots, t_k, 0, \dots)$ such that

$$\|(c^* - t_\varepsilon) - r(q^*)\| = \big\|\big(c^* - r(q^*)\big) - t_\varepsilon\big\| < \varepsilon.$$

Since Φ is a local isomorphism, for some $\varepsilon > 0$ we have

$$(h^*; u^*; c^* - t_\varepsilon) = \big(h^*; u^*; (c_0^* - t_0, \dots, c_k^* - t_k, c_{k+1}^*, c_{k+2}^*, \dots)\big) \in \Phi(\mathbf{H}_+).$$

It means that $(h^*; u^*; c^* - t_\varepsilon) = \Phi(q_{k+1})$ for some $q_{k+1} \in \mathbf{H}_+$. Using Lemma 5.1 step by step, we construct the sequence of potentials

$$q_j = (q_{j+1})_j^{t_j} \in \mathbf{H}_+, \quad j = k, k-1, \dots, 1, 0,$$

such that

$$\Phi(q_j) = \big(h^*; u^*; (c_0^* - t_0, \dots, c_{j-1}^* - t_{j-1}, c_j^*, c_{j+1}^*, \dots)\big).$$

Then, $\Phi(q_0) = (h^*; u^*; c^*)$. □

A. Appendix

Here we collect some technical results from [4], [5] which are essentially used above.

A.1 The unperturbed equation

For each $\lambda \in \mathbb{C}$ the equation $-\psi'' + x^2\psi = \lambda\psi$ has the solution $\psi_+^0(x,\lambda) = D_{\frac{\lambda-1}{2}}(\sqrt{2}x)$, where $D_\mu(x)$ is the Weber function (or the parabolic cylinder function, see [1]). For each x the functions $\psi_+^0(x,\cdot)$ and $(\psi_+^0)'(x,\cdot)$ are entire and the following asymptotics are fulfilled:

$$\begin{aligned} \psi_+^0(x,\lambda) &= (\sqrt{2}x)^{\frac{\lambda-1}{2}} e^{-\frac{x^2}{2}} \big(1 + O(x^{-2})\big), \quad x \to +\infty, \\ (\psi_+^0)'(x,\lambda) &= -\frac{1}{\sqrt{2}}(\sqrt{2}x)^{\frac{\lambda+1}{2}} e^{-\frac{x^2}{2}} \big(1 + O(x^{-2})\big), \quad x \to +\infty, \end{aligned} \tag{A.1}$$

uniformly with respect to λ on bounded domains. Note that (see [1])

$$\begin{aligned}\psi_+^0(0,\lambda) = D_{\frac{\lambda-1}{2}}(0) &= 2^{\frac{\lambda-1}{4}}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3-\lambda}{4})}\\ &= \cos\frac{(\lambda-1)\pi}{4}\cdot\frac{2^{\frac{\lambda-1}{4}}}{\sqrt{\pi}}\,\Gamma\Big(\frac{\lambda+1}{4}\Big)\,,\\ (\psi_+^0)'(0,\lambda) = \sqrt{2}D'_{\frac{\lambda-1}{2}}(0) &= 2^{\frac{\lambda-1}{4}}\,\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1-\lambda}{4})}\\ &= \sin\frac{(\lambda-1)\pi}{4}\cdot\frac{2^{\frac{\lambda+3}{4}}}{\sqrt{\pi}}\,\Gamma\Big(\frac{\lambda+3}{4}\Big)\,.\end{aligned} \tag{A.2}$$

Let $J^0(x,t;\lambda)$ be the solution of $-\psi''+x^2\psi = \lambda\psi$ such that $J^0(t,t;\lambda) = 0$, $(J^0)'_x(t,t;\lambda) = 1$. Then

$$\begin{aligned}J^0(0,t;\lambda) &= -\varphi^0(t,\lambda) = -\varphi(t,\lambda,0)\,,\\ (J^0)'_x(0,t;\lambda) &= \vartheta^0(t,\lambda) = \vartheta(t,\lambda,0)\,.\end{aligned} \tag{A.3}$$

In order to estimate ψ_+^0 and J^0, we introduce real-valued functions

$$\begin{aligned}a(\lambda) &= \left|\frac{\lambda}{2e}\right|^{\frac{\operatorname{Re}\lambda}{4}} e^{\frac{\pi-\phi}{4}\operatorname{Im}\lambda}\,,\\ \lambda &= |\lambda|e^{i\phi},\quad \phi\in[0,2\pi)\,,\\ \rho(x,\lambda) &= 1+|\lambda|^{1/12}+|x^2-\lambda|^{1/4}\,,\\ \sigma(x,\lambda) &= \operatorname{Re}\int_0^x\sqrt{y^2-\lambda}\,dy\,,\quad x\ge 0\,,\end{aligned} \tag{A.4}$$

where $\sqrt{y^2-\lambda} = y+o(1)$ as $y\to+\infty$ (it is equivalent to $\operatorname{Re}\sqrt{y^2-\lambda}\ge 0$, if $y\ge 0$).

Lemma A.1. *For all $(x,t,\lambda)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{C}$ the following estimates are fulfilled:*

$$\begin{aligned}|\psi_+^0(x,\lambda)| &\le C_0 a(\lambda)\cdot\frac{e^{-\sigma(x,\lambda)}}{\rho(x,\lambda)}\,,\\ |(\psi_+^0)'(x,\lambda)| &\le C_0 a(\lambda)\cdot\rho(x,\lambda)e^{-\sigma(x,\lambda)}\,,\\ |J^0(x,t;\lambda)| &\le \frac{C_1}{\rho(x,\lambda)\rho(t,\lambda)}\,e^{|\sigma(x,\lambda)-\sigma(t,\lambda)|}\,,\\ |(J^0)'_x(x,t;\lambda)| &\le C_1\frac{\rho(x,\lambda)}{\rho(t,\lambda)}\,e^{|\sigma(x,\lambda)-\sigma(t,\lambda)|}\,,\end{aligned} \tag{A.5}$$

where $C_0,\ C_1$ are some absolute constants.

Proof. See Lemmas 2.1 and 2.3 [4]. Note that the proof is based on the result of [12]. □

Remark. If $x = 0$ and $|\lambda| \geq 1$, then[9] $\sigma(0,\lambda) = 0$ and $\rho(0,\lambda) \asymp |\lambda|^{1/4}$. It follows from identities (A.2) and routine calculations that

$$\begin{aligned} |\psi_+^0(0,\lambda)| &\asymp |\lambda|^{-1/4} a(\lambda)\,, \quad \text{if } \ |\lambda| = k \neq \lambda_{2n+1}^0\,, \\ |(\psi_+^0)'(0,\lambda)| &\asymp |\lambda|^{1/4} a(\lambda)\,, \quad \text{if } \ |\lambda| = k \neq \lambda_{2n}^0\,, \end{aligned} \qquad k,n \in \mathbb{N}\,. \tag{A.6}$$

In other words, the estimates (A.5) of $|\psi_+(0,\lambda)|$ and $|\psi_+'(0,\lambda)|$ are exact on these contours.

A.2 The perturbed equation

The solutions ψ_+ , ϑ , φ of the perturbed equation $-\psi''+x^2\psi+q(x)\psi = \lambda\psi$, $\lambda \in \mathbb{C}$, can be constructed by iterations:

$$\begin{aligned} \psi_+(x,\lambda,q) &= \sum_{n\geq 0} \psi_+^{(n)}(x,\lambda,q)\,, \\ \psi_+^{(n+1)}(x,\lambda,q) &= -\int_x^{+\infty} J^0(x,t;\lambda)\psi_+^{(n)}(t,\lambda,q)q(t)dt\,, \end{aligned} \tag{A.7}$$

$$\begin{aligned} \vartheta_{1,2}(x,\lambda,q) &= \sum_{n\geq 0} \vartheta_{1,2}^{(n)}(x,\lambda,q)\,, \\ \vartheta_{1,2}^{(n+1)}(x,\lambda,q) &= \int_0^x J^0(x,t;\lambda)\vartheta_{1,2}^{(n)}(t,\lambda,q)q(t)dt\,, \end{aligned} \tag{A.8}$$

where we use the notations $\vartheta_1 = \vartheta$ and $\vartheta_2 = \varphi$ for short and $\vartheta_1^{(0)} = \vartheta^0, \vartheta_2^{(0)} = \varphi^0$, see (A.3).

Introduce functions

$$\beta_+(x,\lambda,q) = C_1 \int_x^{+\infty} \frac{|q(t)|dt}{\rho^2(t,\lambda)}\,,$$

$$\beta_0(x,\lambda,q) = C_1 \int_0^x \frac{|q(t)|dt}{\rho^2(t,\lambda)} dt\,.$$

It is easy to see ([5] Lemma 5.5) that

$$\beta(\lambda,q) = \beta_+(x,\lambda,q) + \beta_0(x,\lambda,q) = C_1 \int_0^{+\infty} \frac{|q(t)|dt}{\rho^2(t,\lambda)} = O\big(|\lambda|^{-1/2}\|q\|_{\mathbf{H}_+}\big)\,. \tag{A.9}$$

Lemma A.2. *For all $(x,\lambda,q) \in \mathbb{R}_+ \times \mathbb{C} \times \mathbf{H}_{+\mathbb{C}}$ the following estimates are fulfilled:*

$$|\psi_+^{(n)}(x,\lambda,q)| \leq C_0 a(\lambda) \frac{e^{-\sigma(x,\lambda)}}{\rho(x,\lambda)} \cdot \frac{\beta_+^n(x,\lambda,q)}{n!}\,,$$

$$|\vartheta_j^{(n)}(x,\lambda,q)| \leq \frac{2C_1}{(1+|\lambda|^{1/4})^{2j-3}} \cdot \frac{e^{\sigma(x,\lambda)}}{\rho(x,\lambda)} \cdot \frac{\beta_0^n(x,\lambda,q)}{n!}\,, \quad j = 1,2\,.$$

In particular, series (A.7), (A.8) converge uniformly on bounded subsets of $\mathbb{R}_+ \times \mathbb{C} \times \mathbf{H}_{+\mathbb{C}}$. Moreover, the similar estimates with $\rho(x,\lambda)$ instead of $\frac{1}{\rho(x,\lambda)}$ in right-hand sides hold true for the values $|(\psi_\pm^{(n)})'(x,\lambda,q)|$ and $|(\vartheta_j^{(n)})'(x,\lambda,q)|$.

[9] Here and below $f \asymp g$ means that $C_1|f| \leq |g| \leq C_2|f|$ for some absolute constants $C_1, C_2 > 0$.

Proof. See [4] Lemma 3.1 and [5] Lemmas 5.2, 5.3. □

Corollary A.3. *For all $(\lambda, q) \in \mathbb{C} \times \mathbf{H}_{+\mathbb{C}}$, $n, m \geq 0$ and some absolute constant $C > 0$ the following estimates are fulfilled:*

$$\left|\frac{\partial^m \psi_+^{(n)}(0,\lambda,q)}{\partial\lambda^m}\right| \leq \frac{m! C^{n+m+1}\|q\|^n_{\mathbf{H}_+}}{n!} \cdot \frac{\log^m(|\lambda|+2)\cdot a(\lambda)}{(|\lambda|+1)^{\frac{n}{2}+\frac{1}{4}}},$$
$$\left|\frac{\partial^m (\psi_+^{(n)})'(0,\lambda,q)}{\partial\lambda^m}\right| \leq \frac{m! C^{n+m+1}\|q\|^n_{\mathbf{H}_+}}{n!} \cdot \frac{\log^m(|\lambda|+2)\cdot a(\lambda)}{(|\lambda|+1)^{\frac{n}{2}-\frac{1}{4}}}. \tag{A.10}$$

Proof. Note that $\sigma(0,\lambda) = 0$ and $\rho(0,\lambda) \asymp 1 + |\lambda|^{1/4}$. Hence, Lemma (A.2) and (A.9) give (A.10) for $m = 0$. Recall that $\psi_+^{(n)}(0,\lambda,q)$, $(\psi_+^{(n)})'(0,\lambda,q)$ are entire functions. Therefore, the simple estimate

$$a\big(\lambda(\phi)\big) = O\big(a(\lambda)\big), \quad \text{if} \quad \lambda(\phi) = \lambda + e^{i\phi}\log^{-1}\big(|\lambda|+2\big),$$

and the integration over the contour $\lambda(\phi)$, $\phi \in [0, 2\pi]$, imply (A.10) in the case $m > 0$. □

Let

$$\widetilde{\beta}_+(x,q) = \frac{1}{x^2+1} + \int_x^{+\infty}\left|\frac{q(t)}{t}\right| dt.$$

The following asymptotics as $x \to +\infty$ are fulfilled uniformly on bounded subsets of $\mathbb{C} \times \mathbf{H}_{+\mathbb{C}}$ (see [5] p. 139 and p. 169):

$$\psi_+(x,\lambda,q) = (\sqrt{2}x)^{\frac{\lambda-1}{2}} e^{-\frac{x^2}{2}}\Big(1 + O\big(\widetilde{\beta}_+(x,q)\big)\Big),$$
$$\psi_+'(x,\lambda,q) = -\frac{1}{\sqrt{2}}(\sqrt{2}x)^{\frac{\lambda+1}{2}} e^{-\frac{x^2}{2}}\Big(1 + O\big(\widetilde{\beta}_+(x,q)\big)\Big). \tag{A.11}$$

Moreover, if $\chi_+(x,\lambda,q)$ is a solution of $-\psi'' + x^2\psi + q(x)\psi = \lambda\psi$ such that $k = \{\chi_+, \psi_+\} \neq 0$, then

$$\chi_+(x,\lambda,q) = -\frac{k}{\sqrt{2}}(\sqrt{2}x)^{\frac{-\lambda-1}{2}} e^{\frac{x^2}{2}}\Big(1 + O\big(\widetilde{\beta}_+(x,q)\big)\Big),$$
$$\chi_+'(x,\lambda,q) = -\frac{k}{2}(\sqrt{2}x)^{\frac{-\lambda+1}{2}} e^{\frac{x^2}{2}}\Big(1 + O\big(\widetilde{\beta}_+(x,q)\big)\Big). \tag{A.12}$$

Remark. If $q \in \mathbf{H}_{+\mathbb{C}}$, then (A.11), (A.12) give $(\psi_+\chi_+)' \in \mathbf{H}_{+\mathbb{C}}$ (see [5] p. 172).

A.3 Analyticity of spectral data and its gradients

Recall that $\mathbf{H}_{+\mathbb{C}}$ is the complexification of the space $\mathbf{H}_+$.

Lemma A.4.

(i) *There exist absolute constants $N_0, r_0 > 0$ such that for any $q \in \mathbf{H}_{+\mathbb{C}}$ and $n > N_0\|q\|_{\mathbf{H}_{+\mathbb{C}}}$ the function $\psi_+(0,\cdot,q)$ has exactly n roots, counted with multiplicities, in the disc $\{\lambda : |\lambda| < 4n\}$ and exactly one simple root in the disc $\{\lambda : |\lambda - \sigma_n^0| < r_0 n^{-1/2}\}$.*

(ii) *For each real potential $q \in \mathbf{H}_+$ all eigenvalues $\sigma_n(q)$ extend analytically to some complex ball $\{p \in \mathbf{H}_{+\mathbb{C}} : \|p-q\|_{\mathbf{H}_{+\mathbb{C}}} < R(q)\}$. Its gradients*[10] *are given by*

$$\frac{\partial \sigma_n(q)}{\partial q(t)} = \psi_{n,D}^2(t,q)\,,$$

where $\psi_{n,D}$ is the n-th normalized eigenfunction of T_D. Moreover,

$$\frac{\partial \log \Big[(-1)^n \psi_+'\big(0,\sigma_n(q),q\big)\Big]}{\partial q(t)} = -(\psi_{n,D}\chi_{n,D})(t,q)\,,$$

where

$$\chi_{n,D}(t,q) = \frac{\vartheta\big(t,\sigma_n(q),q\big)}{\psi_{n,D}'(0,q)} - \frac{\dot\psi_+'}{\psi_+'}\big(0,\sigma_n(q),q\big)\cdot \psi_{n,D}(t,q)\,.$$

Proof. (i) The proof repeats the proof of [4] Lemma 4.1 (see also [3] Lemma 3.1).

(ii) The proof of the analyticity repeats the proof of [5] Lemma 2.3 (p. 172). In order to calculate gradients note that the standard arguments (see [5] Lemma 5.6) give

$$\frac{\partial \psi_+(0,\lambda,q)}{\partial q(t)} = (\varphi\psi_+)(t,\lambda,q)$$
$$\frac{\partial \psi_+'(0,\lambda,q)}{\partial q(t)} = -(\vartheta\psi_+)(t,\lambda,q)\,, \quad t \ge 0\,.$$

Applying the Implicit Function Theorem to the equation $\psi_+(0,\sigma_n(q),q) = 0$ and using the identity $\int_{\mathbb{R}_+} \psi_+^2(t,\lambda,q)dt = \{\psi_+,\dot\psi_+\}(0,\lambda,q)$, we obtain

$$\frac{\partial \sigma_n(q)}{\partial q(t)} = -\frac{\partial\psi_+(0)/\partial q(t)}{\partial \psi_+(0)/\partial\lambda} = -\frac{(\varphi\psi_+)(t)}{\dot\psi_+(0)}$$
$$= -\frac{\psi_+^2(t)}{\psi_+'(0)\dot\psi_+(0)} = \frac{\psi_+^2(t)}{\{\psi_+,\dot\psi_+\}(0)} = \psi_{n,D}^2(t)$$

and

$$\frac{\partial \log \Big[(-1)^n \psi_+'\big(0,\sigma_n(q),q\big)\Big]}{\partial q(t)} = \frac{-(\vartheta\psi_+)(t) + \dot\psi_+'(0)\cdot\psi_{n,D}^2(t)}{\psi_+'(0)} = -(\psi_{n,D}\chi_{n,D})(t)\,,$$

where we omit $\sigma_n(q)$ and q for short. □

[10] Recall that $\partial\xi(q)/\partial q = \zeta(q)$ means that for any $v \in L^2$ the equation $(d_q\xi)(v) = (v,\overline{\zeta})_{L^2}$ holds true.

Lemma A.5. (i) *For each $q \in \mathbf{H}_+$ and $n, m \geq 0$ the following identities are fulfilled:*

$$\begin{aligned}
\big((\psi_{n,D}^2)', \psi_{m,D}^2\big)_+ &= 0\,, \\
\big((\psi_{n,D}\chi_{n,D})', \psi_{m,D}^2\big)_+ &= -\frac{1}{2}\,\delta_{mn}\,, \\
\big((\psi_{n,D}^2)', \psi_{m,D}\chi_{m,D}\big)_+ &= \frac{1}{2}\,\delta_{mn}\,, \\
\big((\psi_{n,D}\chi_{n,D})', \psi_{m,D}\chi_{m,D}\big)_+ &= 0\,.
\end{aligned}$$

Remark. This Lemma is similar to [5] Lemma 2.6 (see also [13, p. 44–45]).

Proof. For instance, we prove the third identity. Integration by parts gives

$$\begin{aligned}
I_{nm} &= \int_{\mathbb{R}_+} (\psi_{n,D}^2)'(t,q)(\psi_{m,D}\chi_{m,D})(t,q)dt \\
&= \frac{1}{2}\int_{\mathbb{R}_+} \big\{\psi_{m,D}\chi_{m,D}\,, (\psi_{n,D}^2)\big\}(t,q)dt \\
&= \frac{1}{2}\int_{\mathbb{R}_+} \big(\chi_{m,D}\psi_{n,D}\{\psi_{m,D}\,, \psi_{n,D}\} + \psi_{m,D}\psi_{n,D}\{\chi_{m,D}\,, \psi_{n,D}\}\big)(t,q)dt\,.
\end{aligned}$$

For $n \neq m$, this implies

$$I_{nm} = \frac{1}{2\big(\sigma_m(q) - \sigma_n(q)\big)}\big(\{\psi_{m,D}\,, \psi_{n,D}\}\{\chi_{m,D}\,, \psi_{n,D}\}\big)'(x,q,t)\Big|_{x=0}^{+\infty} = 0\,.$$

If $n = m$, then $\{\psi_{m,D}\,, \psi_{n,D}\} = 0$, $\{\chi_{m,D}\,, \psi_{n,D}\} = 1$. Hence, $I_{nn} = 1/2\int_{\mathbb{R}_+}\psi_{n,D}^2(t,q)dt = 1/2$. □

A.4 The leading terms of asymptotics of $\psi_+(0,\lambda)$ and $\psi_+'(0,\lambda)$

Recall that $\kappa_n = \psi_+^0(0,\lambda_n^0)$, $\kappa_n' = (\psi_+^0)'(0,\lambda_n^0)$, $\dot\kappa_n = \dot\psi_+^0(0,\lambda_n^0)$ and so on. (A.2) yields

$$\kappa_{2n+1} = 0\,, \quad \kappa_{2n+1}' \asymp |\lambda_{2n+1}^0|^{1/4}\cdot a(\lambda_{2n+1}^0)\,, \quad \dot\kappa_{2n+1} \asymp |\lambda_{2n+1}^0|^{-1/4}\cdot a(\lambda_{2n+1}^0)\,. \tag{A.13}$$

Lemma A.6. *The following asymptotics are fulfilled:*

$$\begin{aligned}
\frac{\dot\kappa_{2n+1}'}{\kappa_{2n+1}'} - \frac{\ddot\kappa_{2n+1}}{2\dot\kappa_{2n+1}} &= O(n^{-1})\,, \\
\frac{\ddot\kappa_{2n+1}'}{\kappa_{2n+1}'} - \frac{(\dot\kappa_{2n+1}')^2}{(\kappa_{2n+1}')^2} + \frac{\pi^2}{16} &= O(n^{-1})\,, \\
\frac{\dddot\kappa_{2n+1}}{3\dot\kappa_{2n+1}} - \frac{(\ddot\kappa_{2n+1})^2}{4(\dot\kappa_{2n+1})^2} + \frac{\pi^2}{48} &= O(n^{-1}) \quad \text{as} \quad n \to \infty\,.
\end{aligned}$$

Proof. We prove the last asymptotics, the others are similar. Let $f(\lambda)=\pi^{-1/2}2^{\frac{\lambda-1}{4}}\Gamma(\frac{\lambda+1}{4})$. Identity (A.2) yields

$$\dot{\kappa}_{2n+1}=\frac{(-1)^{n+1}\pi}{4}\,f(4n+3)\,,$$
$$\ddot{\kappa}_{2n+1}=2\cdot\frac{(-1)^{n+1}\pi}{4}\cdot\frac{df(\lambda)}{d\lambda}\Big|_{\lambda=4n+3}$$

and

$$\dddot{\kappa}_{2n+1}=\frac{(-1)^{n}\pi^3}{64}\,f(4n+3)+3\cdot\frac{(-1)^{n+1}\pi}{4}\cdot\frac{d^2f(\lambda)}{d\lambda^2}\Big|_{\lambda=4n+3}.$$

Hence,

$$\frac{\dddot{\kappa}_{2n+1}}{3\dot{\kappa}_{2n+1}}-\frac{(\ddot{\kappa}_{2n+1})^2}{4(\dot{\kappa}_{2n+1})^2}+\frac{\pi^2}{48}=\left[\frac{d^2f(\lambda)}{d\lambda^2}-\left(\frac{df(\lambda)}{d\lambda}\right)^2\right]\Big|_{\lambda=4n+3}$$
$$=\frac{d^2}{d\lambda^2}\,\log f(\lambda)\Big|_{\lambda=4n+3}=O(n^{-1})$$

since $\frac{d^2}{dx^2}\log\Gamma(x)=O(x^{-1})$ as $x\to+\infty$. □

Lemma A.7. *Let $\psi_+^{(1)}=\psi_+^{(1)}(0,\sigma_n^0,q)$, $\dot{\psi}_+^{(1)}=\dot{\psi}_+^{(1)}(0,\sigma_n^0,q)$ and so on. For some absolute constant $\delta>0$ and all $q\in\mathbf{H}_+$ the following identities and asymptotics are fulfilled:*

$$\psi_+^{(1)}=-2\dot{\kappa}_{2n+1}\hat{q}_{2n+1}\,,$$
$$\dot{\psi}_+^{(1)}=-2\dot{\kappa}_{2n+1}\left(\frac{\dot{\kappa}'_{2n+1}}{\kappa'_{2n+1}}\,\hat{q}_{2n+1}+\frac{1}{2}\,\check{q}_{2n+1}+\ell^2_{\frac14+\delta}(n)\right),$$
$$\left(\psi_+^{(1)}\right)'=-\kappa'_{2n+1}\cdot\check{q}_{2n+1}\,,$$
$$\left(\dot{\psi}_+^{(1)}\right)'=\kappa'_{2n+1}\left(-\frac{\dot{\kappa}'_{2n+1}}{\kappa'_{2n+1}}\,\check{q}_{2n+1}+\frac{\pi^2}{8}\,\hat{q}_{2n+1}+\ell^2_{\frac14+\delta}(n)\right),$$
$$\psi_+^{(2)}=-2\dot{\kappa}_{2n+1}\left(-\check{q}_{2n+1}\hat{q}_{2n+1}+\ell^2_{\frac34+\delta}(n)\right),$$
$$\left(\psi_+^{(2)}\right)'=\kappa'_{2n+1}\left(\frac{1}{2}(\check{q}_{2n+1})^2-\frac{\pi^2}{8}(\hat{q}_{2n+1})^2+\ell^2_{\frac34+\delta}(n)\right),$$

uniformly on bounded subsets of $\mathbf{H}_+$.

Proof. See [5] Lemmas 5.11, 5.12 and [5] Theorem 6.4. □

A.5 Three technical lemmas

Lemma A.8. *Let $q,p\in\mathbf{H}_+$. Then for all $\varepsilon\in(0,1/8)$ the following asymptotics is fulfilled:*

$$a_{nm}=\big(q,(\psi_n\psi_m)(p)\big)_+^2=\begin{cases}O(n^{-\frac12}\,m^{-\frac12}) & \textit{for all } n,m\ge 0\,,\\ O(n^{-\frac12-\frac{\varepsilon}{2}}m^{-\frac12})\,, & \textit{if } m\ge n+n^{\frac12+\varepsilon}.\end{cases}\qquad\text{(A.14)}$$

Proof. Using Lemma (A.2), Corollary (A.3) and asymptotics (A.6), we obtain

$$\psi_n(x,p) = \sqrt{2}\psi_n^0(x) + O\big(n^{-\frac{1}{2}}\log n \cdot \rho^{-1}(x,\lambda_n^0)\big)\,,$$
$$\psi_n^0(x) = O\big(\rho^{-1}(x,\lambda_n^0)\big)\,,$$

where ψ_n^0 is the n-th unperturbed eigenfunction of the harmonic oscillator on $\mathbb{R}$. Note that

$$\int_0^{+\infty}\frac{dt}{(t^{1+\varepsilon}+1)\cdot\rho^4(t,\lambda_n^0)} \le \int_0^{+\infty}\frac{dt}{(t^{1+\varepsilon}+1)(1+|\lambda_n^0-t^2|)} = O(n^{-1})\,.$$

Therefore, for each $R\ge 0$ we have

$$\left[\int_R^{+\infty}\frac{|q(t)|dt}{\rho(t,\lambda_n^0)\rho(t,\lambda_m^0)}\right]^2 \le O(n^{-\frac{1}{2}}m^{-\frac{1}{2}})\cdot$$
$$\int_R^{+\infty}(t^{1+\varepsilon}+1)\,|q(t)|^2dt = O\left(n^{-\frac{1}{2}}m^{-\frac{1}{2}}(R+1)^{-(1-\varepsilon)}\right)\,.$$

If $R=0$, this gives $a_{nm}=O(n^{-1/2}m^{-1/2})$. Let $m-n\ge n^{1/2+\varepsilon}$. In this case we put $R=n^\varepsilon$ and obtain

$$a_{nm} = \big(q,(\psi_n^0\psi_m^0)\big)_+^2 + O(\log n\cdot n^{-1}m^{-\frac{1}{2}})$$
$$= \left[\int_0^{n^\varepsilon} q(t)\psi_n^0(t)\psi_m^0(t)dt\right]^2 + O\left(n^{-\frac{1}{2}-(1-\varepsilon)\varepsilon}m^{-\frac{1}{2}}\right)\,.$$

Using WKB-bounds, it is easy to see (e.g., see [5] Lemma 6.7) that

$$\psi_n^0(t) = \sqrt{2/\pi}\cdot(\lambda_n^0)^{-\frac{1}{4}}\cos\big(\sqrt{\lambda_n^0}\cdot t - \tfrac{\pi n}{2}\big) + O(n^{-\frac{3}{4}+3\varepsilon})\,,\quad |t|\le n^\varepsilon\,.$$

Since $m\ge n+n^{1/2+\varepsilon}$, we have $\sqrt{\lambda_m^0}-\sqrt{\lambda_n^0}\ge n^\varepsilon$. Integration by parts and $q'\in L^2(\mathbb{R})$ imply

$$\int_0^{n^\varepsilon} q(t)\psi_n^0(t)\psi_m^0(t)dt = O\left(n^{-\frac{1}{4}-\varepsilon}m^{-\frac{1}{4}}\right)\cdot\int_0^{n^\varepsilon}|q'(t)|dt + O(n^{-\frac{3}{4}+3\varepsilon}m^{-\frac{1}{4}})$$
$$= O\left(n^{-\frac{1}{4}-\frac{\varepsilon}{2}}m^{-\frac{1}{4}}\right)\,.$$

Therefore, $a_{nm}=O(n^{-1/2-\varepsilon}m^{-1/2})+O(n^{-1/2-(1-\varepsilon)\varepsilon}m^{-1/2})=O(n^{-1/2-\varepsilon/2}m^{-1/2})$. □

Lemma A.9. *Let* $\{a_{nm}\}_{n,m\ge 0}$ *satisfy asymptotics* (A.14) *for some* $\varepsilon\in(0,1/2)$. *Then*

$$S_k = \sum_{n=0}^{k}\sum_{m=k+1}^{+\infty}\frac{a_{nm}}{m-n} \to 0 \quad as \quad k\to\infty\,.$$

Proof. Let

$$S_k^{(1)} = \sum_{n=0}^{k}\sum_{m=k+1}^{k+k^{\frac{1}{2}+\varepsilon}}\frac{a_{nm}}{m-n} \quad\text{and}\quad S_k^{(2)} = \sum_{n=0}^{k}\sum_{m=k+k^{\frac{1}{2}+\varepsilon}}^{+\infty}\frac{a_{nm}}{m-n}\,.$$

Using the simple estimate

$$\sum_{m=k+1}^{k+k^{\frac{1}{2}+\varepsilon}} \frac{1}{m-n} = O\biggl(\log\frac{k+k^{\frac{1}{2}+\varepsilon}-n}{k+1-n}\biggr) = O\biggl(\frac{k^{\frac{1}{2}+\varepsilon}}{k+1-n}\biggr),$$

we get

$$S_k^{(1)} = O(k^{\varepsilon})\sum_{n=0}^{k} O\bigl(n^{-\frac{1}{2}}(k+1-n)^{-1}\bigr) = O\bigl(k^{-\frac{1}{2}+\varepsilon}\log k\bigr).$$

Also, we have

$$S_k^{(2)} = \sum_{n=0}^{k}\ \sum_{m=k+k^{\frac{1}{2}+\varepsilon}}^{+\infty} \frac{O(n^{-\frac{1}{2}-\frac{\varepsilon}{2}}m^{-\frac{1}{2}})}{m-n} \le \sum_{n=0}^{k} O(n^{-\frac{1}{2}-\frac{\varepsilon}{2}}) \sum_{m=k+1}^{+\infty} \frac{O(m^{-\frac{1}{2}})}{m-n}.$$

Note that $\sum_{n=0}^{k} O(n^{-1/2-\varepsilon/2}) = O(k^{1/2-\varepsilon/2})$ and $\sum_{m=k+1}^{+\infty}\frac{O(m^{-1/2})}{m-n} = O(\log k \cdot k^{-1/2})$. Hence, $S_k^{(2)} = O(k^{-\varepsilon/2}\log k)$ as $k\to\infty$. Summarizing, we obtain $S_k \to 0$ as $k\to\infty$. □

Lemma A.10. *Let $h_n = O(n^{-\beta})$ and $h_n = v\cdot(n+1)^{-\beta} + \ell^2_{\beta-1/4}(n)$, where $v\in\mathbb{R}$ and $\beta\in[1/2,1]$. Then,*

$$\sum_{m:m\ne n}\frac{h_m}{(n-m)^2} = \frac{\pi^2 v}{3(n+1)^{\beta}} + \ell^2_{\beta-\frac{1}{4}}(n).$$

Proof. Let

$$\sum_{m:m\ne n}\frac{1}{(m+1)^{\beta}(n-m)^2} = \sum_{m:|m-n|\le\sqrt{n}} + \sum_{m:|m-n|>\sqrt{n}} = S_1 + S_2.$$

Since $\sum_{m:|m-n|\le\sqrt{n}}(n-m)^{-2} = \frac{1}{3}\pi^2 + O(n^{-1/2})$, we have

$$S_1 = \left(\frac{1}{(n+1)^{\beta}} + O\left(n^{-\beta-\frac{1}{2}}\right)\right)\cdot\left(\frac{\pi^2}{3} + O(n^{-\frac{1}{2}})\right) = \frac{\pi^2}{3(n+1)^{\beta}} + O\left(n^{-\beta-\frac{1}{2}}\right).$$

Note that $S_2 = O(n^{-1-\beta})$, if $\beta<1$, and $S_2 = O(n^{-2}\log n)$, if $\beta = 1$. In any case, we obtain

$$\sum_{m:m\ne n}\frac{v\cdot(m+1)^{-\beta}}{(n-m)^2} = \frac{\pi^2 v}{3(n+1)^{\beta}} + \ell^2_{\beta-\frac{1}{4}}(n).$$

Let $\widetilde{h}_m = h_m - v\cdot(m+1)^{-\beta}$. We have

$$\left(\sum_{m:m\ne n}\frac{\widetilde{h}_m}{(n-m)^2}\right)^2 \le \sum_{m:m\ne n}\frac{1}{(m+1)^{2\beta-\frac{1}{2}}(n-m)^2}\cdot\sum_{m:m\ne n}\frac{(m+1)^{2\beta-\frac{1}{2}}\,\widetilde{h}_m^2}{(n-m)^2}.$$

Using the simple estimate $\sum_{m:m\neq n}(m+1)^{-2\beta+1/2}(n-m)^{-2}=O((n+1)^{-2\beta+1/2})$, we deduce that

$$\sum_{n\geq 0}\left((n+1)^{\beta-\frac{1}{4}}\sum_{m:m\neq n}\frac{\widetilde{h}_m}{(n-m)^2}\right)^2\leq O(1)\cdot\sum_{m\geq 0}(m+1)^{2\beta-\frac{1}{2}}\,\widetilde{h}_m^2\sum_{n:n\neq m}\frac{1}{(n-m)^2}=O(1)$$

since $\widetilde{h}_m=\ell^2_{\beta-1/4}(m)$. Therefore, $\sum_{m:m\neq n}\widetilde{h}_m\cdot(n-m)^{-2}=\ell^2_{\beta-1/4}(n)$. □

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