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# Sugawara and Vertex Operator Constructions for Deformed Virasoro Algebras

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*Dedicated to our friend Daniel Arnaudon*

**Abstract.** From the defining exchange relations of the $\mathcal{A}_{q,p}(\widehat{gl}_N)$ elliptic quantum algebra, we construct subalgebras which can be characterized as $q$-deformed $W_N$ algebras. The consistency conditions relating the parameters $p, q, N$ and the central charge $c$ are shown to be related to the singularity structure of the functional coefficients defining the exchange relations of specific vertex operators representations of $\mathcal{A}_{q,p}(\widehat{gl}_N)$ available when $N = 2$.

## 1. Introduction

The notion of $q$-deformed Virasoro, and its natural extensions the $q$-deformed $W_n$ algebras, covers a number of algebraic structures characterized by quadratic exchange relations such that one can define a semi-classical limit as a Poisson structure with a very specific form. This latter exhibits in turn a deformation parameter $q$ such that the suitably defined limit $q \to 1$ leads to the usual classical Virasoro and $W_n$ Poisson algebras. The original Poisson structure was proposed in [1] as originating on an extended center of the affine quantum algebra $U_q(\widehat{sl}_2)_c$ for the critical value $c = -2$ of the central charge. Quantization of this structure has proved to be an intriguing task [2, 3, 4]. Indeed, deep relationship was found among the Virasoro and $W_n$ algebras, algebras of screening currents, the quantum Miura transformation, and the Macdonald polynomials or the quantum $N$-body trigonometric Ruijsenaars–Schneider model.

Another connection between $q$-deformed Virasoro and $W_N$ algebras, and the vertex-type elliptic quantum algebras $\mathcal{A}_{q,p}(\widehat{sl}_2)$ of [5], was established in a series of papers [6, 7, 8]. It was shown that the formal defining exchange relations of the vertex elliptic algebras, parametrized by elliptic $R$ matrices of Baxter–Belavin [9],

allowed to define certain operators, bilinear in the quantum Lax matrix, realizing quadratic exchange relations of $q$-$W_N$ type (i.e., with suitable semi-classical limits), whenever particular commensurability relations existed between the elliptic module $p$, the deformation parameter $q$ and the central charge $c$. Supplementary commensurability relations could then lead to commuting operators, allowing for a definition of consistent Poisson structures reproducing the classical $q$-$W_N$ algebra.

At that point, it may be useful to make a comparison with the constructions done in the undeformed case. The first approach (based on MacDonald polynomials and deformed bosons) can be seen as a deformation of the Miura construction for Virasoro and $W_N$ algebras [10], which realizes these algebras in terms of free fields. The second approach (based on quadratics of $\mathcal{A}_{q,p}(\widehat{sl}_2)$ generators) can be viewed as a deformation of the Sugawara construction [11, 12, 13], which realizes the Virasoro and $W_N$ algebras in terms of affine Lie algebras. In the undeformed case, these two constructions lead to the same algebras, and their connection is done when considering Sugawara construction for an affine Lie algebra, itself realized in terms of vertex operators. When considering the deformations of these approaches, one gets *two* types of algebras, with different structure constants, but same classical limit. Moreover, these constructions have remained until now at the level of formal manipulations since explicit realizations of the elliptic algebra were lacking at the time. The main objective of this paper is to present a first contact point between these two approaches at the quantum level.

Indeed, the recent construction of explicit vertex operator (V.O.) representations of the elliptic vertex algebra (for particular values of $c$ and relations between $p$ and $q$), and the identification of specific fused operators in this vertex algebra [14], lead us to reconsider and generalize our former construction of $q$-$W_N$ generators from this point of view. New sets of commensurability relations have now been identified, both for $\mathcal{A}_{q,p}(\widehat{sl}_2)$ and $\mathcal{A}_{q,p}(\widehat{sl}_N)$. In addition it is now possible to express our formal bilinear operators in terms of this explicit vertex algebra construction, at least for $sl(2)$, and therefore to examine the meaning of the commensurability relations also in the framework of this V.O. representation. It turns out as we shall now see, that these relations are interpreted as characterizing locii for simultaneous singularities in the exchange relations of the two types of V.O. The bilinear operators generating our $q$-$W_N$ algebras are then represented as products of residues of both V.O. at these singular points.

The plan of the paper is as follows. After a reminder of the definition and some useful properties of $\mathcal{A}_{q,p}(\widehat{gl}_2)$ in Section 2, we define, in Section 3, operators labelled by two integers $\ell$ and $\ell'$ realizing an $\mathcal{A}_{q,p}(\widehat{gl}_2)$ subalgebra when the parameters $p, q$ and the central charge $c$ obey certain consistency conditions. The semi-classical limit yields the classical deformed Virasoro algebra introduced by [1]. In Section 4, we recall the level one vertex operator representation of $\mathcal{A}_{q,p}(\widehat{gl}_2)$ introduced by [15, 14] and we connect these vertex operators with the above subalgebra. The generalization to $\mathcal{A}_{q,p}(\widehat{gl}_N)$ is considered in Section 5.

## 2. The elliptic algebra $\mathcal{A}_{q,p}(\widehat{gl}_2)$

The quantum affine elliptic algebra $\mathcal{A}_{q,p}(\widehat{gl}_2)$ is described in the RLL formalism [16]. Its R-matrix is constructed from the Boltzmann weights of the eight vertex model:

$$\mathcal{W}(z,q,p) = \rho(z) \begin{pmatrix} a(z) & 0 & 0 & d(z) \\ 0 & b(z) & c(z) & 0 \\ 0 & c(z) & b(z) & 0 \\ d(z) & 0 & 0 & a(z) \end{pmatrix} \tag{2.1}$$

with

$$a(z) = z^{-1} \frac{\Theta_{p^2}(q^2 z^2)\,\Theta_{p^2}(pq^2)}{\Theta_{p^2}(pq^2 z^2)\,\Theta_{p^2}(q^2)} \qquad\qquad d(z) = -\frac{p^{\frac{1}{2}}}{qz^2} \frac{\Theta_{p^2}(z^2)\,\Theta_{p^2}(q^2 z^2)}{\Theta_{p^2}(pz^2)\,\Theta_{p^2}(pq^2 z^2)}$$

$$b(z) = qz^{-1} \frac{\Theta_{p^2}(z^2)\,\Theta_{p^2}(pq^2)}{\Theta_{p^2}(pz^2)\,\Theta_{p^2}(q^2)} \qquad\qquad c(z) = 1 \tag{2.2}$$

with $\Theta_p(z) = (z;p)_\infty\,(pz^{-1};p)_\infty\,(p;p)_\infty$
and $(z; a_1,\ldots,a_m)_\infty = \prod_{n_i \geq 0}(1 - za_1^{n_1}\ldots a_m^{n_m})$.
The normalization factor $\rho(z)$ is chosen as follows:

$$\rho(z) = \frac{(p^2;p^2)_\infty}{(p;p)_\infty^2} \frac{\Theta_{p^2}(pz^2)\Theta_{p^2}(q^2)}{\Theta_{p^2}(q^2 z^2)} \frac{\xi(z^2;p,q^4)}{\xi(z^{-2};p,q^4)} \tag{2.3}$$

where we have introduced

$$\xi(z;p,q^4) = \frac{(q^2 z;p,q^4)_\infty\,(pq^2 z;p,q^4)_\infty}{(q^4 z;p,q^4)_\infty\,(pz;p,q^4)_\infty}\,. \tag{2.4}$$

**Property 2.1.** *The matrix* (2.1) *has the following properties:*
*YBE:*

$$\mathcal{W}_{12}(z_1)\,\mathcal{W}_{13}(z_2)\,\mathcal{W}_{23}(z_2/z_1) = \mathcal{W}_{23}(z_2/z_1)\,\mathcal{W}_{13}(z_2)\,\mathcal{W}_{12}(z_1) \tag{2.5}$$

*unitarity:*

$$\mathcal{W}_{12}(z)\,\mathcal{W}_{21}(z^{-1}) = 1 \tag{2.6}$$

*crossing symmetry:*

$$\mathcal{W}_{21}^{t_1}(\frac{1}{z}) = (\sigma_x \otimes 1)\mathcal{W}_{12}(-\frac{z}{q})(\sigma_x \otimes 1) = (1 \otimes \sigma_x)\mathcal{W}_{12}(-\frac{z}{q})(1 \otimes \sigma_x) \tag{2.7}$$

*antisymmetry:*

$$\mathcal{W}_{12}(-z) = -(\sigma_z \otimes 1)\mathcal{W}_{12}(z)(\sigma_z \otimes 1) = -(1 \otimes \sigma_z)\mathcal{W}_{12}(z)(1 \otimes \sigma_z) \tag{2.8}$$

*where* $\sigma_x, \sigma_y, \sigma_z$ *are the* $2 \times 2$ *Pauli matrices and* $t_i$ *denotes the transposition in space i.*

The definition of the quantum affine elliptic algebra $\mathcal{A}_{q,p}(\widehat{gl}_2)$ requires the use of a slightly modified R-matrix $R_{12}(z)$, which differs from (2.1) by a suitable normalization factor:

$$R_{12}(z,q,p) \equiv R_{12}(z) = \tau(q^{\frac{1}{2}}z^{-1})\mathcal{W}_{12}(z,q,q).\tag{2.9}$$

The factor $\tau(z)$ is given by

$$\tau(z) = z^{-1}\,\frac{\Theta_{q^4}(qz^2)}{\Theta_{q^4}(qz^{-2})}\tag{2.10}$$

which is $q^2$-periodic and satisfies $\tau(z)\tau(z^{-1}) = 1$ and $\tau(q^{\frac{1}{2}}z^{-1}) = -\tau(q^{\frac{1}{2}}z)$. The R-matrix $R_{12}(z)$ is no longer unitary but verifies

$$R_{12}(z)\,R_{21}(z^{-1}) = \widetilde{\tau}(z)\tag{2.11}$$

where the $\widetilde{\tau}$ function is given by

$$\widetilde{\tau}(z) = \tau(q^{\frac{1}{2}}z^{-1})\,\tau(q^{\frac{1}{2}}z) = -\tau(q^{\frac{1}{2}}z)^2 = -\left[q^{-\frac{1}{2}}z\,\frac{\Theta_{q^4}(q^2z^2)}{\Theta_{q^4}(z^2)}\right]^2.\tag{2.12}$$

The R-matrix $R_{12}(z)$ obeys a quasi-periodicity property

$$R_{12}(-p^{\frac{1}{2}}z) = (\sigma_x \otimes 1)\left(R_{21}(z^{-1})\right)^{-1}(\sigma_x \otimes 1) = \widetilde{\tau}(z)^{-1}\,(\sigma_x \otimes 1)\,R_{12}(z)\,(\sigma_x \otimes 1),\tag{2.13}$$

so that a recursive use of this formula leads to

$$R_{12}\left((-p^{\frac{1}{2}})^{-\ell}z\right) = F(\ell,z)\,(\sigma_x^\ell \otimes 1)\,R_{12}(z)\,(\sigma_x^\ell \otimes 1)\tag{2.14}$$

where

$$F(\ell,z) = \begin{cases} \displaystyle\prod_{s=1}^{\ell}\widetilde{\tau}\left((-p^{\frac{1}{2}})^{-s}z\right) & \text{for } \ell \geq 0 \\[4ex] \displaystyle\prod_{s=1}^{|\ell|}\widetilde{\tau}\left((-p^{\frac{1}{2}})^{|\ell|-s}z\right)^{-1} & \text{for } \ell \leq 0 \end{cases}\tag{2.15}$$

The function $F(\ell,z)$ is $q^2$-periodic (due to the $q^2$-periodicity of the $\tau$ function) and satisfies the following relations $(\ell, n \in \mathbb{Z})$

$$F(\ell,z)F(-\ell,z^{-1}) = \frac{\widetilde{\tau}\left((-p^{\frac{1}{2}})^{-\ell}z\right)}{\widetilde{\tau}(z)}\tag{2.16}$$

$$F(\ell,(-p^{\frac{1}{2}})^{-n}z) = \frac{F(\ell+n,z)}{F(n,z)}.\tag{2.17}$$

In particular, one has

$$F(-\ell,z) = \frac{1}{F(\ell,(-p^{\frac{1}{2}})^\ell z)}.\tag{2.18}$$

The crossing symmetry and the unitarity properties of $\mathcal{W}_{12}$ then allow one to exchange inversion and transposition for the matrix $R_{12}$:

$$\left(R_{12}(z)^{t_2}\right)^{-1} = \left(R_{12}(q^2 z)^{-1}\right)^{t_2}. \tag{2.19}$$

The quantum affine elliptic algebra $\mathcal{A}_{q,p}(\widehat{gl}_2)$ is defined as a formal algebra of operators

$$L(z) = \sum_{n \in \mathbb{Z}} L_n\, z^{-n} = \left( \begin{array}{cc} L_{++}(z) & L_{+-}(z) \\ L_{-+}(z) & L_{--}(z) \end{array} \right) = \sum_{i,j=\pm} L_{ij}(z) E_{ij} \tag{2.20}$$

where the functions $L_{++}$ and $L_{--}$ are even while $L_{+-}$ and $L_{-+}$ are odd in the variable $z$, and obey the following relations:

$$R_{12}(z_1/z_2)\, L_1(z_1)\, L_2(z_2) = L_2(z_2)\, L_1(z_1)\, R^*_{12}(z_1/z_2) \tag{2.21}$$

with $L_1(z) = L(z) \otimes 1$, $L_2(z) = 1 \otimes L(z)$ and $R^*_{12}(z) \equiv R_{12}(z, q, p^* = pq^{-2c})$. The quantum determinant of $L(z)$ given by [15]

$$\text{q-det}\, L(z) = L_{++}(q^{-1}z)L_{--}(z) - L_{-+}(q^{-1}z)L_{+-}(z) \tag{2.22}$$

is in the center of $\mathcal{A}_{q,p}(\widehat{gl}_2)$. It can be factored out, and set to the value $q^{\frac{c}{2}}$ ($c$ being the central charge) so as to get

$$\mathcal{A}_{q,p}(\widehat{sl}_2) = \mathcal{A}_{q,p}(\widehat{gl}_2)/\langle \text{q-det}\, L(z) - q^{\frac{c}{2}}\rangle. \tag{2.23}$$

## 3. Exchange algebras

### 3.1. Deformed Virasoro algebras

**Proposition 3.1.** *The operators $T_{\ell\ell'}(z)$ defined by $(\ell, \ell' \in \mathbb{Z})$*

$$T_{\ell\ell'}(z) \equiv \text{Tr}\left(L(\gamma z)^{-1}\, \sigma_x^\ell\, L(z)\, \sigma_x^{\ell'}\right) = \text{Tr}\left(\sigma_x^\ell\, \left(L(\gamma z)^{-1}\right)^t\, \sigma_x^{\ell'}\, L(z)^t\right) \tag{3.1}$$

*have exchange relations with the generators $L(z)$ of $\mathcal{A}_{q,p}(\widehat{gl}_2)$:*

$$L(z_2)\, T_{\ell\ell'}(z_1) = f_{\ell\ell'}(z_1/z_2)\, T_{\ell\ell'}(z_1)\, L(z_2) \tag{3.2}$$

*if the conditions*

$$\gamma = \gamma_{\ell\ell'} \equiv (-p^{\frac{1}{2}})^{-\ell} = (-p^{*\frac{1}{2}})^{-\ell'}\, q^2 \tag{3.3}$$

*are fulfilled.*
*The exchange function $f_{\ell\ell'}(z)$ is given by*

$$f_{\ell\ell'}(z) = \frac{F^*(\ell', z)}{F(\ell, z)} \tag{3.4}$$

*where $F(\ell, z)$ is given by (2.15) and $F^*(\ell, z)$ is obtained from $F(\ell, z)$ through the shift $p \to p^* = pq^{-2c}$.*

*Proof.* One has

$$L_2(z_2)T_{1\ell\ell'}(z_1) = \text{Tr}_1\left(L_2(z_2)\sigma_{1x}^\ell\left(L_1(\gamma z_1)^{-1}\right)^{t_1}\sigma_{1x}^{\ell'}L_1(z_1)^{t_1}\right)$$

$$= \text{Tr}_1\left(\sigma_{1x}^\ell(R_{12}^{-1}(\gamma z_1/z_2))^{t_1}(L_1(\gamma z_1)^{-1})^{t_1}\right.$$
$$\left.\times L_2(z_2)((R_{12}^{*-1}(\gamma z_1/z_2))^{t_1})^{-1}\sigma_{1x}^{\ell'}L_1(z_1)^{t_1}\right)$$

$$= \text{Tr}_1\left(\sigma_{1x}^\ell(R_{12}^{-1}(\gamma z_1/z_2))^{t_1}(L_1(\gamma z_1)^{-1})^{t_1}\right.$$
$$\left.\times L_2(z_2)(R_{12}^*(q^{-2}\gamma z_1/z_2))^{t_1}\sigma_{1x}^{\ell'}L_1(z_1)^{t_1}\right)$$

where the crossing-unitarity property (2.19) has been used in the last equality. Imposing the relation $q^{-2}\gamma = (-p^{*\frac{1}{2}})^{-\ell'}$ and using the quasi-periodicity property (2.14) of $R$ allows one to further exchange $L_2(z_2)$ and $L_1(z_1)^{t_1}$:

$$L_2(z_2)\,T_{1\ell\ell'}(z_1) = F^*(\ell', z_1/z_2)\,\text{Tr}_1\left(\sigma_{1x}^\ell\,(R_{12}^{-1}(\gamma z_1/z_2))^{t_1}\,(L_1(\gamma z_1)^{-1})^{t_1}\,\sigma_{1x}^{\ell'}\right.$$
$$\left.L_2(z_2)\,(R_{12}^*(z_1/z_2))^{t_1}\,L_1(z_1)^{t_1}\right)$$

$$= F^*(\ell', z_1/z_2)\,\text{Tr}_1\left(\sigma_{1x}^\ell\,(R_{12}^{-1}(\gamma z_1/z_2))^{t_1}\,(L_1(\gamma z_1)^{-1})^{t_1}\,\sigma_{1x}^{\ell'}\right.$$
$$\left.L_1(z_1)^{t_1}\,(R_{12}(z_1/z_2))^{t_1}\,L_2(z_2)\right). \quad (3.5)$$

Imposing now the relation $\gamma = (-p^{\frac{1}{2}})^{-\ell}$ and using once again the relation (2.14), one gets

$$L_2(z_2)\,T_{1\ell\ell'}(z_1) = F^*(\ell', z_1/z_2)\,F(\ell, z_1/z_2)^{-1}$$
$$\text{Tr}_1\left((R_{12}^{-1}(z_1/z_2))^{t_1}\,\sigma_{1x}^\ell\,(L_1(\gamma z_1)^{-1})^{t_1}\,\sigma_{1x}^{\ell'}\,L_1(z_1)^{t_1}\,(R_{12}(z_1/z_2))^{t_1}\,L_2(z_2)\right).$$
$$(3.6)$$

Using the fact that under a trace over the space 1 one has $\text{Tr}_1\left(R_{12}Q_1R'_{12}\right) = \text{Tr}_1\left(Q_1R'^{t_2}_{12}R^{t_2}_{12}\right)^{t_2}$, one gets

$$L_2(z_2)\,T_{1\ell\ell'}(z_1) = F^*(\ell', z_1/z_2)\,F(\ell, z_1/z_2)^{-1}\,\text{Tr}_1\left(\sigma_{1x}^\ell\,(L_1(\gamma z_1)^{-1})^{t_1}\,\sigma_{1x}^{\ell'}\right.$$
$$\left.L_1(z_1)^{t_1}\,(R_{12}(z_1/z_2))^{t_1 t_2}\,(R_{12}^{-1}(z_1/z_2))^{t_1 t_2}\right)^{t_2}L_2(z_2) \quad (3.7)$$

which leaves a trivial dependence in space 2 under the trace in space 1 and therefore leads to the exchange relation between $T_{1\ell\ell'}(z_1)$ and $L_2(z_2)$:

$$L_2(z_2)\,T_{1\ell\ell'}(z_1) = F^*(\ell', z_1/z_2)\,F(\ell, z_1/z_2)^{-1}\,T_{1\ell\ell'}(z_1)\,L_2(z_2). \quad (3.8)$$
$$\square$$

Let us stress that although $T_{\ell\ell'}(z)$ seems to depend on $\ell$ and $\ell'$ only modulo 2, it really depends on them because of the form of $\gamma$ imposed by conditions (3.3).

In the same way, one can establish:

**Proposition 3.2.** *The operators $S_{\ell\ell'}(z)$ defined by ($\ell, \ell' \in \mathbb{Z}$)*

$$S_{\ell\ell'}(z) \equiv \mathrm{Tr}\left(\sigma_x^\ell \, L(z) \, \sigma_x^{\ell'} \, (L(\gamma^{-1}q^2 z)^{-1})\right) = \mathrm{Tr}\left(L(z)^t \, \sigma_x^\ell \, (L(\gamma^{-1}q^2 z)^{-1})^t \, \sigma_x^{\ell'}\right)$$

$$(3.9)$$

*have exchange relations with the generators $L(z)$ of $\mathcal{A}_{q,p}(\widehat{gl}_2)$:*

$$L(z_2) \, S_{\ell\ell'}(z_1) = f_{\ell\ell'}(z_1/z_2) \, S_{\ell\ell'}(z_1) \, L(z_2) \tag{3.10}$$

*if the conditions*

$$\gamma = \gamma_{-\ell-\ell'} \equiv (-p^{\frac{1}{2}})^\ell = (-p^{*\frac{1}{2}})^{\ell'} \, q^2 \tag{3.11}$$

*are fulfilled.*

*Proof.* The proof follows along the same lines as for Proposition 3.1.         □

Let us remark that the above calculations rely only on the exchange relation (2.21). As such, these formal manipulations do not allow us to determine potential 'delta-type' terms. These terms can be obtained through explicit realizations of the $\mathcal{A}_{q,p}(\widehat{gl}_2)$ algebra. Unfortunately, they are still missing.

Note that the relations (3.11) are deduced from the relations (3.3) through the change $(\ell, \ell') \to (-\ell, -\ell')$. The above results suggest the following definition:

**Definition 3.1.** *In the parameter space $(q, p, c)$, the surface $\mathcal{P}_{\ell\ell'}$ is defined by the relation*

$$(-p^{\frac{1}{2}})^{-\ell} = (-p^{*\frac{1}{2}})^{-\ell'} \, q^2 \,, \quad \text{with } p^* = p \, q^{-2c} \,. \tag{3.12}$$

*The operators $T_{\ell\ell'}$ and $S_{-\ell,-\ell'}$ defined on this surface are given by relations (3.1) and (3.9) with $\gamma \equiv \gamma_{\ell\ell'} = (-p^{\frac{1}{2}})^{-\ell}$.*

Since $T_{\ell\ell'}$ and $S_{\ell\ell'}$ have the same exchange relations with the $L(z)$ operators, one can say that $T_{\ell\ell'}$ and $S_{\ell\ell'}$ represent the same formal algebra on respectively the surfaces $\mathcal{P}_{\ell\ell'}$ and $\mathcal{P}_{-\ell,-\ell'}$.

From Propositions 3.1 and 3.2, we get immediately:

**Proposition 3.3.** *When $c$ is a rational number, several surfaces $\mathcal{P}_{\ell\ell'}$ can be defined simultaneously (i.e., have non-trivial intersection).*
*On the surface $\mathcal{P}_{\ell\ell'} \cap \mathcal{P}_{\lambda\lambda'}$, the operators $T_{\ell\ell'}$ and $T_{\lambda\lambda'}$ satisfy the following exchange algebra*

$$T_{\ell\ell'}(z_1) \, T_{\lambda\lambda'}(z_2) = \mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z_1/z_2) \, T_{\lambda\lambda'}(z_2) \, T_{\ell\ell'}(z_1) \,. \tag{3.13}$$

*On the surface $\mathcal{P}_{-\ell,-\ell'} \cap \mathcal{P}_{-\lambda,-\lambda'}$, the operators $S_{\ell\ell'}$ and $S_{\lambda\lambda'}$ satisfy the following exchange algebra*

$$S_{\ell\ell'}(z_1) \, S_{\lambda\lambda'}(z_2) = \mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z_1/z_2) \, S_{\lambda\lambda'}(z_2) \, S_{\ell\ell'}(z_1) \,. \tag{3.14}$$

*On the surface $\mathcal{P}_{\ell,\ell'} \cap \mathcal{P}_{-\lambda,-\lambda'}$, the operators $T_{\ell\ell'}$ and $S_{\lambda\lambda'}$ satisfy the following exchange algebra*

$$T_{\ell\ell'}(z_1) \, S_{\lambda\lambda'}(z_2) = \mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z_1/z_2) \, S_{\lambda\lambda'}(z_2) \, T_{\ell\ell'}(z_1) \,. \tag{3.15}$$

*The exchange function* $\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z)$ *is given by*

$$\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z) = \frac{F(\ell,z)}{F^*(\ell',z)} \frac{F^*(\ell',\gamma_{\lambda\lambda'}^{-1}z)}{F(\ell,\gamma_{\lambda\lambda'}^{-1}z)} . \tag{3.16}$$

Note that when $c$ is rational, the surface condition implies the existence of integers $r,s$ such that $p^r = q^s$.

Let us remark that eqs. (3.13) and (3.14) imply the following compatibility condition:

$$\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z)\,\mathbf{F}_{\lambda\lambda'}^{\ell\ell'}(z^{-1}) = 1 \tag{3.17}$$

which is trivially satisfied thanks to the properties of the function $F(\ell,z)$ and the surface conditions.

In the particular case $(\lambda,\lambda') = (\ell,\ell')$ which will be studied below, formula (3.16) simplifies. Indeed, using the "inversion" formula (2.18) and the $q^2$-periodicity of the function $F(\ell,z)$, one gets

$$\mathbf{F}_{\ell\ell'}^{\ell\ell'}(z) = \frac{F(|\ell|,z)}{F(|\ell|,(-p^{\frac{1}{2}})^{|\ell|}z)} \frac{F^*(|\ell'|,q^2(-p^{*\frac{1}{2}})^{|\ell'|}z)}{F^*(|\ell'|,z)} . \tag{3.18}$$

Propositions 3.1 to 3.3 generalize the results obtained in [6, 8] for the particular values $(\ell,\ell') = (\lambda,\lambda') = (2m+1,1)$.

It is possible to define a semi-classical limit of these exchange algebras. Indeed, the exchange function $\mathbf{F}_{\ell\ell'}^{\ell\ell'}(z)$ degenerates to 1 when $-p^{\frac{1}{2}} = q^k$ ($k \in \mathbb{Z}$) and $c$ integer. Setting $-p^{\frac{1}{2}} = q^{k-\beta/\eta}$ for some integer $k \in \mathbb{Z}$ and $\eta = (\ell-\ell')(\ell+\ell'-1)$, a Poisson structure, when $\ell \neq \ell'$, can now be defined by the following limit[1]:

$$\left\{T_{\ell\ell'}(z_1),\,T_{\ell\ell'}(z_2)\right\} \equiv \lim_{\beta\to 0} \frac{1}{\beta}\left(T_{\ell\ell'}(z_1)\,T_{\ell\ell'}(z_2) - T_{\ell\ell'}(z_2)\,T_{\ell\ell'}(z_1)\right). \tag{3.19}$$

In fact, when computing this limit, one finds that it does not depend on $k$, and the result is

$$\left\{T_{\ell\ell'}(z_1),\,T_{\ell\ell'}(z_2)\right\} = h(z_2/z_1)\,T_{\ell\ell'}(z_1)\,T_{\ell\ell'}(z_2) \tag{3.20}$$

where

$$h(x) = 2\ln q\left[\frac{1+x^2}{1-x^2} + 2\sum_{n=0}^{\infty}\left(\frac{1}{1-q^{4n+2}x^2} - \frac{1}{1-q^{4n}x^2} + \right.\right.$$
$$\left.\left.\frac{1}{1-q^{4n}x^{-2}} - \frac{1}{1-q^{4n+2}x^{-2}}\right)\right] . \tag{3.21}$$

This Poisson structure is identical to the classical deformed Virasoro algebra constructed in [1]. It is therefore consistent to consider all these quantum exchange algebras as "quantum deformed Virasoro algebras".

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[1]When $\ell = \ell'$, the exchange function $\mathbf{F}_{\ell\ell'}^{\ell\ell'}(z)$ degenerates to 1 for any $p$ and $q$ and $c = -2/\ell$. The Poisson structure in this case is obtained by expanding around this "critical" value of $c$, see [1, 6].

### 3.2. Example: $c = 1$ and $p = q^3$

We consider the case where $c = 1$ and the parameters $p$ and $q$ are related by $p = q^3$. The structure constants $\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z)$ depend only on the congruency classes $\overline{\ell}$ and $\overline{\lambda}$ modulo 4 of the integers $\ell$ and $\lambda$, the surface conditions implying that the integers $\ell'$ and $\lambda'$ are given by

$$\ell' = 3\ell + 4 \quad ; \quad \lambda' = 3\lambda + 4 \tag{3.22}$$

$$\overline{\ell}' \equiv 3\overline{\ell} \mod 4 \quad ; \quad \overline{\lambda}' \equiv 3\overline{\lambda} \mod 4 . \tag{3.23}$$

When $(\overline{\lambda}, \overline{\lambda}') = (0,0)$ or $(\overline{\ell}, \overline{\ell}') = (0,0)$, one gets $\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z) = 1$. The non-trivial structure constants $\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z)$ are summarized in the following tableau ($\Theta(x) \equiv \Theta_{q^4}(x)$):

| $(\overline{\ell}, \overline{\ell}')$ \ $(\overline{\lambda}, \overline{\lambda}')$ | $(1, 3)$ | $(2, 2)$ | $(3, 1)$ |
|---|---|---|---|
| $(1, 3)$ | $q^2 \dfrac{\Theta(q^3 z^2)^2 \, \Theta(q^{-1} z^2)^2}{\Theta(qz^2)^4}$ | $q^2 \dfrac{\Theta(q^2 z^2)^4 \, \Theta(q^{-1} z^2)^4}{\Theta(z^2)^4 \, \Theta(qz^2)^4}$ | $q^2 \dfrac{\Theta(q^2 z^2)^2 \, \Theta(q^{-2} z^2)^2}{\Theta(z^2)^4}$ |
| $(2, 2)$ | $q^2 \dfrac{\Theta(z^2)^4 \, \Theta(q^3 z^2)^4}{\Theta(q^2 z^2)^4 \, \Theta(qz^2)^4}$ | $q^4 \dfrac{\Theta(q^3 z^2)^4 \, \Theta(q^{-1} z^2)^4}{\Theta(qz^2)^8}$ | $q^2 \dfrac{\Theta(q^2 z^2)^4 \, \Theta(q^{-1} z^2)^4}{\Theta(z^2)^4 \, \Theta(qz^2)^4}$ |
| $(3, 1)$ | $q^{-2} \dfrac{\Theta(z^2)^4}{\Theta(q^2 z^2)^2 \, \Theta(q^{-2} z^2)^2}$ | $q^2 \dfrac{\Theta(z^2)^4 \, \Theta(q^3 z^2)^4}{\Theta(q^2 z^2)^4 \, \Theta(qz^2)^4}$ | $q^2 \dfrac{\Theta(q^3 z^2)^2 \, \Theta(q^{-1} z^2)^2}{\Theta(qz^2)^4}$ |

### 3.3. Riemann–Hilbert splitting

In order to define the algebra generated by $T_{\ell\ell'}(z)$ and its exchange relations, one has to introduce the modes of the generators $T_{\ell\ell'}(z)$, i.e., $T_{\ell\ell'}(z) = \sum_{n \in \mathbb{Z}} T_{\ell\ell'}(n) z^{-n}$, and use the exchange relations (3.13) (with $\lambda, \lambda' = \ell, \ell'$) as ordering relations among the modes $T_{\ell\ell'}(n)$ (see, e.g., [15]). For this purpose, we need to prepare a Riemann–Hilbert splitting of the exchange function in (3.16), i.e., to factorize the exchange function into a function analytic around $z = 0$ and a function analytic around $z^{-1} = 0$:

$$\mathbf{F}_{\ell\ell'}^{\ell\ell'}(z) = \boldsymbol{\varphi}_+(z) \, \boldsymbol{\varphi}_-^{-1}(z^{-1}) \tag{3.24}$$

$$\boldsymbol{\varphi}_\pm(z) = 1 + O(z^{\pm 1}) \tag{3.25}$$

in the neighborhood of a circle $C$ of radius $R$. $\boldsymbol{\varphi}_+$ and $\boldsymbol{\varphi}_-$ are respectively analytic for $|z| < R$ and $|z| > R$. It may be possible to choose $\boldsymbol{\varphi}_+ = \boldsymbol{\varphi}_- = \varphi$ in the sense of analytic continuation. Indeed, in our case, the exchange relation (3.13) reads, after the Riemann–Hilbert splitting, as

$$\varphi_{\ell\ell'}(z_2/z_1) \, T_{\ell\ell'}(z_1) \, T_{\ell\ell'}(z_2) = \varphi_{\ell\ell'}(z_1/z_2) \, T_{\ell\ell'}(z_2) \, T_{\ell\ell'}(z_1) \tag{3.26}$$

where

$$
\varphi_{\ell\ell'}(x) = \left[ \frac{1}{(1-x^2)^{|\ell|-|\ell'|}} \prod_{s=1}^{|\ell|-1} \frac{(p^s x^2; q^4)_\infty \, (q^2 p^{-s} x^2; q^4)_\infty}{(q^4 p^{-s} x^2; q^4)_\infty \, (q^2 p^s x^2; q^4)_\infty} \right.
$$

$$
\left. \prod_{s=1}^{|\ell'|-1} \frac{(q^4 p^{*-s} x^2; q^4)_\infty \, (q^2 p^{*s} x^2; q^4)_\infty}{(p^{*s} x^2; q^4)_\infty \, (q^2 p^{*-s} x^2; q^4)_\infty} \right]^2.
$$

This choice of analyticity properties for the exchange functions guarantees the existence of a consistent normal ordering procedure based on the reordering by increasing values of the mode indices. This allows one in turn to define a suitable Poincaré–Birkhoff–Witt basis for the algebra.

Once this algebra is well defined, one can wonder whether it admits a central extension term. From the results of [17], one already knows that the central extension will be different from the one computed in this reference. Indeed, it was proved there that this central extension uniquely determine the exchange function, which is not of the type $\phi_{\ell\ell'}(x)$. However, in our case, other kinds of term may arise. Hence, this question remains open.

## 4. Vertex operators

An interesting interpretation of the surfaces arises when considering vertex operator representations [15, 14] of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}_2)$ at $c = 1$. These surfaces are related to coincident singularities in the Riemann–Hilbert splitted exchange relations of the vertex operators.

### 4.1. Level one vertex operators

The well-known Verma modules and irreducible highest weight modules of the affine Lie algebras are expected to have deformations to the elliptic case. Denoting by $V(\Lambda_i)$ and $V(\Lambda_{1-i})$ $(i = 0, 1)$ the $\mathcal{A}_{q,p}(\widehat{sl}_2)$-modules corresponding to the level one irreducible highest weight modules of $\mathcal{U}_q(\widehat{sl}_2)$, one introduces the level one vertex operators defined as intertwiners between $\mathcal{A}_{q,p}(\widehat{sl}_2)$-modules (see [15, 14]):

$$
\Phi^{(i)}(z) \colon V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_z \tag{4.1}
$$

$$
\Psi^{*(i)}(z) \colon V_z \otimes V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \tag{4.2}
$$

where $V_z = \mathbb{C}[z, z^{-1}] \otimes (\mathbb{C}v_+ \oplus \mathbb{C}v_-)$ is the spin $1/2$ evaluation module.

One assumes existence and uniqueness of these operators in the elliptic case. $\Phi$ and $\Psi^*$ are called type I and type II vertex operators. We will use the following decomposition

$$
\Phi^{(i)}(z) = \Phi^{(i)}_+(z) \otimes v_+ + \Phi^{(i)}_-(z) \otimes v_- \quad \text{with} \quad \Phi^{(i)}_\varepsilon(-z) = -(-1)^i \, \varepsilon \, \Phi^{(i)}_\varepsilon(z) \tag{4.3}
$$

$$
\Psi^{*(i)}(z) = \Psi^{*(i)}_+(z) + \Psi^{*(i)}_-(z) \quad \text{with} \quad \Psi^{*(i)}_\varepsilon(-z) = (-1)^i \, \varepsilon \, \Psi^{*(i)}_\varepsilon(z). \tag{4.4}
$$

It is conjectured[2] that the commutation relations between the vertex operators are given by

$$\Phi^{(1-i)}_{\varepsilon_2}(z_2)\,\Phi^{(i)}_{\varepsilon_1}(z_1) = \sum_{\varepsilon'_1,\varepsilon'_2=\pm} \mathcal{W}^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(z_1/z_2)\,\Phi^{(1-i)}_{\varepsilon'_1}(z_1)\,\Phi^{(i)}_{\varepsilon'_2}(z_2) \qquad (4.5)$$

$$\Psi^{*(1-i)}_{\varepsilon_1}(z_1)\,\Psi^{*(i)}_{\varepsilon_2}(z_2) = -\sum_{\varepsilon'_1,\varepsilon'_2=\pm} \Psi^{*(1-i)}_{\varepsilon'_2}(z_2)\,\Psi^{*(i)}_{\varepsilon'_1}(z_1)\,\mathcal{W}^{*\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}(z_1/z_2) \qquad (4.6)$$

$$\Phi^{(1-i)}_{\varepsilon_1}(z_1)\,\Psi^{*(i)}_{\varepsilon_2}(z_2) = \tau(z_1/z_2)\,\Psi^{*(1-i)}_{\varepsilon_2}(z_2)\,\Phi^{(i)}_{\varepsilon_1}(z_1)\,. \qquad (4.7)$$

In terms of the vertex operators, the generators $L(z)$ are given by

$$L_{\varepsilon\varepsilon'}(z) = \kappa\Psi^*_{\varepsilon'}(q^{-\frac{1}{2}}z)\Phi_{\varepsilon}(z) \qquad (4.8)$$

while their inverse read

$$L^{-1}_{\varepsilon\varepsilon'}(z) = \kappa^{-1}\Psi^*_{-\varepsilon}(-q^{-\frac{3}{2}}z)\Phi_{-\varepsilon'}(-q^{-1}z)\,. \qquad (4.9)$$

### 4.2. Connection with the surface conditions

From the relations

$$\rho(z)\big(a(z)\pm d(z)\big) = z^{-1}\frac{\alpha^{\pm}(z^{-1})}{\alpha^{\pm}(z)}\,, \qquad \alpha^{\pm}(z) = \frac{(\pm p^{\frac{1}{2}}qz;p)_{\infty}}{(\pm p^{\frac{1}{2}}q^{-1}z;p)_{\infty}}\,\xi(z^2;p,q^4)^{-1} \qquad (4.10)$$

$$\rho(z)\big(b(z)\pm c(z)\big) = \frac{\beta^{\pm}(z^{-1})}{\beta^{\pm}(z)}\,, \qquad \beta^{\pm}(z) = \frac{(\mp qz;p)_{\infty}}{(\mp pq^{-1}z;p)_{\infty}}\,\xi(z^2;p,q^4)^{-1} \qquad (4.11)$$

we have the following Riemann–Hilbert splittings [15, 14]:

$$z_1^{-1}\alpha^{\pm}(z_2/z_1)\,\Big(\Phi_+(z_1)\Phi_+(z_2)\pm\Phi_-(z_1)\Phi_-(z_2)\Big)$$
$$= z_2^{-1}\alpha^{\pm}(z_1/z_2)\,\Big(\Phi_+(z_2)\Phi_+(z_1)\pm\Phi_-(z_2)\Phi_-(z_1)\Big) \qquad (4.12)$$

$$\beta^+(z_2/z_1)\,\Big(\Phi_+(z_1)\Phi_-(z_2)+\Phi_-(z_1)\Phi_+(z_2)\Big)$$
$$= \beta^+(z_1/z_2)\,\Big(\Phi_+(z_2)\Phi_-(z_1)+\Phi_-(z_2)\Phi_+(z_1)\Big) \qquad (4.13)$$

and

$$\frac{z_1^{-1}}{1-z_2^2/z_1^2}\,\frac{1}{\alpha^{*\pm}(z_2/z_1)}\,\Big(\Psi^*_+(z_1)\Psi^*_+(z_2)\pm\Psi^*_-(z_1)\Psi^*_-(z_2)\Big)$$
$$= \frac{z_2^{-1}}{1-z_1^2/z_2^2}\,\frac{1}{\alpha^{*\pm}(z_1/z_2)}\,\Big(\Psi^*_+(z_2)\Psi^*_+(z_1)\pm\Psi^*_-(z_2)\Psi^*_-(z_1)\Big) \qquad (4.14)$$

---

[2]It was shown in [5] that such relations are valid when $p$ is infinitesimally small. Their validity for finite $p$ remains an open question.

$$\frac{(1 + q^{-1}z_2/z_1)(1 + qz_2/z_1)}{1 - z_2^2/z_1^2} \frac{1}{\beta^{*+}(z_2/z_1)} \left(\Psi_+^*(z_1)\Psi_-^*(z_2) + \Psi_-^*(z_1)\Psi_+^*(z_2)\right)$$
$$= \frac{(1 + q^{-1}z_1/z_2)(1 + qz_1/z_2)}{1 - z_1^2/z_2^2} \frac{1}{\beta^{*+}(z_1/z_2)} \left(\Psi_+^*(z_2)\Psi_-^*(z_1) + \Psi_-^*(z_2)\Psi_+^*(z_1)\right) \tag{4.15}$$

$$z_1 \frac{(qz_2^2/z_1^2; q^4)_\infty}{(q^3z_2^2/z_1^2; q^4)_\infty} \Phi_{\varepsilon_1}(z_1)\Psi_{\varepsilon_2}^*(z_2) = z_2 \frac{(qz_1^2/z_2^2; q^4)_\infty}{(q^3z_1^2/z_2^2; q^4)_\infty} \Psi_{\varepsilon_2}^*(z_2)\Phi_{\varepsilon_1}(z_1). \tag{4.16}$$

As before, $\alpha^{*\pm}(z)$ and $\beta^{*\pm}(z)$ correspond to $\alpha^\pm(z)$ and $\beta^\pm(z)$ with $p \to p^* = pq^{-2c}$.

Equations (4.12)–(4.15) all have the form

$$c(z_2/z_1)\, \mathcal{O}(z_1)\, \mathcal{O}(z_2) = c(z_1/z_2)\, \mathcal{O}(z_2)\, \mathcal{O}(z_1). \tag{4.17}$$

Our interpretation of the surface conditions will be based on the following reading of any exchange relation (4.17). Suppose that the coefficient $c(z_1/z_2)$ of the r.h.s. of (4.17) exhibits a zero at some position $z_1/z_2 = z_0$ and that the coefficient $c(z_2/z_1)$ of the l.h.s. of (4.17) does not have any zero nor pole at $z_1/z_2 = z_0$. Thus, for the equality to hold, the operator $\mathcal{O}(z_2)\, \mathcal{O}(z_1)$ on the r.h.s. must have a pole at $z_1/z_2 = z_0$ and the operator $\mathcal{O}(z_1)\, \mathcal{O}(z_2)\big|_{z_1/z_2=z_0}$ on the l.h.s. is then interpreted as a *residue* operator. In the same way, when the coefficient $c(z_2/z_1)$ of the l.h.s. of (4.17) has a pole at some position $z_1/z_2 = z_0'$ for which the coefficient $c(z_1/z_2)$ of the r.h.s. is regular, the bilinear $\mathcal{O}(z_2)\, \mathcal{O}(z_1)$ on the r.h.s. has a pole at the same location, and $\mathcal{O}(z_1)\, \mathcal{O}(z_2)\big|_{z_1/z_2=z_0'}$ on the l.h.s. is interpreted as a residue operator. In both cases, it may also occur that $\mathcal{O}(z_2)\, \mathcal{O}(z_1)$ be regular and $\mathcal{O}(z_1)\, \mathcal{O}(z_2)\big|_{z_1/z_2=z_0 \text{ or } z_0'}$ degenerate to zero, still interpreted as a "residue" of a regular operator.

The aim of this paragraph is to propose an interpretation of the operators $T_{\ell\ell'}$ and $S_{\ell\ell'}$ as bilinear of such residue operators in the context of the level one vertex operator representation. For such a purpose, let us define the following operator

$$\mathcal{T}_{\bar\ell\bar\ell'}(z_1, z_2) \equiv \mathrm{Tr}\left(L(z_1)^{-1}\, \sigma_x^\ell\, L(z_2)\, \sigma_x^{\ell'}\right) \qquad \ell,\, \ell' \in \mathbb{Z}. \tag{4.18}$$

Because of the property of Pauli matrices, it depends on $\ell$ and $\ell'$ only modulo 2, hence the notation $\bar\ell$ and $\bar\ell'$. This operator is formally related to $T_{\ell\ell'}$:

$$T_{\ell\ell'}(z) = \mathcal{T}_{\bar\ell\bar\ell'}(\gamma_{\ell\ell'} z, z). \tag{4.19}$$

It is this formal relation which we shall now clarify.
In terms of type I and type II vertex operators, one gets for $\ell$ and $\ell'$ even,

$$\begin{aligned}
\mathcal{T}_{00}(z_1, z_2) = {} & \Psi_+^*(-q^{-\frac{3}{2}}z_1)\, \Phi_+(-q^{-1}z_1)\, \Psi_-^*(q^{-\frac{1}{2}}z_2)\, \Phi_-(z_2) \\
& + \Psi_+^*(-q^{-\frac{3}{2}}z_1)\, \Phi_-(-q^{-1}z_1)\, \Psi_-^*(q^{-\frac{1}{2}}z_2)\, \Phi_+(z_2) \\
& + \Psi_-^*(-q^{-\frac{3}{2}}z_1)\, \Phi_+(-q^{-1}z_1)\, \Psi_+^*(q^{-\frac{1}{2}}z_2)\, \Phi_-(z_2) \\
& + \Psi_-^*(-q^{-\frac{3}{2}}z_1)\, \Phi_-(-q^{-1}z_1)\, \Psi_+^*(q^{-\frac{1}{2}}z_2)\, \Phi_+(z_2).
\end{aligned} \tag{4.20}$$

Then using eq. (4.16), one obtains

$$\frac{1}{q^{-\frac{1}{2}}z_2} \frac{(q^2 z_2^2/z_1^2; q^4)_\infty}{(q^4 z_2^2/z_1^2; q^4)_\infty} \, \mathcal{T}_{00}(z_1, z_2) = \frac{1}{q^{-1}z_1} \frac{(z_1^2/z_2^2; q^4)_\infty}{(q^2 z_1^2/z_2^2; q^4)_\infty}$$

$$\times \left( \Psi_+^*(-q^{-\frac{3}{2}}z_1) \, \Psi_-^*(q^{-\frac{1}{2}}z_2) + \Psi_-^*(-q^{-\frac{3}{2}}z_1) \, \Psi_+^*(q^{-\frac{1}{2}}z_2) \right)$$

$$\times \left( \Phi_+(-q^{-1}z_1) \, \Phi_-(z_2) + \Phi_-(-q^{-1}z_1) \, \Phi_+(z_2) \right). \qquad (4.21)$$

We observe that $\mathcal{T}_{00}(z_1, z_2)$ is expressed in terms of the bilinear of vertex operators $\Phi$ and $\Psi$, which appear in the l.h.s. of the Riemann–Hilbert splittings (4.12)–(4.15). Our previous discussion on the meaning of eq. (4.17) indicates that we must now carefully analyze the pole and zero structure of the coefficients of (4.12)–(4.15). The operators $T_{\ell\ell'}$ and the associated surface conditions will in fact appear from the requirement of simultaneous singularities in both bilinears $\Phi\Phi$ and $\Psi\Psi$.

We must first of all refocus the singularity analysis on modified exchange relations (4.12)–(4.15). The structure functions $\alpha^\pm(z)$ and $\beta^\pm(z)$ all contain the same normalization function $\xi(z^2; p, q^4)$. As a working hypothesis, we will not consider the poles arising from this function. A heuristic argument for such a point of view can be sketched as follows. The function $\xi(z^2; p, q^4)$ arises from the normalization coefficient of the elliptic $R$ matrix and can be viewed as the contribution of a $U(1)$ current. However, this $U(1)$ current must not be confused with the (elliptic analogue of the) $gl(1)$ current completing $sl(2)$ into $gl(2)$. Indeed, in our context, this latter current has vanishing commutation relations with the $\mathcal{A}_{q,p}(\widehat{gl}_2)$ currents (including itself). It is possible to use the $U(1)$ current in a redefinition of the vertex operators to eliminate the $\xi$ factor. This redefinition implies for the vertex operators $\Phi$ and $\Psi$ of $\mathcal{A}_{q,p}(\widehat{sl}_2)$ an extension of the Verma module to incorporate this extra $U(1)$ current. For the sake of simplicity, we shall not implement this extension here, but its existence shows that one can consistently restrict the analysis of the poles and zeroes in the exchange relations (4.12)–(4.15) to the part of the $\alpha^\pm(z)$ and $\beta^\pm(z)$ functions factoring the $\xi$ function. From now on, this restriction will be implicit.

Consider first equation (4.13):

$$\beta^+(-qz_2/z_1) \left( \Phi_+(-q^{-1}z_1) \, \Phi_-(z_2) + \Phi_-(-q^{-1}z_1) \, \Phi_+(z_2) \right) =$$

$$\beta^+(-q^{-1}z_1/z_2) \left( \Phi_+(z_2) \, \Phi_-(-q^{-1}z_1) + \Phi_-(z_2) \, \Phi_+(-q^{-1}z_1) \right). \quad (4.22)$$

At the values $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$ where $\ell \in 2\mathbb{Z}_{\geq 0}$, the function $\beta^+$ of the r.h.s. of eq. (4.22) has a zero while the function $\beta^+$ of the l.h.s. has neither poles nor zeroes. Similarly, at the values $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$ where $\ell \in 2\mathbb{Z}_{<0}$, the function $\beta^+$ of the l.h.s. of eq. (4.22) has a pole, while the function $\beta^+$ of the l.h.s. is regular. Hence the operator $\Phi_+(z_2) \, \Phi_-(-q^{-1}z_1) + \Phi_-(z_2) \, \Phi_+(-q^{-1}z_1)$ must have a pole located at $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$, $\ell \in 2\mathbb{Z}$, while $\Phi_+(-q^{-1}z_1) \, \Phi_-(z_2) + \Phi_-(-q^{-1}z_1) \, \Phi_+(z_2)$ is interpreted as a residue operator.

In the same way, from equation (4.15), one has

$$\frac{(1 - z_2/z_1)(1 - q^2 z_2/z_1)}{(1 - q^2 z_2^2/z_1^2)\,\beta^{*+}(-q z_2/z_1)}$$
$$\left( \Psi_+^*(-q^{-\frac{3}{2}} z_1)\Psi_-^*(q^{-\frac{1}{2}} z_2) + \Psi_-^*(-q^{-\frac{3}{2}} z_1)\Psi_+^*(q^{-\frac{1}{2}} z_2) \right) =$$
$$\frac{(1 - z_1/z_2)(1 - q^{-2} z_1/z_2)}{(1 - q^{-2} z_1^2/z_2^2)\,\beta^{*+}(-q^{-1} z_1/z_2)}$$
$$\left( \Psi_+^*(q^{-\frac{1}{2}} z_2)\Psi_-^*(-q^{-\frac{3}{2}} z_1) + \Psi_-^*(q^{-\frac{1}{2}} z_2)\Psi_+^*(-q^{-\frac{3}{2}} z_1) \right). \quad (4.23)$$

At the values $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$ where $\ell' \in 2\mathbb{Z}_{\geq 0}$, the function $1/\beta^{*+}$ of the r.h.s. of eq. (4.23) has a zero while the function $1/\beta^{*+}$ of the l.h.s. has no poles nor zeroes. Similarly, at the values $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$ where $\ell' \in 2\mathbb{Z}_{<0}$, the function $1/\beta^{*+}$ of the r.h.s. of eq. (4.23) has a pole while the function $1/\beta^{*+}$ of the l.h.s. is regular. It follows that in both cases the quantity $\Psi_+^*(q^{-\frac{1}{2}} z_2)\Psi_-^*(-q^{-\frac{3}{2}} z_1) + \Psi_-^*(q^{-\frac{1}{2}} z_2)\Psi_+^*(-q^{-\frac{3}{2}} z_1)$ must have a pole at $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$, $\ell' \in 2\mathbb{Z}$, while $\Psi_+^*(-q^{-\frac{3}{2}} z_1)\Psi_-^*(q^{-\frac{1}{2}} z_2) + \Psi_-^*(-q^{-\frac{3}{2}} z_1)\Psi_+^*(q^{-\frac{1}{2}} z_2)$ is interpreted as a residue operator.

Gathering these analyses, one concludes that the operator $T_{\ell\ell'}(z)$ has to be interpreted as a product of residue operators from expression (4.20) when both conditions $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$ and $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$, with $\ell$ and $\ell'$ even, are fulfilled.

Consider now the case where $\ell$ and $\ell'$ are odd. In terms of type I and type II vertex operators, one gets

$$\mathcal{T}_{11}(z_1, z_2) = \Psi_-^*(-q^{-\frac{3}{2}} z_1)\,\Phi_-(-q^{-1} z_1)\,\Psi_-^*(q^{-\frac{1}{2}} z_2)\,\Phi_-(z_2)$$
$$+ \Psi_+^*(-q^{-\frac{3}{2}} z_1)\,\Phi_+(-q^{-1} z_1)\,\Psi_+^*(q^{-\frac{1}{2}} z_2)\,\Phi_+(z_2)$$
$$+ \Psi_-^*(-q^{-\frac{3}{2}} z_1)\,\Phi_+(-q^{-1} z_1)\,\Psi_-^*(q^{-\frac{1}{2}} z_2)\,\Phi_+(z_2)$$
$$+ \Psi_+^*(-q^{-\frac{3}{2}} z_1)\,\Phi_-(-q^{-1} z_1)\,\Psi_+^*(q^{-\frac{1}{2}} z_2)\,\Phi_-(z_2) \quad (4.24)$$

and using again (4.16), one obtains

$$\frac{1}{q^{-\frac{1}{2}} z_2}\,\frac{(q^2 z_2^2/z_1^2; q^4)_\infty}{(q^4 z_2^2/z_1^2; q^4)_\infty}\,\mathcal{T}_{11}(z_1, z_2) = \frac{1}{q^{-1} z_1}\,\frac{(z_1^2/z_2^2; q^4)_\infty}{(q^2 z_1^2/z_2^2; q^4)_\infty}$$
$$\times \left( \Psi_+^*(-q^{-\frac{3}{2}} z_1)\,\Psi_+^*(q^{-\frac{1}{2}} z_2) + \Psi_-^*(-q^{-\frac{3}{2}} z_1)\,\Psi_-^*(q^{-\frac{1}{2}} z_2) \right)$$
$$\times \left( \Phi_+(-q^{-1} z_1)\,\Phi_+(z_2) + \Phi_-(-q^{-1} z_1)\,\Phi_-(z_2) \right). \quad (4.25)$$

From equation (4.12), one has

$$\frac{q}{z_1}\,\alpha^+(-q z_2/z_1)\left( \Phi_+(-q^{-1} z_1)\Phi_+(z_2) + \Phi_-(-q^{-1} z_1)\Phi_-(z_2) \right) =$$
$$\frac{1}{z_2}\,\alpha^+(-q^{-1} z_1/z_2)\left( \Phi_+(z_2)\Phi_+(-q^{-1} z_1) + \Phi_-(z_2)\Phi_-(-q^{-1} z_1) \right). \quad (4.26)$$

At the values $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$ where $\ell \in \mathbb{Z}_{\geq 0}$, $\ell$ odd, the function $\alpha^+$ of the r.h.s. of eq. (4.26) has a zero while the function $\alpha^+$ of the l.h.s. has neither poles nor zeroes. In addition, at the values $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$ where $\ell \in \mathbb{Z}_{<0}$, $\ell$ odd, the function $\alpha^+$ of the l.h.s. of eq. (4.26) has a pole while the function $\alpha^+$ of the r.h.s. is regular. Hence the operator $\Phi_+(z_2)\,\Phi_+(-q^{-1}z_1) + \Phi_-(z_2)\,\Phi_-(-q^{-1}z_1)$ has a pole at $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$, $\ell + 1 \in 2\mathbb{Z}$, while $\Phi_+(-q^{-1}z_1)\,\Phi_+(z_2) + \Phi_-(-q^{-1}z_1)\,\Phi_-(z_2)$ is interpreted as a residue operator.

Similarly, from equation (4.14), one has

$$\frac{z_1^{-1}}{1 - q^2 z_2^2/z_1^2}\, \frac{1}{\alpha^{*+}(-qz_2/z_1)}$$
$$\left( \Psi_+^*(-q^{-\frac{3}{2}}z_1)\Psi_+^*(q^{-\frac{1}{2}}z_2) + \Psi_-^*(-q^{-\frac{3}{2}}z_1)\Psi_-^*(q^{-\frac{1}{2}}z_2) \right) =$$
$$\frac{-q^{-1}z_2^{-1}}{1 - q^{-2}z_1^2/z_2^2}\, \frac{1}{\alpha^{*+}(-q^{-1}z_1/z_2)}$$
$$\left( \Psi_+^*(q^{-\frac{1}{2}}z_2)\Psi_+^*(-q^{-\frac{3}{2}}z_1) + \Psi_-^*(q^{-\frac{1}{2}}z_2)\Psi_-^*(-q^{-\frac{3}{2}}z_1) \right). \quad (4.27)$$

At the values $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$ where $\ell' \in \mathbb{Z}_{\geq 0}$, $\ell'$ odd, the function $1/\alpha^{*+}$ of the r.h.s. of eq. (4.27) has a zero while the function $1/\alpha^{*+}$ of the l.h.s. has neither poles nor zeroes. at the values $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$ where $\ell' \in \mathbb{Z}_{<0}$, $\ell'$ odd, the function $1/\alpha^{*+}$ of the l.h.s. of eq. (4.27) has a pole while the function $1/\alpha^{*+}$ of the r.h.s. is regular. It follows that the operator $\Psi_+^*(q^{-\frac{1}{2}}z_2)\Psi_+^*(-q^{-\frac{3}{2}}z_1) + \Psi_-^*(q^{-\frac{1}{2}}z_2)\Psi_-^*(-q^{-\frac{3}{2}}z_1)$ must have a pole at $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$, $\ell' + 1 \in 2\mathbb{Z}$, while $\Psi_+^*(-q^{-\frac{3}{2}}z_1)\Psi_+^*(q^{-\frac{1}{2}}z_2) + \Psi_-^*(-q^{-\frac{3}{2}}z_1)\Psi_-^*(q^{-\frac{1}{2}}z_2)$ is interpreted as a residue operator.

Finally, one concludes that the operator $T_{\ell\ell'}(z)$ has to be interpreted as a product of residue operators from expression (4.24) when both conditions $z_1/z_2 = (-p^{\frac{1}{2}})^{-\ell}$ and $z_1/z_2 = q^2(-p^{*\frac{1}{2}})^{-\ell'}$, with $\ell$ and $\ell'$ odd, are fulfilled.

One deals with the cases $\ell + \ell'$ odd along the same lines. Therefore, one can state:

**Theorem 4.1.** *At $c = 1$, the surface conditions $(-p^{\frac{1}{2}})^{-\ell} = q^2\,(-p^{*\frac{1}{2}})^{-\ell'}$ where $\ell, \ell' \in \mathbb{Z}$, correspond to simultaneous existence of zeroes in the coefficients of the r.h.s. (when $\ell$ and/or $\ell' \in \mathbb{Z}_{\geq 0}$) or poles in the coefficients of the l.h.s. (when $\ell$ and/or $\ell' \in \mathbb{Z}_{<0}$) of the Riemann–Hilbert splitting of both products of vertex operators $\Phi$ on the one hand and $\Psi^*$ on the other hand.*
*The operators $T_{\ell\ell'}(z)$ are then interpreted as residue operators of $\mathcal{T}_{\bar{\ell}\bar{\ell}'}(z_1, z_2)$ (expressed in term of vertex operators) at the point $z_1 = \gamma_{\ell\ell'} z_2$.*

Interpretation of the operators $S_{\ell\ell'}(z)$ follows along the same lines, introducing

$$\mathcal{S}_{\bar{\ell}\bar{\ell}'}(z_1, z_2) \equiv \mathrm{Tr}\left( \sigma_x^\ell\, L(z_1)\, \sigma_x^{\ell'}\, L(z_2)^{-1} \right) \qquad \ell,\, \ell' \in \mathbb{Z} \qquad (4.28)$$

which is related to the operator $S_{\ell\ell'}(z)$ by $S_{\ell\ell'}(z) = \mathcal{S}_{\bar{\ell}\bar{\ell}'}(z, q^2 \gamma^{-1}_{-\ell-\ell'} z)$. Performing the same analysis, one deduces similar conclusions for the $S_{\ell\ell'}$ type operators, summarized in:

**Theorem 4.2.** *At $c = 1$, the surface conditions $(-p^{\frac{1}{2}})^{\ell} = q^2 (-p^{*\frac{1}{2}})^{\ell'}$ where $\ell, \ell' \in \mathbb{Z}$, correspond to simultaneous existence of zeroes in the coefficients of the r.h.s. (when $\ell$ and/or $\ell' \in \mathbb{Z}_{<0}$) or poles of the coefficients of the l.h.s. (when $\ell$ and/or $\ell' \in \mathbb{Z}_{\geq 0}$) of the Riemann–Hilbert splitting of both products of vertex operators $\Phi$ on the one hand and $\Psi^*$ on the other hand.*
*The operators $S_{\ell\ell'}(z)$ are then interpreted as residue operators of $\mathcal{S}_{\bar{\ell}\bar{\ell}'}(z_1, z_2)$ (expressed in term of vertex operators) at the point $z_1 = q^{-2}\gamma_{-\ell-\ell'} z_2$.*

# 5. Generalization to $\mathcal{A}_{q,p}(\widehat{gl}_N)$

## 5.1. The algebra $\mathcal{A}_{q,p}(\widehat{gl}_N)$

We start with the Boltzmann weights matrix for $\mathbb{Z}_N$-vertex model [9, 18]:

$$\mathcal{W}(z, q, p) = z^{2/N-2} \frac{1}{\kappa(z^2)} \frac{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\mu, \tau)}{\vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\xi + \mu, \tau)}$$

$$\sum_{(\alpha_1, \alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N} W_{(\alpha_1, \alpha_2)}(\xi, \mu, \tau) \, I_{(\alpha_1, \alpha_2)} \otimes I^{-1}_{(\alpha_1, \alpha_2)} \quad (5.1)$$

where the variables $z, q, p$ are related to the variables $\xi, \mu, \tau$ by

$$z = e^{i\pi\xi}, \qquad q = e^{i\pi\mu}, \qquad p = e^{2i\pi\tau}. \quad (5.2)$$

The Jacobi theta functions with rational characteristics $\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}(\xi, \tau)$ are defined in Appendix A.

The normalization factor is chosen as follows:

$$\frac{1}{\kappa(z^2)} = \frac{(q^{2N}z^{-2}; p, q^{2N})_\infty (q^2 z^2; p, q^{2N})_\infty (pz^{-2}; p, q^{2N})_\infty (pq^{2N-2}z^2; p, q^{2N})_\infty}{(q^{2N}z^2; p, q^{2N})_\infty (q^2 z^{-2}; p, q^{2N})_\infty (pz^2; p, q^{2N})_\infty (pq^{2N-2}z^{-2}; p, q^{2N})_\infty}.$$
$$(5.3)$$

The functions $W_{(\alpha_1, \alpha_2)}$ are given by

$$W_{(\alpha_1, \alpha_2)}(\xi, \mu, \tau) = \frac{\vartheta \begin{bmatrix} \frac{1}{2} + \alpha_1/N \\ \frac{1}{2} + \alpha_2/N \end{bmatrix}(\xi + \mu/N, \tau)}{N\vartheta \begin{bmatrix} \frac{1}{2} + \alpha_1/N \\ \frac{1}{2} + \alpha_2/N \end{bmatrix}(\mu/N, \tau)}. \quad (5.4)$$

The matrices $I_{(\alpha_1, \alpha_2)}$ are defined as follows:

$$I_{(\alpha_1, \alpha_2)} = g^{\frac{1}{2}}(g^{\alpha_2} h^{\alpha_1})g^{-\frac{1}{2}} \quad (5.5)$$

where the $N \times N$ matrices $g$ and $h$ are given by $g_{ij} = \omega^i \delta_{ij}$ and $h_{ij} = \delta_{i+1,j}$, the addition of indices being understood modulo $N$. They satisfy $hg = \omega gh$.

The matrix (5.1) is $\mathbb{Z}_N$-symmetric, that is $\mathcal{W}_{c+s,\,d+s}^{a+s,\,b+s} = \mathcal{W}_{c,\,d}^{a,\,b}$ for any indices $a, b, c, d, s \in \mathbb{Z}_N$ (the addition of indices being understood modulo $N$) and the non-vanishing elements of the $\mathcal{W}$ matrix are of the type $\mathcal{W}_{c,\,a+b}^{a,\,c+b}$.

To define the elliptic quantum algebra $\mathcal{A}_{q,p}(\widehat{gl}_N)$, we introduce the following matrix, which differs from (5.1) by a suitable normalization factor:

$$R_{12}(z) \equiv R_{12}(z, q, p) = \tau_N(q^{\frac{1}{2}} z^{-1}) \, \mathcal{W}_{12}(z, q, p) \tag{5.6}$$

where the function $\tau_N(z)$ is defined by

$$\tau_N(z) = z^{\frac{2}{N}-2} \, \frac{\Theta_{q^{2N}}(qz^2)}{\Theta_{q^{2N}}(qz^{-2})} \,. \tag{5.7}$$

The function $\tau_N(z)$ is $q^N$-periodic and satisfies $\tau_N(z) \, \tau_N(z^{-1}) = 1$.

The matrix $R_{12}$ is crossing-unitary [19, 7]:

$$\left( R_{12}(z)^{t_2} \right)^{-1} = \left( R_{12}(q^N z)^{-1} \right)^{t_2} \tag{5.8}$$

and obeys a quasi-periodicity property [7]:

$$R_{12}(-p^{\frac{1}{2}} z) = (g^{\frac{1}{2}} h g^{\frac{1}{2}} \otimes 1)^{-1} \, R_{21}(z^{-1})^{-1} \, (g^{\frac{1}{2}} h g^{\frac{1}{2}} \otimes 1)$$
$$= \widetilde{\tau}_N(z)^{-1} \, (g^{\frac{1}{2}} h g^{\frac{1}{2}} \otimes 1) \, R_{12}(z) \, (g^{\frac{1}{2}} h g^{\frac{1}{2}} \otimes 1) \tag{5.9}$$

where

$$\widetilde{\tau}(z) = \tau(q^{\frac{1}{2}} z^{-1}) \, \tau(q^{\frac{1}{2}} z) = q^{2/N-2} \, \frac{\Theta_{q^{2N}}(q^2 z^2) \, \Theta_{q^{2N}}(q^2 z^{-2})}{\Theta_{q^{2N}}(z^2) \, \Theta_{q^{2N}}(z^{-2})} \,. \tag{5.10}$$

Hence, we get

$$R_{12}\left( (-p^{\frac{1}{2}})^{-\ell} z \right) = F(\ell, z) \, ((g^{\frac{1}{2}} h g^{\frac{1}{2}})^\ell \otimes 1) \, R_{12}(z) \, ((g^{\frac{1}{2}} h g^{\frac{1}{2}})^\ell \otimes 1) \tag{5.11}$$

where, for $\ell \geq 0$,

$$F(\ell, z) = \prod_{s=1}^{\ell} \widetilde{\tau}_N((-p^{\frac{1}{2}})^{-s} z) \tag{5.12}$$

$$F(-\ell, z) = \frac{1}{F(\ell, (-p^{\frac{1}{2}})^\ell z)} \,. \tag{5.13}$$

We now define the elliptic quantum algebra $\mathcal{A}_{q,p}(\widehat{gl}_N)$ [16, 5] as an algebra of operators $L_{ij}(z) \equiv \sum_{n \in \mathbb{Z}} L_{ij}(n) \, z^n$ where $i, j \in \mathbb{Z}/N\mathbb{Z}$, encapsulated into a $N \times N$ matrix

$$L(z) = \begin{pmatrix} L_{11}(z) & \cdots & L_{1N}(z) \\ \vdots & & \vdots \\ L_{N1}(z) & \cdots & L_{NN}(z) \end{pmatrix} \,. \tag{5.14}$$

One defines $\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$ by imposing the following constraints on the $L_{ij}(z)$ (with the matrix $R_{12}$ given by eq. (5.6)):

$$R_{12}(z_1/z_2)\, L_1(z_1)\, L_2(z_2) = L_2(z_2)\, L_1(z_1)\, R_{12}^*(z_1/z_2) \qquad (5.15)$$

where $L_1(z) \equiv L(z) \otimes \mathbb{I}$, $L_2(z) \equiv \mathbb{I} \otimes L(z)$ and $R_{12}^*$ is defined by $R_{12}^*(z,q,p) \equiv R_{12}(z,q,p^* = pq^{-2c})$.

The matrix $R_{12}^*$ obeys also the properties of crossing-unitarity (5.8) and quasi-periodicity (5.9), this last one being understood with the modified elliptic nome $p^*$. The $q$-determinant $q$-det $L(z)$ given by

$$q\text{-}\det L(z) \equiv \sum_{\sigma \in \mathfrak{S}_N} \varepsilon(\sigma) \prod_{i=1}^{N} L_{i,\sigma(i)}(zq^{i-N-1}) \qquad (5.16)$$

($\varepsilon(\sigma)$ being the signature of the permutation $\sigma$) is in the center of $\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$. It can be set to the value $q^{\frac{c}{2}}$ so as to get

$$\mathcal{A}_{q,p}(\widehat{sl}(N)_c) = \mathcal{A}_{q,p}(\widehat{gl}(N)_c)/\langle q\text{-}\det L - q^{\frac{c}{2}}\rangle. \qquad (5.17)$$

### 5.2. Exchange algebras

**Proposition 5.1.** *We define the operators $T_{\ell\ell'}(z)$ and $S_{\ell\ell'}(z)$, $\ell, \ell' \in \mathbb{Z}$, by*

$$\begin{aligned}
T_{\ell\ell'}(z) &\equiv \mathrm{Tr}\left( L(\gamma z)^{-1}\, (g^{\frac{1}{2}}hg^{\frac{1}{2}})^{\ell}\, L(z)\, (g^{\frac{1}{2}}hg^{\frac{1}{2}})^{\ell'} \right) \\
&= \mathrm{Tr}\left( (g^{\frac{1}{2}}h^t g^{\frac{1}{2}})^{\ell}\, (L(\gamma z)^{-1})^t\, (g^{\frac{1}{2}}h^t g^{\frac{1}{2}})^{\ell'}\, L(z)^t \right)
\end{aligned} \qquad (5.18)$$

$$\begin{aligned}
S_{\ell\ell'}(z) &\equiv \mathrm{Tr}\left( (g^{\frac{1}{2}}hg^{\frac{1}{2}})^{\ell}\, L(z)\, (g^{\frac{1}{2}}hg^{\frac{1}{2}})^{\ell'}\, (L(\gamma^{-1}q^{-N}z)^{-1}) \right) \\
&= \mathrm{Tr}\left( L(z)^t\, (g^{\frac{1}{2}}h^t g^{\frac{1}{2}})^{\ell}\, (L(\gamma^{-1}q^{-N}z)^{-1})^t\, (g^{\frac{1}{2}}h^t g^{\frac{1}{2}})^{\ell'} \right).
\end{aligned} \qquad (5.19)$$

*The operators $T_{\ell\ell'}(z)$ have exchange relations with the generators $L(z)$ of $\mathcal{A}_{q,p}(\widehat{gl}_N)$:*

$$L(z_2)\, T_{\ell\ell'}(z_1) = f_{\ell\ell'}(z_1/z_2)\, T_{\ell\ell'}(z_1)\, L(z_2) \qquad (5.20)$$

*if the conditions*

$$\gamma = (-p^{\frac{1}{2}})^{-\ell} = (-p^{*\frac{1}{2}})^{-\ell'}\, q^N \qquad (5.21)$$

*are fulfilled.*

*In the same way, operators $S_{\ell\ell'}(z)$ have exchange relations with the generators $L(z)$ of $\mathcal{A}_{q,p}(\widehat{gl}_N)$:*

$$L(z_2)\, S_{\ell\ell'}(z_1) = f_{\ell\ell'}(z_1/z_2)\, S_{\ell\ell'}(z_1)\, L(z_2) \qquad (5.22)$$

*if the conditions*

$$\gamma = (-p^{\frac{1}{2}})^{\ell} = (-p^{*\frac{1}{2}})^{\ell'}\, q^N \qquad (5.23)$$

*are fulfilled.*

*The exchange function $f_{\ell\ell'}(z)$ is given by*

$$f_{\ell\ell'}(z) = \frac{F^*(\ell', z)}{F(\ell, z)} \qquad (5.24)$$

where $F(\ell, z)$ is now given by (5.12)–(5.13) and $F^*(\ell, z)$ is obtained from $F(\ell, z)$ by $p \to p^*$.

*Proof.* The proof is completely algebraic and follows exactly the same lines as the one of Proposition 3.1. $\qquad\square$

Following the steps of Section 3.1, we introduce:

**Definition 5.1.** *In the parameter space $(q, p, c)$, the surface $\mathcal{P}_{\ell\ell'}^{(N)}$ is defined by the relation*

$$(-p^{\frac{1}{2}})^{-\ell} = (-p^{*\frac{1}{2}})^{-\ell'} q^N \,, \quad \text{with } p^* = p\, q^{-2c} \,. \tag{5.25}$$

*The operators $T_{\ell\ell'}$ and $S_{-\ell, -\ell'}$ defined on this surface are given by relations (5.18) and (5.19) with $\gamma \equiv \gamma_{\ell\ell'} = (-p^{\frac{1}{2}})^{-\ell}$.*

**Proposition 5.2.** *When $c$ is a rational number, several surfaces $\mathcal{P}_{\ell\ell'}^{(N)}$ can be defined simultaneously.*
*On the surface $\mathcal{P}_{\ell\ell'}^{(N)} \cap \mathcal{P}_{\lambda\lambda'}^{(N)}$, the operators $T_{\ell\ell'}$ and $T_{\lambda\lambda'}$ satisfy the following exchange algebra*

$$T_{\ell\ell'}(z_1)\, T_{\lambda\lambda'}(z_2) = \mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z_1/z_2)\, T_{\lambda\lambda'}(z_2)\, T_{\ell\ell'}(z_1) \,. \tag{5.26}$$

*On the surface $\mathcal{P}_{-\ell, -\ell'}^{(N)} \cap \mathcal{P}_{-\lambda, -\lambda'}^{(N)}$, the operators $S_{\ell\ell'}$ and $S_{\lambda\lambda'}$ satisfy the following exchange algebra*

$$S_{\ell\ell'}(z_1)\, S_{\lambda\lambda'}(z_2) = \mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z_1/z_2)\, S_{\lambda\lambda'}(z_2)\, S_{\ell\ell'}(z_1) \,. \tag{5.27}$$

*On the surface $\mathcal{P}_{\ell\ell'}^{(N)} \cap \mathcal{P}_{-\lambda, -\lambda'}^{(N)}$, the operators $T_{\ell\ell'}$ and $S_{\lambda\lambda'}$ satisfy the following exchange algebra*

$$T_{\ell\ell'}(z_1)\, S_{\lambda\lambda'}(z_2) = \mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z_1/z_2)\, S_{\lambda\lambda'}(z_2)\, T_{\ell\ell'}(z_1) \,. \tag{5.28}$$

*The exchange function $\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z)$ is given by*

$$\mathbf{F}_{\ell\ell'}^{\lambda\lambda'}(z) = \frac{F(\ell, z)}{F^*(\ell', z)} \, \frac{F^*(\ell', \gamma_{\lambda\lambda'}^{-1} z)}{F(\ell, \gamma_{\lambda\lambda'}^{-1} z)} \tag{5.29}$$

*where $F(\ell, z)$ is given by (5.12)–(5.13).*

*Proof.* The proof follows the same lines as the one of Proposition 3.3. $\qquad\square$

Similarly to the $sl(2)$ case, we need to introduce a suitable Riemann–Hilbert splitting in order to be able to define ordering relations among the component operators $T_{\ell\ell'}(z)$. It can be consistently chosen as

$$\varphi_{\ell\ell'}(z_2/z_1)\, T_{\ell\ell'}(z_1)\, T_{\ell\ell'}(z_2) = \varphi_{\ell\ell'}(z_1/z_2)\, T_{\ell\ell'}(z_2)\, T_{\ell\ell'}(z_1) \tag{5.30}$$

where

$$\varphi_{\ell\ell'}(x) = \frac{1}{(1-x^2)^{2|\ell|-2|\ell'|}}$$

$$\times \prod_{s=1}^{|\ell|-1} \frac{(q^2 p^{-s} x^2; q^{2N})_\infty \, (q^{2N-2} p^{-s} x^2; q^{2N})_\infty}{(q^2 p^{s} x^2; q^{2N})_\infty \, (q^{2N-2} p^{s} x^2; q^{2N})_\infty} \left[ \frac{(p^{s} x^2; q^{2N})_\infty}{(q^{2N} p^{-s} x^2; q^{2N})_\infty} \right]^2$$

$$\times \prod_{s=1}^{|\ell'|-1} \frac{(q^2 p^{*s} x^2; q^{2N})_\infty \, (q^{2N-2} p^{*s} x^2; q^{2N})_\infty}{(q^2 p^{*-s} x^2; q^{2N})_\infty \, (q^{2N-2} p^{*-s} x^2; q^{2N})_\infty} \left[ \frac{(q^{2N} p^{*-s} x^2; q^{2N})_\infty}{(p^{*s} x^2; q^{2N})_\infty} \right]^2. \quad (5.31)$$

This allows one to properly define the algebra generated by $T_{\ell\ell'}(z)$. The next step should now be to interpret possible vertex operator representation of $\mathcal{A}_{q,p}(\widehat{sl}_N)$, so as to give a characterization of the surfaces (5.25) on the same lines as the one explicited in Theorems 4.1 and 4.2 for $sl(2)$. To our knowledge, such a representation is not yet available.


## 6. Conclusion

We have established a connection between the vertex operator representations available at $c = 1$ and the so-called "surface conditions" characterizing the existence of subalgebras of $q$-$W_N$ type in $\mathcal{A}_{q,p}(\widehat{sl}_N)$, as defined in [6, 7, 8]. We are now able to define new directions of investigations which will either make use of, or further extend this connection. It will undoubtedly lead to a better understanding of the $q$-$W_N$ algebra structures which we have constructed.

Regarding the first option, it must be indicated here that the VO construction used in [14] leads to the identification of very specific, so-called "fused" operators located at some precise singular points of the VO exchange algebra. However, a deep understanding of the connection between these non-singular fusion locations and our own construction, is at present lacking.

The singular fused operators, by contrast, now allow us to get explicit representations of our generators of $q$-$W_N$-type algebras, thereby enabling us to better understand subtle properties of these algebras. Noticeable amongst them is the existence of consistent central extensions. This delicate question was touched upon in [20], where central extensions were built in a formal way by explicitly solving the coboundary equations (Jacobi identity) for given $q$-Virasoro exchange functions. The relevance, and explicit construction of these formal central extensions, is however disputable: indeed the entanglement between the requirements of normal ordering of $q$-$W_N$ generators (such as discussed in Section 3.3) leading to Riemann–Hilbert splitting of the structure functions, and the resolution of the consistency conditions for the central extensions (depending also on these structure functions), is a delicate issue. Hopefully the study of short-distance expansions of $q$-$W_N$ generators using such explicit examples as are now available in [14], may shed light on this problem.

Regarding the second option, it is obvious that extensions of this connection entail the comparison of our commensurability conditions with VO representations both for $c \neq 1$ in the case of $\mathcal{A}_{q,p}(\widehat{sl}_2)$ and more generically for $\mathcal{A}_{q,p}(\widehat{sl}_N)$, $N \geq 3$.

## Appendix A. Jacobi theta functions with rational characteristics

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the upper half-plane and $\Lambda_\tau = \{\lambda_1 \tau + \lambda_2 \mid \lambda_1, \lambda_2 \in \mathbb{Z}, \tau \in \mathbb{H}\}$ the lattice with basis $(1, \tau)$ in the complex plane. One sets $\omega = e^{2i\pi/N}$.

One defines the Jacobi theta functions with rational characteristics $\gamma = (\gamma_1, \gamma_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$ by:

$$\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau) = \sum_{m \in \mathbb{Z}} \exp \left( i\pi(m + \gamma_1)^2 \tau + 2i\pi(m + \gamma_1)(\xi + \gamma_2) \right). \qquad (A.1)$$

The functions $\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau)$ satisfy the following shift properties:

$$\vartheta \begin{bmatrix} \gamma_1 + \lambda_1 \\ \gamma_2 + \lambda_2 \end{bmatrix} (\xi, \tau) = \exp \left( 2i\pi\gamma_1\lambda_2 \right) \vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau) \qquad (A.2)$$

$$\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi + \lambda_1 \tau + \lambda_2, \tau) =$$
$$\exp \left( -i\pi\lambda_1^2 \tau - 2i\pi\lambda_1\xi \right) \exp \left( 2i\pi(\gamma_1\lambda_2 - \gamma_2\lambda_1) \right) \vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau) \qquad (A.3)$$

where $\gamma = (\gamma_1, \gamma_2) \in \frac{1}{N}\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z} \times \mathbb{Z}$.

Moreover, for arbitrary $\lambda = (\lambda_1, \lambda_2)$ (not necessarily integers), one has the following shift exchange:

$$\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi + \lambda_1 \tau + \lambda_2, \tau) = \exp \left( -i\pi\lambda_1^2 \tau - 2i\pi\lambda_1(\xi + \gamma_2 + \lambda_2) \right) \vartheta \begin{bmatrix} \gamma_1 + \lambda_1 \\ \gamma_2 + \lambda_2 \end{bmatrix} (\xi, \tau). \qquad (A.4)$$

The Jacobi theta functions $\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau)$ with rational characteristics $(\gamma_1, \gamma_2)$ can be expressed in terms of the usual theta function $\Theta_p(z) = (z; p)_\infty \, (pz^{-1}; p)_\infty \, (p; p)_\infty$ as (with $p = e^{2i\pi\tau}$ and $z = e^{i\pi\xi}$):

$$\vartheta \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} (\xi, \tau) = e^{2i\pi\gamma_1\gamma_2} p^{\frac{1}{2}\gamma_1^2} z^{2\gamma_1} \Theta_p(-e^{2i\pi\gamma_2} p^{\gamma_1 + \frac{1}{2}} z^2). \qquad (A.5)$$

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