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Large Time Asymptotics for the BBM–Burgers Equation

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To the memory of Professor Tsutomu Arai

Abstract. We study large time asymptotics of solutions to the BBM–Burgers equation

$$\partial_t (u - u_{xx}) + \beta u_x - \mu u_{xx} + uu_x = 0\,.$$

We are interested in the large time asymptotics for the case, when the initial data have an arbitrary size. Let the initial data $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$, and $\theta = \int_{\mathbf{R}} u_0(x)\, dx \neq 0$. Then we prove that there exists a unique solution $u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})\right)$ to the Cauchy problem for the BBM–Burgers equation. We also find the large time asymptotics for the solutions.

1. Introduction

This paper is devoted to the study of the Cauchy problem for the Benjamin–Bona–Mahony–Burgers (BBM–Burgers) equation

$$\begin{cases} \partial_t (u - u_{xx}) - \mu u_{xx} + \beta u_{xxx} + uu_x = 0\,, & x \in \mathbf{R},\ t > 0\,, \\ u(0,x) = u_0(x)\,, & x \in \mathbf{R}\,, \end{cases} \tag{1.1}$$

where $\mu > 0$, $\beta \in \mathbf{R}$. Usually the BBM–Burgers equation is written as follows

$$\partial_t (v - v_{xx}) + \beta v_x - \mu v_{xx} + vv_x = 0\,,$$

which is equivalent to (1.1) in view of the change $u(t,x) = v(t, x + \beta t)$.

For the global existence and optimal time decay estimates in Sobolev norms of solutions to the Cauchy problem for the BBM–Burgers equation we refer to [1]. In the present paper we are interested to describe the first and second terms in an asymptotic expansion of solutions to the Cauchy problem for the BBM–Burgers equation (1.1) for the case of the initial data having an arbitrary size. Everywhere below we suppose that the total mass of the initial data $\theta = \int_{\mathbf{R}} u_0(x)\, dx \neq 0$.

First let us refer to some known results about the large time asymptotics formulas for solutions to (1.1). In paper [16] it was proved that for small initial data $u_0 \in \mathbf{L}^{1,1}(\mathbf{R}) \cap \mathbf{H}^7(\mathbf{R})$, the solutions of (1.1) have the asymptotics

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + O\left(t^{-\frac{1}{2}-\gamma}\right) \tag{1.2}$$

as $t \to \infty$, where $\gamma \in (0, 1/2)$ and

$$f_\theta(\chi) = -2\sqrt{\mu}\partial_\chi \log\left(\cosh\frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right)\operatorname{Erf}\left(\frac{\chi}{2\sqrt{\mu}}\right)\right) \tag{1.3}$$

is the self-similar solution for the Burgers equation [4]

$$u_t + uu_x - \mu u_{xx} = 0\,, \quad x \in \mathbf{R},\, t > 0\,, \tag{1.4}$$

defined by the total mass $\theta = \int_{\mathbf{R}} u_0(x)\, dx$ of the initial data. Here $\operatorname{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2} dy$ is the error function.

As far as we know there are no results on the large time asymptotics for solutions to the Cauchy problem (1.1) without restrictions on the size of the initial data. To remove the smallness condition on the initial data $u_0(x)$ we use the method of paper [9].

Some other results for dissipative equations with critical nonlinearities were obtained in papers [1–3, 5, 6, 8, 11–18].

Our aim is to prove the following result, where we find the large time asymptotics of solutions to the Cauchy problem for (1.1) in the case of initial data of arbitrary size.

Theorem 1.1. *Let the initial data* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$, *and* $\theta = \int_{\mathbf{R}} u_0(x)\, dx \neq 0$. *Then there exists a unique solution* $u(t,x) \in \mathbf{C}\left([0,\infty);\mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})\right)$ *to the Cauchy problem for the BBM–Burgers equation* (1.1), *which has the asymptotics*

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + o\left(t^{-\frac{1}{2}}\right)$$

as $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where* f_θ *is given by* (1.3). *Moreover if additionally the initial data* $u_0 \in \mathbf{L}^{1,1}(\mathbf{R})$, *then the asymptotics* (1.2) *is true.*

Next we obtain the second term of the large time asymptotic behavior of solutions to the Cauchy problem for (1.1) in the case of the initial data of arbitrary size. A similar result was shown in [12] for the KdV-Burgers equation.

Theorem 1.2. *Let* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,1}(\mathbf{R})$, *and* $\theta = \int_{\mathbf{R}} u_0(x)\, dx \neq 0$. *Then the solution* $u(t)$ *to the Cauchy problem for the BBM–Burgers equation* (1.1), *with the initial condition* u_0 *has the following asymptotics*

$$u(t) = t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + \frac{\log t}{t}\widetilde{f}_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right) + O\left(\frac{1}{t}\right) \tag{1.5}$$

as $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where*

$$\widetilde{f}_\theta(x) = -\frac{\left(f_\theta(x) - \frac{x}{2\sqrt{\mu}}\right)e^{-\frac{x^2}{4\mu}}}{2\sqrt{\pi\mu}H(x)}\int_{\mathbf{R}} f_\theta^3(y)\, H(y)\, dy\,,$$

with

$$H(\chi) = \cosh\frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right)\operatorname{Erf}\left(\frac{\chi}{2\sqrt{\mu}}\right).$$

Below $\mathcal{F}\phi$ or $\hat{\phi}$ is the Fourier transform of ϕ defined by

$$\hat{\phi}(\xi) = (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{-ix\xi}\phi(x)\,dx$$

and

$$\mathcal{F}^{-1}\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{ix\xi}\phi(x)\,dx$$

is the inverse Fourier transform of ϕ. By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. The usual Lebesgue space is denote by $\mathbf{L}^p(\mathbf{R})$, $1 \le p \le \infty$, the weighted Lebesgue space $\mathbf{L}^{p,a}(\mathbf{R})$ is defined by

$$\mathbf{L}^{p,a}(\mathbf{R}) = \left\{\phi \in \mathbf{L}^p(\mathbf{R}); \|\phi\|_{\mathbf{L}^{p,a}} = \|\langle\cdot\rangle^a \phi\|_{\mathbf{L}^p} < \infty\right\},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $a \ge 0$. Weighted Sobolev spaces we define as follows

$$\mathbf{W}_p^{k,a}(\mathbf{R}) = \left\{\phi \in \mathbf{L}^p(\mathbf{R}); \|\phi\|_{\mathbf{W}_p^{k,a}} = \sum_{j=0}^{k} \left\|\partial^j\phi\right\|_{\mathbf{L}^{p,a}} < \infty\right\},$$

where $k \ge 0$, $a \ge 0$, $1 \le p \le \infty$. In a particular case $p = 2$ we denote $\mathbf{H}^{k,a}(\mathbf{R}) = \mathbf{W}_p^{k,a}(\mathbf{R})$. Also we define the norm of the usual Sobolev space $\mathbf{H}^k(\mathbf{R}) = \mathbf{H}^{k,0}(\mathbf{R})$ and $\mathbf{W}_p^k(\mathbf{R}) = \mathbf{W}_p^{k,0}(\mathbf{R})$ as follows $\|\phi\|_{\mathbf{H}^k} = \sum_{j=0}^k \left\|\partial^j\phi\right\|_{\mathbf{L}^2}, \|\phi\|_{\mathbf{W}_p^k} = \sum_{j=0}^k \left\|\partial^j\phi\right\|_{\mathbf{L}^p}$.

The rest of the paper we organize as follows. Some preliminary estimates of the Green operator $\mathcal{G}(t)$ solving the linearized Cauchy problem corresponding to (1.1) are obtained in Section 2. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2.

2. Preliminaries

Consider the linear Cauchy problem

$$\begin{cases} \partial_t(u - u_{xx}) - \mu u_{xx} + \beta u_{xxx} = f(t,x), & x \in \mathbf{R},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbf{R}. \end{cases} \tag{2.1}$$

Using the Duhamel principle we rewrite problem (2.1) in the form

$$u(t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}f(\tau)\,d\tau, \tag{2.2}$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\phi = \mathcal{F}^{-1}e^{-t\left(\mu\xi^2 - i\beta\xi^3\right)\left(1+\xi^2\right)^{-1}}\hat{\phi}(\xi)$$

and (see [7])

$$\mathcal{B}\phi = \left(1-\partial_x^2\right)^{-1}\phi = \mathcal{F}^{-1}\left(1+\xi^2\right)^{-1}\hat{\phi}\left(\xi\right) = \frac{1}{2}\int_{\mathbf{R}} e^{-|x-y|}\phi\left(y\right)dy\,.$$

Note that

$$\begin{aligned}\left\|\mathcal{B}^k\phi\right\|_{\mathbf{W}_p^{2k}} &\le C\left\|\left(1-\partial_x^2\right)^k\mathcal{B}^k\phi\right\|_{\mathbf{L}^p}\\ &= C\left\|\left(1-\partial_x^2\right)^k\left(1-\partial_x^2\right)^{-k}\phi\right\|_{\mathbf{L}^p} = C\left\|\phi\right\|_{\mathbf{L}^p}\end{aligned}$$

for $k\in\mathbf{N}$.

We first collect some preliminary estimates of the Green operator $\mathcal{G}\left(t\right)$ in the norms $\left\|\phi\right\|_{\mathbf{L}^p}$ and $\left\|\phi\right\|_{\mathbf{L}^{1,a}}$, where $a\in[0,1]$, $1\le p\le\infty$. We use the notation $\langle t\rangle=\sqrt{1+t^2}$. Denote the commutator

$$\left[\partial_x^3\mathcal{B}^2\mathcal{G}\left(t\right),\psi\right]\phi \equiv \partial_x^3\mathcal{B}^2\mathcal{G}\left(t\right)\left(\psi\phi\right)-\psi\partial_x^3\mathcal{B}^2\mathcal{G}\left(t\right)\phi\,.$$

Lemma 2.1. *Suppose that the function* $\phi\in\mathbf{L}^\infty\left(\mathbf{R}\right)\cap\mathbf{L}^{1,a}\left(\mathbf{R}\right)$, *where* $a\in[0,1]$. *Then the estimates*

$$\begin{aligned}\left\|\mathcal{G}\left(t\right)\phi\right\|_{\mathbf{L}^p} &\le e^{-\mu t}\left\|\phi\right\|_{\mathbf{L}^p}+C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\left\|\phi\right\|_{\mathbf{L}^r}\,,\\ \left\|\partial_x^k\mathcal{G}\left(t\right)\phi\right\|_{\mathbf{L}^p} &\le C\langle t\rangle^{-\frac{k}{2}}\left\|\phi\right\|_{\mathbf{W}_p^k}\,,\\ \left\|\mathcal{G}\left(t\right)\phi-\vartheta G_0\left(t\right)\right\|_{\mathbf{L}^\infty} &\le Ct^{-\frac{1}{2}-\frac{a}{2}}\left(\left\|\phi\right\|_{\mathbf{L}^\infty}+\left\|\phi\right\|_{\mathbf{L}^{1,a}}\right),\\ \left\||\cdot|^b\left(\mathcal{G}\left(t\right)\phi-\vartheta G_0\left(t\right)\right)\right\|_{\mathbf{L}^1} &\le Ct^{\frac{b-a}{2}}\left\|\phi\right\|_{\mathbf{L}^{1,a}}\end{aligned}$$

and

$$\left\|\left[\partial_x^3\mathcal{B}^2\mathcal{G}\left(t\right),\psi\right]\phi\right\|_{\mathbf{L}^1}\le C\langle t\rangle^{-\frac{1}{2}}t^{-\frac{1}{2}}\left\|\psi_x\right\|_{\mathbf{L}^2}\left\|\phi\right\|_{\mathbf{L}^2}$$

are valid for all $t>0$, *where* $G_0\left(t,x\right)=\left(4\pi\mu t\right)^{-\frac{1}{2}}e^{-\frac{x^2}{4\mu t}}$,

$$k\in\mathbf{N}\,,\quad 1\le r\le p\le\infty\,,\quad 0\le b\le a\,,\quad \vartheta=\int_{\mathbf{R}}\phi\left(x\right)dx\,.$$

Proof. Note that the Green operator $\mathcal{G}\left(t\right)$ can be represented as

$$\begin{aligned}\mathcal{G}\left(t\right)\phi &= \mathcal{F}^{-1}e^{-t\left(\mu\xi^2-i\beta\xi^3\right)\left(1+\xi^2\right)^{-1}}\hat{\phi}\left(\xi\right)\\ &= \mathcal{F}^{-1}e^{-\mu t\xi^2}\hat{\phi}\left(\xi\right)-\mathcal{F}^{-1}e^{-\mu t+i\beta t\xi}\sum_{k=0}^{5}\frac{t^k}{k!}\left(\mu-i\beta\xi\right)^k\left(1+\xi^2\right)^{-k}\hat{\phi}\left(\xi\right)\\ &\quad+\mathcal{F}^{-1}\Bigg(e^{-t\left(\mu\xi^2-i\beta\xi^3\right)\left(1+\xi^2\right)^{-1}}-e^{-\mu t\xi^2}\\ &\quad+e^{-\mu t+i\beta t\xi}\sum_{k=0}^{5}\frac{t^k}{k!}\left(\mu-i\beta\xi\right)^k\left(1+\xi^2\right)^{-k}\Bigg)\hat{\phi}\left(\xi\right)\\ &= \mathcal{G}_0\left(t\right)\phi+e^{-\mu t}\sum_{k=0}^{5}\frac{t^k}{k!}\mathcal{A}^k\left(t\right)\phi+\mathcal{R}\left(t\right)\phi\,,\end{aligned}\qquad(2.3)$$

where

$$\mathcal{G}_0(t)\,\phi = \int_{\mathbf{R}} G_0(t, x-y)\,\phi(y)\,dy\,,$$
$$\mathcal{A}^k(t)\,\phi = \left((\mu - \beta\partial_x)^k\,\mathcal{B}^k\phi\right)(x+\beta t)$$

and the remainder

$$\mathcal{R}(t)\,\phi = \int_{\mathbf{R}} R(t, x-y)\,\phi(y)\,dy$$

with a kernel

$$R(t,x) = (2\pi)^{-\frac{1}{2}}\,\mathcal{F}^{-1}\widehat{R}(t,\xi)\,,$$

where

$$\widehat{R}(t,\xi) = e^{-t\left(\mu\xi^2 - i\beta\xi^3\right)\left(1+\xi^2\right)^{-1}} - e^{-\mu t\xi^2} - e^{-\mu t + i\beta\xi t}\sum_{k=0}^{5}\frac{t^k}{k!}(\mu - i\beta\xi)^k\left(1+\xi^2\right)^{-k}.$$

Note that the Green operator for the heat equation $\mathcal{G}_0(t)$ satisfies all the estimates of the lemma. We have

$$\left\| e^{-\mu t}\sum_{k=0}^{5}\frac{t^k}{k!}\mathcal{A}^k(t)\,\phi\right\|_{\mathbf{L}^p} \le C\langle t\rangle^5 e^{-\mu t}\|\phi\|_{\mathbf{L}^p} \tag{2.4}$$

and

$$\left\| e^{-\mu t}\sum_{k=0}^{5}\frac{t^k}{k!}\mathcal{A}^k(t)\,\phi\right\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^5 e^{-\mu t}\|\phi\|_{\mathbf{L}^{1,a}} \tag{2.5}$$

for all $t > 0$, where $1 \le p \le \infty$.

Now we estimate the remainder $\mathcal{R}(t)$. We represent

$$\widehat{R}(t,\xi) = e^{-t\mu\xi^2\left(1+\xi^2\right)^{-1}}\left(e^{it\beta\xi^3\left(1+\xi^2\right)^{-1}} - e^{-t\mu\xi^4\left(1+\xi^2\right)^{-1}}\right) - e^{-\mu t + i\beta\xi t}\sum_{k=0}^{5}\frac{t^k}{k!}(\mu - i\beta\xi)^k\left(1+\xi^2\right)^{-k}$$

for $|\xi| \le 1$, then applying $\left|\xi^n e^{-t\mu\xi^2\left(1+\xi^2\right)^{-1}}\right| \le C\langle t\rangle^{-\frac{n}{2}} e^{-\frac{\mu}{3}t\xi^2}$ for all $|\xi| \le 1$, $t > 0$, where $n \ge 0$, we see that

$$\left|\xi^k\partial_\xi^j\widehat{R}(t,\xi)\right| \le C\langle t\rangle^{\frac{j-1-k}{2}} e^{-\frac{\mu}{3}t\xi^2} + C\langle t\rangle^8 e^{-\mu t}$$

for all $|\xi| \le 1$, $t > 0$, $0 \le j,k \le 3$. In the domain $|\xi| \ge 1$ we represent

$$\widehat{R}(t,\xi) = -e^{-\mu t\xi^2} + e^{-\mu t + i\beta\xi t}\left(e^{\frac{\mu - i\beta\xi}{1+\xi^2}t} - \sum_{k=0}^{5}\frac{t^k}{k!}(\mu - i\beta\xi)^k\left(1+\xi^2\right)^{-k}\right)$$

for $|\xi| \ge 1$, then

$$\left|\xi^k\partial_\xi^j\widehat{R}(t,\xi)\right| \le Ce^{-\frac{\mu}{2}t} + C\langle t\rangle^8 e^{-\mu t}\xi^{k-5}$$

for all $|\xi| \geq 1$, $t > 0$, $0 \leq j,k \leq 3$. Thus we see that

$$\left|\xi^k \partial_\xi^j \widehat{R}(t,\xi)\right| \leq C \langle t\rangle^{\frac{j-1-k}{2}} e^{-\frac{\mu}{3}t\xi^2} + C\langle t\rangle^8 e^{-\mu t} \langle\xi\rangle^{k-5}$$

for all $\xi \in \mathbf{R}$, $t > 0$, $0 \leq j,k \leq 3$. Hence, taking the inverse Fourier transform, we find for all $|x| \leq \langle t\rangle^{\frac{1}{2}}$

$$\begin{aligned}\left|\partial_x^k R(t,x)\right| &= \left|(2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{i\xi x} \xi^k \widehat{R}(t,\xi)\, d\xi\right| \\ &\leq C\langle t\rangle^{-\frac{1+k}{2}} \left\|e^{-\frac{\mu}{3}t|\xi|^2}\right\|_{\mathbf{L}^1} + C\langle t\rangle^8 e^{-\mu t} \left\|(1+|\xi|)^{-2}\right\|_{\mathbf{L}^1} \\ &\leq C\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-3} \langle t\rangle^{-\frac{1+k}{2}} t^{-\frac{1}{2}}.\end{aligned}$$

For the case of $|x| \geq \langle t\rangle^{\frac{1}{2}}$, integrating three times by parts with respect to ξ we obtain

$$\begin{aligned}\left|\partial_x^k R(t,x)\right| &= C|x|^{-3} \left|\int_{\mathbf{R}} e^{i\xi x} \partial_\xi^3 \left(\xi^k \widehat{R}(t,\xi)\right) d\xi\right| \\ &\leq C|x|^{-3}\langle t\rangle^{1-\frac{k}{2}} \left\|e^{-\frac{\mu}{3}t|\xi|^2}\right\|_{\mathbf{L}^1} + C|x|^{-3}\langle t\rangle^8 e^{-\mu t}\left\|(1+|\xi|)^{-2}\right\|_{\mathbf{L}^1} \\ &\leq C|x|^{-3}\langle t\rangle^{1-\frac{k}{2}} t^{-\frac{1}{2}} \leq C\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-3}\langle t\rangle^{-\frac{1+k}{2}} t^{-\frac{1}{2}}.\end{aligned}$$

Thus we have the inequality

$$\left|\partial_x^k R(t,x)\right| \leq C\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-3}\langle t\rangle^{-\frac{1+k}{2}} t^{-\frac{1}{2}} \tag{2.6}$$

for all $x \in \mathbf{R}$, $t > 0$. Now by the Young inequality for convolutions with $1/p + 1 = 1/r + 1/q$ in view of (2.6) we find

$$\begin{aligned}\left\|\partial_x^k \mathcal{R}(t)\phi\right\|_{\mathbf{L}^p} &= \left\|\int_{\mathbf{R}} \partial_x^k R(t,x-y)\phi(y)\,dy\right\|_{\mathbf{L}^p} \\ &\leq \left\|\partial_x^k R(t)\right\|_{\mathbf{L}^r}\|\phi\|_{\mathbf{L}^q} \\ &\leq C\langle t\rangle^{-\frac{1+k}{2}} t^{-\frac{1}{2}}\|\phi\|_{\mathbf{L}^q}\left(\int_{\mathbf{R}}\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-3r} dx\right)^{\frac{1}{r}} \\ &\leq C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{k}{2}} t^{-\frac{1}{2}}\|\phi\|_{\mathbf{L}^q}\end{aligned}$$

for all $1 \leq q \leq p \leq \infty$ and

$$\begin{aligned}\|\mathcal{R}(t)\phi\|_{\mathbf{L}^{1,b}} &= \left\|\int_{\mathbf{R}} R(t,x-y)\phi(y)\,dy\right\|_{\mathbf{L}^{1,b}} \\ &\leq \|R(t)\|_{\mathbf{L}^{1,b}}\|\phi\|_{\mathbf{L}^1} + \|R(t)\|_{\mathbf{L}^1}\|\phi\|_{\mathbf{L}^{1,b}} \\ &\leq C\langle t\rangle^{-\frac{1}{2}} t^{-\frac{1}{2}} \int_{\mathbf{R}}\left(\|\phi\|_{\mathbf{L}^1}\langle x\rangle^b + \|\phi\|_{\mathbf{L}^{1,b}}\right)\left\langle x\langle t\rangle^{-\frac{1}{2}}\right\rangle^{-3} dx \\ &\leq Ct^{-\frac{1}{2}}\left(\langle t\rangle^{\frac{b}{2}}\|\phi\|_{\mathbf{L}^1} + \|\phi\|_{\mathbf{L}^{1,b}}\right)\end{aligned}$$

for all $t > 0$, $0 \le b \le a \le 1$. Now by representation (2.3) in view of (2.4) and (2.5) the first four estimates of the lemma follow.

Finally we estimate the commutator $\left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t), \psi\right] \phi$. Using the identity

$$\int_{\mathbf{R}} \mathcal{B}^2 \partial_y^3 G_0(t,y)\,(\psi(x-y) - \psi(x))\,\phi(x-y)\,dy$$
$$= \int_{\mathbf{R}} dy \mathcal{B}^2 \partial_y^3 G_0(t,y)\,\phi(x-y) \int_0^y \psi_x(x-z)\,dz\,,$$

by the Cauchy–Schwarz inequality we obtain the estimate

$$\begin{aligned}
&\left\| \int_{\mathbf{R}} \mathcal{B}^2 \partial_y^3 G_0(t,y) \big(\psi(x-y) - \psi(x)\big) \phi(x-y)\,dy \right\|_{\mathbf{L}^1} \\
&\quad = \int_{\mathbf{R}} dy \left|\mathcal{B}^2 \partial_y^3 G_0(t,y)\right| \int_{\mathbf{R}} dx \left|\phi(x-y)\right| \int_0^y \left|\psi_x(x-z)\right| dz \\
&\quad \le \|\phi\|_{\mathbf{L}^2_x} \int_{\mathbf{R}} dy \left|\mathcal{B}^2 \partial_y^3 G_0(t,y)\right| \left\| \int_0^y \psi_x(x-z)\,dz \right\|_{\mathbf{L}^2_x} \\
&\quad \le \|\psi_x\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^2} \int_{\mathbf{R}} \left| y \mathcal{B}^2 \partial_y^3 G_0(t,y)\right| dy \le C \langle t\rangle^{-1} \|\psi_x\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^2}\,.
\end{aligned}$$

Also denoting

$$\begin{aligned}
\mathcal{B}^2 \partial_x^3 \mathcal{A}^k(t)\,\phi &= \Big((\mu - \beta\partial_x)^k \partial_x^3 \mathcal{B}^{k+2} \phi \Big)(x + \beta t) \\
&= \int_{\mathbf{R}} A_k(t,y)\,\phi(x + \beta t - y)\,dy
\end{aligned}$$

we have

$$\begin{aligned}
&\left\| e^{-\mu t} \sum_{k=0}^{5} \frac{t^k}{k!} \left(\mathcal{B}^2 \partial_x^3 \mathcal{A}^k(t)\,\psi\phi - \psi \mathcal{B}^2 \partial_x^3 \mathcal{A}^k(t)\,\phi \right) \right\|_{\mathbf{L}^1} \\
&\quad \le C \|\psi_x\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^2} e^{-\mu t} \sum_{k=0}^{5} \frac{t^k}{k!} \int_{\mathbf{R}} \left|(y - \beta t)\, A_k(t, \beta t - y)\right| dy \\
&\quad \le C \langle t\rangle^{-1} \|\psi_x\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^2}
\end{aligned}$$

and in the same manner using (2.6) we find

$$\begin{aligned}
&\left\| \int_{\mathbf{R}} \mathcal{B}^2 \partial_y^3 R(t,y) \big(\psi(x-y) - \psi(x)\big) \phi(x-y)\,dy \right\|_{\mathbf{L}^1} \\
&\quad \le C \|\psi_x\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^2} \int_{\mathbf{R}} \left| y \mathcal{B}^2 \partial_y^3 R(t,y)\right| dy \le C \langle t\rangle^{-\frac{1}{2}} t^{-\frac{1}{2}} \|\psi_x\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^2}\,.
\end{aligned}$$

Hence

$$\left\| \left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t), \psi\right] \phi \right\|_{\mathbf{L}^1} \le C \langle t\rangle^{-\frac{1}{2}} t^{-\frac{1}{2}} \|\psi_x\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^2}$$

for all $t > 0$. Lemma 2.1 is proved. □

Consider the integral equation associated with the Cauchy problem for the BBM–Burgers equation (1.1)

$$u(t,x) = \mathcal{G}(t)\, u_0 - \int_0^t \mathcal{G}(t-\tau)\, \mathcal{B}\big(u(\tau)\, u_x(\tau)\big) d\tau\,. \tag{2.7}$$

Define the norms

$$\|\phi\|_{p,q} \equiv \Big\| \|\phi(t,x)\|_{\mathbf{L}^q(\mathbf{R}_x)} \Big\|_{\mathbf{L}^p\left(\mathbf{R}_t^+\right)}.$$

First let us prove a global existence result for large initial data.

Proposition 2.2. *Suppose that the initial data $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})$, $a \geq 0$. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$ to the Cauchy problem* (1.1). *Moreover the a-priory estimates of a solution are valid*

$$\|u\|_{\infty,2} + \|u_x\|_{\infty,2} + \|u_x\|_{2,2} \leq C\, \|u_0\|_{\mathbf{H}^1}\,. \tag{2.8}$$

Proof. By using a standard contraction mapping principle we easily can prove that for some $T > 0$ there exists a unique solution $u \in \mathbf{C}\left([0,T]; \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$ to the Cauchy problem (1.1). We now multiply (1.1) by $2u$ and integrate the resulting equation with respect to x over $\mathbf{R}$ to get

$$\frac{d}{dt}\left(\|u(t)\|_{\mathbf{L}^2}^2 + \|u_x(t)\|_{\mathbf{L}^2}^2\right) + 2\mu\, \|u_x(t)\|_{\mathbf{L}^2}^2 = 0\,, \tag{2.9}$$

hence integrating with respect to time $t > 0$ we see that

$$\|u(t)\|_{\mathbf{L}^2}^2 + \|u_x(t)\|_{\mathbf{L}^2}^2 + 2\mu \int_0^t \|u_x(\tau)\|_{\mathbf{L}^2}^2\, d\tau \leq \|u_0\|_{\mathbf{H}^1}^2 \tag{2.10}$$

for all $t \in [0,T]$. Thus, in particular

$$\|u(t)\|_{\mathbf{H}^1} \leq C$$

for all $t \in [0,T]$. Then, applying estimates of Lemma 2.1 we obtain from integral equation (2.7)

$$\begin{aligned}
\|u(t)\|_{\mathbf{W}_1^1} &\leq C\, \|u_0\|_{\mathbf{W}_1^1} + C \int_0^t \left\|\partial_x \mathcal{B} u^2(\tau)\right\|_{\mathbf{W}_1^1} d\tau \\
&\leq C + C \int_0^t \|u(\tau)\|_{\mathbf{L}^2}^2\, d\tau \leq C\,(1+t)\,,
\end{aligned} \tag{2.11}$$

since $\left\|\partial_x \mathcal{B} u^2(\tau)\right\|_{\mathbf{W}_1^1} \leq C \left\|u^2(\tau)\right\|_{\mathbf{L}^1} = C\, \|u(\tau)\|_{\mathbf{L}^2}^2 \leq C$ and

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^{1,a}} &\leq C\, \|u_0\|_{\mathbf{L}^{1,a}} + C \int_0^t \langle t\rangle^{\frac{a}{2}} \left\|\partial_x \mathcal{B} u^2(\tau)\right\|_{\mathbf{L}^1} d\tau \\
&\quad + C \int_0^t \left\|\partial_x \mathcal{B} u^2(\tau)\right\|_{\mathbf{L}^{1,a}} d\tau \\
&\leq C\, \langle t\rangle^{\frac{a}{2}+1} + C \int_0^t \|u(\tau)\|_{\mathbf{L}^{1,a}}\, d\tau\,,
\end{aligned} \tag{2.12}$$

for all $t \in [0,T]$ since $\left\| \partial_x \mathcal{B} u^2(\tau) \right\|_{\mathbf{L}^1} \le C \|u(\tau)\|_{\mathbf{L}^2}^2 \le C$ and by the Sobolev imbedding inequality $\left\| \partial_x \mathcal{B} u^2(\tau) \right\|_{\mathbf{L}^{1,a}} \le C \|u(\tau)\|_{\mathbf{L}^\infty} \|u(\tau)\|_{\mathbf{L}^{1,a}} \le C \|u(\tau)\|_{\mathbf{L}^{1,a}}$. Denote $\int_0^t \|u(\tau)\|_{\mathbf{L}^{1,a}} d\tau = \Phi(t) e^{Ct}$, then by (2.12) we have $\Phi'(t) \le C \langle t \rangle^{\frac{a}{2}+1} e^{-Ct}$. Integration yields $\Phi(t) \le C$, again using (2.12) we get $\|u(t)\|_{\mathbf{L}^{1,a}} \le Ce^{Ct}$ for all $t \in [0,T]$. Combining this result with (2.11) we find the estimate

$$\|u(t)\|_{\mathbf{L}^{1,a}} + \|u(t)\|_{\mathbf{W}_1^1} \le Ce^{Ct}$$

for all $t \in [0,T]$, where $C > 0$ does not depend on T. Therefore by a standard continuation argument we can prolong the local solution to the global one. Thus there exists a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R}) \cap \mathbf{L}^{1,a}(\mathbf{R})\right)$ to the Cauchy problem (1.1). Again we can write (2.9) now for all $t > 0$, which therefore implies inequality (2.10) for all $t > 0$. Hence the solution satisfies the a-priory estimate (2.8). Proposition 2.2 is proved. □

We now estimate the third derivative of the solution. Denote $\Theta(x) = 1$ for all $x > 0$ and $\Theta(x) = -1$ for all $x < 0$; $\Theta(0) = 0$.

Lemma 2.3. *Let the initial data $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. Moreover we assume that the norms of the solutions are bounded*

$$\|u\|_{\infty,2} + \|u_x\|_{\infty,2} + \|u_x\|_{2,2} \le C.$$

Then the estimate is true

$$\left| \int_0^T dt \int_{\mathbf{R}} \Theta\big(u(t,x)\big) \mathcal{B} u_{xxx}(t,x)\, dx \right| \le C \langle T \rangle^{\frac{1}{6}} \log \langle T \rangle$$

for all $T > 0$.

Proof. By the integral equation (2.7) we have

$$\begin{aligned} \mathcal{B} u_{xxx}(t,x) = \partial_x^3 \mathcal{B} \mathcal{G}(t) u_0 &- \int_0^{t-\nu(t)} \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau)\, d\tau \\ &- \int_{t-\nu(t)}^t \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau)\, d\tau, \end{aligned} \tag{2.13}$$

where $\nu(t) = t^{\frac{2}{3}}$ for $t \ge 1$ and $\nu(t) = 0$ for $t \in (0,1)$. The first summand in the right-hand side of (2.13) can be estimated as

$$\left\| \partial_x^3 \mathcal{B} \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^1} \le C \langle t \rangle^{-\frac{3}{2}} \|u_0\|_{\mathbf{W}_1^1}. \tag{2.14}$$

For the second term in the right-hand side of (2.13) changing the order of integration and applying the Cauchy–Schwarz inequality we find

$$\begin{aligned}
&\int_0^T dt \left\| \int_0^{t-\nu(t)} \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) d\tau \right\|_{\mathbf{L}^1} \\
&\quad \le \int_0^T dt \int_{\nu(t)}^t \langle\tau\rangle^{-\frac{3}{2}} \|u(t-\tau)\|_{\mathbf{L}^2} \|u_x(t-\tau)\|_{\mathbf{L}^2} d\tau \\
&\quad \le C \|u\|_{\infty,2} \int_0^T dt \int_{\nu(t)}^t \langle\tau\rangle^{-\frac{3}{2}} \|u_x(t-\tau)\|_{\mathbf{L}^2} d\tau \\
&\quad \le C \int_0^T dt \langle t\rangle^{-\frac{1}{3}} \int_0^T \langle\tau\rangle^{-1} \|u_x(t-\tau)\|_{\mathbf{L}^2} d\tau \\
&\quad \le C \int_0^T d\tau \langle\tau\rangle^{-1} \int_0^T \langle t\rangle^{-\frac{1}{3}} \|u_x(t-\tau)\|_{\mathbf{L}^2} dt \\
&\quad \le C \langle T\rangle^{\frac{1}{6}} \log\langle T\rangle .
\end{aligned} \tag{2.15}$$

We now estimate the third term in the right-hand side of (2.13)

$$\begin{aligned}
&\int_{\mathbf{R}} \Theta\big(u(t,x)\big) \int_{t-\nu(t)}^t \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u(\tau) u_x(\tau) d\tau dx \\
&\quad = \int_{\mathbf{R}} dx\, |u(t,x)| \int_{t-\nu(t)}^t d\tau \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau) \\
&\qquad + \int_{\mathbf{R}} dx \Theta\big(u(t,x)\big) \int_{t-\nu(t)}^t d\tau \left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau), u(\tau)\right] u_x(\tau) \\
&\qquad + \int_{\mathbf{R}} dx \Theta\big(u(t,x)\big) \int_{t-\nu(t)}^t d\tau \big(u(\tau,x) - u(t,x)\big) \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau) \\
&\quad = I_1 + I_2 + I_3 ,
\end{aligned} \tag{2.16}$$

where the commutator

$$\begin{aligned}
&\left[\partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau), u(\tau)\right] \phi(\tau) \\
&\qquad \equiv \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) \big(u(\tau) \phi(\tau)\big) - u(\tau) \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) \phi(\tau) .
\end{aligned}$$

In the integral I_1 we integrate by parts to get

$$\begin{aligned}
I_1 &= \int_{\mathbf{R}} dx\, |u(t,x)| \int_{t-\nu(t)}^t d\tau \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau) \\
&= -\int_{\mathbf{R}} dx u_x(t,x) \Theta\big(u(t,x)\big) \int_{t-\nu(t)}^t d\tau \partial_x^2 \mathcal{B}^2 \mathcal{G}(t-\tau) u_x(\tau) ,
\end{aligned}$$

hence by the Young inequality

$$\begin{aligned}\int_0^T |I_1(t)|\,dt &\le \int_0^T dt\,\|u_x(t)\|_{\mathbf{L}^2} \int_{t-\nu(t)}^t \frac{d\tau}{\langle t-\tau\rangle}\,\|u_x(\tau)\|_{\mathbf{L}^2} \\ &\le C\int_0^T dt\,\|u_x(t)\|_{\mathbf{L}^2}\int_0^T \frac{d\tau}{\langle\tau\rangle}\,\|u_x(t-\tau)\|_{\mathbf{L}^2} \\ &\le C\,\|u_x\|_{2,2}^2 \log(T+1) \le C\log(T+1)\,. \end{aligned} \tag{2.17}$$

For the integral I_2 by Lemma 2.1, via the Young inequality we find

$$\begin{aligned}\int_0^T |I_2(t)|\,dt &\le \int_0^T dt \int_0^t d\tau\,\left\|\left[\partial_x^3\mathcal{B}^2\mathcal{G}(t-\tau)\,,u(\tau)\right]u_x(\tau)\right\|_{\mathbf{L}^1} \\ &\le C\int_0^T dt\int_0^t \langle t-\tau\rangle^{-\frac12}(t-\tau)^{-\frac12}\,\|u_x(\tau)\|_{\mathbf{L}^2}^2\,d\tau \\ &\le C\,\|u_x\|_{2,2}^2\log\langle T\rangle \le C\log\langle T\rangle\,. \end{aligned} \tag{2.18}$$

To estimate I_3 we use integral equation (2.7)

$$\begin{aligned} u(t)-u(t-\tau) &= \int_0^\tau u_t(t-t')\,dt' \\ &= \int_0^\tau dt'\partial_t\mathcal{G}(t-t')\,u_0 - \int_0^\tau dt'\mathcal{B}\big(u(t-t')\,u_x(t-t')\big) \\ &\quad - \int_0^\tau dt'\int_0^{t-t'} d\tau'\partial_t\mathcal{G}(t-t'-\tau')\,\mathcal{B}\big(u(\tau')\,u_x(\tau')\big)\,, \end{aligned}$$

hence

$$\begin{aligned}\big\|u(t)-u(t-\tau)\big\|_{\mathbf{L}^1} &\le \int_0^\tau dt'\,\langle t-t'\rangle^{-1} + \int_0^\tau dt'\,\|u_x(t-t')\|_{\mathbf{L}^2} \\ &\quad + \int_0^\tau dt'\int_0^{t-t'} d\tau'\,\langle\tau'\rangle^{-1}\,\|u_x(t-t'-\tau')\|_{\mathbf{L}^2}\end{aligned}$$

then we have

$$\begin{aligned}\int_0^T \big|I_3(t)\big|dt &\le \int_0^T dt\int_0^{\nu(t)} d\tau\big\|u(t)-u(t-\tau)\big\|_{\mathbf{L}^1}\big\|\partial_x^3\mathcal{B}^2\mathcal{G}(\tau)\,u_x(t-\tau)\big\|_{\mathbf{L}^\infty} \\ &\le C\int_0^T dt\int_0^{\nu(t)} d\tau\,\langle\tau\rangle^{-\frac74}\,\|u_x(t-\tau)\|_{\mathbf{L}^2}\Bigg(\int_0^\tau dt'\,\langle t-t'\rangle^{-1} \\ &\quad + \int_0^\tau dt'\,\|u_x(t-t')\|_{\mathbf{L}^2} \\ &\quad + \int_0^\tau dt'\int_0^{t-t'} d\tau'\,\langle\tau'\rangle^{-1}\,\|u_x(t-t'-\tau')\|_{\mathbf{L}^2}\Bigg). \end{aligned}$$

Therefore changing the order of integration we obtain

$$\int_0^T \left|I_3(t)\right| dt \le \int_0^{T^{\frac{2}{3}}} dt' \int_{t'}^{T^{\frac{2}{3}}} d\tau \left\langle \tau \right\rangle^{-\frac{7}{4}} \int_{\tau^{\frac{3}{2}}}^{T} dt \left\|u_x(t-\tau)\right\|_{\mathbf{L}^2} \Bigg(\left\langle t-t' \right\rangle^{-1} + \left\|u_x(t-t')\right\|_{\mathbf{L}^2} + \int_0^{t-t'} d\tau' \left\langle \tau' \right\rangle^{-1} \left\|u_x(t-t'-\tau')\right\|_{\mathbf{L}^2} \Bigg)$$
$$\le C \log\langle T\rangle \int_0^{T^{\frac{2}{3}}} dt' \int_{t'}^{T^{\frac{2}{3}}} d\tau \left\langle \tau \right\rangle^{-\frac{7}{4}} \le C \log\langle T\rangle \int_0^{T^{\frac{2}{3}}} dt' \left\langle t' \right\rangle^{-\frac{3}{4}}$$
$$\le C \langle T\rangle^{\frac{1}{6}} \log\langle T\rangle . \tag{2.19}$$

Substitution of estimates (2.17)–(2.19) into (2.16) yields

$$\left| \int_{\mathbf{R}} \Theta(u(t,x)) \int_{t-\nu(t)}^{t} \partial_x^3 \mathcal{B}^2 \mathcal{G}(t-\tau)\, u(\tau)\, u_x(\tau)\, d\tau dx \right| \le C \langle T\rangle^{\frac{1}{6}} \log\langle T\rangle . \tag{2.20}$$

Now from (2.14), (2.15) and (2.20) we get the result of the lemma. Lemma 2.3 is proved. □

Now we give estimates for the third derivative of the solution.

Lemma 2.4. *Let the initial data $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. Moreover we assume that the norms of the solutions are bounded*

$$\left\|u(t)\right\|_{\mathbf{L}^2} \le C(1+t)^{\sigma-\frac{1}{4}}$$

for all $t > 0$, where $\sigma \in (0, 1/4]$. Then the estimate is true

$$\left\|\mathcal{B}u_{xxx}\right\|_{s,1} \le C$$

where $s > \max\left(1, (9/8 - 3/2\sigma)^{-1}\right)$ for $\sigma \in [1/12, 1/4]$, and $s = 1$ if $\sigma \in (0, 1/12)$. Moreover we have

$$\left\|\partial_x^2 \mathcal{B} u u_x\right\|_{s,1} \le C \tag{2.21}$$

where $s > 1$ for $\sigma \in [1/12, 1/4]$, and $s = 1$ if $\sigma \in (0, 1/12)$.

Proof. In view of the integral equation (2.1) we find

$$\left\|\mathcal{B}u_{xx}(t)\right\|_{\mathbf{L}^1} \le \left\|\mathcal{B}\partial_x^2 \mathcal{G}(t) u_0\right\|_{\mathbf{L}^1} + \left\| \int_0^t \mathcal{B}\partial_x^2 \mathcal{G}(t-\tau)\, \mathcal{B}u(\tau)\, u_x(\tau)\, d\tau \right\|_{\mathbf{L}^1}$$
$$\le \left\|\mathcal{B}\partial_x^2 \mathcal{G}(t) u_0\right\|_{\mathbf{L}^1} + C \int_0^t \langle t-\tau\rangle^{-1} \left\|u(\tau)\right\|_{\mathbf{L}^2} \left\|u_x(\tau)\right\|_{\mathbf{L}^2} d\tau .$$

We first note that

$$\left\|\mathcal{B}\partial_x^2 \mathcal{G}(t) u_0\right\|_{\mathbf{L}^1} \le C \langle t\rangle^{-1} \left\|u_0\right\|_{\mathbf{L}^1} .$$

By the Young inequality we obtain

$$\left\| \int_0^t \langle t-\tau \rangle^{-1} \left\| u(\tau) \right\|_{\mathbf{L}^2} \| u_x(\tau) \|_{\mathbf{L}^2} \, d\tau \right\|_{\mathbf{L}^s_t(0,\infty)}$$

$$\leq C \left\| \int_0^t \langle t-\tau \rangle^{-1} \langle \tau \rangle^{\sigma - \frac{1}{4}} \| u_x(\tau) \|_{\mathbf{L}^2} \, d\tau \right\|_{\mathbf{L}^s_t(0,\infty)}$$

$$\leq C \left\| \langle t \rangle^{-1} \right\|_{\mathbf{L}^{s_1}_t(0,\infty)} \left\| \langle t \rangle^{\sigma - \frac{1}{4}} \right\|_{\mathbf{L}^{s_2}_t(0,\infty)} \| u_x \|_{2,2} \,,$$

for all $t > 0$ where $1/s + 1 = 1/s_1 + 1/s_2 + 1/2$, $s_1 > 1, s_2 > (1/4 - \sigma)^{-1}, s > (3/4 - \sigma)^{-1}$. Collecting these estimates we get

$$\| \mathcal{B} u_{xx} \|_{s,1} \leq C$$

for all $s > (3/4 - \sigma)^{-1}$.

In the same manner we estimate the third derivative

$$\| \mathcal{B} u_{xxx}(t) \|_{\mathbf{L}^1} \leq \left\| \mathcal{B} \partial_x^3 \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^1} + \left\| \int_0^{\frac{t}{2}} \mathcal{B}^2 \partial_x^4 \mathcal{G}(t-\tau) u^2(\tau) \, d\tau \right\|_{\mathbf{L}^1}$$

$$+ \left\| \int_{\frac{t}{2}}^t \mathcal{B} \partial_x^2 \mathcal{G}(t-\tau) \partial_x \mathcal{B} u(\tau) u_x(\tau) \, d\tau \right\|_{\mathbf{L}^1} .$$

By Lemma 2.1 we find

$$\left\| \mathcal{B} \partial_x^3 \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^1} \leq C \langle t \rangle^{-\frac{3}{2}} \| u_0 \|_{\mathbf{W}^1_1} .$$

And also

$$\| \partial_x \mathcal{B} u(\tau) u_x(\tau) \|_{\mathbf{L}^1} \leq \left\| \mathcal{B} u_x^2(\tau) \right\|_{\mathbf{L}^1} + \| u(\tau) \mathcal{B} u_{xx}(\tau) \|_{\mathbf{L}^1} + \| [u(\tau), \mathcal{B}] u_{xx}(\tau) \|_{\mathbf{L}^1} ,$$

where

$$\big[u(\tau), \mathcal{B} \big] \partial_x^2 u(\tau) = \frac{1}{2} \int_{\mathbf{R}} e^{-|x-y|} \big(u(\tau, x) - u(\tau, y) \big) u_{yy}(\tau, y) \, dy \,.$$

By the Cauchy–Schwarz inequality

$$\left\| \big[u(\tau), \mathcal{B} \big] u_{xx}(\tau) \right\|_{\mathbf{L}^1}$$

$$= \frac{1}{2} \left\| \int_{\mathbf{R}} e^{-|x-y|} \big(u(\tau, x) - u(\tau, y) \big) u_{yy}(\tau, y) \, dy \right\|_{\mathbf{L}^1}$$

$$\leq \left\| \int_{\mathbf{R}} \Theta(x-y) e^{-|x-y|} \big(u(\tau, x) - u(\tau, y) \big) u_y(\tau, y) \, dy \right\|_{\mathbf{L}^1}$$

$$+ \left\| \int_{\mathbf{R}} e^{-|x-y|} u_y^2(\tau, y) \, dy \right\|_{\mathbf{L}^1}$$

$$\leq C \int_{\mathbf{R}} dx \int_{\mathbf{R}} dy e^{-|x-y|} \Theta(x-y) \left| u_y(\tau, y) \right| \int_0^{x-y} \left| u_x(\tau, y+z) \right| dz$$

$$
\begin{aligned}
&+ C \|u_x(\tau)\|^2_{\mathbf{L}^2} \\
&\le C \int_{\mathbf{R}} d\xi e^{-|\xi|} |\xi| \int_{\mathbf{R}} dy \, |u_y(\tau, y)| \int_0^{\xi} |u_x(\tau, y+z)| \, dz \\
&\le C \|u_x(\tau)\|^2_{\mathbf{L}^2} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|\mathcal{B}u_{xxx}(t)\|_{\mathbf{L}^1} &\le C \langle t \rangle^{-\frac{3}{2}} \|u_0\|_{\mathbf{W}^1_1} + C \int_0^{\frac{t}{2}} \langle t-\tau \rangle^{-2} \langle \tau \rangle^{2\sigma - \frac{1}{2}} d\tau \\
&+ C \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-1} \left(\|u_x(\tau)\|^2_{\mathbf{L}^2} + \langle \tau \rangle^{\frac{\sigma}{2} - \frac{1}{8}} \|u_x(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^2} \|\mathcal{B}u_{xx}(\tau)\|_{\mathbf{L}^1} \right) d\tau ,
\end{aligned}
$$

for all $t > 0$. We have

$$
\left\| \int_0^t \langle t-\tau \rangle^{-1} \|u_x(\tau)\|^2_{\mathbf{L}^2} d\tau \right\|_{\mathbf{L}^s_t(0,\infty)} \le C
$$

for $s > 1$. Using Lemma 2.1 and the Young inequality we obtain

$$
\begin{aligned}
&\left\| \int_0^t \langle t-\tau \rangle^{-1} \|u(\tau)\|_{\mathbf{L}^\infty} \|\mathcal{B}u_{xx}(\tau)\|_{\mathbf{L}^1} d\tau \right\|_{\mathbf{L}^s_t(0,\infty)} \\
&\qquad \le C \left\| \langle t \rangle^{-1} \right\|_{\mathbf{L}^{s_1}_t(0,\infty)} \|u\|^{\frac{1}{2}}_{\infty,2} \|u_x\|^{\frac{1}{2}}_{2,2} \|\mathcal{B}u_{xx}\|_{s_2,1}
\end{aligned}
$$

where $1/s = 1/s_1 + 1/s_2 + 1/4 - 1 < 3/4$, since $s_1 > 1$, $s_2 > 2$.

Collecting these estimates we get

$$
\|\mathcal{B}u_{xxx}\|_{s,1} \le C
$$

for all $s > \max\left(1, (9/8 - 3/2\sigma)^{-1}\right)$ for $\sigma \in [1/12, 1/4]$.

Consider now the case $\sigma \in (0, 1/12)$. Then we can obtain a better decay estimate for $\|u_x(t)\|_{\mathbf{L}^2}$. In view of the integral equation (2.1) we find

$$
\begin{aligned}
\|u_x(t)\|_{\mathbf{L}^2} &\le \|\partial_x \mathcal{G}(t) u_0\|_{\mathbf{L}^2} + \left\| \int_0^t \mathcal{B}\partial_x^2 \mathcal{G}(t-\tau) u^2(\tau) d\tau \right\|_{\mathbf{L}^2} \\
&\le C \langle t \rangle^{-\frac{3}{4}} + C \int_0^t \langle t-\tau \rangle^{-1} \|u(\tau)\|^{\frac{3}{2}}_{\mathbf{L}^2} \|u_x(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^2} d\tau \\
&\le C \langle t \rangle^{-\frac{3}{4}} + C t^{-1} \int_0^{\frac{t}{2}} \langle \tau \rangle^{\frac{3}{2}\sigma - \frac{3}{8}} \|u_x(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^2} d\tau \\
&\quad + C \langle t \rangle^{\frac{3}{2}\sigma - \frac{3}{8}} \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-1} \|u_x(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^2} d\tau \\
&\le C \langle t \rangle^{\frac{3}{2}\sigma - \frac{5}{8}} + \frac{C}{\varepsilon} \langle t \rangle^{3\sigma - \frac{3}{4}} \log \langle t \rangle \\
&\quad + \frac{C\varepsilon}{\log \langle t \rangle} \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-1} \|u_x(\tau)\|_{\mathbf{L}^2} d\tau
\end{aligned}
$$

then choosing a sufficiently small $\varepsilon > 0$, by the Gronwall inequality we obtain

$$\|u_x\|_{\mathbf{L}^2} \le C \langle t \rangle^{\frac{3}{2}\sigma - \frac{5}{8}}$$

for all $t > 0$. By applying this estimate, we obtain

$$\begin{aligned} \|u_x(t)\|_{\mathbf{L}^2} &\le C \langle t \rangle^{-\frac{3}{4}} + C \int_0^t \langle t - \tau \rangle^{-1} \|u(\tau)\|_{\mathbf{L}^2}^{\frac{3}{2}} \|u_x(\tau)\|_{\mathbf{L}^2}^{\frac{1}{2}} d\tau \\ &\le C \langle t \rangle^{-\frac{3}{4}} + C t^{-1} \int_0^{\frac{t}{2}} \langle \tau \rangle^{\frac{9}{4}\sigma - \frac{11}{16}} d\tau + C \langle t \rangle^{\frac{9}{4}\sigma - \frac{11}{16}} \int_{\frac{t}{2}}^t \langle t - \tau \rangle^{-1} d\tau \\ &\le C \langle t \rangle^{-\frac{3}{4}} + C \langle t \rangle^{\frac{9}{4}\sigma - \frac{11}{16}} \log \langle t \rangle . \end{aligned}$$

Iterating the same procedure, we obtain

$$\|u_x(t)\|_{\mathbf{L}^2} \le C \langle t \rangle^{-\frac{3}{4}} + C \langle t \rangle^{3\sigma - \frac{3}{4}} \log \langle t \rangle .$$

By the Young inequality we obtain

$$\int_0^t \langle t - \tau \rangle^{-1} \|u(\tau)\|_{\mathbf{L}^2} \|u_x(\tau)\|_{\mathbf{L}^2} d\tau \le C \int_0^t \langle t - \tau \rangle^{-1} \langle \tau \rangle^{4\sigma - 1} d\tau$$

for all $t > 0$. Collecting these estimates we get

$$\|\mathcal{B} u_{xx}(t)\|_{\mathbf{L}^1} \le C \langle t \rangle^{4\sigma - 1} \log^2 \langle t \rangle$$

for all $t > 0$. In the same manner we estimate the third derivative

$$\begin{aligned} \|\mathcal{B} u_{xxx}(t)\|_{\mathbf{L}^1} \le \left\| \mathcal{B} \partial_x^3 \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^1} &+ \left\| \int_0^{\frac{t}{2}} \mathcal{B} \partial_x^4 \mathcal{G}(t - \tau) \mathcal{B} u^2(\tau) d\tau \right\|_{\mathbf{L}^1} \\ &+ \left\| \int_{\frac{t}{2}}^t \mathcal{B} \partial_x^2 \mathcal{G}(t - \tau) \partial_x \mathcal{B} u(\tau) u_x(\tau) d\tau \right\|_{\mathbf{L}^1} . \end{aligned}$$

We have

$$\left\| \mathcal{B} \partial_x^3 \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^1} \le C \langle t \rangle^{-\frac{3}{2}}$$

hence

$$\begin{aligned} \|\mathcal{B} u_{xxx}(t)\|_{\mathbf{L}^1} &\le C \langle t \rangle^{-\frac{3}{2}} + C \int_0^{\frac{t}{2}} \langle t - \tau \rangle^{-2} \langle \tau \rangle^{2\sigma - \frac{1}{2}} d\tau \\ &\quad + C \int_{\frac{t}{2}}^t \langle t - \tau \rangle^{-1} \langle \tau \rangle^{6\sigma - \frac{3}{2}} \log^2 \langle \tau \rangle d\tau \le C \langle t \rangle^{6\sigma - \frac{3}{2}} \log^3 \langle t \rangle \end{aligned}$$

for all $t > 0$. By this estimate we get

$$\|\mathcal{B} u_{xxx}\|_{1,1} \le C$$

if $\sigma \in (0, 1/12)$. Estimate (2.21) is proved in the same manner since we have

$$\begin{aligned} \left\| \partial_x^2 \mathcal{B} u(\tau) u_x(\tau) \right\|_{\mathbf{L}^1} \le C \left\| \partial_x \mathcal{B} u_x^2(\tau) \right\|_{\mathbf{L}^1} &+ \|u(\tau) \mathcal{B} u_{xxx}(\tau)\|_{\mathbf{L}^1} \\ &+ \|[u(\tau), \mathcal{B}] u_{xxx}(\tau)\|_{\mathbf{L}^1} , \end{aligned}$$

where integrating by parts and by the Cauchy–Schwarz inequality

$$\begin{aligned}
&\left\| \left[u(\tau), \mathcal{B}\right] u_{xxx}(\tau) \right\|_{\mathbf{L}^1} \\
&\quad = \frac{1}{2} \left\| \int_{\mathbf{R}} e^{-|x-y|} \big(u(\tau,x) - u(\tau,y)\big) u_{yyy}(\tau,y)\, dy \right\|_{\mathbf{L}^1} \\
&\quad \leq \left\| \int_{\mathbf{R}} \Theta(x-y)\, e^{-|x-y|} \big(u(\tau,x) - u(\tau,y)\big) u_{yy}(\tau,y)\, dy \right\|_{\mathbf{L}^1} \\
&\quad\quad + \left\| \int_{\mathbf{R}} e^{-|x-y|} u_y(\tau,y)\, u_{yy}(\tau,y)\, dy \right\|_{\mathbf{L}^1} \\
&\quad \leq C \left\| \int_{\mathbf{R}} e^{-|x-y|} \big(u(\tau,x) - u(\tau,y)\big) u_y(\tau,y)\, dy \right\|_{\mathbf{L}^1} \\
&\quad\quad + C \left\| \int_{\mathbf{R}} e^{-|x-y|} u_y^2(\tau,y)\, dy \right\|_{\mathbf{L}^1} \leq C \left\| u_x(\tau) \right\|_{\mathbf{L}^2}^2
\end{aligned}$$

as in the previous case. Lemma 2.4 is proved. □

Now we estimate the decay rate of the $\mathbf{L}^2(\mathbf{R})$-norm of the solutions.

Lemma 2.5. *Let $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. Assume that*

$$\left| \int_0^t d\tau \int_{\mathbf{R}} \Theta\big(u(\tau,x)\big) \mathcal{B} u_{xxx}(\tau,x)\, dx \right| + \int_0^t d\tau \left\| \partial_x^2 \mathcal{B}\big(u(\tau)\, u_x(\tau)\big) \right\|_{\mathbf{L}^1} \leq C \langle t \rangle^{\sigma} \tag{2.22}$$

for all $t > 0$, where $\sigma \in [0, 1/4)$. Then the estimates are valid

$$\| u(t) \|_{\mathbf{L}^p} \leq C \langle t \rangle^{\sigma - \frac{1}{2}\left(1 - \frac{1}{p}\right)}$$

for all $t > 0$, and $1 \leq p \leq 2$.

Proof. Applying operator $\mathcal{B} = \left(1 - \partial_x^2\right)^{-1}$ to (1.1) we get

$$u_t = \mu(1 - \mathcal{B})\, u + \beta \partial_x^3 \mathcal{B} u + \mathcal{B} u u_x \tag{2.23}$$

since $\mathcal{B} = 1 - \partial_x^2 \mathcal{B}$. We estimate the $\mathbf{L}^1(\mathbf{R})$-norm. We multiply (2.23) by $\Theta(u(t,x))$ and integrate with respect to x over $\mathbf{R}$ to get

$$\begin{aligned}
\int_{\mathbf{R}} \partial_t |u(t,x)|\, dx = {} & \mu \int_{\mathbf{R}} \Theta\big(u(t,x)\big) (1 - \mathcal{B})\, u dx \\
& + \beta \int_{\mathbf{R}} \Theta\big(u(t,x)\big) \partial_x^3 \mathcal{B} u dx + \int_{\mathbf{R}} |u(t,x)|\, u_x dx \\
& - \int_{\mathbf{R}} \Theta\big(u(t,x)\big) \partial_x^2 \mathcal{B} u u_x dx\,.
\end{aligned}$$

We have

$$\int_{\mathbf{R}} \partial_t |u(t,x)|\, dx = \frac{d}{dt} \|u(t)\|_{\mathbf{L}^1},$$
$$\int_{\mathbf{R}} |u(t,x)|\, u_x dx = 0,$$
$$\int_{\mathbf{R}} \Theta\big(u(t,x)\big)\mathcal{B}u dx \le \int_{\mathbf{R}} \mathcal{B}|u|\, dx \le \|u(t)\|_{\mathbf{L}^1}.$$

Therefore we find

$$\|u(t)\|_{\mathbf{L}^1} \le \|u_0\|_{\mathbf{L}^1} + \left|\beta \int_0^t dt \int_{\mathbf{R}} \Theta(u(t,x))\, \partial_x^3 \mathcal{B}u dx\right| + \left|\int_0^t dt \int_{\mathbf{R}} \Theta\big(u(t,x)\big)\partial_x^2 \mathcal{B}u u_x dx\right|. \tag{2.24}$$

In view of estimate (2.22) yields

$$\sup_{\xi\in\mathbf{R}} |\widehat{u}(t,\xi)| \le C\|u(t)\|_{\mathbf{L}^1} \le \|u_0\|_{\mathbf{L}^1} + C\langle t\rangle^{\sigma} \le C\langle t\rangle^{\sigma} \tag{2.25}$$

for all $t > 0$. Thus the estimate of the lemma with $p = 1$ is fulfilled.

We now multiply (1.1) by $2u$, then integrating with respect to $x \in \mathbf{R}$ we get

$$\frac{d}{dt}\Big(\|u(t)\|^2_{\mathbf{L}^2} + \|u_x(t)\|^2_{\mathbf{L}^2}\Big) = -2\mu \|u_x(t)\|^2_{\mathbf{L}^2}. \tag{2.26}$$

By the Plancherel theorem using the Fourier splitting method due to [17], we have

$$\|u_x(t)\|^2_{\mathbf{L}^2} = \|\xi\widehat{u}(t)\|^2_{\mathbf{L}^2} = \int_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2 \xi^2 d\xi + \int_{|\xi|\ge\delta} |\widehat{u}(t,\xi)|^2 \xi^2 d\xi$$
$$\ge \delta^2 \|u(t)\|^2_{\mathbf{L}^2} - 2\delta^3 \sup_{|\xi|\le\delta} |\widehat{u}(t,\xi)|^2,$$

where $\delta > 0$. Thus from (2.26) we have the inequality

$$\frac{d}{dt}\|u(t)\|^2_{\mathbf{H}^1} \le -\mu\delta^2 \|u(t)\|^2_{\mathbf{H}^1} + 4\mu\delta^3 \sup_{\xi\le\delta} |\widehat{u}(t,\xi)|^2. \tag{2.27}$$

We choose $\mu\delta^2 = 2(1+t)^{-1}$ and change $\|u(t)\|^2_{\mathbf{H}^1} = (1+t)^{-2} W(t)$. Then via (2.25) we get from (2.27)

$$\frac{d}{dt} W(t) \le C(1+t)^{2\sigma+\frac{1}{2}}. \tag{2.28}$$

Integration of (2.28) with respect to time yields

$$W(t) \le \|u_0\|^2_{\mathbf{H}^1} + C\Big((1+t)^{\frac{3}{2}+2\sigma} - 1\Big).$$

Therefore we obtain a time decay estimate of the $\mathbf{L}^2$-norm

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{\sigma-\frac{1}{4}} \tag{2.29}$$

for all $t > 0$. Lemma 2.5 is proved. □

Proposition 2.6. *Suppose that the initial data* $u_0 \in \mathbf{H}^1(\mathbf{R}) \cap \mathbf{W}_1^1(\mathbf{R})$. *Then the estimates for the solution are valid*

$$\|u(t)\|_{\mathbf{L}^p} + \langle t\rangle \left\|\partial_x^2 \mathcal{B}u(t)\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, *where* $1 \le p \le \infty$.

Proof. By Proposition 2.2 we have estimate

$$\|u\|_{\infty,2} + \|u_x\|_{\infty,2} + \|u_x\|_{2,2} \le C\|u_0\|_{\mathbf{H}^1}.$$

Now applying Lemma 2.3 we get

$$\left|\int_0^t d\tau \int_{\mathbf{R}} \Theta\big(u(\tau,x)\big)\mathcal{B}u_{xxx}(\tau,x)\,dx\right| \le C\langle t\rangle^{\frac{1}{6}} \log\langle t\rangle \le C\langle t\rangle^{\sigma_0} \tag{2.30}$$

for all $t > 0$, where $\sigma_0 = 1/6 + \gamma$, $\gamma > 0$ is small. And by Lemma 2.4 we have

$$\int_0^t d\tau \left\|\partial_x^2 \mathcal{B}\big(u(\tau)\,u_x(\tau)\big)\right\|_{\mathbf{L}^1} \le C\langle t\rangle^{\frac{1}{16}}$$

for all $t > 0$. Then by Lemma 2.5 we find the time decay of the $\mathbf{L}^2(\mathbf{R})$-norm

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{\sigma_0 - \frac{1}{4}} \tag{2.31}$$

for all $t > 0$. Applying Lemma 2.4, by the Hölder inequality we obtain

$$\left|\int_0^t d\tau \int_{\mathbf{R}} \Theta\big(u(\tau,x)\big)\mathcal{B}u_{xxx}(\tau,x)\,dx\right| \le Ct^{1-\frac{1}{s_0}}\|u_{xxx}\|_{s_0,1} \le Ct^{1-\frac{1}{s_0}},$$

for all $t > 0$, where $s_0 > \max\Big(1, (9/8 - 3/2\sigma_0)^{-1}\Big)$. Hence we arrive at estimate (2.30) with σ_0 replaced by $\sigma_1 = 1 - 1/s_0 = 1/8 + O(\gamma)$. We again apply Lemma 2.5 to get a better time decay of the $\mathbf{L}^2(\mathbf{R})$-norm (2.31) with σ_0 replaced by $\sigma_1 = 1/8 + O(\gamma)$. Namely

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{\sigma_1 - \frac{1}{4}}.$$

Then Lemma 2.4 yields estimate (2.30) with σ_0 replaced by $\sigma_2 = 1/16 + O(\gamma)$. Now by Lemma 2.5 we get time decay estimate (2.31) with σ_0 replaced by $\sigma_2 = 1/16 + O(\gamma)$. Lemma 2.4 gives us estimate (2.30) with $\sigma_0 = 0$. Therefore by virtue of Lemma 2.5 we obtain an optimal time decay estimate of the $\mathbf{L}^2(\mathbf{R})$-norm

$$\|u(t)\|_{\mathbf{L}^2} \le C(1+t)^{-\frac{1}{4}} \tag{2.32}$$

for all $t > 0$. Using (2.32) we can prove the following optimal time decay estimates

$$\|u(t)\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{2.33}$$

for all $t > 0$, where $1 \le p \le \infty$. For $1 \le p \le 2$ estimate (2.33) follows from (2.32), Lemma 2.5 and the Hölder inequality. Let us prove (2.33) for $p = \infty$. By the integral equation (2.7) applying Hölder inequality we get

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^\infty} &\le \|\mathcal{G}(t)u_0\|_{\mathbf{L}^\infty} + \frac{1}{2}\int_0^{\frac{t}{2}} \|\partial_x \mathcal{B} G(t-\tau)\|_{\mathbf{L}^\infty} \|u(\tau)\|^2_{\mathbf{L}^2}\, d\tau \\ &\quad + \frac{1}{2}\int_{\frac{t}{2}}^{t} \|\partial_x \mathcal{B} G(t-\tau)\|_{\mathbf{L}^4} \left\|u^2(\tau)\right\|_{\mathbf{L}^{\frac{4}{3}}} d\tau \\ &\le Ct^{-\frac{1}{2}} + Ct^{-1}\int_0^{\frac{t}{2}} \langle\tau\rangle^{-\frac{1}{2}}\, d\tau \\ &\quad + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^\infty} \|u(\tau)\|^{\frac{3}{2}}_{\mathbf{L}^2}\, d\tau\end{aligned}$$

hence

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^\infty} &\le Ct^{-\frac{1}{2}} + Ct^{-\frac{3}{8}}\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|^{\frac{1}{2}}_{\mathbf{L}^\infty}\, d\tau \\ &\le Ct^{-\frac{1}{2}} + C\varepsilon t^{-\frac{1}{8}}\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{7}{8}} \|u(\tau)\|_{\mathbf{L}^\infty}\, d\tau \\ &\quad + \frac{C}{\varepsilon} t^{-\frac{5}{8}}\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{7}{8}}\, d\tau\,,\end{aligned}$$

hence by the Gronwall lemma it follows that $\|u(t)\|_{\mathbf{L}^\infty} \le \frac{C}{\varepsilon} t^{-\frac{1}{2}}$. We find (2.33) for all $2 \le p \le \infty$ via the Hölder inequality. In the same manner we get the estimates

$$\left\|\partial_x^2 \mathcal{B} u(t)\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)-1}$$

for all $t > 0$, $1 \le p \le \infty$. Proposition 2.6 is proved. □

3. Proof of Theorem 1.1

Now we obtain the large time asymptotic formulas for solutions to the Cauchy problem (1.1). Let us take a sufficiently large initial time $T > 0$ and define $v(t,x)$ as a solution to the Cauchy problem for the Burgers equation with $u(T,x)$ as the initial data

$$\begin{cases} v_t + vv_x - \mu v_{xx} = 0\,, & t > T,\ x \in \mathbf{R}\,, \\ v(T,x) = u(T,x)\,, & x \in \mathbf{R}\,. \end{cases} \tag{3.1}$$

By the Hopf–Cole [10] transformation $v(t,x) = -2\mu\frac{\partial}{\partial x}\log Z(t,x)$ it is converted to the heat equation $Z_t = \mu Z_{xx}$. So we have the solution explicitly

$$Z(t,x) = \int_{\mathbf{R}} dy G_0(t,x-y)\exp\left(-\frac{1}{2\mu}\int_{-\infty}^{y} u(T,\xi)\, d\xi\right), \tag{3.2}$$

where $G_0(t,x) = (4\pi\mu t)^{-\frac{1}{2}} e^{-\frac{x^2}{4\mu t}}$ is the Green function for the heat equation. Note that the following estimates are true

$$\|v(t)\|_{\mathbf{L}^p} + \langle t\rangle \left\|\partial_x^2 \mathcal{B} v(t)\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \tag{3.3}$$

for all $t > T$, $1 \le p \le \infty$.

Consider now the difference $w(t,x) = u(t,x) - v(t,x)$ for all $t > T$. By (1.1) and (3.1) we get the Cauchy problem

$$\begin{cases} w_t + \partial_x (vw) + \frac{1}{2}\frac{\partial}{\partial x} w^2 - \mu w_{xx} + h_x = 0\,, & t > T,\ x \in \mathbf{R}\,, \\ w(T,x) = 0\,, & x \in \mathbf{R}\,, \end{cases} \tag{3.4}$$

where

$$h = \left(\beta \partial_x^2 + \mu \partial_x^3\right) \mathcal{B} u - \partial_x \mathcal{B} u u_x\,.$$

Since we consider the large initial data, we need to eliminate the linear term $\frac{\partial}{\partial x}(vw)$. We make a change $\partial_x^{-1} w = \int_{-\infty}^{x} w(t,y)\,dy = \mu g / Z$, then from (3.4) we obtain the Cauchy problem

$$\begin{cases} g_t - \mu g_{xx} + F = 0\,, & t > T,\ x \in \mathbf{R}\,, \\ g(T,x) = 0\,, & x \in \mathbf{R}\,, \end{cases} \tag{3.5}$$

where

$$F = \frac{\mu}{2Z}\left(g_x + \frac{1}{2\mu} g v\right)^2 + Zh\,.$$

By virtue of estimates of Proposition 2.6 and (3.3) we have

$$\|Z(t)\|_{\mathbf{L}^\infty} + \left\|Z^{-1}(t)\right\|_{\mathbf{L}^\infty} \le C \tag{3.6}$$

for all $t \ge T$ and a rough time decay estimate

$$\|w(t)\|_{\mathbf{L}^p} + \langle t \rangle \left\|\partial_x^2 \mathcal{B} w(t)\right\|_{\mathbf{L}^p} \le C \langle t \rangle^{-\frac{1}{2}\left(1 - \frac{1}{p}\right)} \tag{3.7}$$

for all $t \ge T$, $1 \le p \le \infty$. Let us prove the estimate

$$\|g(t)\|_{\mathbf{L}^p} + \langle t \rangle^{\frac{1}{2}} \|g_x(t)\|_{\mathbf{L}^p} < C \langle t \rangle^{-\gamma + \frac{1}{2p}} \tag{3.8}$$

for all $t \ge T$, $2 \le p \le \infty$, where $\gamma \in (0, 1/2)$. By contradiction, suppose that for some $t = T_1$ estimate (3.8) is violated, i.e., we have

$$\|g(t)\|_{\mathbf{L}^p} + \langle t \rangle^{\frac{1}{2}} \|g_x(t)\|_{\mathbf{L}^p} \le C \langle t \rangle^{-\gamma + \frac{1}{2p}} \tag{3.9}$$

for all $t \in [T, T_1]$, $1 \le p \le \infty$. In view of (3.3), (3.6), (3.8) and (3.9) we find

$$\begin{aligned} \|F(t)\|_{\mathbf{L}^p} &\le C \|g_x\|_{\mathbf{L}^\infty} \|g_x\|_{\mathbf{L}^p} + C \|g\|_{\mathbf{L}^\infty}^2 \left\|v^2\right\|_{\mathbf{L}^p} + C \|Zh\|_{\mathbf{L}^p} \\ &\le C \left(\langle t \rangle^{-1 - 2\gamma + \frac{1}{2p}} + \langle t \rangle^{-\frac{3}{2} + \frac{1}{2p}} \right) \\ &\le C \max\left\{ T^{\gamma - \frac{1}{2}}, T^{-\gamma} \right\} \langle t \rangle^{-\gamma - 1 + \frac{1}{2p}} \end{aligned} \tag{3.10}$$

for all $t \in [T, T_1]$, $1 \leq p \leq \infty$. Using the integral equation associated with (3.5) in view of estimate (3.10) we find

$$\begin{aligned} \|g(t)\|_{\mathbf{L}^p} &\leq \int_T^t d\tau \, \|G_0(t-\tau)\|_{\mathbf{L}^p} \|F(\tau)\|_{\mathbf{L}^1} \\ &\leq C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \int_T^t (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}} \langle\tau\rangle^{-\frac{1}{2}-\gamma} d\tau \\ &\leq C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \langle t\rangle^{-\gamma+\frac{1}{2p}} < C \langle t\rangle^{-\gamma+\frac{1}{2p}} \end{aligned}$$

for all $t \in [T, T_1]$, $1 \leq p \leq \infty$, since T is sufficiently large. In the same manner we have

$$\begin{aligned} \|g_x(t)\|_{\mathbf{L}^p} \leq & \int_T^{\frac{t+T}{2}} d\tau \, \|\partial_x G_0(t-\tau)\|_{\mathbf{L}^p} \|F(\tau)\|_{\mathbf{L}^1} \\ &+ \int_{\frac{t+T}{2}}^t d\tau \, \|\partial_x G_0(t-\tau)\|_{\mathbf{L}^1} \|F(\tau)\|_{\mathbf{L}^p} \\ \leq & \, C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \int_T^{\frac{t+T}{2}} (t-\tau)^{-1+\frac{1}{2p}} \langle\tau\rangle^{-\frac{1}{2}-\gamma} d\tau \\ &+ C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \int_{\frac{t+T}{2}}^t (t-\tau)^{-\frac{1}{2}} \langle\tau\rangle^{-1-\gamma+\frac{1}{2p}} d\tau \\ \leq & \, C \max\left\{T^{\gamma-\frac{1}{2}}, T^{-\gamma}\right\} \langle t\rangle^{-\gamma-\frac{1}{2}+\frac{1}{2p}} < C \langle t\rangle^{-\gamma-\frac{1}{2}+\frac{1}{2p}} \end{aligned}$$

for all $t \in [T, T_1]$, $1 \leq p \leq \infty$. The contradiction obtained proves estimate (3.8) for all $t \geq T$. Since

$$w = Z^{-1}\left(\mu g_x + \frac{1}{2} g v\right)$$

estimate (3.8) implies

$$\|u(t) - v(t)\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-\gamma-\frac{1}{2}}$$

for all $t > T$. It is known that if $x u_0 \in \mathbf{L}^1(\mathbf{R})$, then

$$\left\| v(t) - t^{-\frac{1}{2}} f_\theta\left((\cdot)\, t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-\gamma-\frac{1}{2}}.$$

Therefore the estimate of the theorem follows. Theorem 1.1 is proved.

4. Proof of Theorem 1.2

As in the proof of Proposition 2.6 we can obtain the following estimate of the $\mathbf{L}^{1,1}(\mathbf{R})$-norm of solutions of the Cauchy problem (1.1)

$$\|u(t)\|_{\mathbf{L}^{1,1}} \leq C \langle t\rangle^{\frac{1}{2}}. \tag{4.1}$$

Indeed applying estimates of Lemma 2.1 to the integral equation (2.7) we get

$$\begin{aligned}
\|u(t)\|_{\mathbf{L}^{1,1}} &\le \|\mathcal{G}(t)u_0\|_{\mathbf{L}^{1,1}} + C\int_0^{\frac{t}{2}} \left\|\partial_x \mathcal{B}\mathcal{G}(t-\tau)u^2(\tau)\right\|_{\mathbf{L}^{1,1}} d\tau \\
&\quad + C\int_{\frac{t}{2}}^{t} \|\mathcal{G}(t-\tau)\mathcal{B}u(\tau)u_x(\tau)\|_{\mathbf{L}^{1,1}} d\tau \\
&\le C\langle t\rangle^{\frac{1}{2}} + C\int_0^{\frac{t}{2}} \left\|u^2(\tau)\right\|_{\mathbf{L}^1} d\tau \\
&\quad + C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{1}{2}} \left\|u^2(\tau)\right\|_{\mathbf{L}^{1,1}} d\tau \\
&\quad + C\int_{\frac{t}{2}}^{t} \langle t-\tau\rangle^{\frac{1}{2}} \|\mathcal{B}u(\tau)u_x(\tau)\|_{\mathbf{L}^1} d\tau \\
&\quad + C\int_{\frac{t}{2}}^{t} \|\mathcal{B}u(\tau)u_x(\tau)\|_{\mathbf{L}^{1,1}} d\tau \\
&\le C\langle t\rangle^{\frac{1}{2}} + C\langle t\rangle^{-\frac{1}{2}} \int_0^t \langle\tau\rangle^{-\frac{1}{2}} \|u(\tau)\|_{\mathbf{L}^{1,1}} d\tau
\end{aligned}$$

hence by the Gronwall lemma estimate (4.1) follows for all $t > 0$.

Now we obtain the second term of the large time asymptotics as $t \to \infty$ of solutions $u(t,x)$ to the Cauchy problem (1.1). As in the previous section we take a sufficiently large initial time $T > 0$ and consider the Cauchy problem for the Burgers equation (3.1). Then for the difference $w(t,x) = u(t,x) - v(t,x)$ by (1.1) and (3.1) we get the Cauchy problem (3.4) with estimates (3.8).

Consider now the linear Cauchy problem

$$\begin{cases} \varphi_t + \partial_x(\varphi v) - \mu\varphi_{xx} + \beta v_{xxx} = 0\,, & t > T,\ x \in \mathbf{R}\,, \\ \varphi(T,x) = 0\,, & x \in \mathbf{R}\,. \end{cases} \tag{4.2}$$

To eliminate the second term from (4.2), we integrate (4.2) with respect to x and make the substitution

$$\int_{-\infty}^{x} \varphi(t,y)dy = \mu\frac{s(t,x)}{Z(t,x)}\,,$$

where $Z(t,x)$ is defined by (3.2). We obtain

$$\begin{cases} s_t - \mu s_{xx} + \beta Z v_{xx} = 0\,, & t > T,\ x \in \mathbf{R}\,, \\ s(T,x) = 0\,, & x \in \mathbf{R}\,. \end{cases} \tag{4.3}$$

It is easy to integrate (4.3) to get

$$s(t,x) = -\beta\int_T^t d\tau \int_{\mathbf{R}} dy G_0(t-\tau, x-y)\, Z(\tau,y)\, v_{yy}(\tau,y)\,. \tag{4.4}$$

Now let us compute the asymptotics of $s(t,x)$ as $t \to \infty$. We integrate by parts with respect to y to obtain

$$\begin{aligned} s(t,x) = &-\beta \int_T^t d\tau \int_0^\infty dy \partial_x G_0(t-\tau, x-y) \int_y^\infty Z(\tau,\eta) v_{\eta\eta}(\tau,\eta)\, d\eta \\ &+ \beta \int_T^t d\tau \int_{-\infty}^0 dy \partial_x G_0(t-\tau, x-y) \int_{-\infty}^y Z(\tau,\eta) v_{\eta\eta}(\tau,\eta)\, d\eta \\ &- \beta \int_T^t d\tau G_0(t-\tau, x) \int_{\mathbf{R}} Z(\tau,\eta) v_{\eta\eta}(\tau,\eta)\, d\eta \\ \equiv\, & I_1 + I_2 + I_3 \,. \end{aligned}$$

Since

$$\|x v_{xx}(t)\|_{\mathbf{L}^1} \le C \langle t \rangle^{-\frac{1}{2}} ,$$

we obtain

$$\begin{aligned} |I_1| &\le |\beta| \int_T^t d\tau \int_0^\infty dy \, |\partial_x G_0(t-\tau, x-y)| \int_y^\infty Z(\tau,\eta) \, |v_{\eta\eta}(\tau,\eta)| \, d\eta \\ &\le C t^{-1} \int_T^{\frac{t+T}{2}} d\tau \, \|x v_{xx}(\tau)\|_{\mathbf{L}^1(\mathbf{R})} + C \int_{\frac{t+T}{2}}^t d\tau \, (t-\tau)^{-\frac{1}{2}} \|v_{xx}(\tau)\|_{\mathbf{L}^1(\mathbf{R})} \\ &\le C t^{-1} \int_T^{\frac{t+T}{2}} \langle \tau \rangle^{-\frac{1}{2}} d\tau + C \int_{\frac{t+T}{2}}^t (t-\tau)^{-\frac{1}{2}} \langle \tau \rangle^{-1} d\tau = O(t^{-\frac{1}{2}}) \,. \end{aligned} \tag{4.5}$$

The integral I_2 is estimated in the same way. Now we consider I_3. We have the asymptotics

$$Z(t,x) = H(\chi) + O\left(t^{-\frac{1}{2}}\right) ,$$

where

$$H(\chi) = \cosh \frac{\theta}{4} - \sinh\left(\frac{\theta}{4}\right) \operatorname{Erf}\left(\frac{\chi}{2\sqrt{\mu}}\right)$$

with $\chi = x/\sqrt{t}$. Then in view of the identity

$$Z_x = -\frac{1}{2\mu} Z v \,,$$

integration by parts yields

$$\begin{aligned} \int_{\mathbf{R}} Z(\tau,\eta) v_{\eta\eta}(\tau,\eta) \, d\eta &= -\frac{1}{2} \int_{\mathbf{R}} v^3(\tau,y) Z(\tau,y) dy \\ &= -\frac{1}{2\tau} \int_{\mathbf{R}} f_\theta^3(y) H(y) dy + O(\tau^{-3/2}) \,. \end{aligned}$$

Therefore

$$
\begin{aligned}
I_3 &= -\beta \int_T^t d\tau G_0\left(t-\tau, x\right) \int_{\mathbf{R}} Z\left(\tau, \eta\right) v_{\eta\eta}\left(\tau, \eta\right) d\eta \\
&= \frac{\beta}{2\sqrt{4\pi\mu}} \int_{\mathbf{R}} f_\theta^3(y) H(y) dy \int_T^t \frac{d\tau}{\tau\sqrt{t-\tau}} e^{-\frac{x^2}{4\mu(t-\tau)}} + O\left(t^{-\frac{1}{2}}\right) \\
&= \frac{\beta}{2} G_0\left(t, x\right) \log t \int_{\mathbf{R}} f_\theta^3(y) H(y) dy + O\left(t^{-\frac{1}{2}}\right). \qquad (4.6)
\end{aligned}
$$

Hence the asymptotics is true

$$
s\left(t, x\right) = \frac{\beta}{2\sqrt{4\pi\mu t}} e^{-\frac{\xi^2}{4\mu}} \log t \int_{\mathbf{R}} f_\theta^3(y) H(y) dy + O\left(t^{-\frac{1}{2}}\right)
$$

for large $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$. This formula can be differentiated with respect to x. Hence we see that

$$
\varphi\left(t, x\right) = \mu \partial_x \left(\frac{s\left(t, x\right)}{Z\left(t, x\right)}\right) = \frac{\log t}{t} \widetilde{f}_\theta\left(\chi\right) + O\left(\frac{1}{t}\right)
$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\chi = x/\sqrt{t}$ and

$$
\widetilde{f}_\theta\left(x\right) = -\frac{\beta\left(f_\theta\left(x\right) - \frac{x}{2\sqrt{\mu}}\right) e^{-\frac{x^2}{4\mu}}}{2\sqrt{\pi\mu} H\left(x\right)} \int_{\mathbf{R}} f_\theta^3\left(y\right) H\left(y\right) dy\,.
$$

It follows from (3.4) and (4.2) that the remainder $\psi\left(t, x\right) = w(t, x) - \varphi(t, x)$ is the solution to the Cauchy problem

$$
\begin{cases}
\psi_t + \partial_x\left(v\psi\right) - \mu\psi_{xx} + \frac{1}{2}\partial_x w^2 + \beta w_{xxx} + \partial_x h_1 = 0\,, & t > T,\ x \in \mathbf{R}\,, \\
\psi\left(T, x\right) = 0\,, & x \in \mathbf{R}\,,
\end{cases} \qquad (4.7)
$$

where

$$
h_1 = \left(-\beta\partial_x^4 + \mu\partial_x^3\right) \mathcal{B}u - \partial_x \mathcal{B}uu_x\,.
$$

To eliminate the second term from (4.7) as above we integrate this equation with respect to x and introduce the new unknown function

$$
r(t, x) = \frac{Z(t, x)}{\mu} \int_{-\infty}^x \psi(t, y) dy\,.
$$

Then we obtain

$$
\begin{cases}
r_t - \mu r_{xx} + F_1 = 0\,, & t > T,\ x \in \mathbf{R}\,, \\
r\left(T, x\right) = 0\,, & x \in \mathbf{R}\,,
\end{cases} \qquad (4.8)
$$

where

$$
F_1 = \frac{1}{2} Z w^2 + Z\beta w_{xx} + Z h_1\,.
$$

In view of (3.3) and (3.8) we find

$$
\begin{aligned}
\left\|F_1\left(t\right)\right\|_{\mathbf{L}^p} &\le C \left\|w\right\|_{\mathbf{L}^\infty} \left\|w\right\|_{\mathbf{L}^p} + C \left\|w_{xx}\right\|_{\mathbf{L}^p} + \left\|h_1\right\|_{\mathbf{L}^p} \\
&\le C \left\langle t\right\rangle^{-1-2\gamma+\frac{1}{2p}} + C \left\langle t\right\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} \le C \left\langle t\right\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} \qquad (4.9)
\end{aligned}
$$

for all $t \geq T$, $1 \leq p \leq \infty$ if we choose $\gamma \in (1/4, 1/2)$. Using the integral equation associated with (4.8) we obtain in view of (4.9)

$$\begin{aligned}\|r(t)\|_{\mathbf{L}^p} &\leq \int_T^{\frac{t+T}{2}} d\tau \, \|G(t-\tau)\|_{\mathbf{L}^p} \|F_1(\tau)\|_{\mathbf{L}^1} \\ &\quad + \int_{\frac{t+T}{2}}^t d\tau \, \|G(t-\tau)\|_{\mathbf{L}^1} \|F_1(\tau)\|_{\mathbf{L}^p} \\ &\leq C \int_T^{\frac{t+T}{2}} (t-\tau)^{-\frac{1}{2}+\frac{1}{2p}} \langle\tau\rangle^{-1-\gamma} d\tau \\ &\quad + C \int_{\frac{t+T}{2}}^t \langle\tau\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} d\tau \leq C \langle t\rangle^{-\frac{1}{2}+\frac{1}{2p}}\end{aligned}$$

for all $t \geq T$, $1 \leq p \leq \infty$. In the same manner we have

$$\begin{aligned}\|r_x(t)\|_{\mathbf{L}^p} &\leq \int_T^{\frac{t+T}{2}} d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^p} \|F_1(\tau)\|_{\mathbf{L}^1} \\ &\quad + \int_{\frac{t+T}{2}}^t d\tau \, \|\partial_x G(t-\tau)\|_{\mathbf{L}^1} \|F_1(\tau)\|_{\mathbf{L}^p} \\ &\leq C \int_T^{\frac{t+T}{2}} (t-\tau)^{-1+\frac{1}{2p}} \langle\tau\rangle^{-1-\gamma} d\tau \\ &\quad + C \int_{\frac{t+T}{2}}^t (t-\tau)^{-\frac{1}{2}} \langle\tau\rangle^{-\frac{3}{2}-\gamma+\frac{1}{2p}} d\tau \leq C \langle t\rangle^{-1+\frac{1}{2p}}\end{aligned}$$

for all $t \geq T$, $1 \leq p \leq \infty$. Then by the identity

$$\psi = Z^{-1}\left(r_x + \frac{1}{2} r v\right)$$

we obtain the estimate

$$\|u(t) - v(t) - \varphi(t)\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-1}$$

for all $t \geq T$. Theorem 1.2 is proved.

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