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Non-Commutative Ricci and Calabi Flows

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Dedicated to the memory of Daniel Arnaudon

Abstract. Starting from an improved version of the bicomplex structure associated the continual Lie algebra with non-commutative base algebra, we obtain dynamical systems resulting from the bicomplex conditions. General expressions for conserved currents associated to a continual Lie algebra bicomplex are found explicitly in two first orders. The Moyal-product counterparts for two-dimensional Ricci and Calabi flow equations depending on non-commutative variables are introduced.

1. Introduction

Having appeared a few years ego [18, 5, 10, 8, 7], non-commutative deformations of low-dimensional integrable models remain in the focus of investigations aiming at possible non-commutative generalization of solvability in the field theory. One way to treat solvable models in this direction, is to introduce non-commutative variables, and study resulting powerful algebraic structures underlying symmetries of a dynamical system. Though deep relations to similar attempts arising from consideration of integrable models occurring in non-commutative geometry are still quite vague, certain progress has been made.

In this paper, we continue to use quite attractive properties of continual Lie algebras [16] that proved to be a very convenient tool in formulation of new types of completely integrable and exactly solvable models, and turned to be behind algebraic symmetries of group-theoretical methods to various dynamical models. Being combined with the bicomplex approach, continual Lie algebras help us to avoid technical problems in construction of non-commutative deformations of integrable models, which could not be archived in frames of usual (with discrete sets of roots) Kac–Moody Lie algebras. We introduce a new version of the bicomplex structure on continual Lie algebras, which, in contrast to previous works [9, 20], can depend in an arbitrary way on two (possibly non-commuting) variables. Starting with this structure we obtain a dynamical system described by equations following from the bicomplex conditions on corresponding bicomplex mappings. The dynamical system introduced incorporates both a large class of known models which could be obtained from the bicomplex approach applied to discrete Kac–Moody Lie algebras, and certain new equations which appears in consideration of particular examples of continual Lie algebras [20].

Bicomplex structures turned to be a natural object in construction of the Moyal product [4] non-commutative deformations of integrable theories [7, 8]. General bicomplex approach to the construction of conserved currents works also in non-commutative case, and helps to prove classical integrability of a model [8]. Starting from the bicomplex associated to continual Lie algebras, we introduce the Moyal product deformed models that can be considered as non-commutative analogues of the two-dimensional Ricci and Calabi flow equations.

2. Bicomplexes associated to continual Lie algebras

In this paper we introduce an improved version of the continual Lie algebra bicomplex which was used in [20] in order to construct corresponding Toda-type models (see also [9], the case of conformal affine Toda models associated to affine Kac–Moody Lie algebras). In contrast to the original version [20], this modification have weaker restriction on the dependence of base algebra elements parametrizing continual Lie algebra elements. In addition to that, this new construction allows as to overcome problems in finding non-commutative generalization to certain continual Lie algebras.

Consider a continual Lie algebra with a base algebra E endowed with a non-commutative product $*$, with elements $X_\pm$, X_0 (see appendix for details). Elements of the algebra E can be parametrized by a set of parameters, and, in particular, we can choose certain among them a subset of non-commutative variables which play the role of coordinates $z^\pm$ on a non-commutative (two-dimensional) space M. Let $V = \oplus_{r\geq 0} V^r$, be a graded linear space over $\mathbb{C}$. Suppose that ξ^i, $i = 1, 2$, form a basis of V^1, subject to the relations $\xi^1\xi^2 = \xi^2\xi^2 = \xi^1\xi^2 + \xi^2\xi^1 = 0$. Let $v = v_1\ \xi^1 + v_2\ \xi^2 \in V^1$, be an arbitrary element. We start from the following mappings δ, $d\colon V^r \to V^{r+1}$,

$$\begin{aligned} \delta v &= \partial_- v\ \xi^1 - v\ X_-(w_-)\ \xi^2, \\ dv &= vX_+(w_+)\ \xi^1 + (\partial_+ v - v\ X_0(g_*^{-1} * \partial_+ g))\ \xi^2, \end{aligned} \tag{2.1}$$

where $w_\pm = w_\pm(z^\pm, \mathcal{T})$ are arbitrary elements of the algebra E, $g = g(z^+, z^-, \mathcal{T})$ are elements of a group algebra of E, g_*^{-1} denotes the (left- or right-) $*$-inverse to g, and $v = v(z^\pm, \mathcal{T})$ belongs to an appropriate left module of E. All these elements are assumed to depend on non-commutative variables $z^\pm$, and, possibly, on a set of extra parameters $\mathcal{T}$ which commute with $z^\pm$ and with each other. Applying the bicomplex conditions

$$\delta^2 = d^2 = \delta d + d\delta = 0, \tag{2.2}$$

on the mappings the δ, d, we obtain (omitting the $*$ product)

$$\delta^2 v = [\, \partial_-(v\ X_-(w_-)) - (\partial_- v)\ X_-(w_-)]\ \xi^1\ \xi^2 = 0, \tag{2.3}$$

$$\begin{aligned} d^2 v = \big[&-\partial_+ v + v\ X_0(g^{-1}\ \partial_+ g))\ X_+(w_+) \\ &+ (\partial_+(vX_+(w_+)) - v\ X_+(w_+)\ X_0(g^{-1}\ \partial_+ g))\big]\ \xi^1\ \xi^2 = 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &(d\,\delta + \delta\, d)\ v = \\ &\big[v\ X_-(w_-)\ X_+(w_+) + (\partial_+(\partial_- v) - (\partial_- v)\ X_0(g^{-1}\ \partial_+ g)) \\ &+ \partial_-(-\partial_+ v + v\ X_0(g^{-1}\ \partial_+ g)) - v\ X_+(w_+) X_-(w_-)\big]\ \xi^1\ \xi^2 = 0. \end{aligned} \tag{2.5}$$

Suppose that a non-commutative multiplication in the base algebra E of our continual Lie algebra is defined in such a way that the derivatives $\partial_\pm$ (with respect to non-commutative variables $z_\pm$) satisfy Leibniz rule. Then the conditions (2.3)–(2.5) can be rewritten as

$$\delta^2 v = (v\ X_-(\partial_- w_-))\ \xi^1\ \xi^2 = 0, \tag{2.6}$$

$$d^2 v = \big[v\ \big[X_0(g^{-1}\partial_+ g)), X_+(w_+)\big] + v\ X_+(\partial_+ w_+)\big]\ \xi^1\ \xi^2 = 0, \tag{2.7}$$

$$(d\,\delta + \delta\, d)\ v = \big[v\ [X_-(w_-), X_+(w_+)] + v\ X_0(g^{-1}\ \partial_+ g)\big]\,\xi^1\ \xi^2 = 0. \tag{2.8}$$

Using the commutation relations (5.1) for the underlying continual Lie algebra, we obtain from the conditions (2.6)–(2.8) the following system of equations for an arbitrary element v

$$\partial_- w_- = 0, \tag{2.9}$$

$$\partial_+ w_+ + K_+(g_*^{-1} * \partial_+ g, w_+) = 0, \tag{2.10}$$

$$\partial_-(g_*^{-1} * \partial_+ g) - K_0(w_+, w_-) = 0. \tag{2.11}$$

The system of equations (2.9)–(2.11) represents an improved version of a dynamical system constructed in a continual Lie algebra bicomplex approach (which includes a dependence of elements of a continual Lie algebra base algebra on non-commutative dynamical variables $z^\pm$), and generalizes dynamical systems introduced in [20], where interesting dynamical systems associated to various examples of continual Lie algebras and resulting from (2.9)–(2.11) can be found. In a discrete commutative limit of E [16], a continual Lie algebra used in the bicomplex (2.1) represents a Kac–Moody Lie algebra, the system (2.9)–(2.11) delivers corresponding examples of a non-abelian or usual affine Toda models [14].

3. Non-commutative Ricci and Calabi flows

Two dimensional Ricci and Calabi flow equations [1]–[3] can be obtained from the system (2.9)–(2.11) originating from the bicomplex (2.1) in a specific choice of the continual Lie algebra. In this section we show how to introduce modifications of these equations on fields depending on non-commutative variables.

3.1. Ricci flow

As it was discovered in [3], that the two-dimensional Ricci flow equation can be associated to a particular example of continual Lie algebra. In our bicomplex approach, consider a continual Lie algebra defined by the mappings

$$K_0(\phi,\psi) = \partial_t(\phi\cdot\psi), \qquad K_\pm(\phi,\psi) = \mp\phi\cdot\psi, \qquad K_{0\,0}(\phi,\psi) = 0, \tag{3.1}$$

(t denotes an extra real parameter) with the usual commutative product $\cdot$. If we identify w_+ with g, then the equation (2.10) trivializes. With the choice $g = w_+ = e^\phi$, $\phi(z^+, z^-; t)$, the system (2.9)–(2.11) results in

$$\partial_-\partial_+\phi = \partial_t(w_-(z^+;t)\, e^\phi), \qquad \partial_- w_- = 0. \tag{3.2}$$

The first equation is the commutative two-dimensional Ricci flow equation

$$\partial_-\partial_+\phi = w_-\ \partial_t e^\phi, \tag{3.3}$$

when w_- is a constant.

There exist a few ways how one can define a non-commutative counterpart to a two-dimensional integrable model [7, 8, 10, 22]. The main problem in this direction consists in the ambiguity of an interpretation of parts of new equations containing derivatives of the field with respect to non-commutative variables [8]. Since in the commutative case the equation (3.3) can be associated to a continual Lie algebra defined by the mappings (3.1), a natural idea in construction of a non-commutative version of the equation (3.3) would be to find an analog of the continual Lie algebra defined by the mappings (3.1) over a non-commutative algebra, yet this turns to be a non-trivial problem. A naive substitution of the usual multiplication in the definition (3.1) of the mappings K_0, K_+ by a non-commutative one, fails to satisfy the conditions (5.2) which follow from Jacobi identities. Nevertheless, this can be done in principle in a quite indirect way [21]. Instead, in this paper, in order to introduce non-commutative counterparts of two-dimensional Ricci and Calabi flow equations, we use the same continual Lie algebra mappings (3.1), though allowing a $*$-product non-commutativity inside the structure of elements of the algebra E parametrizing elements of the continual Lie algebra.

We endow the algebra E of a continual Lie algebra used in the bicomplex (2.1) with two (not necessary associative with respect to each other) multiplications, the usual commutative $\cdot$, and a non-commutative $*$-products. Then define the mappings determining this continual Lie algebra as in (3.1). These mappings keep the usual commutative product $\cdot$, though elements ϕ, ψ can be $*$-products of some other elements of E. In this paper we consider systems depending on non-commutative variables $z^\pm$ subject to the commutation relation $[z^+, z^-] = i\,\theta$, and, as a non-commutative $*$-multiplication, we take the well-known Moyal product [4] defined by the formula

$$f_1(z^+,z^-) * f_2(z^+,z^-) = m\ \circ\ e^P\ \left[f_1(\xi^+,\xi^-)\otimes f_2(\xi^+,\xi^-)\right]\big|_{z^\pm=\xi^\pm}\,, \tag{3.4}$$

where $P = 1/2\ \theta^{\mu\nu}\ \partial^{(z)}_{+\mu}\otimes\partial^{(\xi)}_{-\nu}$, μ, $\nu = 1,\ldots,N$, and m is the mapping $m\colon f\otimes g \to f\cdot g$. Notice that usual derivatives $\partial_\pm$ are with respect to this product, and Leibniz rule can be used. Thus starting from the bicomplex mappings (2.1) subject to the bicomplex conditions (2.2), we deduce the system (2.9)–(2.11). Then substitute to (2.9)–(2.11) the definitions (3.1) of the mappings K_0, K_+. These mappings act on the elements $w_- = \mathrm{const}$, $g = w_+ = e_*^{\phi} = \sum \frac{1}{n!}\, \phi_*^n$, $\phi_*^n = (\phi * \cdots * \phi)_{n\text{-times}}$, and $g_*^{-1} * \partial_+ g$, which contain the Moyal product in the their structure, though commute to all other elements of the algebra E with respect to the usual multiplication $\cdot$. Implementing then the property of the Moyal product $(e_*^{\phi})_*^{-1} = e_*^{-\phi}$, we obtain from (2.9)–(2.11) the following dynamical system,

$$\partial_-(e_*^{-\phi} * \partial_+ e_*^{\phi}) = \partial_t\, e_*^{\phi} \cdot w_-, \tag{3.5}$$

$$\partial_+ e_*^{\phi} = (e_*^{-\phi} * \partial_+ e_*^{\phi}) \cdot e_*^{\phi}, \tag{3.6}$$

on a field $\phi(z^+, z^-; t)$ depending on non-commutative variables $z^{\pm}$, and a real commutative parameter t. We call this system the non-commutative counterpart of two-dimensional Ricci flow model. The presence of the equation (3.6) which represents a constraint on the main field ϕ reflects a quite usual situation in non-commutative theories introduced with the help of the bicomplex structure [8].

3.2. Calabi flow

As it was shown in [2], the two-dimensional Calabi flow equation

$$-\partial_+\partial_-(e^{-\varphi}\partial_+\partial_-\varphi) = \partial_t e^{\varphi} \tag{3.7}$$

can be written in a form similar to that of the Ricci flow equation. Introducing an anticommutative variable κ, with $\kappa^2 = 0$, a supersymmetric partner to the parameter t, one can consider the equation

$$\partial_+\partial_- \mathcal{F} = \mathcal{D}_T e^{\mathcal{F}}, \tag{3.8}$$

on a mixed superfield $\mathcal{F}(z^+, z^-; t) = \Phi(z^+, z^-; t) + \kappa\Psi(z^+, z^-; t)$, with bosonic components, Φ, Ψ, where $\mathcal{D}_T = \partial_\kappa - \kappa\partial_t$, $\mathcal{D}_T^2 = -\partial_t$, is the superderivative in the superspace with coordinates $T = (t, \kappa)$. The equation (3.8) is a super analog of the Ricci equation with respect to supertime T. Expanding $e^{\mathcal{F}} = (1 + \kappa\Psi)e^{\Phi}$, and substituting to (3.8), one obtains

$$\partial_+\partial_-\Phi = e^{\Phi}\Psi, \qquad -\partial_+\partial_-\Psi = \partial_t e^{\Phi}, \tag{3.9}$$

which is equivalent to (3.7) with $\varphi = \Phi$, when expressing Ψ through Φ, [2]. Therefore we can interprete the two-dimensional Calabi flow equation as a Toda field equation associated with a supercontinual Lie algebra with the same form of commutative relations (5.1), and the mappings

$$K_0(\phi, \psi) = \mathcal{D}_T(\phi \cdot \psi), \qquad K_{\pm}(\phi, \psi) = \mp\phi \cdot \psi, \qquad K_{0\,0}(\phi, \psi) = 0, \tag{3.10}$$

on superfield arguments ϕ, ψ. A mathematically rigorous definition and studies of properties of continual super Lie algebras are planed to be presented in a forthcoming paper [23]. In the next section we discuss the derivation of conserved quantities associated to the bicomplex (2.1). Since the Calabi flow can be represented in the form of the the Ricci flow equation (3.3) on a superfield, and this equation arises from the same bicomplex structure (2.1) with a supercontinual Lie algebra defined by (3.10), we see that the structure of conserved quantities to the Calabi flow equation has to be very close to that of the Ricci flow, up to an obvious change of notations in (2.9)–(2.11).

The methods of non-commutative interpretation of the Calabi flow equation follows exactly the same arguments as in the Ricci flow case, and we call

$$\begin{aligned} \partial_-(e_*^{-\mathcal{F}} * \partial_+ e_*^{\mathcal{F}}) &= \mathcal{D}_T(e_*^{\mathcal{F}} \cdot w_-), \\ \partial_+ e_*^{\mathcal{F}} &= (e_*^{-\mathcal{F}} * \partial_+ e_*^{\mathcal{F}}) \cdot e_*^{\mathcal{F}}, \end{aligned} \tag{3.11}$$

a non-commutative counterpart to the Calabi flow equation.

4. Bicomplex conservation laws

In commutative case, the existence of the bicomplex formulation for the Ricci and Calabi flow equations guarantees [6, 7] the existence of an infinite chain of currents. The bicomplex structure (2.1) associated to a continual Lie algebra with possibly non-commutative algebra E, allows as to define an chain of equations on the infinite number of quantities which are used in construction of conserved currents. In this section we show how to apply this bicomplex procedure and find conserved currents in particular case of the equations (2.9)–(2.11).

Let us remind briefly the idea of the construction of a set of conserved quantities associated to a system of equations dictated by the bicomplex conditions on mappings d, δ [6]. Assume that for some $r \in \mathbf{N}$, there exists a (non-vanishing) element $\chi^{(0)} \in V^{r-1}$, such that $dJ^{(0)} = 0$, $J^{(0)} = \delta\chi^{(0)}$. Then define $J^{(1)} = d\chi^{(0)}$. From the bicomplex definition it follows that $\delta J^{(1)} = -d\delta\chi^{(0)} = 0$. If δ-closed element $J^{(1)}$ is δ-exact, then there exists some $\chi^{(1)} \in V^{r-1}$, such that $J^{(1)} = \delta\chi^{(1)}$. Define $J^{(2)} = d\chi^{(1)}$. Then $\delta J^{(2)} = -d\delta\chi^{(1)} = -dJ^{(1)} = -d^2\chi^{(0)} = 0$. This procedure leads to a (possibly infinite) chain of elements $\chi^{(n)} \in V^{r-1}$, and conserved quantities $J^{(n)} \in V^r$, $J^{(n+1)} = d\chi^{(n)} = \delta\,\chi^{(n+1)}$, $dJ^{(n+1)} = 0$, $\delta J^{(n+1)} = 0$, provided that the cohomology $H^r_\delta(V) = 0$. The relation

$$\delta(\chi - \chi^{(0)}) = \lambda\, d\chi, \tag{4.1}$$

where λ is a complex parameter, is called the linear equation associated to the bicomplex. With no loss of generality we can put $\delta\chi^{(0)} = 0$.

In the case of the bicomplex (2.1), one obtains from (4.1)

$$\partial_-\chi = \lambda\chi X_+(w_+), \qquad -\chi X_-(w_-) = \lambda(\partial_+\chi - \chi X_0(g^{-1}\partial_+ g)). \tag{4.2}$$

Introduce two currents $j_\pm = \chi^{-1}\partial_\pm\chi$. Thus from (4.2) we find

$$j_- = \lambda X_+(g), \qquad \lambda j_+ = \lambda X_0(g^{-1}\partial_+ g) - X_-(w_-). \tag{4.3}$$

Notice that the identity $\lambda^{-1}\partial_+ j_- - \lambda\partial_- j_+ + [j_+, j_-] = 0$, obviously satisfied by the currents (4.3), implies $\mathrm{tr}(\lambda^{-1}\partial_+ j_-) = \mathrm{tr}(\lambda\partial_- j_+)$, which is in a form of a conservation law. In the last equality we take traces of the matrices expressed in terms elements of a matrix representation of a continual Lie algebra used in the bicomplex (2.1). Differentiating the second equation in (4.2) with respect to z^+, and using (4.3), we find

$$
\begin{aligned}
&- j_+ X_-(w_-) - X_-(\partial_+ w_-) = \\
&\qquad \lambda \left(j_+^2 + \partial_+ j_+ - j_+ X_0(g^{-1}\partial_+ g) - X_0(\partial_+(g^{-1}\partial_+ g)) \right). \qquad (4.4)
\end{aligned}
$$

Now expand $j_\pm = \sum_{n\geq 0} \lambda^n j_\pm^{(n)}$, in terms of λ. Then from the equation (4.4) one obtains an infinite chain of recursive equations

$$
\begin{aligned}
&- j_+^{(n)} X_-(w_-) - \delta_{n,1} X_-(\partial_+ w_-) = \\
&\sum_{k\geq 0} j_+^{(k)} j_+^{(n-k-1)} + \partial_+ j_+^{(n-1)} - j_+^{(n-1)} X_0(g^{-1}\partial_+ g) - \delta_{n,1} X_0(\partial_+(g^{-1}\partial_+ g)). \qquad (4.5)
\end{aligned}
$$

which can be easily solved for an infinite number of $j_+^{(n)}$,$n \in \mathbb{Z}_+$. Then $\mathrm{tr}(j_+^{(n)})$, $n \in \mathbb{N}$, represent conserved quantities associated to the equations (2.9)–(2.11). In particular, for $n = 0, 1, 2$, one obtains

$$
j_+^{(0)} X_-(w_-) = -X_-(\partial_+ w_-), \qquad (4.6)
$$

$$
-j_+^{(1)} X_-(w_-) = \left(j_+^{(0)} \right)^2 + \partial_+ j_+^{(0)} - j_+^{(0)} X_0(g^{-1}\partial_+ g) - X_0(\partial_+(g^{-1}\partial_+ g)), \qquad (4.7)
$$

$$
-j_+^{(2)} X_-(w_-) = j_+^{(0)} j_+^{(1)} + j_+^{(1)} j_+^{(0)} + \partial_+ j_+^{(1)} - j_+^{(1)} X_0(g^{-1}\partial_+ g). \qquad (4.8)
$$

Solving the chain of equations (4.7), (4.8),... recurrently, we find

$$
\begin{aligned}
&j_+^{(1)} X_-(w_-) X_-(w_-) = \\
&\qquad 2 j_+^{(0)} X_-(\partial_+ w_-) + X_-(\partial_+^2 w_-) - X_-(\partial_+ w_-) X_0(g^{-1}\partial_+ g) \\
&\qquad\qquad + j_+^{(0)} X_-((g^{-1}\partial_+ g) \cdot w_-) + X_0(\partial_+(g^{-1}\partial_+ g)) X_-(w_-), \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
&j_+^{(2)} X_-(w_-) X_-(w_-) X_-(w_-) = 2 \left(j_+^{(0)} \right)^2 X_-(\partial_+ w_-) + j_+^{(0)} X_-(\partial_+^2 w_-) \\
&- j_+^{(0)} X_-(\partial_+ w_-) X_0(g^{-1}\partial_+ g) + \left(j_+^{(0)} \right)^2 X_-(g^{-1}\partial_+ g \cdot w_-) \\
&- X_-(\partial_+ w_-) X_0(\partial_+(g^{-1}\partial_+ g)) + j_+^{(0)} X_-(\partial_+(g^{-1}\partial_+ g) \cdot w_-) \\
&+ \left[\left(j_+^{(0)} \right)^2 + \partial_+ j_+^{(0)} - j_+^{(0)} X_0(g^{-1}\partial_+ g) - X_0(\partial_+(g^{-1}\partial_+ g)) \right] X_-(\partial_+ w_-) \\
&+ \Big[2\partial_+ j_+^{(0)} X_-(\partial_+ w_-) + 2\partial_+ j_+^{(0)} X_-(\partial_+^2 w_-) + X_-(\partial_+^3 w_-) \\
&- X_-(\partial_+^2 w_-) X_0(g^{-1}\partial_+ g) - X_-(\partial_+ w_-) X_0(\partial_+(g^{-1}\partial_+ g)) \\
&+ \partial_+ j_+^{(0)} X_-(g^{-1}\partial_+ g \cdot w_-) \partial_+ j_+^{(0)} X_-(\partial_+(g^{-1}\partial_+ g \cdot w_-)) \\
&+ X_0(\partial_+^2(g^{-1}\partial_+ g)) X_-(w_-) + X_0(\partial_+(g^{-1}\partial_+ g)) X_-(\partial_+ w_-) \Big] +
\end{aligned}
$$

$$
\begin{aligned}
&+2\Big[\Big(j_+^{(0)}\Big)^2+\partial_+ j_+^{(0)}-j_+^{(0)}X_0(g^{-1}\partial_+ g)-X_0(\partial_+(g^{-1}\partial_+ g))\Big]X_-(\partial_+ w_-)\\
&-\Big[2j_+^{(0)}X_-(\partial_+ w_-)+X_-(\partial_+^1 w_-)-X_-(\partial_+ w_-)X_0(g^{-1}\partial_+ g)\\
&+j_+^{(0)}X_-(g^{-1}\partial_+ g\cdot w_-)+X_0(\partial_+(g^{-1}\partial_+ g))X_-(w_-)\Big]X_0(g^{-1}\partial_+ g)\\
&+2\Big[\Big(j_+^{(0)}\Big)^2+\partial_+ j_+^{(0)}-j_+^{(0)}X_0(g^{-1}\partial_+ g)-X_0(\partial_+(g^{-1}\partial_+ g))\Big]X_0(g^{-1}\partial_+ g\cdot w_-).
\end{aligned}
\tag{4.10}
$$

Now commute j_- with $X_-(w_-)$, and compare the result with the derivative of $\partial_- j_+$ using the equations (2.9)–(2.11). One obtains the relation $\lambda\partial_- j_+ = [j_-, X_-(w_-)]$, which allows to find $j_- = \sum_{n\geq 0}\lambda^n j_-^{(0)}$, as a series in λ. Then each $j_\pm^{(n)}$, $n \in \mathbb{Z}_+$ can be expressed as elements of a matrix representation of corresponding continual Lie algebra, and the traces of currents $\mathrm{tr}(\lambda^{-1}\partial_+ j_-) = \mathrm{tr}(\lambda\partial_- j_+)$, are conserved. Thus for a dynamical system resulting from a bicomplex of the form (2.1) we find a procedure of conserved quantity derivation.

5. Conclusions

We proposed a new form of the bicomplex structure which essentially uses properties of continual Lie algebras and, under bicomplex conditions, leads to a dynamical system (2.9)–(2.11). The main useful feature of the bicomplex approach is a possibility to find an infinite number of conserved quantities for corresponding dynamical system. Properties of the resulting system crucially depend on properties of mappings of particular continual Lie algebra that can be used in the construction. The bicomplex structure we introduce is a very convenient tool when defining analogues of integrable models depending on non-commutative variables. Though in some cases it is quite difficult to find an appropriate continual Lie algebra which would have a non-commutative base algebra, in the frames of our approach one can still use usual continual Lie algebra mappings for the definition on non-commutative models. In this paper we apply this construction to two examples, the two-dimensional Ricci and Calabi flows. Further applications of group-theoretical methods [11] in models which have internal symmetries related to properties of continual Lie algebras, should definitely include implementation of vertex operator constructions [13, 21, 19], as well as generalization of algebraic constructions [2] of exact special solutions both in commutative and non-commutative cases.

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Appendix: Continual Lie algebras

Continual Lie algebras were introduced in [15] and then studied in [16, 17]. Suppose E is an associative algebra (which we call the *base algebra*) over the field $\mathbb{R}$ or $\mathbb{C}$, and K_0, $K_\pm$, $K_{0,0}\colon E \times E \to E$ are bilinear mappings. The local Lie part of a continual Lie algebra is defined as $\hat{g} = g_{-1} \oplus g_0 \oplus g_{+1}$, where g_i, $i = 0, \pm 1$, are isomorphic to E and parametrized by its elements. The subspaces g_i consist of the elements $\{X_i(\phi), \phi \in E\}$, $i = 0, \pm 1$. The generators $X_i(\phi)$ are subject to the commutation relations

$$\begin{aligned} [X_0(\phi), X_0(\psi)] &= X_0(K_{0,0}(\phi, \psi)), \\ [X_0(\phi), X_{\pm 1}(\psi)] &= X_{\pm 1}(K_\pm(\phi, \psi)), \\ [X_{+1}(\phi), X_{-1}(\psi)] &= X_0(K_0(\phi, \psi)), \end{aligned} \tag{5.1}$$

for all ϕ, $\psi \in E$. It is also assumed that the Jacobi identities are satisfied which lead to the conditions on mappings $K_{0,0}$, $K_{0,\,\pm}$,

$$\begin{aligned} K_\pm(K_{0,0}(\phi, \psi), \chi) &= K_\pm(\phi, K_\pm(\psi, \chi)) - K_\pm(\psi, K_\pm(\phi, \chi)), \\ K_{0,0}(\psi, K_0(\phi, \chi)) &= K_0(K_+(\psi, \phi), \chi)) + K_0(\phi, K_-(\psi, \chi)), \end{aligned} \tag{5.2}$$

for all ϕ, ψ, $\chi \in E$. Then an infinite dimensional algebra $g(E;K) = g'(E;K)/J$ is called a *continual contragredient Lie algebra* where $g'(E;K)$ is a Lie algebra freely generated by $\hat{g}$, and J is the largest homogeneous ideal with trivial intersection with g_0 (consideration of the quotient is equivalent to imposing the Serre relations in an ordinary Lie algebra case) [16, 17]. More general definition of a continual Lie algebra can be found in [12].

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