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Eigenfunction Statistics in the Localized Anderson Model

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Abstract. We consider the localized region of the Anderson model and study the distribution of eigenfunctions simultaneously in space and energy. In a natural scaling limit, we prove convergence to a Poisson process. This provides a counterpoint to recent work, [9], which proves repulsion of the localization centres in a subtly different regime.

1. Introduction

The purpose of this note is to describe the distribution of eigenfunctions (in space and energy) for the Anderson model in the localized regime. We will prove that one obtains a Poisson process in a natural scaling limit.

The Anderson model is an ensemble of random operators on $\ell^2(\mathbb{Z}^d)$,

$$(H\varphi)(x) = \sum_{|y-x|=1} \varphi(y) + V_\omega(x)\varphi(x)\,,$$

where $\{V_\omega(x) : x \in \mathbb{Z}^d\}$ are a family of independent identically distributed random variables. We will assume throughout that the common probability distribution is absolutely continuous with density $\rho \in L^\infty$. Expectation over the random potential will be denoted $\mathbb{E}$.

For this model, there is always an interval of energies exhibiting Anderson localization, that is, dense pure point spectrum with exponentially decaying eigenfunctions, with probability one. The size and location of this interval depend on the dimension d and the strength of the random potential. By simple ergodicity arguments, the spectrum (as a set) and the spectral type are almost surely constant.

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Early proofs of localization can be found in [3,4]; however, the fractional moment method introduced by Aizenman and Molchanov, [1], is better suited to the matters we wish to discuss. Indeed, for the model we treat, it is not unreasonable to *redefine* Anderson localization as exponential decay of fractional moments of the resolvent. (As described in [2, §4.4] this conclusion can be deduced from other natural notions of localization.) With this in mind, we make the

Definition 1.1. The ensemble of operators H obeys *FM-localization* on $[a,b]$ if this interval belongs to the spectrum of H (with probability one) and for all $E \in [a,b]$ and all $\epsilon > 0$,

$$\mathbb{E}\big\{\big|\langle \delta_y|(H-E-i\epsilon)^{-1}\delta_x\rangle\big|^s\big\} \le Ae^{-\mu|x-y|}, \qquad \forall\, x,y \in \mathbb{Z}^d\,, \tag{1.1}$$

with fixed $s \in (0,1)$, $\mu > 0$, and $A > 0$.

The natural way to describe the location of eigenvectors simultaneously in space and energy is in terms of the random measure $d\xi$ on $\mathbb{R}^{1+d}$ defined by

$$\int_{\mathbb{R}\times\mathbb{R}^d} f(E)g(x)\,d\xi = \operatorname{Tr}\big(g(x)f(H)\big)\,,$$

for all $f \in C_c(\mathbb{R})$ and all $g \in C_c(\mathbb{R}^d)$. Note that $d\xi$ will be supported only on $\mathbb{R}\times\mathbb{Z}^d$. On the right-hand side of this equation, we are considering $g(x)$ as a multiplication operator on $\ell^2(\mathbb{Z}^d)$. An equivalent and perhaps more appealing definition can be given when H has pure point spectrum:

$$\int_{\mathbb{R}\times\mathbb{R}^d} f(E,x)\,d\xi = \sum_j \sum_{x\in\mathbb{Z}^d} f(E_j,x)|\psi_j(x)|^2\,,$$

where E_j enumerate the eigenvalues (according to multiplicity) and ψ_j, the corresponding ℓ^2-normalized eigenfunctions.

The measure $d\xi$ can be studied in two natural scaling limits. The first is the macroscopic limit where space (and the measure) are rescaled, but not energy. Using ergodicity under space translations, it is not difficult to see that for any $f \in C_c(\mathbb{R}^{1+d})$,

$$L^{-d}\int f(E,x/L)\,d\xi(E,x) \to \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(E,y)\,d\nu(E)\,dy$$

as $L \to \infty$. Here $d\nu$ denotes the density of states measure, which is defined by

$$\int g(E)\,d\nu(E) = \mathbb{E}\left\{\big\langle\delta_x\big|g(H)\,\delta_x\big\rangle\right\},$$

for all $g \in C_c(\mathbb{R})$. Let us note for future reference that for the model we consider the density of states measure is absolutely continuous with bounded density. This follows from Wegner's estimate; see Lemma 2.1.

We will show that in the localized regime, the random measure $d\xi$ converges to a Poisson process in the *microscopic* scaling limit, that is, when both energy and space are rescaled. Given a length scale L, which will eventually be taken to infinity, and a reference energy E_0, we define a rescaled measure $d\xi_L$ by

$$\int f(E,x)\, d\xi_L = \int f\left(L^d(E-E_0), \frac{x}{L}\right) d\xi\,.$$

In order to prove convergence of $d\xi_L$, we need to make a mild assumption on the reference energy, E_0, namely that it is a Lebesgue point of the density of states measure:

$$\frac{d\nu}{dE}(E_0) = \lim_{r\downarrow 0} \frac{1}{r}\nu([E_0, E_0+r]) = \lim_{r\downarrow 0} \frac{1}{r}\nu([E_0-r, E_0]) \tag{1.2}$$

(and both limits exist). This is slightly stronger that the symmetrical version commonly found in textbooks, but still holds Lebesgue almost everywhere.

Theorem 1.1. *Let E_0 be a Lebesgue point of the density of states measure and lie inside an interval of FM-localization, then the random measure $d\xi_L$ converges in distribution to a Poisson point process on $\mathbb{R}\times\mathbb{R}^d$ with intensity $\frac{d\nu}{dE}(E_0)\, dE \otimes dx$.*

Minami, [7], studied the eigenvalue statistics of finite volume Anderson Hamiltonians in this limit and proved convergence to a Poisson process. (This was shown earlier for a related one-dimensional model by Molchanov, [8].) The main observation in this note is that Minami's methods extend without too much difficulty to give a proof of Theorem 1.1.

Our interest in this question stems from earlier work of the second author, [9], which showed that when eigenvalues are anomalously close (relative to the natural scaling) the eigenfunctions must live far apart. Note that such nearby eigenvalues do occur. Minami's result merely says that they are rare in a small neighbourhood of a fixed energy; whereas, [9] considers proximate eigenvalues wherever they lie in the region of localization.

The related result that in the localized regime, eigenvalues are simple (with probability one) was proved by Simon, [10]. A new proof of this, based on Minami's work, can be found in [6].

Perhaps the main message to be taken from this note and its companion, [9], is that the physically natural mantra of eigenfunction repulsion is more subtle than it seems. After all, a Poisson process is the very model of non-repulsion, indeed of total non-interaction.

2. Proof of Theorem 1.1

The key to proving that $d\xi_L$ converges to a Poisson process is the effective independence of distant regions of space, which follows from the exponential decay of the Green function. The way to exploit this is to compare H with a direct sum of finite volume Anderson Hamiltonians.

To each sufficiently large L let us associate an integer l_L; we merely require that $l_L \sim L^\alpha$ for some $0<\alpha<1$. We define H_L by replacing by zero all matrix elements of H that connect distinct boxes

$$B_p(L) = \{x \in \mathbb{Z}^d : x_j \in [p_j l_L, (p_j+1)l_L) \text{ for all } 1 \le j \le d\}, \qquad p \in \mathbb{Z}^d .$$

In this way, H_L is the direct sum of the restrictions $H_{L,p}$ of H to each $B_p(L)$. Note also that these summands are statistically independent.

Just as we defined the family of measures $d\xi_L$ associated to H, we can define a measure associated to each $H_{L,p}$ in the analogous way:

$$\int_{\mathbb{R}\times\mathbb{R}^d} f(E,x)\, d\tilde{\eta}_{L,p} = \sum_j \sum_{x\in B_p(L)} f\left(L^d(E_j - E_0), \frac{x}{L}\right) |\psi_j(x)|^2 \tag{2.1}$$

where E_j enumerate the eigenvalues and ψ_j, the ℓ^2-normalized eigenfunctions of $H_{L,p}$; remember, $H_{L,p}$ is a finite matrix.

It will be convenient to have a notation for the analogue of $B_p(L)$ under the scaling given in (2.1). To this end, we partition $\mathbb{R}^d$ into cubes whose sides have length l_L/L:

$$C_p(L) = \{x \in \mathbb{R}^d : x_j \in [p_j l_L/L, (p_j+1)l_L/L) \text{ for all } 1 \le j \le d\}, \qquad p \in \mathbb{Z}^d .$$

Note that the support of $d\tilde{\eta}_{L,p}$ is contained in $\mathbb{R} \times C_p(L)$.

Wegner's estimate, [11], is useful for bounding various error terms that appear in the proof. We formulate it as follows:

Lemma 2.1. *For any $f \in L^1(\mathbb{R})$ and any $x \in \mathbb{Z}^d$, we have*

$$\int f(E)\, d\nu(E) = \mathbb{E}\left\{\langle \delta_x | f(H)\, \delta_x\rangle\right\} \le \|\rho\|_\infty \|f\|_1 \tag{2.2}$$

and

$$\mathbb{E}\left\{\langle \delta_x | f(H_L)\, \delta_x\rangle\right\} \le \|\rho\|_\infty \|f\|_1 . \tag{2.3}$$

Recall that ρ is the probability density for the random potential.

Proposition 2.1. *Suppose FM-localization holds in a neighbourhood of E_0. Then, for any function $f \in C_c(\mathbb{R}\times\mathbb{R}^d)$,*

$$\mathbb{E}\left\{\left|\int f\, d\xi_L - \sum_p \int f\, d\tilde{\eta}_{L,p}\right|\right\} \to 0 \tag{2.4}$$

as $L \to \infty$. This remains true if f is the characteristic function of a rectangle (with sides parallel to the axes).

Proof. This synthesizes Steps 3 and 5 in Minami's paper [7].

It suffices to prove the result for f of the form $f(E,x) = h(E)g(x)$. Linear combinations of such functions, with f and g continuous, are dense in $C_c(\mathbb{R}\times\mathbb{R}^d)$, while the characteristic function of a rectangle is already of this form. In this case,

$$\left|\int f\, d\xi_L - \sum_p \int f\, d\tilde{\eta}_{L,p}\right| \le \sum_x |g_L(x)|\, \left|\langle \delta_x | [h_L(H) - h_L(H_L)]\delta_x\rangle\right| \tag{2.5}$$

$$\le \|g\|_\infty \sum_{x\in \operatorname{supp}(g_L)} \left|\langle \delta_x | [h_L(H) - h_L(H_L)]\delta_x\rangle\right| \tag{2.6}$$

where $g_L(x) = g(x/L)$ and $h_L(E) = h([E - E_0]L^d)$.

As $\mathrm{supp}(g_L)$ intersects only $O(L^d/l_L^d)$ many cubes $B_p(L)$, the problem reduces to showing

$$\lim_{L\to\infty} \frac{L^d}{l_L^d} \sum_{x\in B_p(L)} \mathbb{E}\Big\{ \big|\langle \delta_x | [h_L(H) - h_L(H_L)]\delta_x\rangle\big| \Big\} = 0\,. \tag{2.7}$$

We need just one further reduction, namely, it is sufficient to prove (2.7) for

$$h(E) = \frac{\tau}{(E-\sigma)^2 + \tau^2}\,.$$

The reason is two-fold: finite linear combinations of these functions are dense in L^1 and by Lemma 2.1, this level of approximation is sufficient. To see this, one should note that $\|L^d h_L(E)\|_1 = \|h(E)\|_1$.

To recap, we need to prove

$$\frac{1}{l_L^d} \sum_{x\in B_p(L)} \mathbb{E}\Big\{ \Big|\mathrm{Im}\, G(x,x;E_0 + zL^{-d}) - \mathrm{Im}\, G_{L,p}(x,x;E_0 + zL^{-d})\Big| \Big\} \to 0 \tag{2.8}$$

where $z = \sigma + i\tau$ and G, G_p denote the Green's functions of H and $H_{L,p}$ respectively. The proof of this can be found in the part of [7] cited above. We will review the argument, which involves breaking the sum into two pieces.

If x is close to the boundary of $B_p(L)$, which means within $\log^2(L)$, we make a very crude estimate. We replace the absolute value of the difference of Green's functions by the sum of their absolute values and then apply Lemma 2.1:

$$\mathbb{E}\Big\{ \Big|\mathrm{Im}\, G(x,x;E_0 + zL^{-d}) - \mathrm{Im}\, G_{L,p}(x,x;E_0 + zL^{-d})\Big| \Big\} \le 2\pi\,.$$

This is satisfactory because the number of $x \in B_p(L)$ that are this close to the boundary is $O(\log^2(L) l_L^{d-1})$ and so $o(l_L^d)$.

For x far from the boundary, we obtain smallness from the exponential decay provided by FM-localization. By the resolvent identity,

$$\begin{aligned} &\Big|\mathrm{Im}\, G(x,x;E_0 + zL^{-d}) - \mathrm{Im}\, G_{L,p}(x,x;E_0 + zL^{-d})\Big| \\ &\quad \le \sum_{y,y'} \Big| G_{L,p}(x,y;E_0 + zL^{-d}) G(y',x;E_0 + zL^{-d})\Big| \end{aligned} \tag{2.9}$$

where the sum is over neighbouring points $y \in B_p(L)$ and $y' \notin B_p(L)$. There are $O(l_L^{d-1})$ such pairs. Freezing one such pair for the moment we have

$$\begin{aligned} &\mathbb{E}\left\{ \Big| G_{L,p}(x,y;E_0 + zL^{-d}) G(y',x;E_0 + zL^{-d})\Big| \right\} \\ &\quad \le \frac{L^d}{\mathrm{Im}\, z} \left(\frac{L^d}{\mathrm{Im}\, z}\right)^{1-s} \mathbb{E}\left\{ \Big| G(y',x;E_0 + zL^{-d})\Big|^s \right\} \end{aligned} \tag{2.10}$$

using $\|(B-z)^{-1}\| \le (\mathrm{Im}\, z)^{-1}$ for any self-adjoint operator B. By hypothesis, the last factor is no larger than $A\exp[-\mu \log^2(L)]$. This is small enough to beat the other contributions (including both sums), which grow as a power of L. $\square$

Because there are clean results in the literature proving the convergence of certain point processes to the Poisson process, it is convenient to approximate $d\xi_L$ by a such a process rather than a general random measure. We implement this by moving all the mass of $d\tilde{\eta}_{L,p}$ into one corner of the cube $C_p(L)$. More precisely, let us define $d\eta_{L,p}$ to be the point process

$$\int_{\mathbb{R}\times\mathbb{R}^d} f(E,x)\,d\eta_{L,p} = \sum_j f\big(L^d(E_j - E_0), p\,l_L/L\big)$$

where E_j are the eigenvalues of $H_{L,p}$. The results of the previous proposition carry over as we now show.

Corollary 2.1. (a) *For each* $f \in C_c(\mathbb{R}^{1+d})$, $\mathbb{E}\{|\int f\,d\xi_L - \sum_p \int f\,d\eta_{L,p}|\} \to 0$ *as* $L \to \infty$.

(b) *Let* $Q \subset \mathbb{R}^d$ *be a rectangle with sides parallel to the axes and let* I *be a finite interval, then* $\mathbb{E}\{|\xi_L(I \times Q) - \sum_p \eta_{L,p}(I \times Q)|\} \to 0$ *as* $L \to \infty$.

(c) *If* $\sum_p d\eta_{L,p}$ *converges in distribution to a Poisson process, then this is also true of* $d\xi_L$.

Proof. In view of Proposition 2.1, we can prove parts (a) and (b) by controlling the difference between $\tilde{\eta}_{L,p}$ and $\eta_{L,p}$.

(a) Choose $a, b \in \mathbb{R}$ so that $f(E,x) = 0$ for all $E \notin [a,b]$. Then

$$\Big|\int f\,d\tilde{\eta}_{L,p} - \int f\,d\eta_{L,p}\Big| \le \omega(l_L/L)\,\eta_{L,p}([a,b]\times\mathbb{R}^d)\,,$$

where $\omega(\delta) = \sup\{|f(E,x) - f(E,y)| : |x-y| < \delta,\ E \in \mathbb{R}\}$. As f is uniformly continuous, $\omega(\delta) = o(1)$ as $\delta \to 0$.

As $\mathbb{E}\{\eta_{L,p}([a,b]\times\mathbb{R}^d)\}$ is the average number of eigenvalues of $H_{L,p}$ in the interval $[E_0 + aL^{-d}, E_0 + bL^{-d}]$, it follows from (2.3) that $\mathbb{E}\{\eta_{L,p}([a,b]\times\mathbb{R}^d)\} = O(l_L^d/L^d)$. In this way, we find that

$$\sum_p \mathbb{E}\left\{\Big|\int f\,d\tilde{\eta}_{L,p} - \int f\,d\eta_{L,p}\Big|\right\} = O(\omega(l_L/L)) = o(1)\,,$$

because the number of cubes $C_p(L)$ that intersect the support of f is $O(L^d/l_L^d)$. This proves (a).

(b) Note that $\tilde{\eta}_{L,p}(I\times Q) - \eta_{L,p}(I\times Q)$ is only non-zero if $C_p(L)$ intersects the boundary of Q. The number of such cubes is $O(L^{d-1}/l_L^{d-1})$, while as we saw in part (a), $\mathbb{E}\{\eta_{L,p}(I\times\mathbb{R}^d)\} = O(l_L^d/L^d)$. Putting these two facts together gives

$$\sum_p \mathbb{E}\left\{\big|\tilde{\eta}_{L,p}(I\times Q) - \eta_{L,p}(I\times Q)\big|\right\} = O(l_L/L)$$

and so the result follows.

(c) As described in [5], convergence in distribution is precisely convergence of Laplace functionals,

$$\mathcal{L}_f(\xi) = \mathbb{E}\Big\{e^{-\int f\,d\xi}\Big\}\,,$$

for each non-negative $f \in C_c(\mathbb{R}^{d+1})$. Using part (a), we have

$$\mathbb{E}\Big\{\Big|e^{-\int f\, d\xi_L} - e^{-\sum_p \int f\, d\eta_{L,p}}\Big|\Big\} \le \mathbb{E}\Big\{\Big|\int f\, d\xi_L - \sum_p \int f\, d\eta_{L,p}\Big|\Big\} = o(1)\,,$$

because $|e^{-x} - e^{-y}| \le |x-y|$ for $x, y \ge 0$. □

From the corollary, we see that $d\xi_L$ can be approximated by a sum of independent point processes. The next proposition provides an important bound on the summands. The statement and proof are taken more or less directly from [7]; it is the most ingenious part of that paper. It is worth noting that while the previous proposition is directly contingent on localization, the next is not – it holds even in the regime where delocalization is expected.

Proposition 2.2. *For each finite interval I,*

$$\mathbb{E}\big\{\eta_{p,L}(I \times \mathbb{R}^d)\big[\eta_{p,L}(I\times\mathbb{R}^d) - 1\big]\big\} = O\left(\frac{l_L^{2d}}{L^{2d}}\right) \tag{2.11}$$

as $L \to \infty$.

Proof. Recall that $d\eta_{L,p}$ is a measure on space and energy, however the statement concerns only its energy marginal. After integrating out the spacial variables, we see that $d\eta_{L,p}$ consists of a unit point mass at each $(E_j - E_0)L^d$ where E_j enumerate the eigenvalues of the finite-volume operator $H_{L,p}$. In particular, if we write $I = [a,b]$, then $\eta_{p,L}(I \times \mathbb{R}^d)$ is the number of eigenvalues of $H_{L,p}$ in the interval

$$I_L = [E_0 + aL^{-d}, E_0 + bL^{-d}]\,.$$

As noted the proof of (2.11) can be found in Minami's paper. In scant detail his argument is as follows: By a two-site spectral averaging argument, [7, Lemma 2], he proves

$$\mathbb{E}\big\{\mathrm{Tr}\big(f(H_{L,p}) \wedge f(H_{L,p})\big)\big\} \le \|\rho\|_\infty^2\, l_L^{2d}$$

for all functions f of the form

$$f(E) = \frac{1}{\pi}\,\frac{\tau}{(E-\sigma)^2 + \tau^2}$$

with $\sigma \in \mathbb{R}$ and $\tau > 0$. (Recall that $A \wedge A$ is the restriction of $A \otimes A$ to antisymmetric two-tensors.) If we choose $\sigma = E_0$ and $\tau = (|a|+|b|)L^{-d}$, then $2\pi\tau f(E) \ge \chi_{I_L}(E)$. Therefore,

$$\begin{aligned} 2 \times \mathrm{LHS}(2.11) &= \mathbb{E}\{\text{number of unordered pairs of eigenvalues of } H_{L,p} \text{ in } I_L\} \\ &= \mathbb{E}\big\{\mathrm{Tr}\big(\chi_{I_L}(H_{L,p}) \wedge \chi_{I_L}(H_{L,p})\big)\big\} \\ &\le 4\pi^2(|a|+|b|)^2\, L^{-2d}\|\rho\|_\infty^2\, l_L^{2d}\,, \end{aligned}$$

which proves (2.11). □

Proof of Theorem 1.1. By Corollary 2.1, it suffices to show that

$$\sum_p \eta_{L,p}$$

converges to a Poisson process. Notice that each summand is independent and identically distributed; moreover as we will prove momentarily, each makes negligible contribution. This is the natural setting for a central limit theorem. The specific version we need is Corollary 7.5 of [5]. It says that it is sufficient to show that for each compact rectangle $I \times Q \subset \mathbb{R} \times \mathbb{R}^d$ with sides parallel to the axes,

$$\sum_p \mathbb{P}\big\{\eta_{p,L}(I \times Q) \geq 1\big\} \to |I| \cdot |Q| \cdot \frac{d\nu}{dE}(E_0) \tag{2.12}$$

and

$$\sum_p \mathbb{P}\big\{\eta_{p,L}(I \times Q) \geq 2\big\} \to 0 \tag{2.13}$$

as $L \to \infty$.

The second requirement follows easily from Minami's estimate, (2.11):

$$\begin{aligned}\sum_p \mathbb{P}\big\{\eta_{p,L}(I \times Q) \geq 2\big\} &\leq \sum_{p\,:\,C_p(L)\cap Q\neq\emptyset} \mathbb{E}\Big\{\eta_{p,L}(I \times \mathbb{R}^d)\big[\eta_{p,L}(I \times \mathbb{R}^d) - 1\big]\Big\} \\ &= O\left(\frac{L^d}{l_L^d} \cdot \frac{l_L^{2d}}{L^{2d}}\right),\end{aligned}$$

which is considerably stronger than (2.13).

In view of the formula immediately above, (2.12) will follow once we prove

$$\sum_p \mathbb{E}\big\{\eta_{p,L}(I \times Q)\big\} \to |I| \cdot |Q| \cdot \frac{d\nu}{dE}(E_0)\,,$$

which can itself be deduced from

$$\mathbb{E}\big\{\xi_L(I \times Q)\big\} \to |I| \cdot |Q| \cdot \frac{d\nu}{dE}(E_0)\,, \tag{2.14}$$

by means of part (b) of Corollary 2.1. The proof of (2.14) requires just two ingredients. First, by (1.2),

$$L^d \nu\big([E_0 + aL^{-d}, E_0 + bL^{-d}]\big) \to (b-a) \cdot \frac{d\nu}{dE}(E_0)\,. \tag{2.15}$$

And secondly, by the definition of the density of states,

$$\mathbb{E}\big\{\xi_L\big([a,b] \times Q\big)\big\} = N_L(Q)\nu\big([E_0 + aL^{-d}, E_0 + bL^{-d}]\big)\,, \tag{2.16}$$

where $N_L(Q) = \#\{x \in \mathbb{Z}^d : L^{-1}x \in Q\}$. It is easy to see that

$$N_L(Q) = |Q| \cdot L^d + O(L^{d-1})\,.$$

Combining this with (2.15) and (2.16) proves (2.14) and so the theorem. $\square$

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