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The Schrödinger-Virasoro Lie Group and Algebra: Representation Theory and Cohomological Study

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Abstract. This article is devoted to an extensive study of an infinite-dimensional Lie algebra $\mathfrak{sv}$, introduced in [14] in the context of non-equilibrium statistical physics, containing as subalgebras both the Lie algebra of invariance of the free Schrödinger equation and the central charge-free Virasoro algebra $\mathrm{Vect}(S^1)$. We call $\mathfrak{sv}$ the *Schrödinger-Virasoro Lie algebra*. We study its representation theory: realizations as Lie symmetries of field equations, coadjoint representation, coinduced representations in connection with Cartan's prolongation method (yielding analogues of the tensor density modules for $\mathrm{Vect}(S^1)$). We also present a detailed cohomological study, providing in particular a classification of deformations and central extensions; there appears a non-local cocycle.

0. Introduction

There is, in the physical literature of the past decades – without mentioning the pioneering works of Wigner for instance –, a deeply rooted belief that physical systems – macroscopic systems for statistical physicists, quantum particles and fields for high energy physicists – could and should be classified according to which group of symmetries acts on them and how this group acts on them.

Let us just point at two very well-known examples: elementary particles on the (3+1)-dimensional Minkowski space-time, and two-dimensional conformal field theory.

From the point of view of 'covariant quantization', introduced at the time of Wigner, elementary particles of relativistic quantum mechanics (of positive mass, say) may be described as irreducible unitary representations of the Poincaré group

$\mathfrak{p}_4 \simeq \mathfrak{so}(3,1) \ltimes \mathbb{R}^4$, which is the semi-direct product of the Lorentz group of rotations and relativistic boosts by space-time translations: that is to say, the physical states of a particle of mass $m > 0$ and spin $s \in \frac{1}{2}\mathbb{N}$ are in bijection with the states of the Hilbert space corresponding to the associated irreducible representation of $\mathfrak{p}_4$; the indices (m, s) characterizing positive square mass representations come from the two Casimir of the enveloping algebra $\mathcal{U}(\mathfrak{p}_4)$.

This 'covariant quantization' has been revisited by the school of Souriau in the 60'es and 70'es as a particular case of geometric quantization; most importantly for us, the physicists J.-J. Lévy-Leblond and C. Duval introduced the so-called *Newton-Cartan* manifolds (which provide the right geometric frame for Newtonian mechanics, just as Lorentz manifolds do for relativistic mechanics) and applied the tools of geometric quantization to construct wave equations in a geometric context.

Two-dimensional conformal field theory is an attempt at understanding the universal behaviour of two-dimensional statistical systems at equilibrium and at the critical temperature, where they undergo a second-order phase transition. Starting from the basic assumption of translational and rotational invariance, together with the fundamental hypothesis (confirmed by the observation of the fractal structure of the systems and the existence of long-range correlations, and made into a cornerstone of renormalization-group theory) that scale invariance holds at criticality, one is[1] naturally led to the idea that invariance under the whole conformal group $\mathrm{Conf}(d)$ should also hold. This group is known to be finite-dimensional as soon as the space dimension d is larger than or equal to three, so physicists became very interested in dimension $d = 2$, where local conformal transformations are given by holomorphic or anti-holomorphic functions. A systematic investigation of the theory of representations of the Virasoro algebra (considered as a central extension of the algebra of infinitesimal holomorphic transformations) in the 80'es led to introduce a class of physical models (called *unitary minimal models*), corresponding to the unitary highest weight representations of the Virasoro algebra with central charge less than one. Miraculously, covariance alone is enough to allow the computation of the statistic correlators – or so-called 'n-point functions' – for these highly constrained models.

A systematic investigation of the consequences of Lie symmetries has been conducted since the mid-nineties (see short survey [17]) in the same spirit in two related fields: strongly anisotropic critical systems and out-of-equilibrium statistical physics (notably ageing phenomena). Theoretical studies and numerical models coming from both fields have been developed, in which invariance under space rotations and *anisotropic dilations* $(t, r) \to (e^{\lambda z} t, e^{\lambda} r)$ $(\lambda \in \mathbb{R})$ plays a central rôle. Here $r \in \mathbb{R}^d$ is considered as a space coordinate and $t \in \mathbb{R}$ is (depending on the context) either the time coordinate or an extra (longitudinal, say) space coordinate; the parameter $z \neq 1$ is called the *anisotropy* or *dynamical exponent*.

Let us restrict to the value $z = 2$. Then the simplest wave equation invariant under translations, rotations *and* anisotropic dilations is the free Schrödinger

[1]for systems with sufficiently short-ranged interactions

equation $2\mathcal{M}\partial_t\psi = \Delta_d\psi$, where $\Delta_d := \sum_{i=1}^d \partial_{r_i}^2$ is the Laplacian in spatial coordinates. So it is natural to believe that this equation should play the same rôle as the Klein-Gordon equation in the study of relativistic quantum particles, or the Laplace equation in conformal field theory, whose maximal group of Lie symmetries is the conformal group; in other words, one may also say that one is looking for symmetry groups arising naturally in a non-relativistic setting, while hoping that their representations might be applied to a classification of non-relativistic systems, or, more or less equivalently, to $(z = 2)$ anisotropic systems.

This program, as we mentioned earlier in this introduction, was partially carried out by Duval, Künzle and others through the 70'es and 80'es (see for instance [6, 7, 8, 9]). We skip the arguments that can be found in [30] or [31] which prove that the maximal group of Lie symmetries of the Schrödinger group, Sch_d, called the *Schrödinger group*, appears in some sense as the natural substitute for the conformal group in Newtonian mechanics. Unfortunately, it is finite-dimensional for every value of d, and its unitary irreducible representations are well-understood and classified (see [27]), giving very interesting though partial informations on two- and three-point functions in anisotropic and out-of-equilibrium statistical physics at criticality that have been systematically pursued in the past ten years of so (see [33, 34, 35, 15, 16, 2]) but relying on rather elementary mathematics, so this story could well have stopped here short of further arguments.

Contrary to the conformal group though, which corresponds to a rather 'rigid' Riemannian or Lorentzian geometry, the Schrödinger group is only one of the groups of symmetries that come out of the much more 'flexible' Newtonian geometry, with its loosely related time and space directions. In particular (restricting here to one space dimension for simplicity, although there are straightforward generalizations in higher dimensions), there arises a new group SV, which will be our main object of study, and that we shall call the *Schrödinger-Virasoro* group for reasons that will become clear shortly. Its Lie algebra $\mathfrak{sv}$ was originally introduced by M. Henkel in 1994 (see [14]) as a by-product of the computation of n-point functions that are covariant under the action of the Schrödinger group. It is given abstractly as

$$\mathfrak{sv} = \langle L_n \rangle_{n\in\mathbb{Z}} \oplus \langle Y_m \rangle_{m\in\frac{1}{2}+\mathbb{Z}} \oplus \langle M_p \rangle_{p\in\mathbb{Z}} \tag{0.1}$$

with relations

$$[L_n, L_p] = (n-p)L_{n+p} \tag{0.2}$$

$$[L_n, Y_m] = (\frac{n}{2} - m)Y_{n+m},\ [L_n, M_p] = -pM_p \tag{0.3}$$

$$[Y_m, Y_{m'}] = (m-m')M_{m+m'},\ [Y_m, M_p] = 0,\ [M_n, M_p] = 0 \tag{0.4}$$

$(n, p \in \mathbb{Z},\ m, m' \in \frac{1}{2} + \mathbb{Z})$. Denoting by $\mathrm{Vect}(S^1) = \langle L_n \rangle_{n\in\mathbb{Z}}$ the Lie algebra of vector fields on the circle (with brackets $[L_n, L_p] = (n-p)L_{n+p}$), $\mathfrak{sv}$ may be viewed as a semi-direct product $\mathfrak{sv} \simeq \mathrm{Vect}(S^1) \ltimes \mathfrak{h}$, where $\mathfrak{h} = \langle Y_m \rangle_{m\in\frac{1}{2}+\mathbb{Z}} \oplus \langle M_p \rangle_{p\in\mathbb{Z}}$ is a two-step nilpotent Lie algebra, isomorphic to $\mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_0$ as a $\mathrm{Vect}(S^1)$-module (see Definition 1.3 for notations).

This article is made of two parts which are relatively independent: the first part (the longer one actually), including Sections 1 through 3, is concerned with the representation theory of the Lie algebra $\mathfrak{sv}$; the second one contains a detailed cohomological study of $\mathfrak{sv}$ with applications to its deformation theory and to the systematic investigation of central extensions.

Section 1 is partially introductory; we define $\mathfrak{sv}$, give some notations and show that $\mathfrak{sv}$ can be integrated to a group.

In Section 2, we decompose $\mathfrak{sv}$ as a sum of tensor density modules for $\mathrm{Vect}(S^1)$. Introducing its central extension $\widehat{\mathfrak{sv}} \simeq \mathrm{Vir} \ltimes \mathfrak{h}$ which contains both the Virasoro algebra and the Schrödinger algebra (hence its name!), we shall study its coadjoint action on its regular dual $(\widehat{\mathfrak{sv}})^*$. We shall also study the action of $\mathfrak{sv}$ on a certain space of Schrödinger-type operators and on some other spaces of operators related to field equations.

In Section 3, we shall see that 'half of $\mathfrak{sv}$' can be interpreted as a Cartan prolongation $\oplus_{k=-1}^{\infty}\mathfrak{g}_k$ with $\mathfrak{g}_{-1} \simeq \mathbb{R}^3$ and $\mathfrak{g}_0$ three-dimensional solvable, and study the related co-induced representations by analogy with the case of the algebra of formal vector fields on $\mathbb{R}$, where this method leads to the tensor density modules of the Virasoro representation theory.

Section 4 is devoted to a systematic study of deformations and central extensions of $\mathfrak{sv}$.

The authors chose to present in this article the most significant mathematical results on the Schrödinger-Virasoro algebra known at the time of writing. A more comprehensive presentation of $\mathfrak{sv}$, in relation with Newton-Cartan geometry (see [6, 7, 8, 9] for instance), conformal structures, Poisson structures and the algebra of pseudo-differential operators on the line can be found in [35, 30] and will appear shortly in a lecture notes or book style, together with a construction of vertex algebra representations of $\mathfrak{sv}$ with applications to strongly anisotropic critical systems and out-of-equilibrium statistical physics (see [31]), and a study of induced representations which would have made this article much too long. One could object that this physical introduction would fit better with a more physically-minded article, but the authors think it would have been a pity to dispense with it and start from a mathematical definition of $\mathfrak{sv}$ out of the blue in this volume of the *Annales de l'Institut Henri Poincaré*, which is after all dedicated to mathematical physics.

1. Definition of the Schrödinger-Virasoro algebra and integration to a group

1.1. Definitions and notations

Let us first recall some well-known facts about the Virasoro algebra, that we shall use throughout the article.

We represent an element of $\mathrm{Vect}(S^1)$ by the vector field $f(z)\partial_z$, where $f \in \mathbb{C}[z, z^{-1}]$ is a Laurent polynomial. Vector field brackets $[f(z)\partial_z, g(z)\partial_z] = (fg' - f'g)\partial_z$, may equivalently be rewritten in the basis $(l_n)_{n\in\mathbb{Z}}$, $l_n = -z^{n+1}\partial_z$ (also called Laurent components), which yields $[\ell_n, \ell_m] = (n-m)\ell_{n+m}$. Notice the unusual choice of signs, justified (among other arguments) by the precedence of [14] on our subject.

The Lie algebra $\mathrm{Vect}(S^1)$ has only one non-trivial central extension (see [12] or [18] for instance), given by the so-called Virasoro cocycle $c \in Z^2(\mathrm{Vect}(S^1), \mathbb{R})$ defined by

$$c(f\partial_z, g\partial_z) = \int_{S^1} f'''(z)g(z)\,dz, \tag{1.1}$$

or, in Laurent components,

$$c(\ell_n, \ell_m) = \delta_{n+m,0}(n+1)n(n-1). \tag{1.2}$$

The resulting centrally extended Lie algebra, called *Virasoro algebra*, will be denoted by $\mathfrak{vir}$.

The Lie algebra $\mathrm{Vect}(S^1)$ has a one-parameter family of representations $\mathcal{F}_\lambda, \lambda \in \mathbb{R}$.

Definition 1.1. *We denote by $\mathcal{F}_\lambda$ the representation of $\mathrm{Vect}(S^1)$ on $\mathbb{C}[z, z^{-1}]$ given by*

$$\ell_n.z^m = (\lambda n - m)z^{n+m}, \quad n, m \in \mathbb{Z}. \tag{1.3}$$

An element of $\mathcal{F}_\lambda$ is naturally understood as a $(-\lambda)$-density $\phi(z)dz^{-\lambda}$, acted by $\mathrm{Vect}(S^1)$ as

$$f(z)\partial_z.\phi(z)dz^{-\lambda} = (f\phi' - \lambda f'\phi)(z)dz^{-\lambda}. \tag{1.4}$$

In the bases $\ell_n = -z^n\partial_z$ and $a_m = z^m dz^{-\lambda}$, one gets $\ell_n.a_m = (\lambda n - m)a_{n+m}$.

Definition 1.2. *We denote by $\mathfrak{sv}$ the Lie algebra with generators L_n, Y_m, M_n ($n \in \mathbb{Z}$, $m \in \frac{1}{2} + \mathbb{Z}$) and following relations (where $n, p \in \mathbb{Z}$, $m, m' \in \frac{1}{2} + \mathbb{Z}$):*

$$\begin{aligned}
[L_n, L_p] &= (n-p)L_{n+p} \\
[L_n, Y_m] &= (\frac{n}{2} - m)Y_{n+m}, \\
[Y_m^i, Y_{m'}^j] &= (m - m')M_{m+m'}, \\
[Y_m, M_p] &= 0, \quad [M_n, M_p] = 0.
\end{aligned}$$

Remark. The Lie subalgebra $\mathfrak{sch} = \langle\langle L_{-1}, L_0, L_1\rangle \ltimes \langle Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0\rangle$ is isomorphic to the Schrödinger Lie algebra in one space dimension.

One sees immediately that $\mathfrak{sv}$ has a semi-direct product structure $\mathfrak{sv} \simeq \mathrm{Vect}(S^1) \ltimes \mathfrak{h}$, with $\mathrm{Vect}(S^1) \simeq \langle L_n\rangle_{n\in\mathbb{Z}}$ and $\mathfrak{h} = \langle Y_m^i\rangle_{m\in\mathbb{Z}, i\le d} \oplus \langle M_p\rangle_{p\in\mathbb{Z}}$. The Lie algebra $\mathfrak{h}$ is a two-step nilpotent infinite dimensional Lie algebra.

The Lie algebra $\mathfrak{sv}$ was originally found in its following realization $d\pi_\lambda$ that we give for future reference (see Sections 2 and 3).

Definition 1.3 (see [33]).

1. *Denote by $d\tilde{\pi}_\lambda$ the representation of $\mathfrak{sv}$ as differential operators of order one on $\mathbb{R}^3$ with coordinates t, r, ζ defined by*

$$d\tilde{\pi}_\lambda(L_n) = -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n r\partial_r - \frac{1}{4}(n+1)nt^{n-1}r^2\partial_\zeta - (n+1)\lambda t^n$$

$$d\tilde{\pi}_\lambda(Y_m) = -t^{m+\frac{1}{2}}\partial_r - (m+\frac{1}{2})t^{m-\frac{1}{2}}r\partial_\zeta$$

$$d\tilde{\pi}_\lambda(M_p) = -t^p\partial_\zeta. \quad (1.5)$$

2. *Denote by $d\pi_\lambda$ the* mass $\mathcal{M}$-representation *obtained from $d\tilde{\pi}_\lambda$ by formally replacing ∂_ζ with $\mathcal{M}$.*

The representation $d\tilde{\pi}_\lambda$ may be derived from $d\pi_\lambda$ by taking a Laplace transform in the mass coordinate, yielding the supplementary coordinate ζ, while the constant λ may be interpreted as the *scaling dimension* of a corresponding physical field (see [33] for details).

1.2. Integration of the Schrödinger-Virasoro algebra to a group

We let $\mathrm{Diff}(S^1)$ be the group of orientation-preserving C^∞-diffeomorphisms of the circle. Orientation is important since we shall need to consider the square-root of the Jacobian of the diffeomorphism (see Proposition 1.2).

Theorem 1.1.

1. *Let $H = C^\infty(S^1) \times C^\infty(S^1)$ be the product of two copies of the space of infinitely differentiable functions on the circle, with its group structure modified as follows:*

$$(\alpha_2, \beta_2).(\alpha_1, \beta_1) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2 + \frac{1}{2}(\alpha_1'\alpha_2 - \alpha_1\alpha_2')). \quad (1.6)$$

   Then H is a Fréchet-Lie group which integrates $\mathfrak{h}$.

2. *Let $SV = \mathrm{Diff}(S^1) \ltimes H$ be the group with semi-direct product given by*

$$(1; (\alpha, \beta)).(\phi; 0) = (\phi; (\alpha, \beta)) \quad (1.7)$$

   and

$$(\phi; 0).(1; (\alpha, \beta)) = (\phi; ((\phi')^{\frac{1}{2}}(\alpha \circ \phi), \beta \circ \phi)). \quad (1.8)$$

   Then SV is a Fréchet-Lie group which integrates $\mathfrak{sv}$.

Proof.

1. From Hamilton (see [13]), one easily sees that H is a Fréchet-Lie group, its underlying manifold being the Fréchet space $C^\infty(S^1) \times C^\infty(S^1)$ itself.

   One sees moreover that its group structure is unipotent.

   By computing commutators

$$(\alpha_2, \beta_2)(\alpha_1, \beta_1)(\alpha_2, \beta_2)^{-1}(\alpha_1, \beta_1)^{-1} = (0, \alpha_1'\alpha_2 - \alpha_1\alpha_2') \quad (1.9)$$

   one recovers the formulas for the nilpotent Lie algebra $\mathfrak{h}$.

2. It is a well-known folk result that the Fréchet-Lie group $\mathrm{Diff}(S^1)$ integrates the Lie algebra $\mathrm{Vect}(S^1)$ (see Hamilton [13], or [12], Chapter 4, for details). Here the group H is realized (as $\mathrm{Diff}(S^1)$-module) as a product of modules of densities $\mathcal{F}_{\frac{1}{2}} \times \mathcal{F}_0$, hence the semi-direct product $\mathrm{Diff}(S^1) \ltimes H$ integrates the semi-direct product $\mathrm{Vect}(S^1) \ltimes \mathfrak{h}$. □

The representation $d\tilde{\pi}_0$, defined in Proposition 1.3, can be exponentiated into a representation of SV, given in the following proposition:

Proposition 1.2 (see [33]**).**

1. *Define* $\tilde{\pi}_0 \colon SV \to \mathrm{Diff}(S^1 \times \mathbb{R}^2)$ *by*
$$\tilde{\pi}(\phi;(\alpha,\beta)) = \tilde{\pi}(1;(\alpha,\beta)).\tilde{\pi}(\phi;0)$$
and
$$\tilde{\pi}(\phi;0)(z,r,\zeta) = (\phi(z), r\sqrt{\phi'(z)}, \zeta - \frac{1}{4}\frac{\phi''(z)}{\phi'(z)}r^2).$$
Then $\tilde{\pi}_0$ *is a representation of* SV.
2. *The infinitesimal representation of* $\tilde{\pi}_0$ *is equal to* $d\tilde{\pi}_0$.

Proof. Point 1. may be checked by direct verification (note that the formulas were originally derived by exponentiating the vector fields in the realization $d\tilde{\pi}$).

For 2., it is plainly enough to show that, for any $f \in C^\infty(S^1)$ and $g, h \in C^\infty(\mathbb{R})$,
$$\frac{d}{du}|_{u=0}\tilde{\pi}(\exp uL_f) = d\tilde{\pi}(L_f),$$
$$\frac{d}{du}|_{u=0}\tilde{\pi}(\exp uY_g) = d\tilde{\pi}(Y_g),$$
$$\frac{d}{du}|_{u=0}\tilde{\pi}(\exp uM_h) = d\tilde{\pi}(M_h).$$
Put $\phi_u = \exp uL_f$, so that $\frac{d}{du}|_{u=0}\phi_u(z) = f(z)$. Then
$$\frac{d}{du}r(\phi'_u)^{\frac{1}{2}} = \frac{1}{2}r(\phi'_u)^{-\frac{1}{2}}\frac{d}{du}\phi'_u \to_{u\to 0} \frac{1}{2}rf'(z),$$
$$\frac{d}{du}(r^2\frac{\phi''_u}{\phi'_u}) = r^2\left(\frac{\frac{d}{du}\phi''_u}{\phi'_u} - \frac{\phi''_u}{(\phi'_u)^2}\frac{d}{du}\phi'_u\right) \to_{u\to 0} r^2 f''(z)$$
so the equality $\frac{d}{du}|_{u=0}\tilde{\pi}(\exp uL_f) = d\tilde{\pi}(L_f)$ holds. The two other equalities can be proved in a similar way. □

Let us introduce other related representations by using the 'triangular' structure of the representation $\tilde{\pi}$. The action $\tilde{\pi} \colon SV \to \mathrm{Diff}(S^1 \times \mathbb{R}^2)$ can be projected onto an action $\bar{\pi} \colon SV \to \mathrm{Diff}(S^1 \times \mathbb{R})$ by 'forgetting' the coordinate ζ, since the way coordinates (t,r) are transformed does not depend on ζ. Note also that $\tilde{\pi}$ acts by (time- and space-dependent) translations on the coordinate ζ, so one may define a function $\Phi\colon SV \to C^\infty(\mathbb{R}^2)$ with coordinates (t,r) by

$$\tilde{\pi}(g)(t,r,\zeta) = (\bar{\pi}(g)(t,r), \zeta + \Phi_g(t,r))$$

(independently of $\zeta \in \mathbb{R}$). This action may be further projected onto $\bar{\pi}_{S^1}\colon SV \to \mathrm{Diff}(S^1)$ by 'forgetting' the second coordinate r this time, so

$$\bar{\pi}_{S^1}(\phi;(\alpha,\beta)) = \phi.$$

Proposition 1.3.

1. *One has the relation*

$$\Phi_{g_2\circ g_1}(t,r) = \Phi_{g_1}(t,r) + \Phi_{g_2}(\bar{\pi}(g_1)(t,r)).$$

   In other words, Φ is a trivial π-cocycle: $\Phi \in Z^1(G, C^\infty(\mathbb{R}^2))$.

2. *The application $\pi_\lambda\colon SV \to \mathrm{Hom}(C^\infty(S^1\times\mathbb{R}), C^\infty(S^1\times\mathbb{R}))$ defined by*

$$\pi_\lambda(g)(\phi)(t,r) = \left(\bar{\pi}'_{S^1} \circ \bar{\pi}_{S^1}^{-1}(t)\right)^\lambda e^{\mathcal{M}\Phi_g(\bar{\pi}(g)^{-1}.(t,r))}\ \phi(\bar{\pi}(g)^{-1}.(t,r))$$

   defines a representation of SV in $C^\infty(S^1\times\mathbb{R})$.

Proof. Straightforward. □

Note that the function Φ comes up naturally when considering projective representations of the Schrödinger group in one space dimension Sch $\simeq SL(2,\mathbb{R}) \ltimes$ Gal, where Gal is the Galilei Lie group (see [27]).

Let us look at the associated infinitesimal representation. Introduce the function Φ' defined by $\Phi'(X) = \frac{d}{du}|_{u=0}\Phi(\exp uX)$, $X \in \mathfrak{sv}$. If now $g = \exp X,\ X \in \mathfrak{sv}$, then

$$\frac{d}{du}|_{u=0}\pi_\lambda(\exp uX)(\phi)(t,r) = (\mathcal{M}\Phi'(X) + \lambda(d\bar{\pi}_{S^1}(X))'(t) + d\bar{\pi}(X))\,\phi(t,r) \tag{1.10}$$

so $\frac{d}{du}|_{u=0}\pi_\lambda(\exp uX)$ may be represented as the differential operator of order one

$$d\bar{\pi}(X) + \mathcal{M}\Phi'(X) + \lambda(d\bar{\pi}_{S^1}(X))'(t).$$

Hence the infinitesimal representation of π_λ is $d\pi_\lambda$ (see Definition 1.3), as should be.

1.3. About graduations and deformations of the Lie algebra $\mathfrak{sv}$

We shall say in this paragraph a little more on the algebraic structure of $\mathfrak{sv}$ and introduce another related Lie algebra $\mathfrak{tsv}$ ('*twisted Schrödinger-Virasoro algebra*').

The reader may wonder why we chose half-integer indices for the field Y. The shift in the indices is due to the fact that Y behaves as a $(-\frac{1}{2})$-density, or, in other words, Y has conformal weight $\frac{3}{2}$ under the action of the Virasoro field L (see, e.g., [18] or [4] for a mathematical introduction to conformal field theory and its terminology).

Note in particular that, although its weight is a half-integer, Y is a bosonic field, which would contradict spin-statistics theorem, were it not for the fact that Y is not meant to represent a relativistic field (and also that we are in a one-dimensional context).

Nevertheless, as in the case of the double Ramond/Neveu-Schwarz superalgebra (see [19]), one may define a 'twisted' Schrödinger-Virasoro algebra $\mathfrak{tsv}$ which is a priori equally interesting, and exhibits to some respects quite different properties (see Section 4).

Definition 1.4. *Let $\mathfrak{tsv}$ be the Lie algebra generated by $(L_n, Y_m, M_p)_{n,m,p\in\mathbb{Z}}$ with relations*

$$[L_n, L_m] = (n-m)L_{n+m},\ [L_n, Y_m] = (\frac{n}{2} - m)Y_{n+m},\ [L_n, M_m] = -mM_{n+m} \tag{1.11}$$

$$[Y_n, Y_m] = (n-m)M_{n+m},\ [Y_n, M_m] = 0, [L_n, Y_m] = 0, \tag{1.12}$$

where n, m are integers.

Notice that the relations are exactly the same as for $\mathfrak{sv}$ (see Definition 1.2), except for the values of the indices.

The simultaneous existence of two linearly independent graduations on $\mathfrak{sv}$ or $\mathfrak{tsv}$ sheds some light on this ambiguity in the definition.

Definition 1.5. *Let δ_1, resp. δ_2, be the graduations on $\mathfrak{sv}$ or $\mathfrak{tsv}$ defined by*

$$\delta_1(L_n) = n,\ \delta_1(Y_m) = m, \delta_1(M_p) = p \tag{1.13}$$

$$\delta_2(L_n) = n,\ \delta_2(Y_m) = m - \frac{1}{2},\ \delta_2(M_p) = p - 1 \tag{1.14}$$

with $n, p \in \mathbb{Z}$ and $m \in \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$.

One immediately checks that both δ_1 and δ_2 define graduations and that they are linearly independent.

Proposition 1.4. *The graduation δ_1, defined either on $\mathfrak{sv}$ or on $\mathfrak{tsv}$, is given by the inner derivation $\delta_1 = \mathrm{ad}(-L_0)$, while δ_2 is an outer derivation, $\delta_2 \in Z^1(\mathfrak{sv}, \mathfrak{sv}) \setminus B^1(\mathfrak{sv}, \mathfrak{sv})$ and $\delta_2 \in Z^1(\mathfrak{tsv}, \mathfrak{tsv}) \setminus B^1(\mathfrak{tsv}, \mathfrak{tsv})$.*

Remark. As we shall see in Section 4, the space $H^1(\mathfrak{sv}, \mathfrak{sv})$ or $H^1(\mathfrak{tsv}, \mathfrak{tsv})$ of outer derivations modulo inner derivations is three-dimensional, but only δ_2 defines a graduation on the basis (L_n, Y_m, M_p).

Proof. The only non-trivial point is to prove that δ_2 is not an inner derivation. Suppose (by absurd) that $\delta_2 = \mathrm{ad}X$, $X \in \mathfrak{sv}$ or $X \in \mathfrak{tsv}$ (we treat both cases simultaneously). Then $\delta_2(M_0) = 0$ since M_0 is central in $\mathfrak{sv}$ and in $\mathfrak{tsv}$. Hence the contradiction. □

Note that the graduation δ_2 is given by the Lie action of the Euler vector field $t\partial_t + r\partial_r + \zeta\partial_\zeta$ in the representation $d\tilde{\pi}_\lambda$ (see Definition 1.3).

Let us introduce a natural deformation of $\mathfrak{sv}$ (we shall need the following definition in Section 4):

Definition 1.6. *Let $\mathfrak{sv}_\varepsilon,\ \varepsilon \in \mathbb{R}$ (resp. $\mathfrak{tsv}_\varepsilon$) be the Lie algebra generated by L_n, Y_m, $M_p,\ n, p \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}$ (resp. $m \in \mathbb{Z}$), with relations*

$$[L_n, L_m] = (n-m)L_{n+m},\ [L_n, Y_m] = (\frac{(1+\varepsilon)n}{2} - m)Y_{n+m}, \qquad [L_n, M_m] = (\varepsilon n - m)M_{n+m}$$

$$[Y_n, Y_m] = (n-m)M_{n+m},\ [Y_n, M_m] = 0, [M_n, M_m] = 0. \tag{1.15}$$

One checks immediately that this defines a Lie algebra, and that $\mathfrak{sv} = \mathfrak{sv}_0$.

All these Lie algebras may be extended by using the trivial extension of the Virasoro cocycle of Section 1.1, yielding Lie algebras denoted by $\widetilde{\mathfrak{sv}}$, $\widetilde{\mathfrak{tsv}}$, $\widetilde{\mathfrak{sv}}_\varepsilon$, $\widetilde{\mathfrak{tsv}}_\varepsilon$.

2. On some natural representations of $\mathfrak{sv}$

We introduce in this section several natural representations of $\mathfrak{sv}$ that split into two classes: the (centrally extended) coadjoint action on the one hand; some apparently unrelated representations on spaces of functions or differential operators that can actually all be obtained as particular cases of the general coinduction method for $\mathfrak{sv}$ (see Section 3).

It is interesting by itself that the coadjoint action should not belong to the same family of representations as the others. We shall come back to this later on in this section.

2.1. Coadjoint action of $\mathfrak{sv}$

Let us recall some facts about coadjoint actions of centrally extended Lie groups and algebras, referring to [12], Chapter 6, for details. So let G be a Lie group with Lie algebra $\mathfrak{g}$, and let us consider central extensions of them, in the categories of groups and algebras respectively:

$$(1) \longrightarrow \mathbb{R} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow (1) \tag{2.1}$$

$$(0) \longrightarrow \mathbb{R} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow (0) \tag{2.2}$$

with $\tilde{\mathfrak{g}} = \mathrm{Lie}(\tilde{G})$, the extension (2.2) representing the tangent spaces at the identity of the extension (2.1) (see [12], II 6.1.1. for explicit formulas). Let $C \in Z^2_{\mathrm{diff}}(G, \mathbb{R})$ and $c \in Z^2(\mathfrak{g}, \mathbb{R})$ the respective cocycles. We want to study the coadjoint action on the dual $\tilde{\mathfrak{g}}^* = \mathfrak{g}^* \times \mathbb{R}$. We shall denote by Ad^* and $\widetilde{\mathrm{Ad}}^*$ the coadjoint actions of G and $\tilde{G}$ respectively, and ad^* and $\widetilde{\mathrm{ad}}^*$ the coadjoint actions of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. One then has the following formulas

$$\widetilde{\mathrm{Ad}}^*(g, \alpha)(u, \lambda) = (\mathrm{Ad}^*(g)u + \lambda\Theta(g), \lambda) \tag{2.3}$$

and

$$\widetilde{\mathrm{ad}}^*(\xi, \alpha)(u, \lambda) = (\mathrm{Ad}^*(\xi)u + \lambda\theta(\xi), 0) \tag{2.4}$$

where $\Theta\colon G \to \hat{\mathfrak{g}}^*$ and $\theta\colon \mathfrak{g} \to \hat{\mathfrak{g}}^*$ are the Souriau cocycles for differentiable and Lie algebra cohomologies respectively; for θ one has the following formula:

$\langle\theta(\xi),\eta\rangle = c(\xi,\eta)$. For details of the proof, as well as 'dictionaries' between the various cocycles, the reader is referred to [12], Chapter 6.

Note that formulas (2.3) and (2.4) define affine actions of G and $\mathfrak{g}$ respectively, different from their coadjoint actions when $\lambda \neq 0$. The actions on hyperplanes $\mathfrak{g}^*_\lambda = \{(u,\lambda) \mid u \in \mathfrak{g}^*\} \subset \tilde{\mathfrak{g}}^*$ with fixed second coordinate will be denoted by ad^*_λ and Ad^*_λ respectively.

Here we shall consider the central extension $\tilde{\mathfrak{sv}}$ of $\mathfrak{sv}$ inherited from Virasoro algebra, defined by the cocycle c such that

$$\begin{aligned} c(L_n, L_p) &= \delta_{n+p,0}\ n(n+1)(n-1) \\ c(L_n, Y_m) = c(L_n, M_p) &= c(Y_m, Y_{m'}) = 0 \end{aligned} \tag{2.5}$$

(with $n,p \in \mathbb{Z}$ and $m,m' \in \frac{1}{2} + \mathbb{Z}$). Note that we shall prove in Section 4 that this central extension is universal (a more 'pedestrian' proof was given in [14]).

As usual in infinite dimension, the algebraic dual of $\tilde{\mathfrak{sv}}$ is untractable, so let us consider the regular dual, consisting of sums of modules of densities of $\mathrm{Vect}(S^1)$ (see Definition 1.3): the dual module $\mathcal{F}^*_\mu$ is identified with $\mathcal{F}_{-1-\mu}$ through

$$\langle u(dx)^{1+\mu}, f\ dx^{-\mu}\rangle = \int_{S^1} u(x)f(x)\ dx. \tag{2.6}$$

So, in particular, $\mathrm{Vect}(S^1)^* \simeq \mathcal{F}_{-2}$, and (as a $\mathrm{Vect}(S^1)$-module)

$$\mathfrak{sv}^* = \mathcal{F}_{-2} \oplus \mathcal{F}_{-\frac{3}{2}} \oplus \mathcal{F}_{-1}; \tag{2.7}$$

we shall identify the element $\Gamma = \gamma_0 dx^2 + \gamma_1 dx^{\frac{3}{2}} + \gamma_2 dx \in \mathfrak{sv}^*$ with the triple $\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} \in (C^\infty(S^1))^3$. In other words,

$$\left\langle \begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, L_{f_0} + Y_{f_1} + M_{f_2}\right\rangle = \sum_{i=0}^{2}\int_{S^1} (\gamma_i f_i)(z)\ dz. \tag{2.8}$$

The following lemma describes the coadjoint representation of a Lie algebra that can be written as a semi-direct product.

Lemma 2.1. *Let $\mathfrak{s} = \mathfrak{s}_0 \ltimes \mathfrak{s}_1$ be a semi-direct product of two Lie algebra $\mathfrak{s}_0$ and $\mathfrak{s}_1$. Then the coadjoint action of $\mathfrak{s}$ on $\mathfrak{s}^*$ is given by*

$$\mathrm{ad}^*_{\mathfrak{s}}(f_0, f_1).(\gamma_0,\gamma_1) = \langle \mathrm{ad}^*_{\mathfrak{s}_0}(f_0)\gamma_0 - \tilde{f}_1.\gamma_1, \tilde{f}^*_0(\gamma_1) + \mathrm{ad}^*_{\mathfrak{s}_1}(f_1)\gamma_1\rangle$$

where by definition

$$\langle \tilde{f}_1.\gamma_1, X_0\rangle_{\mathfrak{s}_0^*\times\mathfrak{s}_0} = \langle\gamma_1, [X_0, f_1]\rangle_{\mathfrak{s}_1^*\times\mathfrak{s}_1}$$

and

$$\langle \tilde{f}^*_0(\gamma_1), X_1\rangle_{\mathfrak{s}_1^*\times\mathfrak{s}_1} = \langle\gamma_1, [f_0, X_1]\rangle_{\mathfrak{s}_1^*\times\mathfrak{s}_1}.$$

Proof. Straightforward. □

Theorem 2.2. *The coadjoint action of* $\mathfrak{sv}$ *on the affine hyperplane* $\mathfrak{sv}^*_\lambda$ *is given by the following formulas:*

$$\mathrm{ad}^*(L_{f_0})\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix} cf_0''' + 2f_0'\gamma_0 + f_0\gamma_0' \\ f_0\gamma_1' + \frac{3}{2}f_0'\gamma_1 \\ f_0\gamma_2' + f_0'\gamma_2\end{pmatrix} \tag{2.9}$$

$$\mathrm{ad}^*(Y_{f_1})\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix} \frac{3}{2}\gamma_1 f_1' + \frac{1}{2}\gamma_1' f_1 \\ 2\gamma_2 f_1' + \gamma_2' f_1 \\ 0\end{pmatrix} \tag{2.10}$$

$$\mathrm{ad}^*(M_{f_2})\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix} -\gamma_2 f_2' \\ 0 \\ 0\end{pmatrix}. \tag{2.11}$$

Proof. The action of $\mathrm{Vect}(S^1) \subset \mathfrak{sv}$ follows from the identification of $\mathfrak{sv}^*_\lambda$ with $\mathfrak{vir}^*_\lambda \oplus \mathcal{F}_{-\frac{3}{2}} \oplus \mathcal{F}_{-1}$.

Applying the preceding lemma, one gets now

$$\begin{aligned}
\langle \mathrm{ad}^*(Y_{f_1}).\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, L_{h_0}\rangle &= -\langle \tilde{Y}_{f_1}.\begin{pmatrix}0\\ \gamma_1\\ \gamma_2\end{pmatrix}, L_{h_0}\rangle \\
&= \langle \begin{pmatrix}0\\ \gamma_1\\ \gamma_2\end{pmatrix}, Y_{\frac{1}{2}h_0' f_1 - h_0 f_1'}\rangle \\
&= \int_{S^1} \gamma_1(\frac{1}{2}h_0' f_1 - h_0 f_1')\, dz \\
&= \int_{S^1} h_0(-\frac{3}{2}\gamma_1 f_1' - \frac{1}{2}\gamma_1' f_1)\, dz;
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{ad}^*(Y_{f_1}).\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, Y_{h_1}\rangle &= \langle \mathrm{ad}^*_{\mathfrak{h}}(Y_{f_1}).\begin{pmatrix}0\\ \gamma_1\\ \gamma_2\end{pmatrix}, Y_{h_1}\rangle \\
&= -\langle \begin{pmatrix}0\\ \gamma_1\\ \gamma_2\end{pmatrix}, M_{f_1' h_1 - f_1 h_1'}\rangle \\
&= -\int_{S^1} \gamma_2(f_1' h_1 - f_1 h_1')\, dz \\
&= \int_{S^1} h_1(-2\gamma_2 f_1' - \gamma_2' f_1)\, dz
\end{aligned}$$

and

$$\langle \mathrm{ad}^*(Y_{f_1})\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, M_{h_2}\rangle = 0.$$

Hence the result for $\mathrm{ad}^*(Y_{f_1})$.

For the action of $\mathrm{ad}^*(M_{f_2})$, one gets similarly

$$\begin{aligned}\langle \mathrm{ad}^*(M_{f_2}).\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, L_{h_0}\rangle &= -\langle \begin{pmatrix}0\\ \gamma_1\\ \gamma_2\end{pmatrix}, M_{f_2' h_0}\rangle\\ &= -\int_{S^1} \gamma_2 f_2' h_0 \; dz\end{aligned}$$

and

$$\langle \mathrm{ad}^*(M_{f_2}).\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, Y_{h_1}\rangle = \langle \mathrm{ad}^*(M_{f_2}).\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, M_{h_2}\rangle = 0.$$

Hence the result for $\mathrm{ad}^*(M_{f_2})$. □

One can now easily construct the coadjoint action of the group SV, which "integrates" the above defined coadjoint action of $\mathfrak{sv}$; as usual in infinite dimension, such an action should not be taken for granted and one has to construct it explicitly case by case. The result is given by the following.

Theorem 2.3. *The coadjoint action of SV on the affine hyperplane $\tilde{\mathfrak{sv}}_\lambda^*$ is given by the following formulas:*

Let $(\varphi, \alpha, \beta) \in SV$, then:

$$\mathrm{Ad}^*(\varphi)\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}\lambda\Theta(\varphi) + (\gamma_0\circ\varphi)(\varphi')^2\\ (\gamma_1\circ\varphi)(\varphi')^{\frac{3}{2}}\\ (\gamma_2\circ\varphi)\varphi'\end{pmatrix} \tag{2.12}$$

$$\mathrm{Ad}^*(\alpha,\beta)\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}\gamma_0 + \frac{3}{2}\gamma_1\alpha' + \frac{\gamma_1'}{2}\alpha + \gamma_2\beta' - \frac{\gamma_2}{2}(3\alpha'^2 + \alpha\alpha'') - \frac{3}{2}\gamma_2'\alpha\alpha' - \frac{\gamma_2''}{4}\alpha^2\\ \gamma_1 + 2\gamma_2\alpha' + \gamma_2'\alpha\\ \gamma_2\end{pmatrix}. \tag{2.13}$$

Proof. The first part (2.12) is easily deduced from the natural action of $\mathrm{Diff}(S^1)$ on $\tilde{\mathfrak{sv}}_\lambda^* = \mathfrak{vir}_\lambda^* \oplus \mathcal{F}_{-3/2} \oplus \mathcal{F}_{-1}$. Here $\Theta(\varphi)$ denotes the Schwarzian derivative of φ. Let's only recall that it is the Souriau cocycle in $H^1(\mathrm{Vect}(S^1), \mathfrak{vir}^*)$ associated to Bott-Virasoro cocycle in $H^2(\mathrm{Diff}(S^1), \mathbb{R})$, referring to [12], Chap. IV, VI for details.

The problem of computing the coadjoint action of $(\alpha, \beta) \in H$ can be split into two pieces; the coadjoint action of H on $\mathfrak{h}^*$ is readily computed and one finds:

$$\mathrm{Ad}^*(\alpha,\beta)\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}\gamma_1 + 2\gamma_2\alpha' + \gamma_2'\alpha\\ \gamma_2\end{pmatrix}.$$

The most delicate part is to compute the part of coadjoint action of $(\alpha,\beta) \in H$ coming from the adjoint action on $\mathrm{Vect}(S^1)$, by using:

$$\langle \mathrm{Ad}^*(\alpha,\beta)\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, f\partial\rangle = \langle \begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix}, \mathrm{Ad}(\alpha,\beta)^{-1}(f\partial,0,0)\rangle.$$

One can now use conjugation in the group SV and one finds

$$\mathrm{Ad}(\alpha,\beta)^{-1}(f\partial,0,0) = \left(f\partial, f\alpha' - \frac{1}{2}\alpha f', f\beta' + \frac{1}{2}(f\alpha''\alpha + \frac{f'}{2}\alpha\alpha' - \frac{\alpha^2}{2}f'' - f\alpha'^2 + \frac{f'}{2}\alpha\alpha'')\right).$$

Now, using integration by part, one finds easily the formula (2.13) above. □

2.2. Action of $\mathfrak{sv}$ on the affine space of Schrödinger operators

The next three sections aim at generalizing an idea that appeared at a crossroads between projective geometry, integrable systems and the theory of representations of $\mathrm{Diff}(S^1)$. We shall, to our own regret, give some new insights on SV from the latter point of view exclusively, leaving aside other aspects of a figure that will hopefully soon emerge.

Let $\partial = \frac{\partial}{\partial x}$ be the derivation operator on the torus $\mathbb{T} = [0,2\pi]$. A *Hill operator* is by definition a second order operator on $\mathbb{T}$ of the form $\mathcal{L}_u := \partial^2 + u$, $u \in C^\infty(\mathbb{T})$. Let π_λ be the representation of $\mathrm{Diff}(S^1)$ on the space of $(-\lambda)$-densities $\mathcal{F}_\lambda$ (see Definition 1.3). One identifies the vector spaces $C^\infty(\mathbb{T})$ and $\mathcal{F}_\lambda$ in the natural way, by associating to $f \in C^\infty(\mathbb{T})$ the density $f dx^{-\lambda}$. Then, for any couple $(\lambda,\mu) \in \mathbb{R}^2$, one has an action $\Pi_{\lambda,\mu}$ of $\mathrm{Diff}(S^1)$ on the space of differential operators on $\mathbb{T}$ through the left-and-right action

$$\Pi_{\lambda,\mu}(\phi)\colon D \to \pi_\lambda(\phi)\circ D\circ \pi_\mu(\phi)^{-1},$$

with corresponding infinitesimal action

$$d\Pi_{\lambda,\mu}(\phi)\colon D \to d\pi_\lambda(\phi)\circ D - D\circ d\pi_\mu(\phi).$$

For a particular choice of λ,μ, namely, $\lambda = -\frac{3}{2}, \mu = \frac{1}{2}$, this representation preserves the affine space of Hill operators; more precisely,

$$\pi_{-3/2}(\phi)\circ(\partial^2+u)\circ\pi_{1/2}(\phi)^{-1} = \partial^2 + (\phi')^2(u\circ\phi') + \frac{1}{2}\Theta(\phi) \tag{2.14}$$

where Θ stands for the Schwarzian derivative. In other words, u transforms as an element of $\mathfrak{vir}^*_{\frac{1}{2}}$ (see Section 2.1). One may also – taking an opposite point of view – say that Hill operators define a $\mathrm{Diff}(S^1)$-equivariant morphism from $\mathcal{F}_{\frac{1}{2}}$ into $\mathcal{F}_{-\frac{3}{2}}$.

This program may be completed for actions of SV on several affine spaces of differential operators. This will lead us to introduce several representations of SV that may all be obtained by the general method of coinduction (see Section 3). Quite remarkably, when one thinks of the analogy with the case of the action of $\mathrm{Diff}(S^1)$ on Hill operators, the (affine) coadjoint action of SV on $\mathfrak{sv}^*_\lambda$ does not appear in this context, and morerover cannot be obtained by the coinduction method, as one concludes easily from the formulas of Section 3 (see Theorem 3.2).

Definition 2.1. *Let $\mathcal{S}^{\rm lin}$ be the vector space of second order operators on $\mathbb{R}^2$ defined by*

$$D \in \mathcal{S}^{\rm lin} \Leftrightarrow D = h(2\mathcal{M}\partial_t - \partial_r^2) + V(r,t), \quad h, V \in C^\infty(\mathbb{R}^2)$$

and $\mathcal{S}^{\rm aff} \subset \mathcal{S}^{\rm lin}$ the affine subspace of 'Schrödinger operators' given by the hyperplane $h = 1$.

In other words, an element of $\mathcal{S}^{\rm aff}$ is the sum of the free Schrödinger operator $\Delta_0 = 2\mathcal{M}\partial_t - \partial_r^2$ and of a potential V.

The following theorem proves that there is a natural family of actions of the group SV on the space $\mathcal{S}^{\rm lin}$: more precisely, for every $\lambda \in \mathbb{R}$, and $g \in SV$, there is a 'scaling function' $F_{g,\lambda} \in C^\infty(S^1)$ such that

$$\pi_\lambda(g)(\Delta_0 + V)\pi_\lambda(g)^{-1} = F_{g,\lambda}(t)(\Delta_0 + V_{g,\lambda}) \tag{2.15}$$

where $V_{g,\lambda} \in C^\infty(\mathbb{R}^2)$ is a 'transformed potential' depending on g and on λ (see Definition 1.3 and Proposition 1.3 and commentaries thereafter for the definition of π_λ and the associated infinitesimal representation $d\pi_\lambda$). Taking the infinitesimal representation of $\mathfrak{sv}$ instead, this is equivalent to demanding that the 'adjoint' action of $d\pi_\lambda(\mathfrak{sv})$ preserve $\mathcal{S}^{\rm lin}$, namely

$$[d\pi_\lambda(X), \Delta_0 + V](t,r) = f_{X,\lambda}(t)(\Delta_0 + V_{X,\lambda}), \quad X \in \mathfrak{sv} \tag{2.16}$$

for a certain infinitesimal 'scaling' function $f_{X,\lambda}$ and with a transformed potential $V_{X,\lambda}$.

We shall actually prove that this last property even characterizes in some sense the differential operators of order one that belong to $d\pi_\lambda(\mathfrak{sv})$.

Theorem 2.4.

1. *The Lie algebra of differential operators of order one $\mathcal{X}$ on $\mathbb{R}^2$ preserving the space $\mathcal{S}^{\rm lin}$, i.e., such that*

$$[\mathcal{X}, \mathcal{S}^{\rm lin}] \subset \mathcal{S}^{\rm lin}$$

is equal to the image of $\mathfrak{sv}$ by the representation $d\pi_\lambda$ (modulo the addition to $\mathcal{X}$ of operators of multiplication by an arbitrary function of t).

2. *The action of $d\pi_{\lambda+1/4}(\mathfrak{sv})$ on the free Schrödinger operator Δ_0 is given by*

$$[d\pi_{\lambda+1/4}(L_f), \Delta_0] = f'\Delta_0 + \frac{\mathcal{M}^2}{2} f''' r^2 + 2\mathcal{M}\lambda f'' \tag{2.17}$$

$$[d\pi_{\lambda+1/4}(Y_g), \Delta_0] = 2\mathcal{M}^2 r g'' \tag{2.18}$$

$$[d\pi_{\lambda+1/4}(M_h), \Delta_0] = 2\mathcal{M}^2 h'. \tag{2.19}$$

Proof. Let $\mathcal{X} = f\partial_t + g\partial_r + h$ preserving the space $\mathcal{S}^{\rm lin}$: this is equivalent to the existence of two functions $\phi(t,r), V(t,r)$ such that $[\mathcal{X}, \Delta_0] = \phi(\Delta_0 + V)$. It is clear that $[h, \mathcal{S}_{\rm lin}] \subset \mathcal{S}_{\rm lin}$ if h is a function of t only.

By considerations of degree, one must then have $[\mathcal{X}, \partial_r] = a(t,r)\partial_r + b(t,r)$, hence f is a function of t only. Then

$$[f\partial_t, 2\mathcal{M}\partial_t - \partial_r^2] = -2\mathcal{M}f'\partial_t \tag{2.20}$$

$$[g\partial_r, 2\mathcal{M}\partial_t - \partial_r^2] = -2\mathcal{M}\partial_t g\partial_r + 2\partial_r g\partial_r^2 + \partial_r^2 g\partial_r \tag{2.21}$$

$$[h, -\partial_r^2] = 2\partial_r h\partial_r + \partial_r^2 h \tag{2.22}$$

so, necessarily,

$$f' = 2\partial_r g = -\phi$$

and

$$(2\mathcal{M}\partial_t - \partial_r^2)g = -2\partial_r h.$$

By putting together these relations, one gets points 1 and 2 simultaneously. □

Using a left-and-right action of $\mathfrak{sv}$ that combines $d\pi_\lambda$ and $d\pi_{1+\lambda}$, one gets a new family of representations $d\sigma_\lambda$ of $\mathfrak{sv}$ which map the affine space $\mathcal{S}^{\mathrm{aff}}$ into differential operators of order zero (that is to say, into functions):

Proposition 2.5. *Let $d\sigma_\lambda\colon \mathfrak{sv} \to \mathrm{Hom}(\mathcal{S}^{\mathrm{lin}}, \mathcal{S}^{\mathrm{lin}})$ defined by the left-and-right infinitesimal action*

$$d\sigma_\lambda(X)\colon D \to d\pi_{1+\lambda}(X) \circ D - D \circ d\pi_\lambda(X).$$

Then $d\sigma_\lambda$ is a representation of $\mathfrak{sv}$ and $d\sigma_\lambda(\mathfrak{sv})(\mathcal{S}^{\mathrm{aff}}) \subset C^\infty(\mathbb{R}^2)$.

Proof. Let $X_1, X_2 \in \mathfrak{sv}$, and put $d\bar{\pi}_{S^1}(X_i) = f_i(t)$, $i = 1, 2$: then, with a slight abuse of notations, $d\sigma_\lambda(X_i) = \mathrm{ad}\, d\pi_\lambda(X_i) + f_i'$, so

$$[d\sigma_\lambda(X_1), d\sigma_\lambda(X_2)] = [\mathrm{ad}(d\pi_\lambda(X_1)) + f_1', \mathrm{ad}(d\pi_\lambda(X_2)) + f_2'] \tag{2.23}$$

$$= \mathrm{ad}\, d\pi_\lambda([X_1, X_2]) + ([d\pi_\lambda(X_1), f_2'(t)] - [d\pi_\lambda(X_2), f_1'(t)])\,. \tag{2.24}$$

Now $\mathrm{ad}\, d\pi_\lambda$ commutes with operators of multiplication by any function of time $g(t)$ if $X \in \mathfrak{h}$, and

$$[d\pi_\lambda(L_f), g(t)] = [f(t)\partial_t, g(t)] = f(t)g'(t)$$

so as a general rule

$$[d\pi_\lambda(X_i), g(t)] = f_i(t)g'(t).$$

Hence

$$[d\sigma_\lambda(X_1), d\sigma_\lambda(X_2)] = \mathrm{ad}\, d\pi_\lambda([X_1, X_2]) + (f_1(t)f_2''(t) - f_2(t)f_1''(t)) \tag{2.25}$$

$$= \mathrm{ad}\, d\pi_\lambda([X_1, X_2]) + (f_1 f_2' - f_2 f_1')'(t) \tag{2.26}$$

$$= d\sigma_\lambda([X_1, X_2]). \tag{2.27}$$

By the preceding theorem, it is now clear that $d\sigma_\lambda(\mathfrak{sv})$ sends $\mathcal{S}^{\mathrm{aff}}$ into differential operators of order zero. □

Remark. Choosing $\lambda = \frac{1}{4}$ leads to a representation of the Schrödinger Lie algebra $\mathfrak{sch}$ (see remark following Definition 1.2) preserving the kernel of Δ_0, see [33]. So, in some sense, $\lambda = \frac{1}{4}$ is the 'best' choice.

Clearly, the (affine) subspace $\mathcal{S}^{\rm aff}_{\le 2} \subset \mathcal{S}^{\rm aff}$ of Schrödinger operators with potentials that are at most quadratic in r, that is,

$$D \in \mathcal{S}^{\rm aff}_{\le 2} \Leftrightarrow D = 2\mathcal{M}\partial_t - \partial_r^2 + g_0(t)r^2 + g_1(t)r + g_2(t)$$

is mapped into potentials of the same form under $d\sigma_\lambda(SV)$.

Let us use the same vector notation for elements of $\mathcal{S}^{\rm aff}_{\le 2}$ and for potentials that are at most quadratic in r (what is precisely meant will be clearly seen from the context): set $D = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$, respectively $V = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$ for $D = \Delta_0 + g_0(t)r^2 + g_1(t)r + g_2(t) \in \mathcal{S}^{\rm aff}_{\le 2}$, respectively $V(t,r) = g_0(t)r^2 + g_1(t)r + g_2(t) \in C^\infty(\mathbb{R}^2)$. Then one can give an explicit formula for the action of $d\sigma_\lambda$ on $\mathcal{S}^{\rm aff}_2$.

Proposition 2.6.

1. *Let $D = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{S}^{\rm aff}_{\le 2}$ and $f_0, f_1, f_2 \in C^\infty(\mathbb{R})$. Then the following formulas hold:*

$$d\sigma_{\lambda+1/4}(L_{f_0})(D) = -\begin{pmatrix} \frac{-\mathcal{M}^2}{2} f_0''' + 2f_0'g_0 + f_0 g_0' \\ f_0 g_1' + \frac{3}{2} f_0' g_1 \\ f_0 g_2' + f_0' g_2 - 2\mathcal{M}\lambda f_0'' \end{pmatrix} \tag{2.28}$$

$$d\sigma_{\lambda+1/4}(Y_{f_1})(D) = -\begin{pmatrix} 0 \\ 2f_1 g_0 - 2\mathcal{M}^2 f_1'' \\ f_1 g_1 \end{pmatrix} \tag{2.29}$$

$$d\sigma_{\lambda+1/4}(M_{f_2})(D) = \begin{pmatrix} 0 \\ 0 \\ 2\mathcal{M}^2 f_2' \end{pmatrix}. \tag{2.30}$$

2. *Consider the restriction of $d\sigma_{1/4}$ to* $\mathrm{Vect}(S^1) \subset \mathfrak{sv}$. *Then $d\sigma_{1/4}|_{\mathrm{Vect}(S^1)}$ acts diagonally on the 3-vectors $\begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}$ and its restriction to the subspaces $\mathcal{S}^{\rm aff}_i := \{\Delta_0 + g(t)r^i \mid g \in C^\infty(\mathbb{R})\}$, $i = 0, 1, 2$, is equal to the coadjoint action of* $\mathrm{Vect}(S^1)$ *on the affine hyperplane $\mathfrak{vir}^*_{1/4}$ ($i = 2$), and to the usual action of* $\mathrm{Vect}(S^1)$ *on $\mathcal{F}_{-3/2} \simeq \mathcal{F}^*_{1/2}$ (when $i = 1$), respectively on $\mathcal{F}_{-1} \simeq \mathcal{F}^*_0$ (when $i = 0$). Taking $\lambda \neq 0$ leads to an affine term proportional to f_0'' on the third coordinate, corresponding to the non-trivial affine cocycle in* $H^1(\mathrm{Vect}(S^1), \mathcal{F}_{-1})$.

In other words, if one identifies $\mathcal{S}^{\rm aff}_{\le 2}$ with $\mathfrak{sv}^*_{\frac{1}{4}}$ by

$$\langle \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} f_0 & f_1 & f_2 \end{pmatrix} \rangle_{\mathcal{S}^{\rm aff}_{\le 2} \times \mathfrak{sv}} = \sum_{i=0}^{2} \int_{S^1} (g_i f_i)(z)\, dz,$$

then the restriction of $d\sigma_{1/4}$ to $\mathrm{Vect}(S^1)$ is equal to the restriction of the coadjoint action of $\mathfrak{sv}$ on $\mathfrak{sv}^*_{\frac{1}{4}}$.

But mind that $d\sigma_{1/4}$ is *not* equal to the coadjoint action of $\mathfrak{sv}$.

Proof. Point 2 is more or less obvious, and we shall only give some of the computations for the first one. One has $[d\pi_{\lambda+1/4}(L_{f_0}), \Delta_0] = f_0'\Delta_0 - \frac{\mathcal{M}}{2} f_0''' r^2 + 2\mathcal{M}\lambda f_0''$, $[d\pi_{\lambda+1/4}(L_{f_0}), g_2(t)] = -f_0(t)g_2'(t)$, $[d\pi_{\lambda+1/4}(L_{f_0}), g_1(t)r] = -(f_0(t)g_1'(t) + \frac{1}{2}f_0'(t)g_1(t))r$, $[d\pi_{\lambda+1/4}(L_{f_0}), g_0(t)r^2] = -(f_0(t)g_0'(t) + f_0'(t)g_0(t))r^2$, so

$$\begin{aligned} d\sigma_{\lambda+1/4}(L_{f_0})(D) &= -f_0'.D + [d\pi_{\lambda+1/4}(L_{f_0}), D] \\ &= -(f_0 g_2' + f_0' g_2 - 2\mathcal{M}\lambda f_0'') - (f_0 g_1' + \frac{3}{2} f_0' g_1) r \\ &\quad - (\frac{\mathcal{M}^2}{2} f_0''' + 2 f_0' g_0 + f_0 g_0') r^2 . \end{aligned}$$

Hence the result for $d\sigma_{\lambda+1/4}(L_{f_0})$. The other computations are similar though somewhat simpler. □

This representation is easily integrated to a representation σ of the group SV. We let $\Theta(\phi) = \frac{\phi'''}{\phi'} - \frac{3}{2}\left(\frac{\phi''}{\phi'}\right)^2$ $(\phi \in \mathrm{Diff}(S^1))$ be the Schwarzian of the function ϕ.

Proposition 2.7. *Let* $D = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{S}^{\mathrm{aff}}_{\le 2}$, *then*

$$\sigma_{\lambda+1/4}(\phi; (a,b))D = \sigma_{\lambda+1/4}(1; (a,b))\sigma_{\lambda+1/4}(\phi; 0)D \tag{2.31}$$

$$\sigma_{\lambda+1/4}(\phi; 1).D = \begin{pmatrix} (\phi')^2.(g_0 \circ \phi) - \frac{\mathcal{M}^2}{2}\Theta(\phi) \\ (\phi')^{\frac{3}{2}}(g_1 \circ \phi) \\ \phi'(g_2 \circ \phi) - 2\mathcal{M}\lambda \frac{\phi''}{\phi} \end{pmatrix} \tag{2.32}$$

$$\sigma_{\lambda+1/4}(1; (a,b)).D = \begin{pmatrix} g_0 \\ g_1 - 2ag_0 + 2\mathcal{M}^2 a'' \\ g_2 - ag_1 + a^2 g_0 + \mathcal{M}^2(2b' - aa'') \end{pmatrix} \tag{2.33}$$

defines a representation of SV *that integrates* $d\sigma$, *and maps the affine space* $\mathcal{S}^{\mathrm{aff}}_{\le 2}$ *into itself.*

In other words, elements of $\mathcal{S}^{\mathrm{aff}}_{\le 2}$ *define an* SV*-equivariant morphism from* $\mathcal{H}_\lambda$ *into* $\mathcal{H}_{\lambda+1}$, *where* $\mathcal{H}_\lambda$, *respectively* $\mathcal{H}_{\lambda+1}$, *is the space* $C^\infty(\mathbb{R}^2)$ *of functions of* t, r *that are at most quadratic in* r, *equipped with the action* π_λ, *respectively* $\pi_{\lambda+1}$ *(see* (1.5)*).*

Proof. Put $SV = G \ltimes H$. Then the restrictions $\sigma|_G$ and $\sigma|_H$ define representations (this is a classical result for the first action, and may be checked by direct computation for the second one). The associated infinitesimal representation of $\mathfrak{sv}$ is easily seen to be equal to $d\sigma$. □

In particular, the orbit of the free Schrödinger operator Δ_0 is given by the remarkable formula

$$\sigma_{\lambda+1/4}(\phi;(a,b))\Delta_0 = \begin{pmatrix} -\frac{\mathcal{M}^2}{2}\Theta(\phi) \\ 2\mathcal{M}^2 a'' \\ \mathcal{M}^2(2b'-aa'') \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ -2\mathcal{M}\frac{\phi''}{\phi} \end{pmatrix}, \tag{2.34}$$

mixing a third-order cocycle with coefficient $\mathcal{M}^2$ which extends the Schwarzian cocycle $\phi \to \Theta(\phi)$ with a second-order cocycle with coefficient $-2\mathcal{M}\lambda$ which extends the well-known cocycle $\phi \to \frac{\phi''}{\phi}$ in $H^1(\mathrm{Vect}(S^1), \mathcal{F}_{-1})$. The following paragraph shows that all affine cocycles of $\mathfrak{sv}$ with coefficients in the (linear) representation space $\mathcal{S}^{\mathrm{aff}}_{\leq 2}$ are of this form.

2.3. Affine cocycles of $\mathfrak{sv}$ and SV on the space of Schrödinger operators

The representations of groups and Lie algebras described above are of affine type; if one has linear representations of the group G and Lie algebra $\mathcal{G}$ on a module M, one can deform these representations into affine ones, using the following construction.

Let $C\colon G \to M$ (resp. $c\colon \mathcal{G} \longrightarrow M$) be a 1-cocycle in $Z^1_{\mathrm{diff}}(G,M)$ (resp. $Z^1(\mathcal{G},M)$); it defines an affine action of G (resp. $\mathcal{G}$) by deforming the linear action as follows:

$$g * m = g.m + C(g)$$
$$\xi * m = \xi.m + c(\xi)$$

respectively. Here the dot indicates the original linear action and $*$ the affine action. One deduces from the formulas given in Propositions 2.5 and 2.6 that the above representations are of this type; the first cohomology of SV (resp. $\mathfrak{sv}$) with coefficients in the module $\mathcal{S}^{\mathrm{aff}}_{\leq 2}$ (equipped with the linear action) classifies all the affine deformations of the action, up to isomorphism. They are given by the following theorem:

Theorem 2.8. *The degree-one cohomology of the group SV (resp. Lie algebra $\mathfrak{sv}$) with coefficients in the module $\mathcal{S}^{\mathrm{aff}}_{\leq 2}$ (equipped with the linear action) is two-dimensional and can be represented by the following cocycles:*

– for SV:

$$C_1(\phi,(a,b)) = \begin{pmatrix} -\frac{1}{2}\Theta(\phi) \\ 2a'' \\ 2b' - aa'' \end{pmatrix}$$

$$C_2(\phi,(a,b)) = \begin{pmatrix} 0 \\ 0 \\ \phi'' \end{pmatrix}$$

– *for* $\mathfrak{sv}$:

$$c_1(L_{f_0} + Y_{f_1} + M_{f_2}) = \begin{pmatrix} \frac{1}{2} f_0''' \\ -2f_1'' \\ -2f_2' \end{pmatrix}$$

$$c_2(L_{f_0} + Y_{f_1} + M_{f_2}) = \begin{pmatrix} 0 \\ 0 \\ f_0'' \end{pmatrix}$$

(*one easily recognizes that* C_1 *and* c_1 *correspond to the representations given in Prop.* 3.7 *and Prop.* 3.6 *respectively.*)

Proof. One shall first make the computations for the Lie algebra and then try to integrate explicitly; here, the "heuristical" version of Van-Est theorem, generalized to the infinite-dimensional case, guarantees the isomorphism between the H^1 groups for SV and $\mathfrak{sv}$ (see [12], Chapter IV).

So let us compute $H^1(\mathfrak{sv}, M) \simeq H^1(\mathcal{G} \ltimes \mathfrak{h}, M)$ for $M = \mathcal{S}^{\text{aff}}_{\leq 2}$ (equipped with the linear action). Let $c\colon \mathcal{G} \times \mathfrak{h} \longrightarrow M$ be a cocycle and set $c = c' + c''$ where $c' = c|_{\mathcal{G}}$ and $c'' = c|_{\mathfrak{h}}$. One has $c' \in Z^1(\mathcal{G}, M)$ and $c'' \in Z^1(\mathfrak{h}, M)$, and these two cocycles are linked together by the compatibility relation

$$c''([X, \alpha]) - X.(c''(\alpha)) + \alpha.(c'(X)) = 0. \tag{2.35}$$

As a $\mathcal{G}$-module, $M = \mathcal{F}_{-2} \oplus \mathcal{F}_{-3/2} \oplus \mathcal{F}_{-1}$, so one determines easily that $H^1(\mathcal{G}, \mathcal{F}_{-2})$ and $H^1(\mathcal{G}, \mathcal{F}_{-1})$ are one-dimensional, generated by $L_{f_0} \longrightarrow f_0''' dx^2$ and $L_{f_0} \longrightarrow f_0'' dx$ respectively, and $H^1(\mathcal{G}, \mathcal{F}_{-3/2}) = 0$ (see [12], Chapter IV). One can now readily compute the 1-cohomology of the nilpotent part $\mathfrak{h}$; one easily remarks that the linear action on $\mathcal{S}^{\text{aff}}_{\leq 2}$ is defined as follows:

$$(Y_{f_1} + M_{f_2}) \,.\, \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2f_1\gamma_0 \\ f_1\gamma_1 \end{pmatrix}.$$

We can then deduce the cocycles in $Z^1(\mathfrak{h}, M)$ through a direct computation. One obtains:

$$c''(Y_{f_1} + M_{f_2}) = \begin{pmatrix} d_0(f_1) \\ d_1(f_1) + \tilde{d}_1(f_2) \\ d_2(f_1) + \tilde{d}_2(f_2) \end{pmatrix}$$

where d_2 is an arbitrary differential operator, and d_0 and $\tilde{d}_1$ on one side, d_1 and $\tilde{d}_2$ on the other, are linked together as follows:

– one has three potential cases for d_0 and $\tilde{d}_1$

- **(i)** $d_0(f_1) = \frac{f_1}{2}$ $\quad \tilde{d}_1(f_2) = 0$
- **(ii)** $d_0(f_1) = \frac{f_1'}{2}$ $\quad \tilde{d}_1(f_2) = f_2$
- **(iii)** $d_0(f_1) = \frac{f_1''}{2}$ $\quad \tilde{d}_1(f_2) = f_2'$

– one has two potential cases for d_1 and $\tilde{d}_2$

- **(iv)** $d_1(f_1) = f_1'$ $\quad \tilde{d}_2(f_2) = f_2$
- **(v)** $d_1(f_1) = f_1''$ $\quad \tilde{d}_2(f_2) = f_2'$

Our theorem then states that only cocycle (v) will survive to compatibility conditions (3.35) above. This can be seen as follows. The compatibility conditions imply that $d_0\colon \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_{-2}$ is $\mathcal{G}$-invariant. As there are no such invariants, the d_0- and $\tilde{d}_1$-terms must cancel. An analogous argument works for d_2. For the last two cases (iv) and (v), the compatibility condition gives:

$$c''([L_{f_0}, Y_{f_1} + M_{f_2}]) - L_{f_0}.(c''(Y_{f_1} + M_{f_2})) = \begin{pmatrix} 0 \\ -f_0' f_1' - \frac{1}{2} f_1 f_0'' \\ -f_0' f_2 \end{pmatrix}$$

for (iv) and

$$c''([L_{f_0}, Y_{f_1} + M_{f_2}]) - L_{f_0}.(c''(Y_{f_1} + M_{f_2})) = \begin{pmatrix} 0 \\ -\frac{1}{2} f_1 f_0''' \\ 0 \end{pmatrix}$$

for (v).

On the other hand one finds:

$$(Y_{f_1} + M_{f_2}).(c'(L_{f_0})) = (Y_{f_1} + M_{f_2}) \cdot \begin{pmatrix} f_0''' \\ 0 \\ f_0'' \end{pmatrix} = \begin{pmatrix} 0 \\ 2 f_1 f_0''' \\ 0 \end{pmatrix}.$$

Hence the result, with the right proportionality coefficients, and one obtains the formula for c_1.

One also remarks that the term with $f_0'' dx$ disappears through the action of $\mathfrak{h}$, so it will induce an independent generator in $H^1(\mathfrak{sv}, \mathcal{S}^{\mathrm{aff}})$, precisely c_2.

Finally the cocycles C_1 and C_2 in $H^1(SV, \mathcal{S}^{\mathrm{aff}}_{\leq 2})$ are not so hard to compute, once we have determined the action of H on $\mathcal{S}^{\mathrm{aff}}_{\leq 2}$, which is unipotent as follows:

$$(a,b).\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 + 2a\gamma_0 \\ \gamma_2 + a\gamma_1 - a^2\gamma_0 \end{pmatrix}.$$

□

2.4. Action on Dirac-Lévy-Leblond operators

Lévy-Leblond introduced in [21] a matrix differential operator $\mathcal{D}_0$ on $\mathbb{R}^{d+1}$ (with coordinates $t, r_1, \ldots, r_d$) of order one, similar to the Dirac operator, whose square is equal to $-\Delta_0 \otimes \mathrm{Id} = -\begin{pmatrix} \Delta_0 & & \\ & \ddots & \\ & & \Delta_0 \end{pmatrix}$ for $\Delta_0 = 2\mathcal{M}\partial_t - \sum_{i=1}^d \partial_{r_i}^2$. So, in some sense, $\mathcal{D}_0$ is a square-root of the free Schrödinger operator, just as the Dirac operator is a square-root of the D'Alembertian. The group of Lie invariance of $\mathcal{D}_0$ has been studied in [36], it is isomorphic to the Schrödinger Lie group in d space dimensions.

Let us restrict to the case $d = 1$ (see [34] for details). Then $\mathcal{D}_0$ acts on spinors, or couples of functions $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ of two variables t, r, and may be written as

$$\mathcal{D}_0 = \begin{pmatrix} \partial_r & -2\mathcal{M} \\ \partial_t & -\partial_r \end{pmatrix}. \tag{2.36}$$

One checks immediately that $\mathcal{D}_0^2 = -\Delta_0 \otimes \mathrm{Id}$.

From the explicit realization of the Schrödinger Lie algebra $\mathfrak{sch}$ on spinors (see [34]), one may easily guess a realization of $\mathfrak{sv}$ that extends the action of $\mathfrak{sch}$, and, more interestingly perhaps, acts on an affine space $\mathcal{D}^{\rm aff}$ of Dirac-Lévy-Leblond operators with potential, in the same spirit as in the previous section. More precisely, one has the following theorem (we need to introduce some notations first).

Definition 2.2. *Let $\mathcal{D}^{\rm lin}$ be the vector space of first order matrix operators on $\mathbb{R}^2$ defined by*

$$D \in \mathcal{D}^{\rm lin} \Leftrightarrow D = h(r,t)\mathcal{D}_0 + \begin{pmatrix} 0 & 0 \\ V(r,t) & 0 \end{pmatrix}, \quad h, V \in C^\infty(\mathbb{R}^2)$$

and $\mathcal{D}^{\rm aff}$, $\mathcal{D}^{\rm aff}_{\le 2}$ be the affine subspaces of $\mathcal{D}^{\rm lin}$ such that

$$D \in \mathcal{D}^{\rm aff} \Leftrightarrow D = \mathcal{D}_0 + \begin{pmatrix} 0 & 0 \\ V(r,t) & 0 \end{pmatrix},$$

$$D \in \mathcal{D}^{\rm aff}_{\le 2} \Leftrightarrow D = \mathcal{D}_0 + \begin{pmatrix} 0 & 0 \\ g_0(t)r^2 + g_1(t)r + g_2(t) & 0 \end{pmatrix}.$$

We shall call *Dirac potential* a matrix of the form $\left(\begin{smallmatrix} 0 & 0 \\ V(r,t) & 0 \end{smallmatrix}\right)$, with $V \in C^\infty(\mathbb{R}^2)$.

Definition 2.3. *Let $d\pi^\sigma_\lambda$ ($\lambda \in \mathbb{C}$) be the infinitesimal representation of $\mathfrak{sv}$ on the space $\widetilde{\mathcal{H}}^\sigma_\lambda \simeq (C^\infty(\mathbb{R}^2))^2$ with coordinates t, r, defined by*

$$\begin{aligned} d\pi^\sigma_\lambda(L_f) &= \left(-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}\mathcal{M}f''(t)r^2\right) \otimes \mathrm{Id} \\ &\quad - f'(t) \otimes \begin{pmatrix} \lambda - \frac{1}{4} & \\ & \lambda + \frac{1}{4} \end{pmatrix} - \frac{1}{2}f''(t)r \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \end{aligned} \tag{2.37}$$

$$d\pi^\sigma_\lambda(Y_g) = (-g(t)\partial_r - \mathcal{M}f'(t)r) \otimes \mathrm{Id} - f'(t) \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{2.38}$$

$$d\pi^\sigma_\lambda(M_h) = -\mathcal{M}f(t) \otimes \mathrm{Id}. \tag{2.39}$$

Theorem 2.9.

1. *Let $d\sigma \colon \mathfrak{sv} \to \mathrm{Hom}(\mathcal{D}^{\rm lin}, \mathcal{D}^{\rm lin})$ defined by the left-and-right infinitesimal action*
$$d\sigma(X) \colon D \to d\pi^\sigma_1(X) \circ D - D \circ d\pi^\sigma_{\frac{1}{2}}(X).$$
Then $d\sigma$ maps $\mathcal{D}^{\rm aff}_{\le 2}$ into the vector space of Dirac potentials.
2. *If one represents the Dirac potential $V = \left(\begin{smallmatrix} 0 & 0 \\ g_0(t)r^2 + g_1(t)r + g_2(t) & 0 \end{smallmatrix}\right)$ or, indifferently, the Dirac operator $\mathcal{D}_0 + V$, by the vector $\left(\begin{smallmatrix} g_0 \\ g_1 \\ g_2 \end{smallmatrix}\right)$, then the action of $d\sigma$ on $\mathcal{D}^{\rm aff}_{\le 2}$ is given by the same formula as in Proposition 2.6, except for the affine terms (with coefficient proportional to $\mathcal{M}$ or $\mathcal{M}^2$) that should all be divided by $2\mathcal{M}$.*

We shall skip the proof (in the same spirit as Theorem 2.4, Proposition 2.5 and Proposition 2.6) which presents no difficulty, partly for lack of space, partly because the action on Dirac operators doesn't give anything new by comparison with the case of Schrödinger operators.

Note that, as in the previous section, one may define a 'shifted' action

$$d\sigma_\lambda(X)\colon D \to d\pi^\sigma_{\lambda+1}(X) \circ D - D \circ d\pi^\sigma_{\lambda+\frac{1}{2}}(X) \tag{2.40}$$

which will only modify the coefficients of the affine cocycles.

As a concluding remark of these two sections, let us emphasize two points:
– contrary to the case of the Hill operators, there is a free parameter λ in the left-and-right actions on the affine space of Schrödinger or Dirac operators;
– looking at the differences of indices between the left action and the right action, one may consider somehow that Schrödinger operators are of order one, while Dirac operators are of order $\frac{1}{2}$! (recall the difference of indices was $2 = \frac{3}{2} - (-\frac{1}{2})$ in the case of the Hill operators, which was the signature of operators of order 2 – see [5]).

So Schrödinger operators are somehow reminiscent of the operators $\partial + u$ of order one on the line, which intertwine $\mathcal{F}_\lambda$ with $\mathcal{F}_{1+\lambda}$ for any value of λ. The case of the Dirac operators, on the other hand, has no counterpart whatsoever for differential operators on the line.

2.5. About multi-diagonal differential operators and some Virasoro-solvable Lie algebras

This paragraph may be skipped in a first approach. We introduce new wave equations and related realizations of $\mathfrak{sv}$ and similar Lie algebras that will also appear as a particular case of the general construction of Section 3.

The original remark that prompted the introduction of multi-diagonal differential operators in our context (see below for a definition) was the following. Consider the space $\mathbb{R}^3$ with coordinates r, t, ζ as in Definition 1.3. We introduce the two-dimensional Dirac operator

$$\widetilde{\mathcal{D}}_0 = \begin{pmatrix} \partial_r & -2\partial_\zeta \\ \partial_t & -\partial_r \end{pmatrix}, \tag{2.41}$$

acting on spinors $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in (C^\infty(\mathbb{R}^3))^2$ – the reader will have noticed that $\widetilde{\mathcal{D}}_0$ can be obtained from the Dirac-Lévy-Leblond operator $\mathcal{D}_0$ of Section 2.3 by taking a formal Laplace transform with respect to the mass. The kernel of $\widetilde{\mathcal{D}}_0$ is given by the equations of motion obtained from the Lagrangian density

$$\big(\bar{\phi}_2(\partial_r\phi_1 - 2\partial_\zeta\phi_2) - \bar{\phi}_1(\partial_t\phi_1 - \partial_r\phi_2)\big)\ dt\ dr\ d\zeta.$$

Let $d\tilde{\pi}^\sigma_{\frac{1}{2}}$ be the Laplace transform with respect to $\mathcal{M}$ of the infinitesimal representation of $\mathfrak{sv}$ given in Definition 2.3. Then $d\tilde{\pi}^\sigma_{\frac{1}{2}}(\mathfrak{sch})$ preserves the space of solutions of the equation $\widetilde{\mathcal{D}}_0 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$, $\phi_1, \phi_2 \in C^\infty(\mathbb{R}^3)$. Now, by computing

$\widetilde{\mathcal{D}}_0\left(d\tilde{\pi}^\sigma_{\frac{1}{2}}(X)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right)$ for $X \in \mathfrak{sv}$ and $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ in the kernel of $\widetilde{\mathcal{D}}_0$, it clearly appears (do it!) that if one adds the constraint $\partial_\zeta\phi_1 = 0$, then $\widetilde{\mathcal{D}}_0\left(d\tilde{\pi}^\sigma_{\frac{1}{2}}(X)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\right) = 0$ for every $X \in \mathfrak{sv}$, and, what is more, the transformed spinor $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = d\tilde{\pi}_{\frac{1}{2}}(X)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ also satisfies the same constraint $\partial_\zeta\psi_1 = 0$. One may realize this constraint by adding to the Lagrangian density the Lagrange multiplier term $(h\partial_\zeta\bar{\phi}_1 - \bar{h}\partial_\zeta\phi_1)\ dt\ dr\ d\zeta$. The new equations of motion read then $\nabla\begin{pmatrix}-h/2\\ \phi_2\\ -\phi_1\end{pmatrix} = 0$, with

$$\nabla = \begin{pmatrix} 2\partial_\zeta & \partial_r & \partial_t \\ & 2\partial_\zeta & \partial_r \\ & & 2\partial_\zeta \end{pmatrix}. \tag{2.42}$$

This is our main example of a multi-diagonal differential operator. Quite generally, we shall call *multi-diagonal* a function- or operator-valued matrix $M = (M_{i,j})_{0\le i,j\le d-1}$ such that $M_{i,j} = M_{i+k,j+k}$ for every admissible triple of indices i,j,k. So M is defined for instance by the d independent coefficients $M_{0,0},\dots,M_{0,j}$, $\dots, M_{0,d-1}$, with $M_{0,j}$ located on the *j-shifted diagonal*.

An obvious generalization in d dimensions leads to the following definition.

Definition 2.4. *Let ∇^d be the $d\times d$ matrix differential operator of order one, acting on d-uples of functions $H = \begin{pmatrix}h_0\\ \vdots\\ h_{d-1}\end{pmatrix}$ on $\mathbb{R}^d$ with coordinates $t = (t_0,\dots,t_{d-1})$, given by*

$$\nabla^d = \begin{pmatrix} \partial_{t_{d-1}} & \partial_{t_{d-2}} & \cdots & \partial_{t_1} & \partial_{t_0} \\ 0 & \ddots & & & \partial_{t_1} \\ \vdots & \ddots & & & \vdots \\ & & & & \partial_{t_{d-2}} \\ 0 & \cdots & & 0 & \partial_{t_{d-1}} \end{pmatrix}. \tag{2.43}$$

So ∇^d is upper-triangular, with coefficients $\nabla^d_{i,j} = \partial_{t_{i-j+d-1}}$, $i \le j$. The kernel of ∇^d is defined by a system of equations linking $h_0,\dots,h_{d-1}$. The set of differential operators of order one of the form

$$X = X_1 + \Lambda = \Big(\sum_{i=0}^{d-1} f_i(t)\partial_{t_i}\Big)\otimes \mathrm{Id} + \Lambda, \quad \Lambda = (\lambda_{i,j}) \in Mat_{d\times d}(C^\infty(\mathbb{R}^d)) \tag{2.44}$$

preserving the equation $\nabla^d H = 0$ forms a Lie algebra, much too large for our purpose.

Suppose now (this is a very restrictive condition) that $\Lambda = \mathrm{diag}(\lambda_0,\dots,\lambda_{d-1})$ is diagonal. Since ∇^d is an operator with constant coefficients, $[X_1,\nabla^d]$ has no term of zero order, whereas $[\Lambda,\nabla^d]_{i,j} = \lambda_i\partial_{t_{i-j+d-1}} - \partial_{t_{i-j+d-1}}\lambda_j$ $(i \le j)$ does have terms of zero order in general. One possibility to solve this constraint, motivated by the preceding examples (see for instance the representation $d\tilde{\pi}_\lambda$ of (1.5)), but also by the theory of scaling in statistical physics (see commentary following Definition 1.3.), is to impose $\lambda_i = \lambda_i(t_0)$, $i = 0, \ldots, d-2$, and $\lambda_{d-1} = 0$. Since $[X, \nabla^d]$ is of first order, preserving $\operatorname{Ker} \nabla^d$ is equivalent to a relation of the type $[X, \nabla^d] = A\nabla^d$, with $A = A(X) \in Mat_{d\times d}(C^\infty(\mathbb{R}^d))$. Then the matrix operator $[X_1, \nabla^d]$ is upper-triangular, and multi-diagonal, so this must also hold for $A\nabla^d - [\Lambda, \nabla^d]$. By looking successively at the coefficients of $\partial_{t_{d-1-l}}$ on the l-shifted diagonals, $l = 0, \ldots, d-1$, one sees easily that A must also be upper-triangular and multi-diagonal, and that one must have $\lambda_i(t_0) - \lambda_{i+1}(t_0) = \lambda(t_0)$ for a certain function λ independent of i, so $\Lambda = \begin{pmatrix} (d-1)\lambda & & & \\ & \ddots & & \\ & & \lambda & \\ & & & 0 \end{pmatrix}$. Also, denoting by $a_0 = A_{0,0}, \ldots, a_{d-1} = A_{0,d-1}$ the coefficients of the first line of the matrix A, one obtains:

$$\partial_{t_i} f_j = 0 \quad (i > j); \tag{2.45}$$

$$a_0 = \partial_{t_0} f_0 + (d-1)\lambda = \partial_{t_1} f_1 + (d-2)\lambda = \cdots = \partial_{t_{d-1}} f_{d-1}; \tag{2.46}$$

$$a_i = \partial_{t_0} f_i = \partial_{t_1} f_{i+1} = \cdots = \partial_{t_{d-i-1}} f_{d-1} \quad (i = 1, \ldots, d-1). \tag{2.47}$$

In particular, f_0 depends only on t_0.

From all these considerations follows quite naturally the following definition. We let $\Lambda_0 \in Mat_{d\times d}(\mathbb{R})$ be the diagonal matrix $\Lambda_0 = \begin{pmatrix} d-1 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$.

Lemma 2.10. *Let $\mathfrak{mv}^d_\varepsilon$ ($\varepsilon \in \mathbb{R}$) be the set of differential operators of order one of the type*

$$X = \left(f_0(t_0)\partial_{t_0} + \sum_{i=1}^{d-1} f_i(t)\partial_{t_i} \right) \otimes \operatorname{Id} - \varepsilon f_0'(t_0) \otimes \Lambda_0 \tag{2.48}$$

preserving $\operatorname{Ker} \nabla^d$.

Then $\mathfrak{mv}_\varepsilon$ forms a Lie algebra.

Proof. Let $\mathcal{X}$ be the Lie algebra of vector fields X of the form

$$X = X_1 + X_0 = \left(f_0(t_0)\partial_{t_0} + \sum_{i=1}^{d-1} f_i(t)\partial_{t_i} \right) \otimes \operatorname{Id} + \Lambda,$$
$$\Lambda = \operatorname{diag}(\lambda_i)_{i=0,\ldots,d-1} \in Mat_{d\times d}(C^\infty(\mathbb{R}^d))$$

preserving $\operatorname{Ker} \nabla^d$. Then the set $\{Y = \sum_{i=0}^{d-1} f_i(t)\partial_{t_i} \mid \exists \lambda \in Mat_{d\times d}(C^\infty(\mathbb{R}^d)),\ Y + \Lambda \in \mathcal{X}\}$ of the differential parts of order one of the elements of $\mathcal{X}$ forms a Lie algebra, say $\mathcal{X}_1$. Define

$$\mathcal{X}_1^\varepsilon := \{\sum_{i=0}^{d-1} f_i(t)\partial_{t_i} - \varepsilon f_0'(t) \otimes \Lambda_0 \mid \sum_{i=0}^{d-1} f_i(t)\partial_{t_i} \in \mathcal{X}_1\}.$$

Let $Y = (\sum f_i(t)\partial_{t_i}) \otimes \operatorname{Id} - \varepsilon f_0'(t) \otimes \Lambda_0$, $Z = (\sum g_i(t)\partial_{t_i}) \otimes \operatorname{Id} - \varepsilon g_0'(t) \otimes \Lambda_0$ be two elements of $\mathcal{X}_1^\varepsilon$: then

$$[Y, Z] = ((f_0 g_0' - f_0' g_0)(t_0)\partial_{t_0} + \cdots) \otimes \operatorname{Id} - \varepsilon (f_0 g_0' - f_0' g_0)'(t_0) \otimes \Lambda_0$$

belongs to $\mathcal{X}_1^\varepsilon$, so $\mathcal{X}_1^\varepsilon$ forms a Lie algebra. Finally, $\mathfrak{md}_\varepsilon^d$ is the Lie subalgebra of $\mathcal{X}_1^\varepsilon$ consisting of all differential operators preserving $\operatorname{Ker} \nabla^d$. $\square$

It is quite possible to give a family of generators and relations for $\mathfrak{md}_\varepsilon^d$. The surprising fact, though, is the following: for $d \geq 4$, one finds by solving the equations that f_0'' is necessarily zero if $\varepsilon \neq 0$ (see proof of Theorem 3.11). So in any case, the only Lie algebra that deserves to be considered for $d \geq 4$ is $\mathfrak{md}_0^d$.

The algebras $\mathfrak{md}_\varepsilon^d$ $(d = 2, 3)$, $\mathfrak{md}_0^d$ $(d \geq 4)$ are semi-direct products of a Lie subalgebra isomorphic to $\operatorname{Vect}(S^1)$, with generators

$$L_{f_0}^{(0)} = (-f_0(t_0)\partial_{t_0} + \cdots) \otimes \mathrm{Id} - \varepsilon f_0'(t_0) \otimes \Lambda_0$$

and commutators $[L_{f_0}^{(0)}, L_{g_0}^{(0)}] = L_{f_0' g_0 - f_0 g_0'}^{(0)}$, with a nilpotent Lie algebra consisting of all generators with coefficient of ∂_{t_0} vanishing. When $d = 2, 3$, one retrieves realizations of the familiar Lie algebras $\operatorname{Vect}(S^1) \ltimes \mathcal{F}_{1+\varepsilon}$ and $\mathfrak{sv}_\varepsilon$.

Theorem 2.11 (structure of $\mathfrak{md}_\varepsilon^d$).

1. (*case* $d = 2$). *Put* $t = t_0, r = t_1$*: then* $\mathfrak{md}_\varepsilon^2 = \langle L_f^{(0)}, L_g^{(1)} \rangle_{f,g \in C^\infty(S^1)}$ *with*

$$L_f^{(0)} = (-f(t)\partial_t - (1+\varepsilon) f'(t) r \partial_r) \otimes \mathrm{Id} + \varepsilon f'(t) \otimes \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \tag{2.49}$$

$$L_g^{(1)} = -g(t)\partial_r. \tag{2.50}$$

It is isomorphic to $\operatorname{Vect}(S^1) \ltimes \mathcal{F}_{1+\varepsilon}$.

2. (*case* $d = 3$). *Put* $t = t_0$, $r = t_1$, $\zeta = t_2$*: then* $\mathfrak{md}_\varepsilon^3 = \langle L_f^{(0)}, L_g^{(1)}, L_h^{(2)} \rangle_{f,g,h \in C^\infty(S^1)}$ *with*

$$L_f^{(0)} = \left(-f(t)\partial_t - (1+\varepsilon) f'(t) r \partial_r - \left[(1+2\varepsilon) f'(t)\zeta + \frac{1+\varepsilon}{2} f''(t) r^2 \right] \partial_\zeta \right) \otimes \mathrm{Id} + \varepsilon f'(t) \otimes \begin{pmatrix} 2 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \tag{2.51}$$

$$L_g^{(1)} = -g(t)\partial_r - g'(t) r \partial_\zeta, \tag{2.52}$$

$$L_h^{(2)} = -h(t)\partial_\zeta. \tag{2.53}$$

The Lie algebra obtained by taking the modes

$$L_n = L_{t^{n+1}}^{(0)}, \ Y_m = L_{t^{m+1+\varepsilon}}^{(1)}, \ M_p = L_{t^{p+1+2\varepsilon}}^{(2)} \tag{2.54}$$

is isomorphic to $\mathfrak{sv}_{1+2\varepsilon}$ (see Definition 1.7). *In particular, the differential parts give three independent copies of the representation* $d\tilde{\pi}$ *of* $\mathfrak{sv}$ *when* $\varepsilon = -\frac{1}{2}$.

3. (*case* $\varepsilon = 0$, $d \geq 2$) *Then* $\mathfrak{mv}_\varepsilon^d \simeq \mathrm{Vect}(S^1) \otimes \mathbb{R}[\eta]/\eta^d$ *is generated by the*

$$L_g^{(k)} = -g(t_0)\partial_{t_k} - \sum_{i=1}^{d-1-k} g^{(i)}(t_0)t_1^{i-1} \left(\frac{1}{i!} t_1 \partial_{t_{i+k}} + \frac{1}{(i-1)!} \sum_{j=2}^{d-i-k} t_j \partial_{t_{i+j+k-1}} \right), \quad g \in C^\infty(S^1) \tag{2.55}$$

$k = 0, \ldots, d-1$, *with commutators* $[L_g^{(i)}, L_h^{(j)}] = L_{g'h-gh'}^{(i+j)}$ *if* $i + j \leq d-1$, 0 *else.*

Proof. Let $X = -\left(f_0(t_0)\partial_{t_0} + \sum_{i=1}^{d-1} f_i(t)\partial_{t_i}\right) \otimes \mathrm{Id} + \varepsilon f_0'(t) \otimes \Lambda_0$: a set of necessary and sufficient conditions for X to be in $\mathfrak{mv}_\varepsilon^d$ has been given before Lemma 3.10, namely

$$\partial_{t_i} f_j = 0 \text{ if } i > j,$$

$$\begin{aligned}(1 + \varepsilon(d-1))f_0'(t_0) &= \partial_{t_1} f_1(t_0, t_1) + \varepsilon(d-2)f_0'(t_0) \\ &= \cdots = \partial_{t_{d-1}} f_{d-1}(t_0, \ldots, t_{d-1})\end{aligned}$$

and

$$\partial_{t_0} f_i = \partial_{t_1} f_{i+1} = \cdots = \partial_{t_{d-i-1}} f_{d-1} \quad (i = 1, \ldots, d-1).$$

Solving successively these equations yields

$$f_i(t_0, \ldots, t_i) = (1 + \varepsilon i)f_0'(t_0) \, . \, t_i + f_i^{[1]}(t_0, \ldots, t_{i-1}), \quad i \geq 1; \tag{2.56}$$

$$f_i^{[1]}(t_0, \ldots, t_{i-1}) = \partial_{t_0} f_1^{[1]}(t_0) \, . \, t_{i-1} + (1 + \varepsilon)f_0''(t_0) \int_0^{t_{i-1}} t_1 dt_{i-1} + f_i^{[2]}(t_0, \ldots, t_{i-2}). \tag{2.57}$$

At the next step, the relation $\partial_{t_0} f_2 = \partial_{t_1} f_3$ yields the equation

$$\begin{aligned}(1 + 2\varepsilon)f_0''(t_0) \, . \, t_2 + (f_1^{[1]})''(t_0) \, . \, t_1 + 2(1 + \varepsilon)f_0'''(t_0) \, . \, t_1 + (f_2^{[2]})'(t_0) \\ = (1 + \varepsilon)f_0''(t_0) \, . \, t_2 + \partial_{t_1} f_3^{[2]}(t_0, t_1)\end{aligned}$$

which has no solution as soon as $\varepsilon \neq 0$ and $f_0'' \neq 0$. So, as we mentioned without proof before the theorem, the most interesting case is $\varepsilon = 0$ when $d \geq 4$.

The previous computations completely solve the cases $d = 2$ and $d = 3$. So let us suppose that $d \geq 4$ and $\varepsilon = 0$.

Then, by solving the next equations, one sees by induction that $f_0, \ldots, f_{d-1}$ may be expressed in terms of d arbitrary functions of t_0, namely, $f_0 = f_0^{[0]}, f_1^{[1]}, f_2^{[2]}, \ldots, f_{d-1}^{[d-1]}$, and that generators satisfying $f_i^{[i]} = 0$ for every $i \neq k$, k fixed, are necessarily of the form

$$f_k^{[k]}(t_0)\partial_{t_k} + \sum_{j=1}^{d-1-k} g_{k+j}(t_0, \ldots, t_j)\partial_{t_{k+j}}$$

for functions g_{k+j} that may be expressed in terms of $f_k^{[k]}$ and its derivatives.

One may then easily check that $L^{(k)}_{-f_k^{[k]}}$ is of this form and satisfies the conditions for being in $\mathfrak{mo}^d_\varepsilon$, so we have proved that the $L^{(k)}_f$, $k = 0, \ldots, d-1$, $f \in C^\infty(S^1)$, generate $\mathfrak{mo}^d_\varepsilon$.

All there remains to be done is to check for commutators. Since $L^{(i)}_f$ is homogeneous of degree $-i$ for the Euler-type operator $\sum_{k=0}^{d-1} kt_k\partial_{t_k}$, one necessarily has $[L^{(i)}_f, L^{(j)}_g] = L^{(i+j)}_{C(f,g)}$ for a certain function C (depending on f and g) of the time-coordinate t_0. One gets immediately $[L^{(0)}_f, L^{(0)}_g] = L^{(0)}_{f'g-fg'}$. Next (supposing $l > 0$), since

$$L^{(0)}_g = -\sum_{i=0}^{l-1} E^0_i(g)\partial_{t_i} - (g'(t_0)t_l + F^0_l(t_0, \ldots, t_{l-1}))\partial_{t_l} + \cdots$$

where $E^0_i(g)$, $i = 0, \ldots, l-1$ do not depend on t_l, and

$$L^{(l)}_h = -h(t_0)\partial_{t_l} + \cdots ,$$

one gets $[L^{(0)}_g, L^{(l)}_h] = (gh' - g'h)(t_0)\partial_{t_l} + \cdots$, so $[L^{(0)}_g, L^{(l)}_h] = L^{(l)}_{g'h-gh'}$. Considering now $k, l > 0$, then one has

$$L^{(k)}_g = -\sum_{i=0}^{l-1} E^k_i(g)\partial_{t_{i+k}} - (h'(t_0)t_l + F^k_l(t_0, \ldots, t_{l-1})\partial_{t_{l+k}}$$

where $E^k_i(g)$, $i = 0, \ldots, l-1$, do not depend on t_l, and a similar formula for $L^{(l)}_h$, which give together the right formula for $[L^{(k)}_g, L^{(l)}_h]$. $\square$

Let us come back to the original motivation, that is, finding new representations of $\mathfrak{sv}$ arising in a geometric context. Denote by $d\pi^{(3,0)}$ the realization of $\mathfrak{sv}$ given in Theorem 2.11.

Definition 2.5. *Let $d\pi^\nabla$ be the infinitesimal representation of $\mathfrak{sv}$ on the space $\widetilde{\mathcal{H}}^\nabla \simeq (C^\infty(\mathbb{R}^2))^3$ with coordinates t, r, defined by*

$$d\tilde{\rho}(L_f) = \left(-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r\right) \otimes \mathrm{Id} + f'(t) \otimes \begin{pmatrix} -1 & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix} \tag{2.58}$$

$$+ \frac{1}{2}f''(t)r \otimes \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \frac{1}{4}f'''(t)r^2 \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}; \tag{2.59}$$

$$d\tilde{\rho}(Y_f) = -f(t)\partial_r \otimes \mathrm{Id} + f'(t) \otimes \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + f''(t)r \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}; \tag{2.60}$$

$$d\tilde{\rho}(M_f) = f'(t) \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}. \tag{2.61}$$

Proposition 2.12. *For every* $X \in \mathfrak{sv}$, $d\pi^{\nabla}(X) \circ \nabla - \nabla \circ d\pi^{(3,0)}(X) = 0$.

Proof. Let $X \in \mathfrak{sv}$; put $d\pi^{(3,0)}(X) = -(f_0(t)\partial_t + f_1(t,r)\partial_r + f_2(t,r,\zeta)\partial_\zeta) \otimes \mathrm{Id} - f_0'(t) \otimes \begin{pmatrix} 1 & & \\ & \frac{1}{2} & \\ & & 0 \end{pmatrix}$.

The computations preceding Lemma 3.10 prove that $[d\pi^{(3,0)}(X), \nabla^d] = A(X)\nabla^d$, $A(X)$ being the upper-triangular, multi-diagonal matrix defined by

$$A(X)_{0,0} = \partial_\zeta f_2,\ A(X)_{0,1} = \partial_r f_2,\ A(X)_{0,2} = \partial_t f_2.$$

Hence one has

$$A(L_f) = \frac{r}{2} f''(t) \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + \frac{r^2}{4} f'''(t) \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix},$$

$$A(Y_g) = g'(t) \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} + rg''(t) \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix},$$

and

$$A(M_h) = f'(t) \otimes \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

Hence the result. $\square$

Remark. Consider the affine space

$$\mathcal{H}_{\nabla}^{\mathrm{aff}} = \{\nabla + \begin{pmatrix} g_0 & g_1 & g_2 \\ & g_0 & g_1 \\ & & g_0 \end{pmatrix} \mid g_0, g_1, g_2 \in C^\infty(S^1 \times \mathbb{R}^2)\}.$$

Then one may define an infinitesimal left-and-right action $d\sigma$ of $\mathfrak{sv}$ on $\mathcal{H}_{\nabla}^{\mathrm{aff}}$ by putting

$$d\sigma(X)(\nabla + V) = d\pi^{\nabla}(X) \circ (\nabla + V) - (\nabla + V) \circ d\pi^{(3,0)}(X),$$

but the action is simply linear this time, since $d\pi^{\nabla} \circ \nabla = \nabla \circ d\pi^{(3,0)}(X)$. So this action is not very interesting and doesn't give anything new.

3. Cartan's prolongation and generalized modules of tensor densities.

3.1. The Lie algebra $\mathfrak{sv}$ as a Cartan prolongation

As in the case of vector fields on the circle, it is natural, starting from the representation $d\tilde{\pi}$ of $\mathfrak{sv}$ given by Definition 1.3, to consider the subalgebra $\mathfrak{fsv} \subset \mathfrak{sv}$ made up of the vector fields with polynomial coefficients. Recall from Definition 1.4 that the outer derivation δ_2 of $\mathfrak{sv}$ is defined by

$$\delta_2(L_n) = n,\ \delta_2(Y_m) = m - \frac{1}{2},\ \delta_2(M_n) = n - 1 \quad (n \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z}). \tag{3.1}$$

and that δ_2 is simply obtained from the Lie action of the Euler operator $t\partial_t + r\partial_r + \zeta\partial_\zeta$ in the representation $d\tilde{\pi}$. The Lie subalgebra $\mathfrak{fsv}$ is given more abstractly, using δ_2, as

$$\mathfrak{fsv} = \oplus_{k=-1}^{+\infty} \mathfrak{sv}_k \tag{3.2}$$

where $\mathfrak{sv}_k = \{X \in \mathfrak{sv} \mid \delta_2(X) = kX\} = \langle L_k, Y_{k+\frac{1}{2}}, M_{k+1}\rangle$ is the eigenspace of δ_2 corresponding to the eigenvalue $k \in \mathbb{Z}$.

Note in particular that $\mathfrak{sv}_{-1} = \langle L_{-1}, Y_{-\frac{1}{2}}, M_0\rangle$ is commutative, generated by the infinitesimal translations $\partial_t, \partial_r, \partial_\zeta$ in the vector field representation, and that $\mathfrak{sv}_0 = \langle L_0, Y_{\frac{1}{2}}, M_1\rangle = \langle L_0\rangle \ltimes \langle Y_{\frac{1}{2}}, M_1\rangle$ is solvable.

Theorem 3.1. *The Lie algebra $\mathfrak{fsv}$ is isomorphic to the Cartan prolongation of $\mathfrak{sv}_{-1} \oplus \mathfrak{sv}_0$ where $\mathfrak{sv}_{-1} \simeq \langle L_{-1}, Y_{-\frac{1}{2}}, M_0\rangle$ and $\mathfrak{sv}_0 \simeq \langle L_0, Y_{\frac{1}{2}}, M_1\rangle$.*

Proof. Let $\mathfrak{sv}_n$ $(n = 1, 2, \ldots)$ the n-th level vector space obtained from Cartan's construction, so that the Cartan prolongation of $\mathfrak{sv}_{-1} \oplus \mathfrak{sv}_0$ is equal to the Lie algebra $\mathfrak{sv}_{-1} \oplus \mathfrak{sv}_0 \oplus \ \oplus_{n\geq 1} \mathfrak{sv}_n$. It will be enough, to establish the required isomorphism, to prove the following. Consider the representation $d\tilde{\pi}$ of $\mathfrak{sv}$. Then the space $\mathfrak{h}_n$ defined through induction on n by

$$\mathfrak{h}_{-1} = \pi(\mathfrak{sv}_{-1}) = \langle \partial_t, \partial_r, \partial_\zeta\rangle \tag{3.3}$$

$$\mathfrak{h}_0 = \pi(\mathfrak{sv}_0) = \langle t\partial_t + \frac{1}{2} r\partial_r, t\partial_r + r\partial_\zeta, t\partial_\zeta\rangle \tag{3.4}$$

$$\mathfrak{h}_{k+1} = \{X \in \mathcal{X}_{k+1} \mid [X, \mathfrak{h}_{-1}] \subset \mathfrak{h}_k\},\ (k \geq 0) \tag{3.5}$$

(where $\mathcal{X}_k$ is the space of vector fields with polynomial coefficients of degree k) is equal to $\pi(\mathfrak{sv}_n)$ for any $n \geq 1$.

So assume that $X = f(t,r,\zeta)\partial_t + g(t,r,\zeta)\partial_r + h(t,r,\zeta)\partial_\zeta$ satisfies

$$\begin{aligned} &[X, \mathfrak{h}_{-1}] \subset \pi(\mathfrak{sv}_n) = \\ &\langle t^{n+1}\partial_t + \frac{1}{2}(n+1)t^n r\partial_r + \frac{1}{4}(n+1)nt^{n-1}r^2\partial_\zeta, t^{n+1}\partial_r + (n+1)t^n r\partial_\zeta, t^{n+1}\partial_\zeta\rangle. \end{aligned} \tag{3.6}$$

In the following lines, C_1, C_2, C_3 are undetermined constants. Then (by comparing the coefficients of ∂_t)

$$f(t,r,\zeta) = C_1 t^{n+2}.$$

By inspection of the coefficients of ∂_r, one gets then

$$\partial_t g(t,r,\zeta) = \frac{C_1}{2}(n+1)(n+2)t^n r + C_2(n+2)t^{n+1}$$

so

$$g(t,r,\zeta) = \frac{C_1}{2}(n+2)t^{n+1}r + C_2 t^{n+2} + G(r,\zeta)$$

with an unknown polynomial $G(r,\zeta)$. But

$$[X, Y_{-\frac{1}{2}}] = \left(\frac{C_1}{2}(n+2)t^{n+1} + \partial_r G(r,\zeta)\right)\partial_r \bmod \partial_\zeta$$

so $\partial_r G(r,\zeta) = 0$.

Finally, by comparing the coefficients of ∂_ζ, one gets

$$[X, L_{-1}] = (n+2)C_1[t^{n+1}\partial_t + \frac{1}{2}(n+1)t^n r\partial_r] + C_2(n+2)t^{n+1}\partial_r + \partial_t h\ \partial_\zeta$$

so

$$\partial_t h(t,r,\zeta) = \frac{C_1}{4}(n+2)(n+1)nt^{n-1}r^2 + C_2(n+2)(n+1)t^n r + C_3(n+2)t^{n+1}$$

whence

$$h(t,r,\zeta) = \frac{C_1}{4}(n+2)(n+1)t^n r^2 + C_2(n+2)t^{n+1}r + C_3 t^{n+2} + H(r,\zeta)$$

where $H(r,\zeta)$ is an unknown polynomial. Also

$$[X, Y_{-\frac{1}{2}}] = \frac{C_1}{2}(n+2)t^{n+1}\partial_r + \frac{C_1}{2}(n+2)(n+1)t^n r\partial_\zeta + C_2(n+2)t^{n+1}\partial_\zeta + \partial_r H(r,\zeta)\partial_\zeta,$$

so $H = H(\zeta)$ does not depend on r; finally

$$[X, M_0] = \frac{dG(\zeta)}{d\zeta}\partial_r + \frac{dH(\zeta)}{d\zeta}\partial_\zeta$$

so $G = H = 0$. □

Remark. By modifying slightly the definition of $\mathfrak{sv}_0$, one gets related Lie algebras. For instance, substituting $L_0^\varepsilon := -t\partial_t - (1+\varepsilon)r\partial_r - (1+2\varepsilon)\zeta\partial_\zeta$ for L_0 leads to the 'polynomial part' of $\mathfrak{sv}_{1+2\varepsilon}$ (see Theorem 3.11 for an explicit realization of $\mathfrak{sv}_{1+2\varepsilon}$).

3.2. Coinduced representations of $\mathfrak{sv}$

In order to classify 'reasonable' representations of the Virasoro algebra, V.G. Kac made the following conjecture: the Harish-Chandra representations, those for which ℓ_0 acts semi-simply with finite-dimensional eigenspaces, are either higher- (or lower-) weight modules, or tensor density modules. As proved in [22] and [23], one has essentially two types of Harish-Chandra representations of the Virasoro algebra:
– Verma modules which are *induced* to $\mathfrak{vir}$ from a character of $\mathfrak{vir}_+ = \langle L_0, L_1, \ldots\rangle$, zero on the subalgebra $\mathfrak{vir}_{\geq 1} = \langle L_1, \ldots\rangle$, and quotients of degenerate Verma modules (see Section 6 for a generalization in our case);
– tensor modules of formal densities which are *coinduced* to the subalgebra of formal or polynomial vector fields $\mathrm{Vect}(S^1)_{\geq -1} = \langle L_{-1}, L_0, \ldots\rangle$ from a character of $\mathrm{Vect}(S^1)_{\geq 0}$ that is zero on the subalgebra $\mathrm{Vect}(S^1)_{\geq 1}$. These modules extend naturally to representations of $\mathrm{Vect}(S^1)$.

We shall generalize in this paragraph this second type of representations to the case of $\mathfrak{sv}$. Note that although we have two natural graduations on $\mathfrak{sv}$, the one given by the structure of Cartan prolongation is most adapted here since $\mathfrak{sv}_{-1}$ is commutative (see [1]).

Let $d\rho$ be a representation of $\mathfrak{sv}_0 = \langle L_0, Y_{\frac{1}{2}}, M_1\rangle$ into a vector space $\mathcal{H}_\rho$. Then $d\rho$ can be trivially extended to $\mathfrak{sv}_+ = \oplus_{i\geq 0}\mathfrak{sv}_i$ by setting $d\rho(\sum_{i>0}\mathfrak{sv}_i) = 0$.

Let $\mathfrak{fsv} = \oplus_{i\geq -1}\mathfrak{sv}_i \subset \mathfrak{sv}$ be the subalgebra of 'formal' vector fields: in the representation $d\pi$, the image of $\mathfrak{fsv}$ is the subset of vector fields that are polynomial in the time coordinate.

Let us now define the representation of $\mathfrak{fsv}$ coinduced from $d\rho$.

Definition 3.1. *The* ρ-formal density module $(\tilde{\mathcal{H}}_\rho, d\tilde{\rho})$ *is the coinduced module*

$$\tilde{\mathcal{H}}_\rho = \mathrm{Hom}_{\mathcal{U}(\mathfrak{sv}_+)}(\mathcal{U}(\mathfrak{fsv}), \mathcal{H}_\rho) \tag{3.7}$$

$$= \{\phi\colon \mathcal{U}(\mathfrak{fsv}) \to \mathcal{H}_\rho \text{ linear} \mid \phi(U_0 V) = d\rho(U_0).\phi(V), \quad U_0 \in \mathcal{U}_{\mathfrak{sv}_+}, V \in \mathcal{U}(\mathfrak{fsv})\} \tag{3.8}$$

with the natural action of $\mathcal{U}(\mathfrak{fsv})$ *on the right*

$$(d\tilde{\rho}(U).\phi)(V) = \phi(VU), \quad U, V \in \mathcal{U}(\mathfrak{fsv}). \tag{3.9}$$

By Poincaré-Birkhoff-Witt's theorem, this space can be identified with

$$\mathrm{Hom}(\mathcal{U}(\mathfrak{sv}_+) \setminus \mathcal{U}(\mathfrak{fsv}), \mathcal{H}_\rho) \simeq \mathrm{Hom}(\mathrm{Sym}(\mathfrak{sv}_{-1}), \mathcal{H}_\rho)$$

(linear applications from the symmetric algebra on $\mathfrak{sv}_{-1}$ into $\mathcal{H}_\rho$), and this last space is in turn isomorphic with the space $\mathcal{H}_\rho \otimes \mathbb{R}[[t,r,\zeta]]$ of $\mathcal{H}_\rho$-valued functions of t, r, ζ, through the application

$$\mathcal{H}_\rho \otimes \mathbb{R}[[t,r,\zeta]] \longrightarrow \mathrm{Hom}(\mathrm{Sym}(\mathfrak{sv}_{-1}), \mathcal{H}_\rho) \tag{3.10}$$

$$F(t,r,\zeta) \longrightarrow \phi_F : (U \to \partial_U F|_{t=0,r=0,\zeta=0}) \tag{3.11}$$

where ∂_U stands for the product derivative $\partial_{L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l} = (-\partial_t)^j (-\partial_r)^k (-\partial_\zeta)^l$ (note our choice of signs!).

We shall really be interested in the action of $\mathfrak{fsv}$ on functions $F(t,r,\zeta)$ that we shall denote by $d\sigma_\rho$, or $d\sigma$ for short.

The above morphisms allow one to compute the action of $\mathfrak{fsv}$ on monomials through the equality

$$\left(\frac{\partial_t^j}{j!}\ \frac{\partial_r^k}{k!}\ \frac{\partial_\zeta^l}{l!}\right)|_{t=0,r=0,\zeta=0}(d\sigma(X).F) = \frac{(-1)^{j+k+l}}{j!k!l!}(d\tilde{\rho}(X).\phi_F)(L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l) \tag{3.12}$$

$$= \frac{(-1)^{j+k+l}}{j!k!l!}\phi_F(L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l X), \quad X \in \mathfrak{fsv}. \tag{3.13}$$

In particular,

$$\partial_t^j \partial_r^k \partial_\zeta^l|_{t=0,r=0,\zeta=0}(d\sigma(L_{-1}).F) = -\partial_t^{j+1}\partial_r^k \partial_\zeta^l|_{t=0,r=0,\zeta=0} F$$

so $d\sigma(L_{-1}).F = -\partial_t F$; similarly, $d\sigma(Y_{-\frac{1}{2}}).F = -\partial_r F$ and $d\sigma(M_0).F = -\partial_\zeta F$.

So one may assume that $X \in \mathfrak{sv}_+$: by Poincaré-Birkhoff-Witt's theorem, $L_{-1}^j Y_{-\frac{1}{2}}^k M_0^l X$ can be rewritten as $U + V$ with

$$U \in \mathfrak{sv}_{>0}\mathcal{U}(\mathfrak{fsv})$$

and

$$V = V_1 V_2, \quad V_1 \in \mathcal{U}(\mathfrak{sv}_0), V_2 \in \mathcal{U}(\mathfrak{sv}_{-1}).$$

Then $\phi_F(U) = 0$ by definition of $\tilde{\mathcal{H}}_\rho$, and $\phi_F(V)$ may easily be computed as $\phi_F(V) = d\rho(V_1) \otimes \partial_{V_2}|_{t=0,r=0,\zeta=0} F$.

Theorem 3.2. *Let $f \in \mathbb{R}[t]$, the coinduced representation $d\tilde{\rho}$ is given by the action of the following matrix differential operators on functions:*

$$d\tilde{\rho}(L_f) = \left(-f(t)\partial_t - \frac{1}{2}f'(t)r\partial_r - \frac{1}{4}f''(t)r^2\partial_\zeta\right) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)d\rho(L_0) + \frac{1}{2}f''(t)rd\rho(Y_{\frac{1}{2}}) + \frac{1}{4}f'''(t)r^2 d\rho(M_1); \quad (3.14)$$

$$d\tilde{\rho}(Y_f) = (-f(t)\partial_r - f'(t)r\partial_\zeta) \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)d\rho(Y_{\frac{1}{2}}) + f''(t)r\ d\rho(M_1); \quad (3.15)$$

$$d\tilde{\rho}(M_f) = -f(t)\partial_\zeta \otimes \mathrm{Id}_{\mathcal{H}_\rho} + f'(t)\ d\rho(M_1). \quad (3.16)$$

Proof. One easily checks that these formulas define a representation of $\mathfrak{fsv}$. Since $(L_{-1}, Y_{-\frac{1}{2}}, M_0, L_0, L_1, L_2)$ generated $\mathfrak{fsv}$ as a Lie algebra, it is sufficient to check the above formulas for L_0, L_1, L_2 (they are obviously correct for $L_{-1}, Y_{-\frac{1}{2}}, M_0$).

Note first that M_0 is central in $\mathfrak{fsv}$, so

$$\partial_\zeta^l (d\sigma(X).F) = d\sigma(X).(\partial_\zeta^l F).$$

Hence it will be enough to compute the action on monomials of the form $t^j r^l \otimes v$, $v \in \mathcal{H}_\rho$.

We shall give a detailed proof since the computations in $\mathcal{U}(\mathfrak{fsv})$ are rather involved.

Let us first compute $d\sigma(L_0)$: one has

$$\begin{aligned}
(-\partial_t)^j(-\partial_r)^k|_{t=0,r=0}(d\sigma(L_0).F) &= \phi_F(L_{-1}^j Y_{-\frac{1}{2}}^k L_0) \\
&= \phi_F(L_{-1}^j L_0 Y_{-\frac{1}{2}}^k - \frac{k}{2} L_{-1}^j Y_{-\frac{1}{2}}^k) \\
&= \phi_F(L_0 L_{-1}^j Y_{-\frac{1}{2}}^k - (j + \frac{k}{2}) L_{-1}^j Y_{-\frac{1}{2}}^k) \\
&= \left[d\rho(L_0)(-\partial_t)^j(-\partial_r)^k - (j+\frac{k}{2})(-\partial_t)^j(-\partial_r)^k\right] F(0)
\end{aligned}$$

so

$$d\sigma(L_0) = -t\partial_t - \frac{1}{2}r\partial_r + d\rho(L_0).$$

Next,

$$\phi(L_{-1}^j Y_{-\frac{1}{2}}^k L_1) = \phi(L_{-1}^j L_1 Y_{-\frac{1}{2}}^k - k L_{-1}^j Y_{\frac{1}{2}} Y_{-\frac{1}{2}}^{k-1} + \frac{k(k-1)}{2} L_{-1}^j Y_{-\frac{1}{2}}^{k-2} M_0) =$$

$$\begin{aligned}
&= \phi((-2jL_0L_{-1}^{j-1} + j(j-1)L_{-1}^{j-1})Y_{-\frac12}^k) - k\phi(Y_{\frac12}L_{-1}^jY_{-\frac12}^{k-1})\\
&+ jk\phi(L_{-1}^{j-1}Y_{-\frac12}^k) + \frac{k(k-1)}{2}\phi(L_{-1}^jY_{-\frac12}^{k-2}M_0)\\
&= \big[(-2j)(-\partial_r)^k(-\partial_t)^{j-1}d\rho(L_0) + j(j-1)(-\partial_r)^k(-\partial_t)^{j-1}\\
&- k(-\partial_t)^j(-\partial_r)^{k-1}d\rho(Y_{\frac12})\\
&+ jk(-\partial_t)^{j-1}(-\partial_r)^k - \frac{k(k-1)}{2}\partial_\zeta(-\partial_t)^j(-\partial_r)^{k-2}\big]F(0)
\end{aligned}$$

hence the result for $d\sigma(L_1)$.
Finally,

$$\begin{aligned}
&\phi(L_{-1}^jY_{-\frac12}^kL_2)\\
&\quad= \phi(L_{-1}^jL_2Y_{-\frac12}^k - \frac32 kL_{-1}^jY_{\frac32}Y_{-\frac12}^{k-1} + 3\frac{k(k-1)}{2}L_{-1}^jM_1Y_{-\frac12}^{k-2})\\
&\quad= \phi((-3jL_1L_{-1}^{j-1} + 3j(j-1)L_0L_{-1}^{j-2} - j(j-1)(j-2)L_{-1}^{j-3})Y_{-\frac12}^k)\\
&\quad- \frac32 k\phi(-2jY_{\frac12}L_{-1}^{j-1}Y_{-\frac12}^{k-1} + j(j-1)L_{-1}^{j-2}Y_{-\frac12}^k)\\
&\quad+ \frac32 k(k-1)\phi(M_1L_{-1}^jY_{-\frac12}^{k-2} - jM_0L_{-1}^{j-1}Y_{-\frac12}^{k-2})\\
&\quad= \big[\big(3j(j-1)d\rho(L_0)(-\partial_t)^{j-2} - j(j-1)(j-2)(-\partial_t)^{j-2}\big)(-\partial_r)^k\\
&\quad- \frac32 k\big(-2jd\rho(Y_{\frac12})(-\partial_t)^{j-1}(-\partial_r)^{k-1} + j(j-1)(-\partial_t)^{j-2}(-\partial_r)^k\big)\\
&\quad+ \frac32 k(k-1)\big(d\rho(M_1)(-\partial_t)^j(-\partial_r)^{k-2} + j\partial_\zeta(-\partial_t)^{j-1}(-\partial_r)^{k-2}\big)\big]F(0).
\end{aligned}$$

Hence

$$d\sigma(L_2) = -t^3\partial_t - \frac32 t^2r\partial_r - \frac32 tr^2\partial_\zeta + 3t^2d\rho(L_0) + \frac32 t^2r\ d\rho(Y_{\frac12}) + \frac32 r^2\ d\rho(M_1).$$

□

Let us see how all actions defined in Section 2 (except for the coadjoint action!) derive from this construction.

Example 1. Take $\mathcal{H}_{\rho_\lambda} = \mathbb{R}$, $d\rho_\lambda(L_0) = -\lambda$, $d\rho_\lambda(Y_{\frac12}) = d\rho_\lambda(M_1) = 0$ ($\lambda \in \mathbb{R}$). Then $d\tilde\rho_\lambda = d\tilde\pi_\lambda$ (see Definition 1.3 for a definition of $d\tilde\pi_\lambda$).

Example 2. The linear part of the infinitesimal action on the affine space of Schrödinger operators (see Proposition 2.4) is given by the restriction of $d\tilde\rho_{-1}$ to functions of the type $g_0(t)r^2 + g_1(t)r + g_2(t)$.

Example 3. Take $\mathcal{H}_{\rho_\lambda} = \mathbb{R}^2$, $d\rho_\lambda(L_0) = \begin{pmatrix} 1/4 & \\ & -1/4 \end{pmatrix} - \lambda\mathrm{Id}$, $d\rho_\lambda(Y_{\frac12}) = -\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $d\rho_\lambda(M_1) = 0$. Then the infinitesimal representation $d\pi_\lambda^\sigma$ of Definition 2.3 (associated with the action on Dirac operators) is equal to $d\tilde\rho_\lambda$ (up to a Laplace transform in the mass).

Example 4. (action on multi-diagonal matrix differential operators) Take $\mathcal{H}_\rho = \mathbb{R}^3$, $d\rho(L_0) = \begin{pmatrix} -1 & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix}$, $d\rho(Y_{\frac{1}{2}}) = d\rho(M_1) = 0$ on the one hand;

$$\mathcal{H}_\sigma = \mathbb{R}^3,\ d\rho(L_0) = \begin{pmatrix} -1 & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix},\ d\rho(Y_{\frac{1}{2}}) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$$

and $d\rho(M_1) = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}$ on the other. Then $d\pi^{(3,0)} = d\tilde{\rho}$ and $d\pi^{\nabla} = d\tilde{\sigma}$ (see Proposition 3.11 in Section 2.4).

The fact that the coadjoint action cannot be obtained by this construction follows easily by comparing the formula for the action of the Y and M generators of Theorem 2.2 and Theorem 3.2: the second derivative f'' does not appear in $\mathrm{ad}^*(Y_f)$, while it does in $d\tilde{\rho}(Y_f)$ for any representation ρ such that $d\rho(M_1) \neq 0$; if $d\rho(M_1) = 0$, then, on the contrary, there's no way to account for the first derivative f' in $\mathrm{ad}^*(M_f)$.

Remark. The problem of classifying all coinduced representations is hence reduced to the problem of classifying the representations $d\rho$ of the Lie algebra $\langle L_0, Y_{\frac{1}{2}}, M_1\rangle$. This is *a priori* an untractable problem (due to the non-semi-simplicity of this Lie algebra), even if one is satisfied with finite-dimensional representations. An interesting class of examples (to which Examples 1 through 4 belong) is provided by extending a (finite-dimensional, say) representation $d\rho$ of the $(ax+b)$-type Lie algebra $\langle L_0, Y_{\frac{1}{2}}\rangle$ to $\langle L_0, Y_{\frac{1}{2}}, M_1\rangle$ by putting $d\rho(M_1) = C d\rho(Y_{\frac{1}{2}})^2$, where C is a constant. In particular, one may consider the spin s-representation $d\sigma$ of $\mathfrak{sl}(2,\mathbb{R})$, restrict it to the Borel subalgebra considered as $\langle L_0, Y_{\frac{1}{2}}\rangle$, 'twist' it by putting $d\sigma^\lambda := d\sigma + \lambda \mathrm{Id}$ and extend it to $\langle L_0, Y_{\frac{1}{2}}, M_1\rangle$ as we just explained.

4. Cohomology of $\mathfrak{sv}$ and $\mathfrak{tsv}$ and applications to central extensions and deformations

Cohomological computations for Lie algebras are mainly motivated by the search for deformations and central extensions. We concentrate on $\mathfrak{tsv}$ in the first three paragraphs of this section, because the generators of $\mathfrak{tsv}$ bear integer indices, which is more natural for computations. The main theorem is Theorem 4.1 in Paragraph 4.1, which classifies all deformations of $\mathfrak{tsv}$; Theorem 4.5 shows that all the infinitesimal deformations obtained in Paragraph 4.1 give rise to genuine deformations. One particularly interesting family of deformations is provided by the Lie algebras $\mathfrak{tsv}_\lambda$ ($\lambda \in \mathbb{R}$), which were introduced in Definition 1.5. We compute their central extensions in Paragraph 4.2, and compute in Paragraph 4.3 their deformations in the particular case $\lambda = 1$, for which $\mathfrak{tsv}_1$ is the tensor product of $\mathrm{Vect}(S^1)$ with a nilpotent associative and commutative algebra. Finally, in Paragraph 4.4, we come back to the original Schrödinger-Virasoro algebra and compute its deformations, as well as the central extensions of the family of deformed Lie algebras $\mathfrak{sv}_\lambda$.

4.1. Classifying deformations of $\mathfrak{tsv}$

We shall be interested in the classification of all formal deformations of $\mathfrak{tsv}$, following the now classical scheme of Nijenhuis and Richardson: deformation of a Lie algebra $\mathcal{G}$ means that one has a formal family of Lie brackets on $\mathcal{G}$, denoted $[\ ,\ ]_t$, inducing a Lie algebra structure on the extended Lie algebra $\mathcal{G} \bigotimes_k k[[t]] = \mathcal{G}[[t]]$.

As well known, one has to study the cohomology of $\mathcal{G}$ with coefficients in the adjoint representation; degree-two cohomology $H^2(\mathcal{G}, \mathcal{G})$ classifies the infinitesimal deformations (the terms of order one in the expected formal deformations) and $H^3(\mathcal{G}, \mathcal{G})$ contains the potential obstructions to a further prolongation of the deformations. So we shall naturally begin with the computation of $H^2(\mathfrak{tsv}, \mathfrak{tsv})$ (as usual, we shall consider only local cochains, equivalently given by differential operators, or polynomial in the modes):

Theorem 4.1. *One has* $\dim H^2(\mathfrak{tsv}, \mathfrak{tsv}) = 3$. *A set of generators is provided by the cohomology classes of the cocycles* c_1, c_2 *and* c_3, *defined as follows in terms of modes* (*the missing components of the cocycles are meant to vanish*):
$c_1(L_n, Y_m) = -\frac{n}{2} Y_{n+m}$, $\quad c_1(L_n, M_n) = -nM_{n+m}$
$c_2(L_n, Y_m) = Y_{n+m}$ $\quad c_2(L_n, M_m) = 2M_{n+m}$
$c_3(L_n, L_m) = (m - n)M_{n+m}$.

Remarks.

1. The cocycle c_1 gives rise to the family of Lie algebras $\mathfrak{tsv}_\varepsilon$ described in Definition 1.5.

2. The cocycle c_3 can be described globally as $c_3\colon \operatorname{Vect}(S^1) \times \operatorname{Vect}(S^1) \longrightarrow \mathcal{F}_0$ given by

$$c_3(f\partial, g\partial) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}.$$

This cocycle appeared in [10] and has been used in a different context in [12].

Before entering the technicalities of the proof, we shall indicate precisely, for the comfort of the reader, some cohomological results on $\mathcal{G} = \operatorname{Vect}(S^1)$ which will be extensively used in the sequel.

Proposition 4.2. (see [10], or [12], Chap. IV for a more elementary approach)

(1) $\operatorname{Inv}_{\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathcal{F}_\mu) = 0$ *unless* $\mu = -1 - \lambda$ *and* $\mathcal{F}_\mu = \mathcal{F}_\lambda^*$; *then* $\operatorname{Inv}_{\mathcal{G}}(\mathcal{F}_\lambda \otimes \mathcal{F}_\lambda^*)$ *is one-dimensional, generated by the identity mapping.*

(2) $H^i(\mathcal{G}, \mathcal{F}_\lambda \otimes \mathcal{F}_\mu) \equiv 0$ *if* $\lambda \neq 1 - \mu$ *and* λ *or* μ *are not integers.*
(1) and (2) can be immediately deduced from [10], Theorem 2.3.5 pp. 136–137.

(3) *Let* W_1 *be the Lie algebra of formal vector fields on the line, its cohomology represents the algebraic part of the cohomology of* $\mathcal{G} = \operatorname{Vect}(S^1)$ (see again

[10], Theorem 2.4.12). *Then* $H^1(W_1, \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\lambda))$ *is one-dimensional, generated by the cocycle* $(f\partial, adx^{-\lambda}) \longrightarrow f'adx^{-\lambda}$ (cocycle I_λ in [10], p. 138).

(4) *Invariant antisymmetric bilinear operators* $\mathcal{F}_\lambda \times \mathcal{F}_\mu \longrightarrow \mathcal{F}_\nu$ *between densities have been determined by P. Grozman* (see [11], p. 280).
They are of the following type:
  (a) *the Poisson bracket for* $\nu = \lambda + \mu - 1$, *defined by*

$$\{fdx^{-\lambda}, gdx^{-\mu}\} = (\lambda f g' - \mu f' g)dx^{-(\lambda+\mu-1)}$$

  (b) *the following three exceptional brackets:*
  $\mathcal{F}_{1/2} \times \mathcal{F}_{1/2} \to \mathcal{F}_{-1}$ *given by* $(f\partial^{1/2}, g\partial^{1/2}) \to \frac{1}{2}(fg'' - gf'')dx$;
  $\mathcal{F}_0 \times \mathcal{F}_0 \to \mathcal{F}_{-3}$ *given by* $(f, g) \to (f''g' - g''f')dx^3$;
  and an operator $\mathcal{F}_{2/3} \times \mathcal{F}_{2/3} \to \mathcal{F}_{-\frac{5}{3}}$ *called the Grozman bracket* (see [11], p. 274).

Proof of Theorem 4.1. We shall use standard techniques in Lie algebra cohomology; the proof will be rather technical, but without specific difficulties. Let us fix the notations: set $\mathfrak{tsv} = \mathcal{G} \ltimes \mathfrak{h}$ where $\mathcal{G} = \mathrm{Vect}(S^1)$ and $\mathfrak{h}$ is the nilpotent part of $\mathfrak{tsv}$.

One can consider the exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathcal{G} \ltimes \mathfrak{h} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{4.1}$$

as a short exact sequence of $\mathcal{G} \ltimes \mathfrak{h}$ modules, thus inducing a long exact sequence in cohomology:

$$\begin{aligned}\cdots \longrightarrow H^1(\mathfrak{tsv}, \mathcal{G}) \longrightarrow H^2(\mathfrak{tsv}, \mathfrak{h}) \longrightarrow H^2(\mathfrak{tsv}, \mathfrak{tsv})& \\ \longrightarrow H^2(\mathfrak{tsv}, \mathcal{G}) \longrightarrow H^3(\mathfrak{tsv}, \mathfrak{h}) \longrightarrow \cdots&\end{aligned} \tag{4.2}$$

So we shall consider $H^*(\mathfrak{tsv}, \mathcal{G})$ and $H^*(\mathfrak{tsv}, \mathfrak{h})$ separately.

Lemma 4.3. $H^*(\mathfrak{tsv}, \mathcal{G}) = 0$ *for* $* = 0, 1, 2$.

Proof of Lemma 4.3. One uses the Hochschild-Serre spectral sequence associated with the exact sequence (4.1). Let us remark first that $H^*(\mathcal{G}, H^*(\mathfrak{h}, \mathcal{G})) = H^*(\mathcal{G}, H^*(\mathfrak{h}) \otimes \mathcal{G})$ since $\mathfrak{h}$ acts trivially on $\mathcal{G}$. So one has to understand $H^*(\mathfrak{h})$ in low dimensions; let us consider the exact sequence $0 \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{h} \longrightarrow y \longrightarrow 0$, where $\mathfrak{n} = [\mathfrak{h}, \mathfrak{h}]$. As $\mathcal{G}$-modules, these algebras are density modules, more precisely $\mathfrak{n} = \mathcal{F}_0$ and $y = \mathcal{F}_{1/2}$. So $H^1(\mathfrak{h}) = y^* = \mathcal{F}_{-3/2}$ as a $\mathcal{G}$-module. Let us recall that, as a module on itself, $\mathcal{G} = \mathcal{F}_1$. One gets: $E_2^{p,0} = H^p(\mathcal{G}, \mathcal{G}) = 0$ as well known (see [10]),

$$E_2^{1,1} = H^1(\mathcal{G}, H^1(\mathfrak{h}) \otimes \mathcal{G}) = H^1(\mathcal{G}, \mathcal{F}_{-3/2} \otimes \mathcal{F}_1).$$

The determination of cohomologies of $\mathrm{Vect}(S^1)$ with coefficients in tensor products of modules of densities has been done by Fuks (see [10], Chap. 2, Thm. 2.3.5, or Proposition 4.2 (2) above), in this case everything vanishes and $E_2^{1,1} = 0$.

One has now to compute $H^2(\mathfrak{h})$ in order to get $E_2^{0,2} = \mathrm{Inv}_{\mathcal{G}}(H^2(\mathfrak{h}) \otimes \mathcal{G})$. We shall use the decomposition of cochains on $\mathfrak{h}$ induced by its splitting into vector subspaces: $\mathfrak{h} = \mathfrak{n} \oplus y$. So $C^1(\mathfrak{h}) = \mathfrak{n}^* \oplus y^*$ and $C^2(\mathfrak{h}) = \Lambda^2\mathfrak{n}^* \oplus \Lambda^2 y^* \oplus y^* \wedge \mathfrak{n}^*$. The coboundary ∂ is induced by the only non-vanishing part $\partial\colon \mathfrak{n}^* \longrightarrow \Lambda^2 y^*$ which is dual to the bracket $\Lambda^2 y \longrightarrow \mathfrak{n}$. So the cohomological complex splits into three subcomplexes and one deduces the following exact sequences:

$$0 \longrightarrow \mathfrak{n}^* \xrightarrow{\partial} \Lambda^2 y^* \longrightarrow M_1 \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow \Lambda^2 \mathfrak{n}^* \xrightarrow{\partial} \Lambda^2 y^* \otimes \mathfrak{n}^*$$

$$0 \longrightarrow M_3 \longrightarrow y^* \wedge \mathfrak{n}^* \xrightarrow{\partial} \Lambda^3 y^*$$

and $H^2(\mathfrak{h}) = M_1 \oplus M_2 \oplus M_3$. One can then easily deduce the invariants $\mathrm{Inv}_{\mathcal{G}}(H^2(\mathfrak{h}) \otimes \mathcal{G}) = \bigoplus_{i=1}^{3} \mathrm{Inv}_{\mathcal{G}}(M_i \otimes \mathcal{G})$ from the cohomological exact sequences associated with the above short exact sequences. One has:

$$0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}(M_2 \otimes \mathcal{G}) \longrightarrow \mathrm{Inv}_{\mathcal{G}}(\Lambda^2 \mathfrak{n}^* \otimes \mathcal{G}) = 0$$

and

$$0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}(M_3 \otimes \mathcal{G}) \longrightarrow \mathrm{Inv}_{\mathcal{G}}(y^* \wedge \mathfrak{n}^* \otimes \mathcal{G}) = 0$$

from Proposition 4.2;

$$\cdots \longrightarrow \mathrm{Inv}_{\mathcal{G}}(\Lambda^2 y^* \otimes \mathcal{G}) \longrightarrow \mathrm{Inv}_{\mathcal{G}}(M_1 \otimes \mathcal{G}) \longrightarrow H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathcal{G}) \xrightarrow{\partial_*} H^1(\mathcal{G}, \Lambda^2 y^* \otimes \mathcal{G}) \longrightarrow \cdots$$

From the same proposition, one gets $\mathrm{Inv}_{\mathcal{G}}(\Lambda^2 y^* \otimes \mathcal{G}) = 0$ and we shall see later (see the last part of the proof) that ∂_* is an isomorphism. So $\mathrm{Inv}_{\mathcal{G}}(H^2(\mathfrak{h}) \otimes \mathcal{G}) = 0$ and $E_2^{0,2} = 0$. The same argument shows that $E_2^{0,1} = 0$, which ends the proof of the lemma. □

From the long exact sequence (4.2) one has now: $H^*(\mathfrak{tsv}, \mathfrak{tsv}) = H^*(\mathfrak{tsv}, \mathfrak{h})$ for $* = 0, 1, 2$. We shall compute $H^*(\mathfrak{tsv}, \mathfrak{h})$ by using the Hochschild-Serre spectral sequence once more; there are three terms to compute.

1. First $E_2^{2,0} = H^2(\mathcal{G}, H^0(\mathfrak{h}, \mathfrak{h}))$, but $H^0(\mathfrak{h}, \mathfrak{h}) = Z(\mathfrak{h}) = \mathfrak{n} = \mathcal{F}_0$ as $\mathcal{G}$-module. So $E_2^{2,0} = H^2(\mathcal{G}, \mathcal{F}_0)$ which is one-dimensional, given by $c_3(f\partial, g\partial) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$, or in terms of modes $c_3(L_n, L_m) = (m-n)M_{n+m}$. Hence we have found one of the classes announced in the theorem.

2. One must now compute $E_2^{1,1} = H^1(\mathcal{G}, H^1(\mathfrak{h}, \mathfrak{h}))$. The following lemma will be useful for this purpose, and also for the last part of the proof.

Lemma 4.4 (identification of $H^1(\mathfrak{h}, \mathfrak{h})$ as a $\mathcal{G}$-module). *The space $H^1(\mathfrak{h}, \mathfrak{h})$ splits into the direct sum of two $\mathcal{G}$-modules $H^1(\mathfrak{h}, \mathfrak{h}) = \mathcal{H}_1 \oplus \mathcal{H}_2$ such that*

1. $\mathrm{Inv}_{\mathcal{G}}\, \mathcal{H}_2 = 0$, $H^1(\mathcal{G}, \mathcal{H}_2) = 0$;
2. $\mathrm{Inv}_{\mathcal{G}}\, \mathcal{H}_1$ *is one-dimensional, generated by the 'constant multiplication' cocycle l defined by*

$$l(Y_n) = Y_n, \quad l(M_n) = 2M_n. \tag{4.3}$$

3. $H^1(\mathcal{G},\mathcal{H}_1)$ *is two-dimensional, generated by two cocycles* c_1, c_2 *defined by*
$$c_1(f\partial, g\partial^{1/2}) = f'g\partial^{1/2}, \quad c_1(f\partial, g) = 2f'g$$
and
$$c_2(f\partial, g\partial^{1/2}) = fg\partial^{1/2}, \quad c_2(f\partial, g) = 2fg.$$
4. $H^2(\mathcal{G},\mathcal{H}_1)$ *is one-dimensional, generated by the cocycle* c_{12} *defined by*
$$c_{12}(f\partial, g\partial, h\partial^{1/2}) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h\partial^{1/2}, \quad c_{12}(f\partial, g\partial, h) = 2\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h.$$

Proof of Lemma 4.4. We shall split the cochains according to the decomposition $\mathfrak{h} = y \oplus \mathfrak{n}$. Set $C^1(\mathfrak{h},\mathfrak{h}) = C_1 \oplus C_2$, where:
$$C_1 = (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y) \qquad C_2 = (\mathfrak{n}^* \otimes y) \oplus (y^* \otimes \mathfrak{n}).$$
So one readily obtains the splitting $H^1(\mathfrak{h},\mathfrak{h}) = \mathcal{H}_1 \oplus \mathcal{H}_2$ where
$$0 \longrightarrow \mathcal{H}_1 \longrightarrow (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y) \stackrel{\partial}{\longrightarrow} \Lambda^2 y^* \otimes \mathfrak{n}$$
$$0 \longrightarrow y \stackrel{\partial}{\longrightarrow} y^* \otimes \mathfrak{n} \longrightarrow \mathcal{H}_2 \longrightarrow 0$$
∂ being the coboundary on the space of cochains on $\mathfrak{h}$ with coefficients into itself. Its non vanishing pieces in degrees $0, 1$ and 2 are the following ones: $y \stackrel{\partial}{\longrightarrow} y^* \otimes \mathfrak{n}$, $\mathfrak{n}^* \otimes \mathfrak{n} \stackrel{\partial}{\longrightarrow} \Lambda^2 y^* \otimes \mathfrak{n}$, $y^* \otimes y \stackrel{\partial}{\longrightarrow} \Lambda^2 y^* \otimes \mathfrak{n}$. We can now describe the second exact sequence in terms of densities as follows:
$$0 \longrightarrow \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_{-3/2} \otimes \mathcal{F}_0 \longrightarrow \mathcal{H}_2 \longrightarrow 0. \tag{4.4}$$

From Proposition 4.2, one has $\mathrm{Inv}_{\mathcal{G}}(\mathcal{F}_{-3/2} \otimes \mathcal{F}_0) = 0$ as well as $H^i(\mathcal{G},\mathcal{F}_{1/2}) = 0$, for $i = 0, 1, 2$, and $H^1(\mathcal{G},\mathcal{F}_{-3/2} \otimes \mathcal{F}_0) = 0$. So the long exact sequence in cohomology associated with (4.3) gives $\mathrm{Inv}_{\mathcal{G}}(\mathcal{H}_2) = 0$ and $H^1(\mathcal{G},\mathcal{H}_2) = 0$.

For $\mathcal{H}_1$, one has to analyse the cocycles by direct computation. So let $l \in C_1$ given by $l(Y_n) = a_n(k)Y_{n+k}$, $l(M_n) = b_n(k)M_{n+k}$. The cocycle conditions are given by:
$$\partial l(Y_n, Y_m) = l((m-n)M_{n+m}) - Y_n \cdot l(Y_m) + Y_m \cdot l(Y_n) = 0$$
for all $(n,m) \in \mathbb{Z}^2$. So identifying the term in M_{n+m+k}, one obtains:
$$(m-n)b_{n+m}(k) = (m-n+k)a_m(k) - (n-m+k)a_n(k)$$
so $b_{n+m}(k) = a_m(k) + a_n(k) + \frac{k}{m-n}(a_m(k) - a_n(k)) = f(n,m,k)$.
One can now determine the $a_n(k)$, remarking that the function $f(n,m,k)$ depends only on k and $(n+m)$. One then obtains that $a_n(k)$ must be affine in n so:
$$a_n(k) = n\lambda(k) + \mu(k)$$
$$b_n(k) = n\lambda(k) + k\lambda(k) + 2\mu(k).$$
So, as a vector space $\mathcal{H}_1$ is isomorphic to $\bigoplus_k \mathbb{C}(\lambda(k)) \bigoplus_k \mathbb{C}(\mu(k))$, two copies of an infinite direct sum of a numerable family of one-dimensional vector spaces.

Now we have to compute the action of $\mathcal{G}$ on $\mathcal{H}_1$; let $L_p \in \mathcal{G}$, one has

$$\begin{aligned}(L_p \cdot l)(Y_n) &= ((n-\frac{p}{2})a_{n+p}(k) - (n+k-\frac{p}{2})a_n(k))Y_{n+p+k} \\ &= \left(n(p-k)\lambda(k) - (\frac{p^2}{2}\lambda(k) + k\mu(k))\right) Y_{n+p+k}.\end{aligned}$$

So if one sets $(L_p \cdot l)(Y_n) = (n(L_p \cdot \lambda)(k+p) + (L_p \cdot \mu)(k+p))Y_{n+p+k}$ one obtains:

$$\begin{aligned}(L_p \cdot \lambda)(k+p) &= (p-k)\lambda(k) \\ (L_p \cdot \mu)(k+p) &= -\frac{p^2}{2}\lambda(k) + k\mu(k).\end{aligned}$$

Finally, $\mathcal{H}_1$ appears as an extension of modules of densities of the following type: $0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{H}_1 \longrightarrow \mathcal{F}_1 \longrightarrow 0$, in which $\mathcal{F}_0$ corresponds to $\bigoplus_k \mathbb{C}(\mu(k))$ and $\mathcal{F}_1$ to $\bigoplus_k \mathbb{C}(\lambda(k))$.

There is a non-trivial extension cocycle γ in $\mathrm{Ext}^1_{\mathcal{G}}(\mathcal{F}_1, \mathcal{F}_0) = H^1(\mathcal{G}, \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_0))$, given by $\gamma(f\partial)(g\partial) = f''g$; this cocycle corresponds to the term in p^2 in the above formula. In any case one has a long exact sequence in cohomology

$$\cdots \longrightarrow H^i(\mathcal{G}, \mathcal{F}_0) \longrightarrow H^i(\mathcal{G}, \mathcal{H}_1) \longrightarrow H^i(\mathcal{G}, \mathcal{F}_1) \longrightarrow H^{i+1}(\mathcal{G}, \mathcal{F}_0) \longrightarrow \cdots$$

As well known, $H^*(\mathcal{G}, \mathcal{F}_1) = H^*(\mathcal{G}, \mathcal{G})$ is trivial, and finally $H^i(\mathcal{G}, \mathcal{F}_0)$ is isomorphic to $H^i(\mathcal{G}, \mathcal{H}_1)$. In particular $H^0(\mathcal{G}, \mathcal{H}_1) = H^0(\mathcal{G}, \mathcal{F}_0)$ is one-dimensional, given by the constants; a scalar μ induces an invariant cocycle as $l(Y_n) = \mu Y_n,\ l(M_n) = 2\mu M_n$.

Moreover, $H^1(\mathcal{G}, \mathcal{F}_0)$ has dimension 2: it is generated by the cocycles $\overline{c}_1$ and $\overline{c}_2$, defined by $\overline{c}_1(f\partial) = f'$ and $\overline{c}_2(f\partial) = f$ respectively. So one obtains two generators of $H^1(\mathcal{G}, \mathcal{H}_1)$ given by

$$c_1(f\partial, g\partial^{1/2}) = f'g\partial^{1/2}, \quad c_1(f\partial, g) = 2f'g$$

and

$$c_2(f\partial, g\partial^{1/2}) = fg\partial^{1/2}, \quad c_2(f\partial, g) = 2fg$$

respectively.

Finally $H^2(\mathcal{G}, \mathcal{F}_0)$ is one-dimensional, with the cup-product $\overline{c}_{12}$ of $\overline{c}_1$ and $\overline{c}_2$ as generator (see [10], p. 177), so $\overline{c}_{12}(f\partial, g\partial) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$, and one deduces the formula for the corresponding cocycle c_{12} in $H^2(\mathcal{G}, \mathcal{H}_1)$:

$$c_{12}(f\partial, g\partial, h\partial^{1/2}) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h\partial^{1/2}, \quad c_{12}(f\partial, g\partial, h) = 2\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} h$$

This finishes the proof of Lemma 4.4. □

So, from Lemma 4.4, we have computed $E_2^{1,1} = H^1(\mathcal{G}, H^1(\mathfrak{h},\mathfrak{h}))$; it is two-dimensional, generated by c_1 and c_2, while earlier we had $H^2(\mathcal{G}, H^0(\mathfrak{h},\mathfrak{h})) = E_2^{2,0}$, a one-dimensional vector space generated by c_3. We have to check now that these cohomology classes shall not disappear in the spectral sequence; the only potentially non-vanishing differentials are $E_2^{0,1} \longrightarrow E_2^{2,0}$ and $E_2^{1,1} \longrightarrow E_2^{3,0}$. One has $E_2^{3,0} = H^3(\mathcal{G},\mathfrak{h}) = H^3(\mathcal{G},\mathfrak{n}) = H^3(\mathcal{G},\mathcal{F}_0) = 0$ (see [10] p. 177); here we consider only local cohomology), then $E_2^{0,1}$ is one-dimensional determined by the constant multiplication (see above) and direct verification shows that $E_2^{0,1} \longrightarrow E_2^{2,0}$ vanishes. So we have just proved that the cocycles c_1, c_2 and c_3 defined in the theorem represent genuinely non-trivial cohomology classes in $H^2(\mathfrak{tsv},\mathfrak{tsv})$.

3. In order to finish the proof, we still have to prove that there does not exist any other non-trivial class in the last piece of the Hochschild-Serre spectral sequence. We shall thus prove that $E_2^{0,2} = \mathrm{Inv}_{\mathcal{G}}\, H^2(\mathfrak{h},\mathfrak{h}) = 0$ As in the proofs of the previous lemmas, we shall use decompositions of the cohomological complex of $\mathfrak{h}$ with coefficients into itself as sums of $\mathcal{G}$-modules.

The space of adjoint cochains $C^2(\mathfrak{h},\mathfrak{h})$ will split into six subspaces according to the vector space decomposition $\mathfrak{h} = y \oplus \mathfrak{n}$. So we can as well split the cohomological complex

$$C^1(\mathfrak{h},\mathfrak{h}) \xrightarrow{\partial} C^2(\mathfrak{h},\mathfrak{h}) \xrightarrow{\partial} C^3(\mathfrak{h},\mathfrak{h})$$

into its components, and the coboundary operators will as well split into different components, as we already explained. So one obtains the following families of exact sequences of $\mathcal{G}$-modules:

$$(\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y) \xrightarrow{\partial} \Lambda^2 y^* \otimes \mathfrak{n} \longrightarrow A_1 \longrightarrow 0 \tag{4.5}$$

$$0 \longrightarrow K \longrightarrow (\Lambda^2 y^* \otimes y) \oplus (\mathfrak{n}^* \wedge y^* \otimes y) \xrightarrow{\partial} \Lambda^3 y^* \otimes \mathfrak{n}$$

$$0 \longrightarrow \mathfrak{n}^* \otimes y \xrightarrow{\partial} K \longrightarrow A_2 \longrightarrow 0 \tag{4.6}$$

$$0 \longrightarrow A_3 \longrightarrow \mathfrak{n}^* \wedge y^* \otimes y \xrightarrow{\partial} (\Lambda^3 y^* \otimes y) \oplus (\mathfrak{n}^* \wedge \Lambda^2 y^*) \otimes \mathfrak{n}$$

$$0 \longrightarrow A_4 \longrightarrow \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n} \xrightarrow{\partial} (\mathfrak{n}^* \wedge \Lambda^2 y^*) \otimes \mathfrak{n}$$

$$0 \longrightarrow A_5 \longrightarrow \Lambda^2 \mathfrak{n}^* \otimes y \xrightarrow{\partial} (\mathfrak{n}^* \wedge \Lambda^2 y^*) \otimes y \oplus (\Lambda^2 \mathfrak{n}^* \wedge y^*) \otimes \mathfrak{n}$$

The restrictions of coboundary operators are still denoted by ∂, and the other arrows are either inclusions of subspaces or projections onto quotients. So one has $H^2(\mathfrak{h},\mathfrak{h}) = \bigoplus_{i=1}^{5} A_i$, and our result will follow from $\mathrm{Inv}_{\mathcal{G}}\, A_i = 0,\ i = 1,\dots 5$. For the last three sequences, the result follows immediately from the cohomology long exact sequence by using $\mathrm{Inv}_{\mathcal{G}}(\mathfrak{n}^* \wedge y^* \otimes y) = 0$, $\mathrm{Inv}_{\mathcal{G}}\, \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{n} = 0$, $\mathrm{Inv}_{\mathcal{G}}\, \Lambda^2 \mathfrak{n}^* \otimes y = 0$: there results are deduced from those of Grozman, recalled in Proposition 4.2. (Note that the obviously $\mathcal{G}$-invariant maps $\mathfrak{n} \otimes \mathfrak{n} \longrightarrow \mathfrak{n}$ and $\mathfrak{n} \otimes y \longrightarrow y$ are not antisymmetric!) So one has $\mathrm{Inv}_{\mathcal{G}}\, A_i = 0$ for $i = 3, 4, 5$.

An analogous argument will work for K, since $\mathrm{Inv}_{\mathcal{G}}\,\Lambda^2 y^* \otimes y = 0$ and $\mathrm{Inv}_{\mathcal{G}}(\mathfrak{n}^* \wedge y^*) \otimes y = 0$ from the same results. So the long exact sequence associated with the short sequence (4.5) above will give:

$$0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}(K) \longrightarrow \mathrm{Inv}_{\mathcal{G}}(A_2) \longrightarrow H^1(\mathcal{G}, \mathfrak{n}^* \otimes y).$$

One has $H^1(\mathcal{G}, \mathfrak{n}^* \otimes y) = H^1(\mathcal{G}, \mathcal{F}_{-1} \otimes \mathcal{F}_{1/2}) = 0$ (see Proposition 4.2). So $\mathrm{Inv}_{\mathcal{G}}(A_2) = 0$.

For A_1, we shall require a much more subtle argument. First of all, the sequence (4.4) can be split into two short exact sequences:

$$0 \longrightarrow \mathcal{H}_1 \longrightarrow (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y) \longrightarrow B \longrightarrow 0$$

$$0 \longrightarrow B \longrightarrow \Lambda^2 y^* \otimes \mathfrak{n} \longrightarrow A_1 \longrightarrow 0.$$

Let us consider the long exact sequence associated with the first one:

$$0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}\,\mathcal{H}_1 \longrightarrow \mathrm{Inv}_{\mathcal{G}}(\mathfrak{n}^* \otimes \mathfrak{n}) \oplus \mathrm{Inv}_{\mathcal{G}}(y^* \otimes y) \longrightarrow \mathrm{Inv}_{\mathcal{G}}\,B \longrightarrow \cdots$$

$$\cdots \hookrightarrow H^1(\mathcal{G}, \mathcal{H}_1) \longrightarrow H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) \oplus H^1(\mathcal{G}, y^* \otimes y) \longrightarrow H^1(\mathcal{G}, B) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^2(\mathcal{G}, \mathcal{H}_1) \longrightarrow H^2(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) \oplus H^2(\mathcal{G}, y^* \otimes y) \longrightarrow H^2(\mathcal{G}, B) \longrightarrow \cdots$$

The case of $H^i(\mathcal{G}, \mathcal{H}_1)$, $i = 0, 1, 2$ has been treated in Lemma 4.4, and analogous techniques can be used to study $H^i(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n})$ and $H^i(\mathcal{G}, y^* \otimes y)$ for $i = 0, 1, 2$. The cohomology classes come from the inclusion $\mathcal{F}_0 \subset \mathfrak{n}^* \otimes \mathfrak{n}$, $y^* \otimes y$ or $\mathcal{H}_1$, and from the well-known computation of $H^*(\mathcal{G}, \mathcal{F}_0)$ (Remark: using the results of Fuks [10], Chap. 2, one should keep in mind the fact that he computes cohomologies for W_1, the formal part of $\mathcal{G} = \mathrm{Vect}(S^1)$. To get the cohomologies for $\mathrm{Vect}(S^1)$ one has to add the classes of differentiable order 0 (or "topological" classes), this is the reason for the occurrence of c_2 in Lemma 4.4).

So $H^i(\mathcal{G}, \mathcal{H}_1) = H^i(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) = H^i(\mathcal{G}, y^* \otimes y)$, $i = 0, 1, 2$, and the maps on the modules are naturally defined through the injection $\mathcal{H}_1 \longrightarrow (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y)$: each generator of $H^i(\mathcal{G}, \mathcal{H}_1)$, $i = 0, 1, 2$, say e, will give $(e, -e)$ in the corresponding component of $H^i(\mathcal{G}, (\mathfrak{n}^* \otimes \mathfrak{n}) \oplus (y^* \otimes y))$. So $\mathrm{Inv}_{\mathcal{G}}\,B$ and $H^2(\mathcal{G}, B)$ are one-dimensional and $H^1(\mathcal{G}, B)$ is two-dimensional.

Now we can examine the long exact sequence associated with:

$$0 \longrightarrow B \stackrel{\partial}{\longrightarrow} \Lambda^2 y^* \otimes \mathfrak{n} \longrightarrow A_1 \longrightarrow 0,$$

which is:

$$\begin{aligned} 0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}\,B \stackrel{\partial^*}{\longrightarrow} \mathrm{Inv}_{\mathcal{G}}\,\Lambda^2 y^* \otimes \mathfrak{n} \longrightarrow \mathrm{Inv}_{\mathcal{G}}\,A_1 & \\ \longrightarrow H^1(\mathcal{G}, B) \longrightarrow H^1(\mathcal{G}, \Lambda^2 y^* \otimes \mathfrak{n}) \longrightarrow \cdots & \end{aligned}$$

The generator of $\mathrm{Inv}_{\mathcal{G}}\,B$ comes from the identity map $\mathfrak{n} \longrightarrow \mathfrak{n}$, and $\mathrm{Inv}_{\mathcal{G}}\ \Lambda^2 y^* \otimes \mathfrak{n}$ is generated by the bracket $y \wedge y \longrightarrow \mathfrak{n}$, so ∂^* is an isomorphism in this case. So one has

$$0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}\,A_1 \longrightarrow H^1(\mathcal{G}, B) \stackrel{\partial^*}{\longrightarrow} H^1(\mathcal{G}, \Lambda^2 y^* \otimes \mathfrak{n}).$$

The result will follow from the fact that this ∂^* is also an isomorphism. The two generators in $H^1(\mathcal{G}, B)$ come from the corresponding ones in $H^1(\mathcal{G}, \mathfrak{n}^* \otimes \mathfrak{n}) \oplus H^1(\mathcal{G}, y^* \otimes y)$, modulo the classes coming from $H^1(\mathcal{G}, \mathcal{H}_1)$; so these generators can be described in terms of Yoneda extensions, since $H^1(\mathcal{G}, \mathcal{F}_0^* \otimes \mathcal{F}_0) = \mathrm{Ext}^1_{\mathcal{G}}(\mathcal{F}_0, \mathcal{F}_0)$, as well as $H^1(\mathcal{G}, \mathcal{F}_{1/2}^* \otimes \mathcal{F}_{1/2}) = \mathrm{Ext}^1_{\mathcal{G}}(\mathcal{F}_{1/2}, \mathcal{F}_{1/2})$.

Let us write this extension as $0 \longrightarrow \mathcal{F}_0 \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$; the action on E can be given in terms of modes as follows:

$$e_n^1(f_a, g_b) = (af_{n+a} + ng_{b+n}, bg_{b+n})$$

or

$$e_n^2(f_a, g_b) = (af_{n+a} + g_{b+n}, bg_{b+n}).$$

The images of these classes in $H^1(\mathcal{G}, \Lambda^2 y^* \otimes \mathfrak{n})$ are represented by the extensions obtained through a pull-back

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_0 & \longrightarrow & E & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow [\ ,\ ] & & \\ 0 & \longrightarrow & \mathcal{F}_0 & - - - > & E' & - - - > & \Lambda^2 \mathcal{F}_{1/2} & \longrightarrow & 0 \end{array}$$

where $[\ ,\ ]$ denotes the mapping given by the Lie bracket $\Lambda^2 \mathcal{F}_{1/2} \longrightarrow \mathcal{F}_0$. One can easily check that these extensions are non-trivial, so finally ∂^* is injective and $\mathrm{Inv}_{\mathcal{G}}(A_1) = 0$, which finishes the proof of $E_2^{0,2} = \mathrm{Inv}_{\mathcal{G}}\, H^2(\mathfrak{h}, \mathfrak{h}) = 0$ and the proof of Theorem 4.1. $\square$

Theorem 4.1 implies that we have three independent infinitesimal deformations of $\mathfrak{tsv}$, defined by the cocycles c_1, c_2 and c_3, so the most general infinitesimal deformation of $\mathfrak{tsv}$ is of the following form:

$$[\ ,\ ]_{\lambda,\mu,\nu} = [\ ,\ ] + \lambda c_1 + \mu c_2 + \nu c_3.$$

In order to study further deformations of this bracket, one has to compute the Richardson-Nijenhuis brackets of c_1, c_2 and c_3 in $H^3(\mathfrak{tsv}, \mathfrak{tsv})$. One can compute directly using our explicit formulas and finds $[c_i, c_j] = 0$ in $H^3(\mathfrak{tsv}, \mathfrak{tsv})$ for $i, j = 1, 2, 3$; and even better, the bracket of the cocycles themselves vanish, not only their cohomology classes. So one has the

Theorem 4.5. *The bracket* $[\ ,\ ]_{\lambda,\mu,\nu} = [\ ,\ ] + \lambda c_1 + \mu c_2 + \nu c_3$ *where* $[\ ,\ ]$ *is the Lie bracket on* $\mathfrak{tsv}$ *and* c_i, $i = 1, 2, 3$ *the cocycles given in Theorem* 4.1*, defines a three-parameter family of Lie algebra brackets on* $\mathfrak{tsv}$*.*

For the sake of completeness, we give below the full formulas in terms of modes:

$$[L_n, L_m]_{\lambda,\mu,\nu} = (m-n)L_{n+m} + \nu(m-n)M_{n+m}$$

$$[L_n, Y_m]_{\lambda,\mu,\nu} = (m - \frac{n}{2} - \frac{\lambda n}{2} + \mu)Y_{n+m}$$

$$[L_n, M_m]_{\lambda,\mu,\nu} = (m - \lambda n + 2\mu)M_{n+m}$$

$$[Y_n, Y_m] = (n-m)M_{n+m}$$

All other terms are vanishing.

The term with cocycle c_3 has already been considered in a slightly different context in [12]. The term with c_2 induces only a small change in the action on $\mathfrak{h}$: the modules $\mathcal{F}_{1/2}$ and $\mathcal{F}_0$ are changed into $\mathcal{F}_{1/2,\mu}$ and $\mathcal{F}_{0,\mu}$ (see [10], p. 127), the bracket on $\mathfrak{h}$ being fixed. This is nothing but a reparametrization of the generators in the module, and for integer values of μ, the Lie algebra given by $[\ ,\ ]_{0,\mu,0}$ is isomorphic to the original one; one sees here a subtle phenomenon: the deformation $\mathcal{G} \longrightarrow \mathcal{G}_{0,\mu,0}$ is a non-trivial one, derived from the non-trivial cocycle c_2, but for some exceptional values of μ (actually for all integer values), one gets a Lie algebra which is isomorphic to the original one through a reparametrization. One may represent the Lie algebras $\mathcal{G}_{0,\mu,0}$, $\mu \in \mathbb{R}$ as a path in the space of Lie algebra structures.

We shall focus in the sequel on the term proportional to c_1, and denote by $\mathfrak{tsv}_\lambda$ the one-parameter family of Lie algebra structures on $\mathfrak{tsv}$ given by $[\ ,\ ]_\lambda = [\ ,\ ]_{\lambda,0,0}$, in coherence with Definition 1.7. Inspection of the above formulas shows that $\mathfrak{tsv}_\lambda$ is a semi-direct product $\mathrm{Vect}(S^1) \ltimes \mathfrak{h}_\lambda$ where $\mathfrak{h}_\lambda$ is a deformation of $\mathfrak{h}$ as a $\mathrm{Vect}(S^1)$-module; one has $\mathfrak{h}_\lambda = \mathcal{F}_{\frac{1+\lambda}{2}} \oplus \mathcal{F}_\lambda$, and the bracket $\mathcal{F}_{\frac{1+\lambda}{2}} \times \mathcal{F}_{\frac{1+\lambda}{2}} \longrightarrow \mathcal{F}_\lambda$ is the usual one, induced by the Poisson bracket on the torus.

Now, as a by-product of the above computations, we shall determine explicitly $H^1(\mathfrak{tsv}, \mathfrak{tsv})$.

Theorem 4.6. *$H^1(\mathfrak{tsv}, \mathfrak{tsv})$ is three-dimensional, generated by the following cocycles, given in terms of modes by:*

$$c_1(L_n) = M_n \qquad c_2(L_n) = nM_n$$

$$l(Y_n) = Y_n \qquad l(M_n) = 2M_n.$$

The cocycle l already appeared in Section 1, when we discussed the derivations of $\mathfrak{tsv}$; with the notations of Definition 1.4 one has $l = 2(\delta_2 - \delta_1)$.

Proof. From Lemma 4.3 above, one has $H^1(\mathfrak{tsv}, \mathcal{G}) = 0$, and so $H^1(\mathfrak{tsv}, \mathfrak{tsv}) = H^1(\mathfrak{tsv}, \mathfrak{h})$. One is led to compute the H^1 of a semi-direct product, as already done in Paragraph 3.3. The space $H^1(\mathfrak{tsv}, \mathfrak{h})$ is made from two parts $H^1(\mathcal{G}, \mathfrak{h})$ and $H^1(\mathfrak{h}, \mathfrak{h})$ satisfying the compatibility condition as in Theorem 3.8:

$$c([X, \alpha]) - [X, c(\alpha)] = -[\alpha, c(X)]$$

for $X \in \mathcal{G}$ and $\alpha \in \mathfrak{h}$.

The result is then easily deduced from the previous computations: $H^1(\mathcal{G}, \mathfrak{h}) = H^1(\mathcal{G}, \mathfrak{n})$ is generated by $f\partial \longrightarrow f$ and $f\partial \longrightarrow f'$, which correspond in the mode decomposition to the cocycles c_1 and c_2. As a corollary, one has $[\alpha, c(X)] = 0$ for $X \in \mathcal{G}$ and $\alpha \in \mathfrak{h}$. Hence the compatibility condition reduces to $c([X, \alpha]) = [X, c(\alpha)]$ and thus $c \in \mathrm{Inv}_{\mathcal{G}}\, H^1(\mathfrak{h}, \mathfrak{h})$. It can now be deduced from Lemma 4.4 above, that the latter space is one-dimensional, generated by l.

We shall now determine the central charges of $\mathfrak{tsv}_\lambda$; the computation will shed light on some exceptional values of λ, corresponding to interesting particular cases.

4.2. Computation of $H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$

We shall again make use of the exact sequence decomposition $0 \longrightarrow \mathfrak{h}_\lambda \longrightarrow \mathfrak{tsv}_\lambda \xrightarrow{\pi} \mathcal{G} \longrightarrow 0$, and classify the cocycles with respect to their "type" along this decomposition; trivial coefficients will make computations much easier than in the above case. First of all, $0 \longrightarrow H^2(\mathcal{G}, \mathbb{R}) \xrightarrow{\pi^*} H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$ is an injection. So the Virasoro class $c \in H^2(\mathrm{Vect}(S^1), \mathbb{R})$ always survives in $H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$.

For $\mathfrak{h}_\lambda$, let us use once again the decomposition $0 \longrightarrow \mathfrak{n}_\lambda \longrightarrow \mathfrak{h}_\lambda \longrightarrow y_\lambda \longrightarrow 0$ where $\mathfrak{n}_\lambda = [\mathfrak{h}_\lambda, \mathfrak{h}_\lambda]$. One has: $H^1(\mathcal{G}, H^1(\mathfrak{h}_\lambda)) = H^1(\mathcal{G}, y_\lambda^*) = H^1(\mathcal{G}, \mathcal{F}^*_{\frac{1+\lambda}{2}}) = H^1(\mathcal{G}, \mathcal{F}_{-(\frac{3+\lambda}{2})})$.

The cohomologies of degree one of $\mathrm{Vect}(S^1)$ with coefficients in densities are known (see [10], Theorem 2.4.12): the space $H^1(\mathcal{G}, \mathcal{F}_{-(\frac{3+\lambda}{2})})$ is trivial, except for the three exceptional cases $\lambda = -3, -1, 1$:

$H^1(\mathcal{G}, \mathcal{F}_0)$ is generated by the cocycles $f\partial \longrightarrow f$ and $f\partial \longrightarrow f'$;

$H^1(\mathcal{G}, \mathcal{F}_{-1})$ is generated by the cocycle $f\partial \longrightarrow f'' dx$;

$H^1(\mathcal{G}, \mathcal{F}_{-2})$ is generated by the cocycle $f\partial \longrightarrow f'''(dx)^2$, corresponding to the "Souriau cocycle" associated to the central charge of the Virasoro algebra (see [12], Chapter IV).

In terms of modes, the corresponding cocycles are given by:

$$c_1(L_n, Y_m) = \delta^0_{n+m}, \quad c_2(L_n, Y_m) = n\delta^0_{n+m} \text{ for } \lambda = -3;$$
$$c(L_n, Y_m) = n^2 \delta^0_{n+m} \text{ for } \lambda = -1;$$
$$c(L_n, Y_m) = n^3 \delta^0_{n+m} \text{ for } \lambda = 1.$$

The most delicate part is the investigation of the term $E_2^{0,2} = \mathrm{Inv}_{\mathcal{G}}\, H^2(\mathfrak{h}_\lambda)$ (this refers of course to the Hochschild-Serre spectral sequence associated to the above decomposition). For $H^2(\mathfrak{h}_\lambda)$ we shall use the same short exact sequences as for $\mathfrak{h}$ in the proof of Lemma 4.3:

$$0 \longrightarrow \mathrm{Ker}\,\partial \longrightarrow \Lambda^2 \mathfrak{n}_\lambda^* \xrightarrow{\partial} \Lambda^2 y_\lambda^* \wedge \mathfrak{n}_\lambda^*$$
$$0 \longrightarrow \mathrm{Ker}\,\partial \longrightarrow y_\lambda^* \wedge \mathfrak{n}_\lambda^* \xrightarrow{\partial} \Lambda^3 y_\lambda^*$$
$$0 \longrightarrow \underline{\mathfrak{n}}_\lambda^* \longrightarrow \Lambda^2 y_\lambda^* \longrightarrow \mathrm{Coker}\,\partial \longrightarrow 0$$

(where $\underline{\mathfrak{n}}_\lambda$ stands for $\mathfrak{n}_\lambda$ divided out by the space of constant functions). One readily shows that for the first two sequences one has $\mathrm{Inv}_{\mathcal{G}}\,\mathrm{Ker}\,\partial = 0$. The third one is more complicated; the cohomology exact sequence yields:

$$0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}\,\mathrm{Coker}\,\partial \longrightarrow H^1(\mathcal{G}, \underline{\mathfrak{n}}_\lambda^*) \longrightarrow H^1(\mathcal{G}, \lambda^2 y_\lambda^*) \longrightarrow \cdots$$

The same result as above (see [10], Theorem 2.4.12) shows that:

$$H^1(\mathcal{G}, \underline{\mathfrak{n}}_\lambda^*) = H^1(\mathcal{G}, \mathcal{F}_{(-1-\lambda)}) = 0 \text{ unless } \lambda = 1, -1, 0,$$

and one then has to investigate case by case; set $c(Y_p, Y_q) = a_p \delta^0_{p+q}$ for the potential cochains on y_λ. For each n one has the relation:

$$(\mathrm{ad}_{L_n} c)(Y_p, Y_q) - (q-p)\gamma(L_n)(M_{p+q}) = 0 \quad (4.6)$$

for some 1-cocycle $\gamma\colon \mathcal{G} \longrightarrow \mathfrak{n}^*_\lambda$, and for all (p,q) such that $n+p+q=0$; if $\gamma(L_n)(M_k) = b_n \delta^0_{n+m+k}$, one obtains in terms of modes, using $a_p = -a_{-p}$:

$$(p+q)\left(\frac{1+\lambda}{2}\right)(a_p - a_q) + q a_q - p a_q - (q-p) b_{-(p+q)} = 0.$$

Let us now check the different cases of non-vanishing terms in $H^1(\mathcal{G}, \mathfrak{n}^*_\lambda)$.

For $\lambda = 1$ one has $b_n = n^3$, and one deduces $a_p = p^3$.

For $\lambda = -1$ there are two possible cases $b_n = n$ or $b_n = 1$, the above equation gives

$$q a_p - p a_q = (q-p)(\alpha(p+q) + \beta);$$

the only possible solution would be to set a_p constant, but this is not consistent with $a_p = -a_{-p}$. For $\lambda = 0$, one gets $b_n = n^2$ and the equation gives

$$\left(\frac{p+q}{2}\right)(a_p + a_q) + q a_p - p a_q - (q-p)(p+q)^2 = 0$$

One easily checks that there are no solutions.

Finally, one gets a new cocycle generating an independent class in $H^2(\mathfrak{tsv}_1, \mathbb{R})$, given by the formulas:

$$c(Y_n, Y_m) = n^3 \delta_{n+m},$$
$$c(L_n, M_m) = n^3 \delta_{n+m}$$

Let us summarize our results:

Theorem 4.7. *For $\lambda \neq -3, -1, 1$, $H^2(\mathfrak{tsv}_\lambda, \mathbb{R}) \simeq \mathbb{R}$ is generated by the Virasoro cocycle.*
For $\lambda = -3, -1$, $H^2(\mathfrak{tsv}_\lambda, \mathbb{R}) \simeq \mathbb{R}^2$ is generated by the Virasoro cocycle and an independent cocycle of the form $c(L_n, Y_m) = \delta^0_{n+m}$ for $\lambda = -3$ or $c(L_n, Y_m) = n^2 \delta^0_{n+m}$ for $\lambda = -1$.
For $\lambda = 1$, $H^2(\mathfrak{tsv}_1, \mathbb{R}) \simeq \mathbb{R}^3$ is generated by the Virasoro cocycle and the two independent cocycles c_1 and c_2 defined by (all other components vanishing)

$$c_1(L_n, Y_m) = n^3 \delta^0_{n+m};$$
$$c_2(L_n, M_m) = c_2(Y_n, Y_m) = n^3 \delta^0_{n+m}.$$

Remark. The isomorphism $H^2(\mathfrak{sv}_0, \mathbb{R}) \simeq \mathbb{R}$ has already been proved in [14]. As we shall see in Paragraph 4.4, generally speaking, local cocycles may be carried over from $\mathfrak{tsv}$ to $\mathfrak{sv}$ or from $\mathfrak{sv}$ to $\mathfrak{tsv}$ without any difficulty.

Let us look more carefully at the $\lambda = 1$ case. One has that $\mathfrak{h}_1 = \mathcal{F}_1 \oplus \mathcal{F}_1$ with the obvious bracket $\mathcal{F}_1 \times \mathcal{F}_1 \longrightarrow \mathcal{F}_1$; so, algebraically, $\mathfrak{h}_1 = \mathrm{Vect}(S^1) \otimes \varepsilon\mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$. One deduces immediately that $\mathfrak{tsv}_1 = \mathrm{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$; so the cohomological result for $\mathfrak{tsv}_1$ can be easily reinterpreted. Let $f_\varepsilon \partial$ and $g_\varepsilon \partial$

be two elements in $\mathrm{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$, and compute the Virasoro cocycle $c(f_\varepsilon\partial, g_\varepsilon\partial) = \int_{S^1} f_\varepsilon''' g_\varepsilon dt$ as a truncated polynomial in ε; one has $f_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2$ and $g_\varepsilon = g_0 + \varepsilon g_1 + \varepsilon^2 g_2$ so finally:

$$c(f_\varepsilon\partial, g_\varepsilon\partial) = \int_{S^1} f_0''' g_0 dt + \varepsilon \int_{S^1} (f_0''' g_1 + f_1''' g_0) dt + \varepsilon^2 \int_{S^1} (f_0''' g_2 + f_1''' g_1 + f_2''' g_0) dt.$$

In other terms: $c(f_\varepsilon\partial, g_\varepsilon\partial) = c_0(f_\varepsilon\partial, g_\varepsilon\partial) + \varepsilon c_1(f_\varepsilon\partial, g_\varepsilon\partial) + \varepsilon^2 c_2(f_\varepsilon\partial, g_\varepsilon\partial)$. One can easily identify the c_i, $i = 0, 1, 2$ with the cocycles defined in the above theorem, using a decomposition into modes. This situation can be described by a universal central extension

$$0 \longrightarrow \mathbb{R}^3 \longrightarrow \widehat{\mathfrak{tsv}}_1 \longrightarrow \mathrm{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0) \longrightarrow 0$$

and the formulas of the cocycles show that $\widehat{\mathfrak{tsv}}_1$ is isomorphic to $\mathrm{Vir} \otimes \mathbb{R}[\varepsilon]/(\varepsilon^3 = 0)$.

Remarks.

1. Cohomologies of Lie algebra of type $\mathrm{Vect}(S^1) \bigotimes_{\mathbb{R}} A$, where A is an associative and commutative algebra (the Lie bracket being as usual given by $[f\partial \otimes a, g\partial \otimes b] = (fg' - gf')\partial \otimes ab$), have been studied by C. Sah and collaborators (see [29]). Their result is: $H^2\left(\mathrm{Vect}(S^1) \bigotimes_{\mathbb{R}} A\right) = A'$ where $A' = \mathrm{Hom}_{\mathbb{R}}(A, \mathbb{R})$; all cocycles are given by the Virasoro cocycle composed with the linear form on A. The isomorphism $H^2(\mathfrak{tsv}_1, \mathbb{R}) \simeq \mathbb{R}^3$ (see Theorem 4.7) could have been deduced from this general theorem.
2. One can obtain various generalizations of our algebra $\mathfrak{h}$ as nilpotent Lie algebras with $\mathrm{Vect}(S^1)$-like brackets, such as
$$[Y_n, Y_m] = (m - n) M_{n+m} \tag{4.7}$$
by using the same scheme. Let A be an Artinian ring quotient of some polynomial ring $\mathbb{R}[t_1, \ldots, t_n]$ and $A_0 \subset A$ its maximal ideal; then $\mathrm{Vect}(S^1) \bigotimes_{\mathbb{R}} A_0$ is a nilpotent Lie algebra whose successive brackets are of the same type (4.7). One could speak of a "virasorization" of nilpotent Lie algebras. Explicit examples are provided in the Subsection 2.5 about multi-diagonal operators of the present article.
3. It is interesting in itself to look at how the dimension of $H^2(\mathfrak{tsv}_\lambda, \mathbb{R})$ varies under deformations. For generic values of λ, this dimension is equal to one, and it increases for some exceptional values of λ; one can consider this as an example of so called "Fuks principle" in infinite dimension: deformations can decrease the rank of cohomologies but never increase it.
4. Analogous Lie algebra structures, of the "Virasoro-tensorized" kind, have been considered in a quite different context in algebraic topology by Tamanoi, see [32].

4.3. About deformations of $\mathfrak{tsv}_1$

We must consider the local cochains $C^*_{\mathrm{loc}}(\mathfrak{tsv}_1, \mathfrak{tsv}_1)$. The Lie algebra $\mathfrak{tsv}_1$ admits a graduation mod 3 by the degree of polynomial in ε, the Lie bracket obviously respects this graduation; this graduation induces on the space of local cochains a graduation by weight, and $C^*_{\mathrm{loc}}(\mathfrak{tsv}_1, \mathfrak{tsv}_1)$ splits into direct sum of subcomplexes of homogeneous weight denoted by $C^*_{\mathrm{loc}}(\mathfrak{tsv}_1, \mathfrak{tsv}_1)_{(p)}$. Moreover, as classical in computations for Virasoro algebra, one can use the adjoint action of the zero mode L_0 (corresponding geometrically to the Euler field $z\frac{\partial}{\partial z}$) to reduce cohomology computations to the subcomplexes $C^*_{\mathrm{loc}}(\mathfrak{tsv}_1, \mathfrak{tsv}_1)_{(p)(0)}$ of cochains which are homogeneous of weight 0 with respect to $\mathrm{ad}L_0$ (see, e.g., [12], Chapter IV).

We can use the graduation in ε and consider homogeneous cochains with respect to that graduation. Here is what one gets, according to the weight:

- weight 1: one has cocycles of the form

$$c(L_n, Y_m) = (m-n)M_{n+m}, \; c(L_n, L_m) = (m-n)Y_{n+m}$$

  but if $b(L_n) = Y_n$, then $c = \partial b$.
- weight 0: all cocycles are coboundaries, using the well-known result $H^*(\mathrm{Vect}(S^1), \mathrm{Vect}(S^1)) = 0$.
- weight -1: one has to consider cochains of the following from

$$\begin{aligned} c(Y_n, M_m) &= a(m-n)M_{n+m} \\ c(Y_n, Y_m) &= b(m-n)Y_{n+m} \\ c(L_n, M_m) &= e(m-n)Y_{n+m} \\ c(L_n, Y_m) &= d(m-n)L_{n+m} \end{aligned}$$

  and check that $\partial c = 0$. It readily gives $e = d = 0$.
  If one sets $\widetilde{c}(Y_n) = \alpha L_n$ and $\widetilde{c}(M_n) = \beta Y_n$, then

$$\begin{aligned} \partial\widetilde{c}(Y_n, M_m) &= (\alpha+\beta)(m-n)M_{n+m} \\ \partial\widetilde{c}(Y_n, M_m) &= (2\alpha-\beta)(m-n)Y_{n+m}. \end{aligned}$$

  So all these cocycles are cohomologically trivial,
- weight -2: set

$$\begin{aligned} c(Y_n, M_m) &= \alpha(m-n)Y_{n+m} \\ c(M_n, M_m) &= \beta(m-n)M_{n+m} \\ c(L_n, M_m) &= \gamma(m-n)L_{n+m}. \end{aligned}$$

  Coboundary conditions give $\gamma = \alpha$ and $\beta = \gamma + \alpha$, but if $\overline{c}(M_n) = L_n$, then

$$\begin{aligned} \partial\overline{c}(M_n, M_m) &= (m-n)Y_{n+m} \\ \partial\overline{c}(Y_n, M_m) &= 2(m-n)M_{n+m} \\ \partial\overline{c}(L_n, M_m) &= (m-n)L_{n+m}. \end{aligned}$$

- weight -3: we find for this case the only surviving cocycle.
  One readily checks that $C \in C^*_{\rm loc}(\mathfrak{tsv}_1, \mathfrak{tsv}_1)_{(-3)(0)}$ defined by
  $$C(Y_n, M_m) = (m-n)L_{n+m}$$
  $$C(M_n, M_m) = (m-n)Y_{n+m}$$
  is a cocycle and cannot be a coboundary.

We can describe the cocycle C above more pleasantly by a global formula: let $f_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2$ and $g_\varepsilon = g_0 + \varepsilon g_1 + \varepsilon^2 g_2$, with f_i, g_i elements of $\mathrm{Vect}(S^1)$. The bracket $[\ ,\ ]$ of $\mathfrak{tsv}_1$ is then the following:

$$[f_\varepsilon, g_\varepsilon] = \sum_{k=0}^{2} \sum_{i+j=k} [f_i, g_i] \quad \text{or} \quad (f_\varepsilon g'_\varepsilon - g_\varepsilon f'_\varepsilon)|_{\varepsilon^3=0},$$

and the deformed bracket $[\ ,\ ] + \mu C$ will be $[f_\varepsilon, g_\varepsilon]_\mu = (f_\varepsilon g'_\varepsilon - g_\varepsilon f'_\varepsilon)|_{\varepsilon^3=\mu}$. So we have found that dim $H^2(\mathfrak{tsv}_1, \mathfrak{tsv}_1) = 1$.

In order to construct deformations, we still have to check for the Nijenhuis-Richardson bracket $[C, C]$ in $C^3(\mathfrak{tsv}_1, \mathfrak{tsv}_1)$. The only possibly non-vanishing term is:

$$\begin{aligned}
[C, C](M_n, M_n, M_p) &= \sum_{\text{(cycl)}} C(C(M_n, M_m,), M_p) \\
&= \sum_{\text{(cycl)}} (m-n)C(Y_{n+m}, M_p) \\
&= \sum_{\text{(cycl)}} (pm - pn + n^2 - m^2)L_{n+m+p} = 0.
\end{aligned}$$

So there does not exist any obstruction and we have obtained a genuine deformation. We summarize all these results in the following

Proposition 4.8. *There exists a one-parameter deformation of the Lie algebra* $\mathfrak{tsv}_1$*, as* $\mathfrak{tsv}_{1,\mu} = \mathrm{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/_{(\varepsilon^3=\mu)}$*. This deformation in the only one possible, up to isomorphism.*

If one is interested in central charges, the above-mentioned theorem of C. Sah et al., see [29], shows that $\dim H^2(\mathfrak{tsv}_{1,\mu}, \mathbb{R}) = \mathbb{R}^3$ and the universal central extension $\widehat{\mathfrak{tsv}}_{1,\mu}$ is isomorphic to $\mathrm{Vir} \otimes \mathbb{R}[\varepsilon]/_{(\varepsilon^3=\mu)}$. We did not do the computations, but we conjecture that $\mathfrak{tsv}_{1,\mu}$ is rigid, the ring $\mathbb{R}[\varepsilon]/_{(\varepsilon^3=\mu)}$ being more generic than $\mathbb{R}[\varepsilon]/_{(\varepsilon^3=0)}$. More generally, it could be interesting to study systematically Lie algebras of type $\mathrm{Vect}(S^1) \otimes A$ where A is a commutative ring, their geometric interpretation being "Virasoro current algebras".

4.4. Coming back to the original Schrödinger-Virasoro algebra

The previous results concern the twisted Schrödinger-Virasoro algebra generated by the modes (L_n, Y_m, M_p) for $(n, m, p) \in \mathbb{Z}^3$, which make computations easier and allows direct application of Fuks' techniques. The "actual" Schrödinger-Virasoro algebra is generated by the modes (L_n, Y_m, M_p) for $(n, p) \in \mathbb{Z}^2$ but $m \in \mathbb{Z} + \frac{1}{2}$.

Yet Theorem 4.1 and Theorem 4.5 on deformations of $\mathfrak{tsv}$ are also valid for $\mathfrak{sv}$: one has dim $H^2(\mathfrak{sv},\mathfrak{sv}) = 3$ with the same cocycles c_1, c_2, c_3, since these do not allow 'parity-changing' terms such as $L \times Y \to M$ or $Y \times Y \to Y$ for instance (the (L, M)-generators being considered as 'even' and the Y-generators as 'odd').

But the computation of $H^2(\mathfrak{sv}_\lambda, \mathbb{R})$ will yield very different results compared to Theorem 4.7, since 'parity' is not conserved for all the cocycles we found, so we shall start all over again. Let us use the adjoint action of L_0 to simplify computations: all cohomologies are generated by cocycles c such that $\mathrm{ad}L_0 \, . \, c = 0$, i.e., such that $c(A_k, B_l) = 0$ for $k+l \neq 0$, A and B being L, Y or M. So, for non-trivial cocycles, one must have $c(Y_n, L_p) = 0, c(Y_n, M_p) = 0$ for all Y_n, L_p, M_p; in $H^2(\mathfrak{sv}_\lambda, \mathbb{R})$, terms of the type $H^1(\mathcal{G}, H^1(\mathfrak{h}_\lambda))$ will automatically vanish. The Virasoro class in $H^2(\mathcal{G}, \mathbb{R})$ will always survive, and one has to check what happens with the terms of type $\mathrm{Inv}_{\mathcal{G}}\, H^2(\mathfrak{h}_\lambda)$. As in the proof of Lemma 4.3, the only possibilities come from the short exact sequence:

$$0 \longrightarrow \underline{\mathfrak{n}}^*_\lambda \longrightarrow \Lambda^2 y^*_\lambda \longrightarrow \mathrm{Coker}\,\partial \longrightarrow 0$$

which induces: $0 \longrightarrow \mathrm{Inv}_{\mathcal{G}}\,\mathrm{Coker}\,\partial \longrightarrow H^1(\mathcal{G}, \underline{\mathfrak{n}}^*_\lambda) \longrightarrow H^1(\mathcal{G}, \Lambda^2 y^*_\lambda)$ and one obtains the same equation (4.6) as above:

$$(\mathrm{ad}_{L_n} c)(Y_p, Y_q) + (q-p)\gamma(L_n)(M_{p+q}) = 0.$$

If $c(Y_p, Y_q) = a_p \delta^0_{p+q}$, the equation gives:

$$-a_p\Big(p + \frac{\lambda+3}{2} n\Big) + a_{p+n}\Big(p - \frac{\lambda+1}{2} n\Big) - (p-q)\gamma(L_n)(M_{p+q}) = 0.$$

One finds two exceptional cases with non-trivial solutions:

- for $\lambda = 1$, $a_p = p^3$ and $c(L_n, M_m) = n^3 \delta^0_{n+m}$ gives a two-cocycle, very much analogous to the $\mathrm{Vect}(S^1) \otimes \mathbb{R}[\varepsilon]/_{(\varepsilon^3=0)}$ case, except that one has no term in $c(L_n, Y_p)$.
- for $\lambda = -3$, if $\gamma \equiv 0$, the above equation gives $pa_p = (p+n)a_{p+n}$ for every p and n. So $a_p = \frac{1}{p}$ is a solution, and one sees why this solution was not available in the twisted case.

Let us summarize:

Proposition 4.9. *The space $H^2(\mathfrak{sv}_\lambda, \mathbb{R})$ is one-dimensional, generated by the Virasoro cocycle, save for two exceptional values of λ, for which one has one more independent cocycle, denoted by c_1, with the following non-vanishing components:*

- *for $\lambda = 1$: $c_1(Y_p, Y_q) = p^3 \delta^0_{p+q}$ and $c_1(L_p, M_q) = p^3 \delta^0_{p+q}$,*
- *for $\lambda = -3$: $c_1(Y_p, Y_q) = \frac{\delta^0_{p+q}}{p}$.*

Remark. The latter case is the most surprising one, since it contradicts the well-established dogma asserting that only local classes are interesting. This principle of locality has its roots in quantum field theory (see, e.g., [20] for basic principles of axiomatic field theory); its mathematical status has its foundations in the famous theorem of J. Peetre, asserting that local mappings are given by differential operators, so – in terms of modes – the coefficients are polynomial in n. Moreover, there is a general theorem in the theory of cohomology of Lie algebras of vector fields (see [10]) which states that continuous cohomology is in general multiplicatively generated by local cochains, called diagonal in [10]. Here our cocycle contains an anti-derivative, so there could be applications in integrable systems, considered as Hamiltonian systems, the symplectic manifold given by the dual of (usually centrally extended) infinite dimensional Lie algebras (see for example [12], Chapters VI and X).

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