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On the R-Matrix Realization of Yangians and their Representations

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Dedicated to Daniel Arnaudon

Abstract. We study the Yangians $\mathrm{Y}(\mathfrak{a})$ associated with the simple Lie algebras $\mathfrak{a}$ of type B, C or D. The algebra $\mathrm{Y}(\mathfrak{a})$ can be regarded as a quotient of the extended Yangian $\mathrm{X}(\mathfrak{a})$ whose defining relations are written in an R-matrix form. In this paper we are concerned with the algebraic structure and representations of the algebra $\mathrm{X}(\mathfrak{a})$. We prove an analog of the Poincaré–Birkhoff–Witt theorem for $\mathrm{X}(\mathfrak{a})$ and show that the Yangian $\mathrm{Y}(\mathfrak{a})$ can be realized as a subalgebra of $\mathrm{X}(\mathfrak{a})$. Furthermore, we give an independent proof of the classification theorem for the finite-dimensional irreducible representations of $\mathrm{X}(\mathfrak{a})$ which implies the corresponding theorem of Drinfeld for the Yangians $\mathrm{Y}(\mathfrak{a})$. We also give explicit constructions for all fundamental representation of the Yangians.

1. Introduction

For any simple Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ the corresponding Yangian $\mathrm{Y}(\mathfrak{a})$ is a canonical deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[x])$, $\mathfrak{a}[x] = \mathfrak{a} \otimes \mathbb{C}[x]$ in the class of Hopf algebras; see Drinfeld [10, 11, 12]. In accordance to Drinfeld, each Yangian $\mathrm{Y}(\mathfrak{a})$ has at least three different presentations; see also Chari and Pressley [7, Chapter 12]. In this paper we are concerned with the one commonly known as the RTT-presentation and which preceded the other two historically. It goes back to the work of the St.-Petersburg school on the inverse scattering method; see, e.g., Takhtajan and Faddeev [24], Kulish and Sklyanin [15], Tarasov [25, 26], Reshetikhin, Takhtajan and Faddeev [23]. In the case of A type, i.e., $\mathfrak{a} = \mathfrak{sl}_N$, the RTT-presentation of the corresponding Yangian turns out to be particularly useful in the applications of the R-matrix techniques to the classical Lie algebras; see, e.g., the review paper [17] and references therein. Moreover, this presentation is most convenient for the study of various subalgebras of the A type Yangian which play an important role in the applications to the quantum spin chain models; see, e.g., Arnaudon *et al.* [2, 3, 4], Molev and Ragoucy [19].

In a recent paper by Arnaudon *et al.* [1], the RTT-presentation of the Yangian associated with the B, C or D type Lie algebra $\mathfrak{a}$ was studied. The Yangian $\mathrm{Y}(\mathfrak{a})$ was presented as a quotient of a quadratic algebra whose defining relations are written in the form of an RTT-relation. Below we denote this algebra by $\mathrm{X}(\mathfrak{a})$ and call it the *extended Yangian*. The paper [1] contains an explicit construction of a formal series $z(u)$ whose coefficients belong to the center of $\mathrm{X}(\mathfrak{a})$. As shown in [1], the quotient of $\mathrm{X}(\mathfrak{a})$ by the relations $z(u) = 1$ is isomorphic to $\mathrm{Y}(\mathfrak{a})$. In the orthogonal case $\mathfrak{a} = \mathfrak{o}_N$ (B and D types) this reproduces an earlier result of Drinfeld [10].

Our aim in this paper is to describe the algebraic structure of the extended Yangian $\mathrm{X}(\mathfrak{a})$ for each orthogonal and symplectic Lie algebra $\mathfrak{a} = \mathfrak{o}_N$ and $\mathfrak{a} = \mathfrak{sp}_{2n}$ and classify its finite-dimensional irreducible representations. First, we prove an analog of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{X}(\mathfrak{a})$. Then, following the approach of Molev, Nazarov and Olshanski [18], we define the Yangian $\mathrm{Y}(\mathfrak{a})$ as a subalgebra of $\mathrm{X}(\mathfrak{a})$. In [18], the A type Yangian $\mathrm{Y}(\mathfrak{sl}_N)$ is defined as a subalgebra of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ for the general lineal Lie algebra $\mathfrak{gl}_N$ so that the algebra $\mathrm{X}(\mathfrak{a})$ can be regarded as an analog of $\mathrm{Y}(\mathfrak{gl}_N)$ for the B, C and D types. Furthermore, we show that the coefficients of the series $z(u)$ are algebraically independent and generate the center of $\mathrm{X}(\mathfrak{a})$. This implies that the finite-dimensional irreducible representations of the algebras $\mathrm{X}(\mathfrak{a})$ and $\mathrm{Y}(\mathfrak{a})$ are essentially the same. These representations of the Yangian $\mathrm{Y}(\mathfrak{a})$ were classified by Drinfeld [12]; see also Chari and Pressley [7, Chapter 12]. However, this classification is given in terms of a different presentation (*new realization*) of $\mathrm{Y}(\mathfrak{a})$. At present, no explicit isomorphism between the new realization of the orthogonal or symplectic Yangian $\mathrm{Y}(\mathfrak{a})$ and its RTT-presentation is known. (Such an isomorphism in the case of $\mathrm{Y}(\mathfrak{sl}_N)$ was given by Drinfeld [12], its most recent detailed exposition can be found in [5].) Therefore, the classification results of [12] do not imply an immediate description of the finite-dimensional irreducible representations of the extended Yangian $\mathrm{X}(\mathfrak{a})$.

We develop an independent approach to the representation theory for the algebras $\mathrm{X}(\mathfrak{a})$. We define Verma modules $M(\lambda(u))$ over $\mathrm{X}(\mathfrak{a})$ in a standard way, where $\lambda(u)$ is a tuple of formal series which we call the highest weight. We show that every finite-dimensional irreducible representation of $\mathrm{X}(\mathfrak{a})$ is isomorphic to the unique irreducible quotient $L(\lambda(u))$ of $M(\lambda(u))$. We classify the finite-dimensional irreducible representations of $\mathrm{X}(\mathfrak{a})$ by producing necessary and sufficient conditions on the highest weight $\lambda(u)$ for the module $L(\lambda(u))$ to be finite-dimensional; see Theorem 5.16. Reformulating these conditions for representations of the subalgebra $\mathrm{Y}(\mathfrak{a})$ of $\mathrm{X}(\mathfrak{a})$ we thus obtain another proof of Drinfeld's theorem [12] for the case of the classical Lie algebras $\mathfrak{a} = \mathfrak{o}_N$ and $\mathfrak{sp}_{2n}$.

As a first step, we consider the low-rank cases and construct explicit isomorphisms $\mathrm{Y}(\mathfrak{sp}_2) \simeq \mathrm{Y}(\mathfrak{sl}_2)$, $\mathrm{Y}(\mathfrak{o}_3) \simeq \mathrm{Y}(\mathfrak{sl}_2)$ and $\mathrm{Y}(\mathfrak{o}_4) \simeq \mathrm{Y}(\mathfrak{sl}_2) \otimes \mathrm{Y}(\mathfrak{sl}_2)$. The former is quite immediate while the remaining two require appropriate versions of the fusion procedure for R-matrices. The representations are then described by using the known results for the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$ which are due to Tarasov [25, 26]. For the sake of completeness, we reproduce a proof of those results which is a simpler version of the one contained in [16]. Using the above isomorphisms, we also give explicit formulas for the evaluation homomorphisms from $\mathrm{X}(\mathfrak{a})$ to the universal enveloping algebra $\mathrm{U}(\mathfrak{a})$ for each $\mathfrak{a} = \mathfrak{sp}_2$, $\mathfrak{o}_3$ and $\mathfrak{o}_4$.

In order to establish the necessary conditions for $L(\lambda(u))$ to be finite-dimensional, we use an induction argument which allows us to get the conditions for the rank n Lie algebra $\mathfrak{a}$ from those of rank $n-1$. The sufficient conditions on $\lambda(u)$ are established by producing finite-dimensional modules having $\lambda(u)$ as a highest weight. We do this first for the so-called fundamental modules and then employ the Hopf algebra structure on $\mathrm{X}(\mathfrak{a})$. In particular, this proves that every finite-dimensional irreducible representations of $\mathrm{X}(\mathfrak{a})$ is isomorphic to a subquotient of a tensor product of the corresponding fundamental modules. We also give an explicit construction of all fundamental modules of $\mathrm{X}(\mathfrak{a})$ basically following the approach of Chari and Pressley [6] but avoiding the use of their results on the singularities of the R-matrices. For the applications of the fundamental Yangian modules to the affine Toda field theories see Chari and Pressley [8].

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2. Definitions and preliminaries

We let $\mathfrak{a}$ denote the simple complex Lie algebra of type B_n, C_n, or D_n. That is,

$$\mathfrak{a} = \mathfrak{o}_{2n+1}, \quad \mathfrak{sp}_{2n}, \quad \text{or} \quad \mathfrak{o}_{2n}, \tag{2.1}$$

respectively. Whenever possible, we consider the three cases (2.1) simultaneously, unless otherwise stated. The Lie algebra $\mathfrak{a}$ can be regarded as a subalgebra of the general linear Lie algebra $\mathfrak{gl}_N$, where $N = 2n+1$ or $N = 2n$, respectively. It will be convenient to enumerate the rows and columns of $N \times N$ matrices by the indices $-n, \dots, -1, 1, \dots, n$, if $N = 2n$, and by the indices $-n, \dots, -1, 0, 1, \dots, n$, if $N = 2n+1$. For $-n \leqslant i, j \leqslant n$ set

$$F_{ij} = E_{ij} - \theta_{ij}\, E_{-j,-i} \tag{2.2}$$

where the E_{ij} are the elements of the standard basis of $\mathfrak{gl}_N$ and

$$\theta_{ij} = \begin{cases} 1 & \text{in the orthogonal case,} \\ \operatorname{sgn} i \cdot \operatorname{sgn} j & \text{in the symplectic case.} \end{cases} \tag{2.3}$$

The elements F_{ij} span the Lie algebra $\mathfrak{a}$ and satisfy the relations

$$F_{ij} + \theta_{ij}\, F_{-j,-i} = 0 \tag{2.4}$$

for any $-n \leqslant i, j \leqslant n$, and

$$[F_{ij}, F_{kl}] = \delta_{kj}\, F_{il} - \delta_{il}\, F_{kj} - \delta_{k,-i}\, \theta_{ij}\, F_{-j,l} + \delta_{l,-j}\, \theta_{ij}\, F_{k,-i}. \tag{2.5}$$

For any n-tuple of complex numbers $\mu = (\mu_1, \dots, \mu_n)$ we shall denote by $V(\mu)$ the irreducible representation of the Lie algebra $\mathfrak{a}$ with the highest weight μ. That is, $V(\mu)$ is generated by a nonzero vector ξ such that

$$F_{ij}\,\xi = 0 \qquad \text{for} \quad -n \leqslant i < j \leqslant n, \qquad \text{and}$$
$$F_{ii}\,\xi = \mu_i\,\xi \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

The representation $V(\mu)$ is finite-dimensional if and only if

$$\mu_i - \mu_{i+1} \in \mathbb{Z}_+ \qquad \text{for} \quad i = 1, \dots, n-1$$

and

$$\begin{aligned} -\mu_1 - \mu_2 \in \mathbb{Z}_+ &\qquad \text{if} \quad \mathfrak{a} = \mathfrak{o}_{2n}, \\ -\mu_1 \in \mathbb{Z}_+ &\qquad \text{if} \quad \mathfrak{a} = \mathfrak{sp}_{2n}, \\ -2\,\mu_1 \in \mathbb{Z}_+ &\qquad \text{if} \quad \mathfrak{a} = \mathfrak{o}_{2n+1}. \end{aligned}$$

Consider the endomorphism algebra $\operatorname{End}\mathbb{C}^N$ and let $e_{ij} \in \operatorname{End}\mathbb{C}^N$ be the standard matrix units (we use lower case letters to distinguish the elements of $\operatorname{End}\mathbb{C}^N$ from the basis elements of $\mathfrak{gl}_N$; the latter will also be regarded as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$). We denote by F the $N \times N$ matrix whose ij-th entry is F_{ij}. We shall also regard F as the element

$$F = \sum_{i,j=-n}^{n} e_{ij} \otimes F_{ij} \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{a}). \tag{2.6}$$

We shall use the transposition $t\colon \operatorname{End}\mathbb{C}^N \to \operatorname{End}\mathbb{C}^N$ which is a linear map defined on the basis elements by the rule

$$(e_{ij})^t = \theta_{ij}\, e_{-j,-i}, \tag{2.7}$$

and the standard transposition defined by

$$(e_{ij})' = e_{ji}. \tag{2.8}$$

The permutation operator P is an element of $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$ given by

$$P = \sum_{i,j=-n}^{n} e_{ij} \otimes e_{ji}. \tag{2.9}$$

We let Q denote the transposed operator $Q = P^{\,t_1} = P^{\,t_2}$ with respect to the first or second copy of $\operatorname{End}\mathbb{C}^N$,

$$Q = \sum_{i,j=-n}^{n} \theta_{ij}\, e_{ij} \otimes e_{-i,-j}. \tag{2.10}$$

Whenever the double sign $\pm$ or $\mp$ occurs, the upper sign corresponds to the orthogonal case while the lower sign corresponds to the symplectic case. Note that the operators P and Q satisfy the relations

$$P^2 = 1, \qquad PQ = QP = \pm Q, \qquad Q^2 = N\,Q. \tag{2.11}$$

Set

$$\kappa = N/2 \mp 1. \tag{2.12}$$

The R-matrix $R(u)$ is a rational function in a complex parameter u with values in $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$ defined by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u-\kappa}. \tag{2.13}$$

It is well known that $R(u)$ satisfies the Yang–Baxter equation

$$R_{12}(u)\, R_{13}(u+v)\, R_{23}(v) = R_{23}(v)\, R_{13}(u+v)\, R_{12}(u), \tag{2.14}$$

see [14], [27]. Here both sides take values in $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$ and the subscripts indicate the copies of $\operatorname{End}\mathbb{C}^N$ so that $R_{12}(u) = R(u) \otimes 1$ etc.

Following the general approach of [11] and [23], we define the *extended Yangian* $\mathrm{X}(\mathfrak{a})$ as an associative algebra with generators $t_{ij}^{(r)}$, where $-n \leqslant i, j \leqslant n$ and $r = 1, 2, \dots$ (the zero value of i and j is skipped if $N = 2n$), satisfying certain quadratic relations. In order to write them down, introduce the formal series

$$t_{ij}(u) = \sum_{r=0}^{\infty} t_{ij}^{(r)}\, u^{-r} \in \mathrm{X}(\mathfrak{a})[[u^{-1}]], \qquad t_{ij}^{(0)} = \delta_{ij}, \tag{2.15}$$

and set

$$T(u) = \sum_{i,j=-n}^{n} e_{ij} \otimes t_{ij}(u) \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{X}(\mathfrak{a})[[u^{-1}]]. \tag{2.16}$$

Consider the algebra $\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \mathrm{X}(\mathfrak{a})[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by

$$T_1(u) = \sum_{i,j=-n}^{n} e_{ij} \otimes 1 \otimes t_{ij}(u), \qquad T_2(u) = \sum_{i,j=-n}^{n} 1 \otimes e_{ij} \otimes t_{ij}(u). \tag{2.17}$$

The defining relations for the algebra $\mathrm{X}(\mathfrak{a})$ have the form of an *RTT-relation*:

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v). \tag{2.18}$$

Equivalently, in terms of the series (2.15) they can be written as

$$\begin{aligned}[]
[t_{ij}(u), t_{kl}(v)] &= \frac{1}{u-v}\Big(t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u)\Big)\\
&\quad - \frac{1}{u-v-\kappa}\Big(\delta_{k,-i} \sum_{p=-n}^{n} \theta_{ip}\, t_{pj}(u)\, t_{-p,l}(v) - \delta_{l,-j} \sum_{p=-n}^{n} \theta_{jp}\, t_{k,-p}(v)\, t_{ip}(u)\Big).
\end{aligned} \tag{2.19}$$

Remark 2.1. The above definition of $\mathrm{X}(\mathfrak{a})$ can be extended to the cases $\mathfrak{a} = \mathfrak{o}_1$ and $\mathfrak{o}_2$. However, both algebras $\mathrm{X}(\mathfrak{o}_1)$ and $\mathrm{X}(\mathfrak{o}_2)$ are commutative. In addition, in $\mathrm{X}(\mathfrak{o}_2)$ we have $t_{-1,1}(u) = t_{1,-1}(u) = 0$. In what follows, we only deal with the orthogonal Lie algebras $\mathfrak{o}_N$ for $N \geqslant 3$. □

Consider an arbitrary formal series $f(u)$ of the form

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]. \tag{2.20}$$

Also, let $a \in \mathbb{C}$ be a constant and let B be a matrix with entries in $\mathbb{C}$ such that $BB^t = 1$. It is easily derived from the defining relations for the algebra $\mathrm{X}(\mathfrak{a})$ that each of the mappings

$$\mu_f \colon T(u) \mapsto f(u)\, T(u), \tag{2.21}$$

$$\tau_a \colon T(u) \mapsto T(u-a), \tag{2.22}$$

$$T(u) \mapsto B\, T(u)\, B^t$$

defines an automorphism of $\mathrm{X}(\mathfrak{a})$. Furthermore, each of the mappings

$$T(u) \mapsto T(-u),$$

$$T(u) \mapsto T^{\,t}(u),$$

$$T(u) \mapsto T^{-1}(u),$$

defines an anti-automorphism of $\mathrm{X}(\mathfrak{a})$; cf. [18, Section 1]. This is easily verified with the use of the following property of the R-matrix implied by (2.11):

$$R(u)\, R(-u) = 1 - \frac{1}{u^2}, \tag{2.23}$$

and the fact that $R(u)$ is stable under the composition of the transpositions in the first and the second copies of $\mathrm{End}\, \mathbb{C}^N$.

The extended Yangian $\mathrm{X}(\mathfrak{a})$ is a Hopf algebra with the coproduct

$$\Delta \colon t_{ij}(u) \mapsto \sum_{a=-n}^{n} t_{ia}(u) \otimes t_{aj}(u), \tag{2.24}$$

the antipode

$$\mathrm{S} \colon T(u) \mapsto T^{-1}(u),$$

and the counit

$$\epsilon \colon T(u) \mapsto 1,$$

cf. [23], [18, Section 1].

Multiplying both sides of (2.18) by $u - v - \kappa$, taking $u = v + \kappa$ and replacing v by u we get

$$Q\, T_1(u+\kappa)\, T_2(u) = T_2(u)\, T_1(u+\kappa)\, Q. \tag{2.25}$$

Since Q/N is a projection operator in $\mathbb{C}^N \otimes \mathbb{C}^N$ with a one-dimensional image, the expression on each side of (2.25) must be equal to Q times a series $z(u)$ with coefficients in $\mathrm{X}(\mathfrak{a})$. Since $Q\, T_1(u) = Q\, T_2^t(u)$ and $T_1(u)\, Q = T_2^t(u)\, Q$, we have

$$T^t(u+\kappa)\, T(u) = T(u)\, T^t(u+\kappa) = z(u)\, 1, \tag{2.26}$$

where

$$z(u) = 1 + z_1 u^{-1} + z_2 u^{-2} + \cdots, \qquad z_i \in \mathrm{X}(\mathfrak{a}). \tag{2.27}$$

Taking the kl-th entries in (2.26) we get the formulas

$$\sum_{i=-n}^{n} \theta_{ki}\, t_{-i,-k}(u+\kappa)\, t_{il}(u) = \sum_{i=-n}^{n} \theta_{il}\, t_{ki}(u)\, t_{-l,-i}(u+\kappa) = \delta_{kl}\, z(u). \qquad (2.28)$$

It was shown in [1] that all the coefficients z_i are central in $\mathrm{X}(\mathfrak{a})$, and $z(u)$ has the property[1]

$$\Delta\colon z(u) \mapsto z(u) \otimes z(u). \qquad (2.29)$$

By the Hopf algebra axioms, this implies that the image of $z(u)$ under the antipode S is found by

$$\mathrm{S}\colon z(u) \mapsto z(u)^{-1}. \qquad (2.30)$$

By (2.26), we have

$$\mathrm{S}\colon T(u) \mapsto z(u)^{-1}\, T^t(u+\kappa).$$

Hence, since the transposition is involutive, we conclude that the square of the antipode is the automorphism of $\mathrm{X}(\mathfrak{a})$ given by

$$\mathrm{S}^2\colon T(u) \mapsto \frac{z(u)}{z(u+\kappa)}\, T(u+2\kappa); \qquad (2.31)$$

cf. [18, Section 1].

We define the *Yangian* $\mathrm{Y}(\mathfrak{a})$ *associated with the Lie algebra* $\mathfrak{a}$ as the subalgebra of $\mathrm{X}(\mathfrak{a})$ which consists of the elements stable under all the automorphisms of the form (2.21). It will follow from [1] and the results below that this definition is consistent with the one given by Drinfeld [10]; cf. [18, Section 1].

3. Poincaré–Birkhoff–Witt theorem and the center of the extended Yangian

Let us denote by $\mathrm{ZX}(\mathfrak{a})$ the subalgebra of $\mathrm{X}(\mathfrak{a})$ generated by all the coefficients z_i of the series $z(u)$; see (2.27).

Theorem 3.1. *We have the tensor product decomposition*

$$\mathrm{X}(\mathfrak{a}) = \mathrm{ZX}(\mathfrak{a}) \otimes \mathrm{Y}(\mathfrak{a}). \qquad (3.1)$$

Proof. We follow the argument of [18, Section 2.16]. There exists a unique series $y(u)$ of the form

$$y(u) = 1 + y_1 u^{-1} + y_2 u^{-2} + \cdots, \qquad y_i \in \mathrm{ZX}(\mathfrak{a})$$

such that $y(u)y(u+\kappa) = z(u)$. In order to see this, it suffices to write this relation in terms of the coefficients,

$$z_k = 2\, y_k + A_k(y_1, \dots, y_{k-1}), \qquad k \geqslant 1, \qquad (3.2)$$

where A_k is a quadratic polynomial in $k-1$ variables. By (2.26), the image of the series $z(u)$ under the automorphism (2.21) is $f(u)\, f(u+\kappa)\, z(u)$. Hence, the

[1]Note that the R-matrix considered in [1] coincides with our $R(-u)$.

automorphism (2.21) takes $y(u)$ to $f(u)\,y(u)$. This implies that the series $\tau_{ij}(u)$ defined by

$$\tau_{ij}(u) = y(u)^{-1}\, t_{ij}(u), \qquad i,j = -n,\dots,n, \tag{3.3}$$

are stable under all automorphisms (2.21). Write

$$\tau_{ij}(u) = \delta_{ij} + \tau_{ij}^{(1)} u^{-1} + \tau_{ij}^{(2)} u^{-2} + \cdots .$$

So, the coefficients $\tau_{ij}^{(r)}$ of $\tau_{ij}(u)$ belong to the subalgebra $\mathrm{Y}(\mathfrak{a})$. Now the decomposition $\mathrm{X}(\mathfrak{a}) = \mathrm{ZX}(\mathfrak{a}) \cdot \mathrm{Y}(\mathfrak{a})$ follows from the relation $t_{ij}(u) = y(u)\,\tau_{ij}(u)$.

It remains to demonstrate that the elements z_i are algebraically independent over $\mathrm{Y}(\mathfrak{a})$. Due to (3.2), it suffices to do this for the elements y_i. Suppose on the contrary, that for some positive integer n there exists a nonzero polynomial B in n variables with the coefficients in $\mathrm{Y}(\mathfrak{a})$ such that

$$B(y_1,\dots,y_n) = 0. \tag{3.4}$$

Take the minimal n with this property. The coefficients of B are stable under any automorphism (2.21). Hence, applying the automorphism (2.21) with $f(u) = 1 + a u^{-n}$ and $a \in \mathbb{C}$ to the equality (3.4) we get

$$B(y_1,\dots,y_n + a) = 0$$

for any $a \in \mathbb{C}$. This means that the polynomial B does not depend on its n-th variable, which contradicts the choice of n. □

Corollary 3.2. *The Yangian* $\mathrm{Y}(\mathfrak{a})$ *is isomorphic to the quotient of* $\mathrm{X}(\mathfrak{a})$ *by the ideal generated by the elements* $z_1, z_2, \dots,$ *i.e.,*

$$\mathrm{Y}(\mathfrak{a}) \cong \mathrm{X}(\mathfrak{a})/(z(u) = 1).$$

Equivalently, $\mathrm{Y}(\mathfrak{a})$ *is generated by the elements* $\tau_{ij}^{(r)}$*, where* $-n \leqslant i,j \leqslant n$ *and* $r = 1,2,\dots$ *subject only to the relations*

$$\begin{aligned}[\tau_{ij}(u), \tau_{kl}(v)] &= \frac{1}{u-v}\Big(\tau_{kj}(u)\,\tau_{il}(v) - \tau_{kj}(v)\,\tau_{il}(u)\Big)\\ &- \frac{1}{u-v-\kappa}\Big(\delta_{k,-i}\sum_{p=-n}^{n} \theta_{ip}\,\tau_{pj}(u)\,\tau_{-p,l}(v) - \delta_{l,-j}\sum_{p=-n}^{n} \theta_{jp}\,\tau_{k,-p}(v)\,\tau_{ip}(u)\Big)\end{aligned} \tag{3.5}$$

and

$$\sum_{i=-n}^{n} \theta_{ki}\,\tau_{-i,-k}(u+\kappa)\,\tau_{il}(u) = \delta_{kl}. \tag{3.6}$$

Proof. Let I be the ideal of $\mathrm{X}(\mathfrak{a})$ introduced in the statement of the corollary. Then Theorem 3.1 implies that $\mathrm{X}(\mathfrak{a}) = \mathrm{I} \oplus \mathrm{Y}(\mathfrak{a})$ proving the first statement.

Now, the coefficients $\tau_{ij}^{(r)}$ of the series $\tau_{ij}(u)$ with $i,j = -n,\dots,n$ generate the subalgebra $\mathrm{Y}(\mathfrak{a})$. Indeed, it follows from the proof of Theorem 3.1 that any element $x \in \mathrm{X}(\mathfrak{a})$ can be uniquely written as a polynomial B in $y_1, y_2, \dots$ such that the coefficients of B are elements of the subalgebra of $\mathrm{X}(\mathfrak{a})$ generated by the elements $\tau_{ij}^{(r)}$. On the other hand, if x belongs to the subalgebra $\mathrm{Y}(\mathfrak{a})$ then B cannot depend on the elements y_i because x is stable under all automorphisms (2.21). Hence, x belongs to the subalgebra of $\mathrm{X}(\mathfrak{a})$ generated by the $\tau_{ij}^{(r)}$.

Finally, recall that the coefficients y_i of the series $y(u)$ are central in $\mathrm{X}(\mathfrak{a})$. Hence, we derive from (3.3) that the relation (2.19) will hold if the series $t_{ij}(u)$ are respectively replaced by $\tau_{ij}(u)$ which gives (3.5). Furthermore, (3.6) follows from (2.28). Conversely, (3.5) and (3.6) are defining relations for $\mathrm{Y}(\mathfrak{a})$ because they are respectively equivalent to (2.19) and the relation $z(u) = 1$. □

Proposition 3.3. *The subalgebra* $\mathrm{Y}(\mathfrak{a})$ *of* $\mathrm{X}(\mathfrak{a})$ *is a Hopf algebra whose coproduct, antipode and counit are obtained by restricting those from* $\mathrm{X}(\mathfrak{a})$.

Proof. The relation (2.29) implies that

$$\Delta\colon y(u) \mapsto y(u) \otimes y(u). \tag{3.7}$$

Therefore the image of $\mathrm{Y}(\mathfrak{a})$ under the coproduct on $\mathrm{X}(\mathfrak{a})$ is contained in $\mathrm{Y}(\mathfrak{a}) \otimes \mathrm{Y}(\mathfrak{a})$. By (2.30), the image of $y(u)$ under the antipode S is $y(u)^{-1}$. Hence,

$$\mathrm{S}\colon y(u)^{-1}\, T(u) \mapsto y(u)\, T^{-1}(u).$$

Any automorphism (2.21) leaves the product $y(u)\, T^{-1}(u)$ invariant and so the subalgebra $\mathrm{Y}(\mathfrak{a})$ of $\mathrm{X}(\mathfrak{a})$ is stable under S. □

Introduce an ascending filtration on the extended Yangian $\mathrm{X}(\mathfrak{a})$ by setting

$$\deg t_{kl}^{(r)} = r - 1 \tag{3.8}$$

for any $k, l \in \{-n, \dots, n\}$. Denote by $\bar{t}_{kl}^{(r)}$ and $\bar{z}_r$ the images of the elements $t_{kl}^{(r)}$ and z_r, respectively, in the $(r-1)$-th component of the associated graded algebra $\mathrm{gr}\,\mathrm{X}(\mathfrak{a})$. Then (2.28) gives the relations

$$\bar{t}_{kl}^{(r)} + \theta_{kl}\, \bar{t}_{-l,-k}^{(r)} = \delta_{kl}\, \bar{z}_r. \tag{3.9}$$

Furthermore, (3.3) implies that the degree of each element $\tau_{kl}^{(r)}$ does not exceed $r-1$ and its image $\bar{\tau}_{kl}^{(r)}$ in the $(r-1)$-th component of $\mathrm{gr}\,\mathrm{X}(\mathfrak{a})$ is given by

$$\bar{\tau}_{kl}^{(r)} = \frac{1}{2}\big(\bar{t}_{kl}^{(r)} - \theta_{kl}\, \bar{t}_{-l,-k}^{(r)}\big). \tag{3.10}$$

The ascending filtration on the Yangian $\mathrm{Y}(\mathfrak{a})$ is induced by the one on $\mathrm{X}(\mathfrak{a})$. We denote by $\mathrm{gr}\,\mathrm{Y}(\mathfrak{a})$ the associated graded algebra.

Proposition 3.4. *The mapping*

$$F_{ij}\, x^{r-1} \mapsto \bar{\tau}_{ij}^{(r)} \tag{3.11}$$

defines an algebra homomorphism $\psi\colon \mathrm{U}\big(\mathfrak{a}[x]\big) \to \mathrm{gr}\,\mathrm{Y}(\mathfrak{a})$.

Proof. By (3.10),

$$\bar{\tau}_{kl}^{(r)} + \theta_{kl}\,\bar{\tau}_{-l,-k}^{(r)} = 0$$

for any $-n \leqslant k, l \leqslant n$ and $r \geqslant 1$. Furthermore, using the expansion

$$\frac{1}{u-v} = u^{-1} + u^{-2}v + \cdots,$$

take the coefficients at $u^{-r}v^{-s}$ on both sides of the relation (3.5). Keeping the highest degree terms, we come to

$$[\bar{\tau}_{ij}^{(r)}, \bar{\tau}_{kl}^{(s)}] = \delta_{kj}\,\bar{\tau}_{il}^{(r+s-1)} - \delta_{il}\,\bar{\tau}_{kj}^{(r+s-1)} - \delta_{k,-i}\,\theta_{ij}\,\bar{\tau}_{-j,l}^{(r+s-1)} + \delta_{l,-j}\,\theta_{ij}\,\bar{\tau}_{k,-i}^{(r+s-1)}.$$

It remains to compare these relations with (2.4) and (2.5). □

Since the graded algebra $\operatorname{gr} \mathrm{Y}(\mathfrak{a})$ is generated by the elements $\bar{\tau}_{ij}^{(r)}$, the homomorphism ψ defined in Proposition 3.4 is obviously surjective. Our aim now is to show that ψ is an algebra isomorphism (see Theorem 3.6 below). We shall follow the approach of Nazarov's paper [21, Section 2], where a similar result was established for the Yangian of the queer Lie superalgebra.

Let ρ be the vector representation of the Lie algebra $\mathfrak{a}$ on the vector space $\mathbb{C}^N$. So,

$$\rho\colon F_{ij} \mapsto e_{ij} - \theta_{ij}\,e_{-j,-i}.$$

For any $c \in \mathbb{C}$ consider the corresponding evaluation representation ρ_c of the polynomial current Lie algebra $\mathfrak{a}[x]$ given by

$$\rho_c\colon F_{ij}\,x^s \mapsto c^s \rho(F_{ij}), \qquad s \geqslant 0.$$

For any $c_1, \dots, c_l \in \mathbb{C}$ consider the tensor product of the evaluation representations of $\mathfrak{a}[x]$,

$$\rho_{c_1,\dots,c_l} = \rho_{c_1} \otimes \cdots \otimes \rho_{c_l}.$$

Lemma 3.5. *Let the parameters $c_1, \dots, c_l$ and integer $l \geqslant 0$ vary. Then the intersection in $\mathrm{U}(\mathfrak{a}[x])$ of the kernels of all representations $\rho_{c_1,\dots,c_l}$ is trivial.*

Proof. Choose a basis $Y_1, \dots, Y_M$ of $\mathfrak{a}$, where $M = \dim \mathfrak{a}$, and set $y_i = \rho(Y_i)$. Let A be a nonzero element of $\mathrm{U}(\mathfrak{a}[x])$. Choose a total ordering on the set of basis elements $Y_i x^s$ of $\mathfrak{a}[x]$ and write A as a linear combination of ordered monomials in the basis elements. Let m be the maximal length of monomials which occur in A. For each monomial

$$(Y_{a_1} x^{s_1}) \cdots (Y_{a_m} x^{s_m}) \in \mathrm{U}(\mathfrak{a}[x]) \tag{3.12}$$

occurring in A consider the corresponding symmetrized elements

$$\sum_{q \in \mathfrak{S}_m} (Y_{a_{q(1)}} x^{s_{q(1)}}) \otimes \cdots \otimes (Y_{a_{q(m)}} x^{s_{q(m)}}) \in (\mathfrak{a}[x])^{\otimes m}. \tag{3.13}$$

Regarding $\mathrm{U}(\mathfrak{a}[x])$ as the quotient of the tensor algebra of $\mathfrak{a}[x]$ we derive that the elements (3.13) are linearly independent. Identifying the vector spaces

$$(\mathfrak{a}[x])^{\otimes m} = \mathfrak{a}^{\otimes m}[x_1, \dots, x_m],$$

we can regard the sum (3.13) as a polynomial function in m independent variables $x_1, \ldots, x_m$ with values in the vector space $\mathfrak{a}^{\otimes m}$,

$$\sum_{q\in\mathfrak{S}_m} x_1^{s_{q(1)}} \cdots x_m^{s_{q(m)}}\, Y_{a_{q(1)}} \otimes \cdots \otimes Y_{a_{q(m)}}. \tag{3.14}$$

Note that

$$\rho_{c_1,\ldots,c_l} \colon Y_a x^s \mapsto \sum_{k=1}^{l} c_k^s\, y_a^{[k]}, \qquad y_a^{[k]} = 1^{\otimes (k-1)} \otimes y_a \otimes 1^{\otimes (l-k)}.$$

Hence, the image of the monomial (3.12) under the representation $\rho_{c_1,\ldots,c_m}$ is given by

$$\sum_{k_1,\ldots,k_m=1}^{m} c_{k_1}^{s_1} \cdots c_{k_m}^{s_m}\, y_{a_1}^{[k_1]} \cdots y_{a_m}^{[k_m]} \in \operatorname{End}(\mathbb{C}^N)^{\otimes m}. \tag{3.15}$$

Let us complete the set of matrices $y_1, \ldots, y_M$ to a basis $y_1, \ldots, y_{N^2}$ of $\operatorname{End}\mathbb{C}^N$ in such a way that the identity matrix $1 \in \operatorname{End}\mathbb{C}^N$ occurs as a basis vector y_i for some $i \in \{M+1, \ldots, N^2\}$. Denote by V_m the subspace in $(\operatorname{End}\mathbb{C}^N)^{\otimes m}$ spanned by the basis elements $y_{i_1} \otimes \cdots \otimes y_{i_m}$ where at least one of the tensor factors is 1. Observe that the image under the representation $\rho_{c_1,\ldots,c_m}$ of any monomial of length $< m$ occurring in A is contained in V_m. Furthermore, modulo elements belonging to V_m, the sum (3.15) can be written as

$$\sum_{q\in\mathfrak{S}_m} c_1^{s_{q(1)}} \cdots c_m^{s_{q(m)}}\, y_{a_{q(1)}} \otimes \cdots \otimes y_{a_{q(m)}}. \tag{3.16}$$

This sum is the value of (3.14) under the specialization $x_i = c_i$ and replacement of Y_i with $y_i = \rho(Y_i)$ for all $i = 1, \ldots, m$. However, since ρ is faithful and the elements (3.13) are linearly independent, there exist values of the parameters $c_1, \ldots, c_m$ such that the corresponding sums (3.16) are linearly independent modulo the subspace V_m which completes the proof. □

We are now in a position to prove the following.

Theorem 3.6. *The mapping* $\psi \colon \mathrm{U}\big(\mathfrak{a}[x]\big) \to \operatorname{gr}\mathrm{Y}(\mathfrak{a})$ *defined in* (3.11) *is an algebra isomorphism.*

Proof. Due to Proposition 3.4, we only need to show that the kernel of ψ is trivial. Let C be a nonzero element of $\mathrm{U}\big(\mathfrak{a}[x]\big)$. We shall show that $\psi(C) \neq 0$. The universal enveloping algebra $\mathrm{U}\big(\mathfrak{a}[x]\big)$ has a grading defined on the generators by declaring the degree of $F_{ij}x^s$ to be equal to s. Then ψ is obviously a homomorphism of graded algebras. Hence, we may assume that C is homogeneous of degree, say, d. Write

$$C = \sum C^{\,r_1,\ldots,r_m}_{\,i_1 j_1,\ldots,i_m j_m} (F_{i_1 j_1} x^{r_1 - 1}) \cdots (F_{i_m j_m} x^{r_m - 1}), \tag{3.17}$$

summed over the indices i_a, j_a, r_a such that $r_1 + \cdots + r_m = d + m$.

Consider the element $C' \in \mathrm{Y}(\mathfrak{a})$ given by the formula

$$C' = \sum C^{\,r_1,\ldots,r_m}_{\,i_1 j_1,\ldots,i_m j_m}\, \tau^{(r_1)}_{i_1 j_1} \cdots \tau^{(r_m)}_{i_m j_m},$$

where the summation is taken over the same set of indices as in (3.17) with the same coefficients. Then the image of C' in the d-th component of the graded algebra $\mathrm{gr}\, \mathrm{Y}(\mathfrak{a})$ coincides with $\psi(C)$. So, it suffices to show that $\deg C' = d$.

Applying the standard transposition (2.8) to the third copy of $\mathrm{End}\, \mathbb{C}^N$ in the Yang–Baxter equation (2.14) and using (2.23) we come to the relation

$$R_{12}(u-v)\, R'_{13}(-u)\, R'_{23}(-v) = R'_{23}(-v)\, R'_{13}(-u)\, R_{12}(u-v), \tag{3.18}$$

where

$$R'(u) = 1 - \frac{P'}{u} + \frac{Q'}{u-\kappa}$$

with the transposition applied to the first (or second) copy of $\mathrm{End}\, \mathbb{C}^N$. Hence, by the defining relations (2.18) of the algebra $\mathrm{X}(\mathfrak{a})$ we conclude that the mapping $T(u) \mapsto R'(-u)$ defines a representation of $\mathrm{X}(\mathfrak{a})$ in the space $\mathbb{C}^N$. Taking its composition with the automorphism (2.22) we obtain for any $c \in \mathbb{C}$ the representation $\sigma_c \colon T(u) \mapsto R'(-u+c)$. Equivalently, in terms of the generating series (2.15) we have

$$\sigma_c \colon t_{ij}(u) \mapsto \delta_{ij} + e_{ij}\,(u-c)^{-1} - \theta_{ij}\, e_{-j,-i}\,(u+\kappa-c)^{-1}. \tag{3.19}$$

Since the transpositions (2.7) and (2.8) commute, using (2.26) and the relations

$$(Q')^2 = 1, \qquad P'Q' = Q'P' = \pm P', \qquad (P')^2 = N\, P',$$

we derive that the image of $z(u)$ under σ_c is given by

$$\sigma_c \colon z(u) \mapsto 1 - \frac{1}{(u-c+\kappa)^2}.$$

There exists a unique series $f_c(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that

$$f_c(u)\, f_c(u+\kappa) = \frac{(u-c+\kappa)^2}{(u-c+\kappa)^2 - 1}.$$

Then $\sigma_c \colon y(u)^{-1} \mapsto f_c(u)$ so that due to (3.19), for the image of the series $\tau_{ij}(u)$ under σ_c we have

$$\sigma_c \colon \tau_{ij}(u) \mapsto f_c(u)\Big(\delta_{ij} + e_{ij}\,(u-c)^{-1} - \theta_{ij}\, e_{-j,-i}\,(u+\kappa-c)^{-1}\Big). \tag{3.20}$$

Observe that the coefficient of the series $f_c(u)$ at u^{-k} is a polynomial in c of degree $\leqslant k-1$. Therefore, taking the coefficient at u^{-r} in (3.20) we find that the image of $\tau_{ij}^{(r)}$ under σ_c is a polynomial in c of degree $\leqslant r-1$ with coefficients in $\mathrm{End}\, \mathbb{C}^N$. Moreover, the coefficient of this polynomial at c^{r-1} coincides with $\rho(F_{ij})$.

Using Proposition 3.3, we can construct a representation of $\mathrm{Y}(\mathfrak{a})$ in the space $(\mathbb{C}^N)^{\otimes l}$ by

$$\sigma_{c_1,\dots,c_l} = \sigma_{c_1} \otimes \cdots \otimes \sigma_{c_l}, \qquad c_i \in \mathbb{C}.$$

The image of the element C' under $\sigma_{c_1,\dots,c_l}$ is a polynomial in $c_1,\dots,c_l$ of degree $\leqslant d$. Moreover, the homogeneous component of degree d of this polynomial coincides with $D = \rho_{c_1,\dots,c_l}(C)$. By Lemma 3.5, there exist values of the parameters $c_1,\dots,c_l$ such that $D \neq 0$. This implies that the element C' has degree d and so $\psi(C) \neq 0$. □

The following is an analog of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}(\mathfrak{a})$. It is immediate from Theorem 3.6.

Corollary 3.7. *Given any total ordering on the set of generators* $\tau_{ij}^{(r)}$ *with*

$$i+j>0, \qquad r\geqslant 1, \qquad \text{in the orthogonal case,}$$

and

$$i+j\geqslant 0, \qquad r\geqslant 1, \qquad \text{in the symplectic case,}$$

the ordered monomials in the generators form a basis of $\mathrm{Y}(\mathfrak{a})$. □

Remark 3.8. The algebra $\mathrm{Y}(\mathfrak{a})$ admits another filtration defined by setting the degree of the generator $\tau_{ij}^{(r)}$ to be equal to r. It follows from Corollary 3.2 that the associated graded algebra $\widetilde{\mathrm{gr}}\,\mathrm{Y}(\mathfrak{a})$ is commutative. Let $\widetilde{\tau}_{ij}^{(r)}$ denote the image of $\tau_{ij}^{(r)}$ in the r-th component of $\widetilde{\mathrm{gr}}\,\mathrm{Y}(\mathfrak{a})$. By Corollary 3.7, the graded algebra $\widetilde{\mathrm{gr}}\,\mathrm{Y}(\mathfrak{a})$ is isomorphic to the algebra of polynomials in the variables $\widetilde{\tau}_{ij}^{(r)}$, where the indices i,j,r are subject to the same conditions as in Corollary 3.7. □

Recall that $\mathrm{ZX}(\mathfrak{a})$ is the subalgebra of $\mathrm{X}(\mathfrak{a})$ generated by the coefficients z_i of the series $z(u)$.

Corollary 3.9.

(i) *The center of the algebra* $\mathrm{Y}(\mathfrak{a})$ *is trivial.*
(ii) *The center of the algebra* $\mathrm{X}(\mathfrak{a})$ *coincides with* $\mathrm{ZX}(\mathfrak{a})$.
(iii) *The coefficients* $z_1, z_2, \dots$ *of the series* $z(u)$ *are algebraically independent over* $\mathbb{C}$, *so that the subalgebra* $\mathrm{ZX}(\mathfrak{a})$ *of* $\mathrm{X}(\mathfrak{a})$ *is isomorphic to the algebra of polynomials in countably many variables.*

Proof. It is well known that the center of the universal enveloping algebra $\mathrm{U}\big(\mathfrak{a}[x]\big)$ is trivial; see, e.g., [18, Proposition 2.12]. So (i) and (ii) follow from Theorem 3.6. It is implied by the proof of Theorem 3.6 that the elements $y_1, y_2, \dots$ of the series $y(u)$ are algebraically independent over $\mathbb{C}\subset \mathrm{Y}(\mathfrak{a})$. Hence so are the elements z_i, $i\geqslant 1$. □

We shall also need the following version of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{X}(\mathfrak{a})$.

Corollary 3.10. *Given any total ordering on the set of elements* $t_{ij}^{(r)}$ *and* z_r *with*

$$i+j>0, \qquad r\geqslant 1, \qquad \text{in the orthogonal case,}$$

and

$$i+j\geqslant 0, \qquad r\geqslant 1, \qquad \text{in the symplectic case,}$$

the ordered monomials in these elements form a basis of $\mathrm{X}(\mathfrak{a})$.

Proof. By Theorems 3.1, 3.6 and Corollary 3.9(iii), the graded algebra $\mathrm{gr}\,\mathrm{X}(\mathfrak{a})$ is isomorphic to the tensor product of the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[x])$ and the algebra of polynomials $\mathbb{C}[\zeta_1,\zeta_2,\dots]$ in indeterminates ζ_r. An isomorphism is given by

$$\bar{t}_{ij}^{(r)} \mapsto F_{ij}\, x^{r-1} + \frac{1}{2}\,\delta_{ij}\,\zeta_r,$$

so that ζ_r is the image of $\bar{z}_r$; see (3.9). This implies the statement. □

Proposition 3.11. *The assignment*

$$F_{ij} \mapsto \tau_{ij}^{(1)} \tag{3.21}$$

defines an embedding $\mathrm{U}(\mathfrak{a}) \hookrightarrow \mathrm{Y}(\mathfrak{a})$*, while the assignment*

$$F_{ij} \mapsto \frac{1}{2}\left(t_{ij}^{(1)} - \theta_{ij}\, t_{-j,-i}^{(1)}\right) \tag{3.22}$$

defines an embedding $\mathrm{U}(\mathfrak{a}) \hookrightarrow \mathrm{X}(\mathfrak{a})$.

Proof. The defining relations (3.5) and (3.6) of $\mathrm{Y}(\mathfrak{a})$ imply that the map (3.21) is a homomorphism. Its injectivity follows from Corollary 3.7. Furthermore, by (3.3) we have $\tau_{ij}^{(1)} = t_{ij}^{(1)} - \delta_{ij}\, y_1$. It remains to observe that $2y_1 = z_1 = t_{i\,i}^{(1)} + t_{-i,-i}^{(1)}$ for any i and $t_{ij}^{(1)} = -\theta_{ij}\, t_{-j,-i}^{(1)}$ for $i \neq j$ by (2.28). □

4. Isomorphisms for low rank Yangians

Recall that the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ for the general linear Lie algebra $\mathfrak{gl}_N$ is defined as a unital associative algebra with countably many generators $T_{ij}^{(1)}, T_{ij}^{(2)}, \dots$ where $1 \leqslant i, j \leqslant N$, and the defining relations

$$[T_{ij}^{(r+1)}, T_{kl}^{(s)}] - [T_{ij}^{(r)}, T_{kl}^{(s+1)}] = T_{kj}^{(r)} T_{il}^{(s)} - T_{kj}^{(s)} T_{il}^{(r)}, \tag{4.1}$$

where $r, s \geqslant 0$ and $T_{ij}^{(0)} = \delta_{ij}$. Equivalently, these relations can be written as

$$[T_{ij}^{(r)}, T_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \Big(T_{kj}^{(a-1)} T_{il}^{(r+s-a)} - T_{kj}^{(r+s-a)} T_{il}^{(a-1)}\Big). \tag{4.2}$$

Introducing the generating series,

$$T_{ij}(u) = \delta_{ij} + T_{ij}^{(1)} u^{-1} + T_{ij}^{(2)} u^{-2} + \cdots \in \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]],$$

we can also write (4.1) in the form

$$(u - v)\,[T_{ij}(u), T_{kl}(v)] = T_{kj}(u) T_{il}(v) - T_{kj}(v) T_{il}(u). \tag{4.3}$$

Equivalently, using the notation of Section 2 and introducing the matrices

$$R^{\circ}(u) = 1 - P\, u^{-1} \tag{4.4}$$

and

$$T^{\circ}(u) = \sum_{i,j=1}^{N} e_{ij} \otimes T_{ij}(u) \in \mathrm{End}\,\mathbb{C}^N \otimes \mathrm{Y}(\mathfrak{gl}_N)[[u^{-1}]], \tag{4.5}$$

we can present the defining relations in the form of an RTT-relation

$$R^{\circ}(u - v)\, T_1^{\circ}(u)\, T_2^{\circ}(v) = T_2^{\circ}(v)\, T_1^{\circ}(u)\, R^{\circ}(u - v); \tag{4.6}$$

cf. (2.18). We use the superscript "∘" here to distinguish the objects related to $\mathrm{Y}(\mathfrak{gl}_N)$ from those related to the algebra $\mathrm{X}(\mathfrak{a})$.

The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is a Hopf algebra with the coproduct

$$\Delta\colon T_{ij}(u) \mapsto \sum_{k=1}^{N} T_{ik}(u) \otimes T_{kj}(u). \tag{4.7}$$

An ascending filtration on $\mathrm{Y}(\mathfrak{gl}_N)$ can be defined by setting

$$\deg T_{ij}^{(r)} = r - 1. \tag{4.8}$$

Let $\overline{T}_{ij}^{(r)}$ denote the image of the generator $T_{ij}^{(r)}$ in the $(r-1)$-th component of the associated graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N)$. We have an algebra isomorphism

$$\mathrm{U}(\mathfrak{gl}_N[x]) \to \mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_N), \qquad E_{ij}\, x^{r-1} \mapsto \overline{T}_{ij}^{(r)}. \tag{4.9}$$

The assignment

$$\mathrm{ev}\colon T_{ij}(u) \mapsto \delta_{ij} + E_{ij}\, u^{-1} \tag{4.10}$$

defines a surjective homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{gl}_N)$. Moreover, the assignment $E_{ij} \mapsto T_{ij}^{(1)}$ defines an embedding $\mathrm{U}(\mathfrak{gl}_N) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_N)$.

For any series $g(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ consider the automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$ defined by

$$T_{ij}(u) \mapsto g(u)\, T_{ij}(u). \tag{4.11}$$

The Yangian for $\mathfrak{sl}_N$ is the subalgebra $\mathrm{Y}(\mathfrak{sl}_N)$ of $\mathrm{Y}(\mathfrak{gl}_N)$ which consists of the elements stable under all automorphisms (4.11).

The algebra $\mathrm{Y}(\mathfrak{gl}_N)$ is isomorphic to the tensor product of its subalgebras

$$\mathrm{Y}(\mathfrak{gl}_N) = \mathrm{ZY}(\mathfrak{gl}_N) \otimes \mathrm{Y}(\mathfrak{sl}_N), \tag{4.12}$$

where $\mathrm{ZY}(\mathfrak{gl}_N)$ denotes the center of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. The subalgebra $\mathrm{ZY}(\mathfrak{gl}_N)$ is generated by the coefficients of the series $D(u)$ called the quantum determinant. In the case $N = 2$ it takes the form

$$D(u) = T_{11}(u)\, T_{22}(u-1) - T_{21}(u)\, T_{12}(u-1).$$

Define the series $d(u)$ with coefficients in $\mathrm{ZY}(\mathfrak{gl}_2)$ by the relation $d(u)\, d(u-1) = D(u)$. Then all the coefficients of the series $\mathcal{T}_{ij}(u) = d(u)^{-1}\, T_{ij}(u)$ belong to the subalgebra $\mathrm{Y}(\mathfrak{sl}_2)$. The series $\mathcal{T}_{ij}(u)$ satisfy the relations

$$(u-v)\, [\mathcal{T}_{ij}(u), \mathcal{T}_{kl}(v)] = \mathcal{T}_{kj}(u)\, \mathcal{T}_{il}(v) - \mathcal{T}_{kj}(v)\, \mathcal{T}_{il}(u) \tag{4.13}$$

and

$$\mathcal{T}_{11}(u)\, \mathcal{T}_{22}(u-1) - \mathcal{T}_{21}(u)\, \mathcal{T}_{12}(u-1) = 1 \tag{4.14}$$

which are defining relations for the algebra $\mathrm{Y}(\mathfrak{sl}_2)$. In other words, the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$ is isomorphic to the quotient of $\mathrm{Y}(\mathfrak{gl}_2)$ by the ideal generated by all the coefficients of $D(u)$.

For more details on the algebraic structure of the Yangians $\mathrm{Y}(\mathfrak{gl}_N)$ and $\mathrm{Y}(\mathfrak{sl}_N)$ see, e.g., [18], [5].

4.1. Extended Yangian $\mathrm{X}(\mathfrak{sp}_2)$

Observe that if $N = 2$ then in the symplectic case the operators P and Q satisfy $P + Q = 1$; see (2.9) and (2.10). Therefore, for the corresponding R-matrix (2.13) we have

$$R(u) = \frac{u-1}{u-2}\Big(1 - \frac{2\,P}{u}\Big) = \frac{u-1}{u-2} \cdot R^{\circ}(u/2).$$

This implies the following isomorphism where we adopt the convention of Section 2 for numbering the rows and columns of 2×2 matrices by the indices $\{-1, 1\}$.

Proposition 4.1. *The mapping*

$$t_{ij}(u) \mapsto T_{ij}(u/2), \qquad i, j \in \{-1, 1\} \tag{4.15}$$

defines an isomorphism $\phi\colon \mathrm{X}(\mathfrak{sp}_2) \to \mathrm{Y}(\mathfrak{gl}_2)$.

Proof. This is immediate from the defining relations (2.18) and (4.6). □

Corollary 4.2. *The restriction of the isomorphism* (4.15) *to the subalgebra* $\mathrm{Y}(\mathfrak{sp}_2)$ *of* $\mathrm{X}(\mathfrak{sp}_2)$ *induces an isomorphism* $\mathrm{Y}(\mathfrak{sp}_2) \to \mathrm{Y}(\mathfrak{sl}_2)$.

Proof. Recall that the subalgebra $\mathrm{Y}(\mathfrak{sp}_2)$ consists of the elements stable under all automorphisms of $\mathrm{X}(\mathfrak{sp}_2)$ of the form (2.21). However, given a series $f(u)$ in u^{-1} with complex coefficients, the mapping (4.15) takes $f(u)\, t_{ij}(u)$ to $f(u)\, T_{ij}(u/2)$. So, we have the relation $\phi \circ \mu_f = \mu_g \circ \phi$, and hence $\mu_f \circ \phi^{-1} = \phi^{-1} \circ \mu_g$, where $g(u)$ is the series in u^{-1} defined by $g(u) = f(2u)$. Thus, the image of $\mathrm{Y}(\mathfrak{sp}_2)$ under the isomorphism ϕ coincides with the subalgebra $\mathrm{Y}(\mathfrak{sl}_2)$ of $\mathrm{Y}(\mathfrak{gl}_2)$, yielding the desired isomorphism. □

Corollary 4.3. *The mapping*

$$\mathrm{ev}\colon T(u) \mapsto 1 + F\, u^{-1} \tag{4.16}$$

defines a surjective homomorphism $\mathrm{X}(\mathfrak{sp}_2) \to \mathrm{U}(\mathfrak{sp}_2)$.

Proof. The composition of the evaluation homomorphism (4.10) and the natural projection $\mathfrak{gl}_N \to \mathfrak{sl}_N$ yields a homomorphism $\mathrm{Y}(\mathfrak{gl}_N) \to \mathrm{U}(\mathfrak{sl}_N)$. For $N = 2$ it takes the form

$$\begin{aligned} T_{-1,-1}(u) &\mapsto 1 + \big(E_{-1,-1} - E_{1,1}\big)\,(2u)^{-1}, \qquad & T_{-1,1}(u) &\mapsto E_{-1,1}u^{-1},\\ T_{1,1}(u) &\mapsto 1 + \big(E_{1,1} - E_{-1,-1}\big)\,(2u)^{-1}, \qquad & T_{1,-1}(u) &\mapsto E_{1,-1}u^{-1}. \end{aligned}$$

Applying the isomorphism of Proposition 4.1 and using the generators F_{ij} of $\mathfrak{sp}_2 \cong \mathfrak{sl}_2$ we get a homomorphism $\mathrm{X}(\mathfrak{sp}_2) \to \mathrm{U}(\mathfrak{sp}_2)$ given by

$$\mathrm{ev}\colon t_{ij}(u) \mapsto \delta_{ij} + F_{ij}\, u^{-1}, \qquad i, j \in \{-1, 1\}.$$

Obviously, it is surjective. □

4.2. Extended Yangian $\mathrm{X}(\mathfrak{o}_3)$

We shall now use a more standard notation for the generators of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$, where the indices i, j in the defining relations (4.1) and (4.3) run over the set $\{1, 2\}$. Consider the vector space $\mathbb{C}^2$ with its canonical basis e_1, e_2 and denote by V the three-dimensional subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$ spanned by the vectors

$$v_{-1} = e_1 \otimes e_1, \qquad v_0 = \frac{1}{\sqrt{2}}\big(e_1 \otimes e_2 + e_2 \otimes e_1\big), \qquad v_1 = -e_2 \otimes e_2.$$

We identify V with $\mathbb{C}^3$ regarding v_{-1}, v_0, v_1 as its canonical basis. In particular, the operators P_V and Q_V in $V \otimes V$ will be given by the respective formulas (2.9) and (2.10) so that, for instance, $P_V(v_0 \otimes v_1) = v_1 \otimes v_0$. Similarly, we regard the generator matrix $T(u) = (t_{ij}(u))$ as an element of $\operatorname{End} V \otimes \mathrm{X}(\mathfrak{o}_3)[[u^{-1}]]$. Note that the operator $(1 + P)/2$ is a projection of $(\mathbb{C}^2)^{\otimes 2}$ to the subspace V. Due to (4.6), we have

$$\frac{1+P}{2} \cdot T_1^{\circ}(2u)\, T_2^{\circ}(2u+1) = T_2^{\circ}(2u+1)\, T_1^{\circ}(2u) \cdot \frac{1+P}{2},$$

because $R^{\circ}(-1) = 1 + P$. Therefore, we may regard each side of this relation as an element of $\operatorname{End} V \otimes \mathrm{Y}(\mathfrak{gl}_2)[[u^{-1}]]$.

Proposition 4.4. *The mapping*

$$T(u) \mapsto \frac{1+P}{2} \cdot T_1^{\circ}(2u)\, T_2^{\circ}(2u+1) \tag{4.17}$$

defines an isomorphism $\phi\colon \mathrm{X}(\mathfrak{o}_3) \to \mathrm{Y}(\mathfrak{gl}_2)$*. More explicitly, the images of the generators under the isomorphism are given by the formulas*

$$\begin{aligned}
t_{-1,-1}(u) &\mapsto T_{11}(2u)\, T_{11}(2u+1)\\
t_{-1,0}(u) &\mapsto \frac{1}{\sqrt{2}}\Big(T_{11}(2u)\, T_{12}(2u+1) + T_{12}(2u)\, T_{11}(2u+1)\Big)\\
t_{-1,1}(u) &\mapsto -T_{12}(2u)\, T_{12}(2u+1)\\
t_{0,-1}(u) &\mapsto \frac{1}{\sqrt{2}}\Big(T_{11}(2u)\, T_{21}(2u+1) + T_{21}(2u)\, T_{11}(2u+1)\Big)\\
t_{0,0}(u) &\mapsto T_{11}(2u)\, T_{22}(2u+1) + T_{21}(2u)\, T_{12}(2u+1)\\
t_{0,1}(u) &\mapsto -\frac{1}{\sqrt{2}}\Big(T_{12}(2u)\, T_{22}(2u+1) + T_{22}(2u)\, T_{12}(2u+1)\Big)\\
t_{1,-1}(u) &\mapsto -T_{21}(2u)\, T_{21}(2u+1)\\
t_{1,0}(u) &\mapsto -\frac{1}{\sqrt{2}}\Big(T_{21}(2u)\, T_{22}(2u+1) + T_{22}(2u)\, T_{21}(2u+1)\Big)\\
t_{1,1}(u) &\mapsto T_{22}(2u)\, T_{22}(2u+1).
\end{aligned}$$

Proof. We start by showing that the mapping defines an algebra homomorphism. We use a version of the well-known fusion procedure for R-matrices; see, e.g., [2] and references therein.

Consider the tensor product space $(\mathbb{C}^2)^{\otimes 4}$. As in (2.14), we use subscripts of the R-matrix (4.4) or the permutation operator $P \in \mathrm{End}(\mathbb{C}^2)^{\otimes 2}$ to indicate the copies of $\mathbb{C}^2$ where the operator acts. In the following we consider $V \otimes V$ as a natural subspace of $(\mathbb{C}^2)^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 2}$. Obviously, the operator

$$\frac{1+P_{12}}{2} \cdot \frac{1+P_{34}}{2} = \frac{1}{4} \cdot R^{\circ}_{12}(-1)\, R^{\circ}_{34}(-1)$$

is a projection of $(\mathbb{C}^2)^{\otimes 2} \otimes (\mathbb{C}^2)^{\otimes 2}$ to the subspace $V \otimes V$. Let us set

$$R_V(u) = \frac{1+P_{12}}{2} \cdot \frac{1+P_{34}}{2} \cdot R^{\circ}_{14}(2u-1)\, R^{\circ}_{13}(2u)\, R^{\circ}_{24}(2u)\, R^{\circ}_{23}(2u+1). \tag{4.18}$$

Since the R-matrix $R^{\circ}(u)$ satisfies the Yang–Baxter equation (2.14), we have the following equivalent expression for $R_V(u)$,

$$R_V(u) = R^{\circ}_{23}(2u+1)\, R^{\circ}_{13}(2u)\, R^{\circ}_{24}(2u)\, R^{\circ}_{14}(2u-1) \cdot \frac{1+P_{12}}{2} \cdot \frac{1+P_{34}}{2}. \tag{4.19}$$

Clearly, the subspace $V \otimes V$ is stable under the operator $R_V(u)$.

Lemma 4.5. *We have the equality of operators in $V \otimes V$,*

$$R_V(u) = \frac{2u-1}{2u+1} \cdot \Big(1 - \frac{P_V}{u} + \frac{Q_V}{u-1/2}\Big). \tag{4.20}$$

Proof. Using the formulas of the kind

$$(1+P_{12})\, P_{14}\, P_{24} = (1+P_{12})\, P_{14}$$

and

$$(1+P_{12})\,(1+P_{34})\, P_{14}\, P_{23} = (1+P_{12})\,(1+P_{34})\, P_{13}\, P_{24},$$

it is easy to get a simplified expression for the operator $R_V(u)$,

$$R_V(u) = \frac{1+P_{12}}{2} \cdot \frac{1+P_{34}}{2} \cdot \Big(1 - \frac{P_{14}+P_{24}+P_{13}+P_{23}}{2u+1} + \frac{P_{13}\, P_{24}}{u\,(2u+1)}\Big).$$

The restriction of $R_V(u)$ to the subspace $V \otimes V$ is given by

$$1 - \frac{P_{14}+P_{24}+P_{13}+P_{23}}{2u+1} + \frac{P_{13}\, P_{24}}{u\,(2u+1)} \tag{4.21}$$

so that the proof of the lemma is completed by the application of (4.21) to all basis vectors $v_i \otimes v_j$ of $V \otimes V$. For instance, we have

$$\begin{aligned} R_V(u)&(v_{-1} \otimes v_{-1}) \\ &= \Big(1 - \frac{P_{14}+P_{24}+P_{13}+P_{23}}{2u+1} + \frac{P_{13}\, P_{24}}{u\,(2u+1)}\Big)(e_1 \otimes e_1 \otimes e_1 \otimes e_1) \\ &= \frac{(u-1)(2u-1)}{u\,(2u+1)} \cdot e_1 \otimes e_1 \otimes e_1 \otimes e_1. \end{aligned}$$

Clearly, the application of the operator on the right-hand side of (4.20) to the vector $v_{-1} \otimes v_{-1}$ gives the same result. The remaining cases are verified by the same calculation. □

By the lemma, the element $R_V(u)$ coincides with the R-matrix (2.13) for $\mathfrak{a} = \mathfrak{o}_3$, up to a scalar factor. So, in order to verify that the mapping (4.17) defines a homomorphism $\mathrm{X}(\mathfrak{o}_3) \to \mathrm{Y}(\mathfrak{gl}_2)$ we need to show that the relation

$$R_V(u-v)\, T_{1'}(u)\, T_{2'}(v) = T_{2'}(v)\, T_{1'}(u)\, R_V(u-v) \tag{4.22}$$

remains valid when $T(u)$ is replaced by its image. Here we use primed indices to indicate the copies of the space V in the tensor product $V \otimes V$. We reserve unprimed indices for the copies of $\mathbb{C}^2$ in the tensor product $(\mathbb{C}^2)^{\otimes 4}$. The left-hand side of (4.22) reads

$$\begin{aligned}\frac{1+P_{12}}{2}\cdot\frac{1+P_{34}}{2}\cdot R^\circ_{14}(2u-2v-1)\, R^\circ_{13}(2u-2v)\, R^\circ_{24}(2u-2v)\, R^\circ_{23}(2u-2v+1)\\ \times\frac{1+P_{12}}{2}\cdot T^\circ_1(2u)\, T^\circ_2(2u+1)\cdot\frac{1+P_{34}}{2}\cdot T^\circ_3(2v)\, T^\circ_4(2v+1).\end{aligned}$$

Writing the product of R-matrices in the equivalent form (4.19), we simplify this expression to

$$\begin{aligned}\frac{1+P_{12}}{2}\cdot\frac{1+P_{34}}{2}\cdot R^\circ_{14}(2u-2v-1)\, R^\circ_{13}(2u-2v)\, R^\circ_{24}(2u-2v)\, R^\circ_{23}(2u-2v+1)\\ \times T^\circ_1(2u)\, T^\circ_2(2u+1)\, T^\circ_3(2v)\, T^\circ_4(2v+1).\end{aligned}$$

Now apply the RTT-relation (4.6) repeatedly to bring this expression to the form

$$\begin{aligned}\frac{1+P_{12}}{2}\cdot\frac{1+P_{34}}{2}\cdot T^\circ_3(2v)\, T^\circ_4(2v+1)\, T^\circ_1(2u)\, T^\circ_2(2u+1)\\ \times R^\circ_{14}(2u-2v-1)\, R^\circ_{13}(2u-2v)\, R^\circ_{24}(2u-2v)\, R^\circ_{23}(2u-2v+1).\end{aligned}$$

Finally, since $R^\circ_{12}(-1)/2$ is a projection, we derive from (4.6) that

$$\frac{1+P_{12}}{2}\cdot T^\circ_1(2u)\, T^\circ_2(2u+1) = \frac{1+P_{12}}{2}\cdot T^\circ_1(2u)\, T^\circ_2(2u+1)\cdot\frac{1+P_{12}}{2}.$$

Using the same property of $R^\circ_{34}(-1)/2$ we obtain that the resulting expression coincides with the right-hand side of (4.22), where $T(u)$ is replaced with its image in accordance with (4.17).

The explicit images of the generators of $\mathrm{X}(\mathfrak{o}_3)$ are found by taking the matrix elements in (4.17). Indeed, the application of $T(u)$ to the basis vector v_{-1} of V gives

$$T(u)(v_{-1}) = t_{-1,-1}(u)\, v_{-1} + t_{0,-1}(u)\, v_0 + t_{1,-1}(u)\, v_1,$$

while

$$\begin{aligned}&\frac{1+P_{12}}{2}\cdot T^\circ_1(2u)\, T^\circ_2(2u+1)(v_{-1}) = \frac{1+P_{12}}{2}\cdot T^\circ_1(2u)\, T^\circ_2(2u+1)(e_1\otimes e_1)\\ &= \frac{1}{2}\sum_{a,b=1}^{2} T_{a1}(2u)\, T_{b1}(2u+1)\,(e_a\otimes e_b + e_b\otimes e_a) = T_{11}(2u)\, T_{11}(2u+1)\, v_{-1}\\ &+\frac{1}{\sqrt{2}}\Big(T_{11}(2u)\, T_{21}(2u+1) + T_{21}(2u)\, T_{11}(2u+1)\Big)\, v_0 - T_{21}(2u)\, T_{21}(2u+1)\, v_1.\end{aligned}$$

This agrees with the formulas for the images of the series $t_{a,-1}(u)$ for $a = -1, 0, 1$ given in the statement. The remaining formulas are verified in the same way. Note also that the image of the series $t_{0,0}(u)$ can be equivalently written as

$$T_{12}(2u)\, T_{21}(2u+1) + T_{22}(2u)\, T_{11}(2u+1)$$

due to the defining relation in the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$.

In order to complete the proof of the proposition we now verify that the homomorphism $\mathrm{X}(\mathfrak{o}_3) \to \mathrm{Y}(\mathfrak{gl}_2)$ given by (4.17) is bijective. Taking the coefficient at u^{-r} in $t_{-1,-1}(u)$ we find that for any $r \geqslant 1$

$$t^{(r)}_{-1,-1} \mapsto 2^{-r+1}\, T^{(r)}_{11} + A_{r-1}(T^{(1)}_{11}, \dots, T^{(r-1)}_{11}),$$

where A_{r-1} stands for a quadratic polynomial in the generators $T^{(1)}_{11}, \dots, T^{(r-1)}_{11}$. The obvious induction on r shows that each generator $T^{(r)}_{11}$ with $r \geqslant 1$ belongs to the image of the homomorphism. Similarly, taking the image of $t^{(r)}_{1,1}$ we find that each generator $T^{(r)}_{22}$ with $r \geqslant 1$ also belongs to the image. Then taking the images of $t^{(r)}_{-1,0}$ and $t^{(r)}_{0,-1}$ we derive the same property of the generators $T^{(r)}_{12}$ and $T^{(r)}_{21}$ with $r \geqslant 1$. This proves that the homomorphism is surjective.

Finally, observe that the homomorphism preserves the respective filtrations on $\mathrm{X}(\mathfrak{o}_3)$ and $\mathrm{Y}(\mathfrak{gl}_2)$. Hence, we have a homomorphism of the associated graded algebras $\mathrm{gr}\, \mathrm{X}(\mathfrak{o}_3) \to \mathrm{gr}\, \mathrm{Y}(\mathfrak{gl}_2)$. It suffices to show that this homomorphism is injective. Identifying $\mathrm{gr}\, \mathrm{Y}(\mathfrak{gl}_2)$ with the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_2[x])$ via the isomorphism (4.9), we get

$$\bar{t}^{(r)}_{0,1} \mapsto -\frac{1}{\sqrt{2}}\, E_{12}\, (x/2)^{r-1}, \quad \bar{t}^{(r)}_{1,0} \mapsto -\frac{1}{\sqrt{2}}\, E_{21}\, (x/2)^{r-1}, \quad \bar{t}^{(r)}_{1,1} \mapsto E_{22}\, (x/2)^{r-1}$$

and

$$\bar{z}_r \mapsto (E_{11} + E_{22})\, (x/2)^{r-1}.$$

Therefore, the injectivity of the homomorphism follows from Corollary 3.10. □

Corollary 4.6. *The restriction of the isomorphism* $\phi\colon \mathrm{X}(\mathfrak{o}_3) \to \mathrm{Y}(\mathfrak{gl}_2)$ *to the subalgebra* $\mathrm{Y}(\mathfrak{o}_3)$ *induces an isomorphism* $\mathrm{Y}(\mathfrak{o}_3) \to \mathrm{Y}(\mathfrak{sl}_2)$.

Proof. Recall that the subalgebra $\mathrm{Y}(\mathfrak{o}_3)$ consists of the elements stable under all automorphisms of $\mathrm{X}(\mathfrak{o}_3)$ of the form (2.21). For any series $f(u)$ of the form (2.20) there exists a unique series

$$g(u) = 1 + g_1 u^{-1} + g_2 u^{-2} + \cdots \in \mathbb{C}[[u^{-1}]]$$

such that $f(u) = g(2u)\, g(2u+1)$. By Proposition 4.4, we have the relation $\phi \circ \mu_f = \mu_g \circ \phi$, and hence $\mu_f \circ \phi^{-1} = \phi^{-1} \circ \mu_g$. This implies that the image of $\mathrm{Y}(\mathfrak{o}_3)$ under the isomorphism ϕ coincides with the subalgebra $\mathrm{Y}(\mathfrak{sl}_2)$ of $\mathrm{Y}(\mathfrak{gl}_2)$ thus yielding the desired isomorphism. □

Let us denote by c the Casimir element for the Lie algebra $\mathfrak{o}_3$,

$$c = \frac{1}{2}\big(F^2_{11} - F_{11}\big) + F_{10}F_{01}.$$

In the following we use notation (2.6).

Corollary 4.7. *The mapping*

$$\mathrm{ev}\colon T(u) \mapsto 1 + \frac{F}{u} + \frac{F^2 - c\,1}{u\,(2u-1)} \tag{4.23}$$

defines a surjective homomorphism $\mathrm{X}(\mathfrak{o}_3) \to \mathrm{U}(\mathfrak{o}_3)$.

Proof. Writing the homomorphism $\mathrm{Y}(\mathfrak{gl}_2) \to \mathrm{U}(\mathfrak{sl}_2)$ used in the proof of Corollary 4.3 in the current notation we get

$$T_{11}(u) \mapsto 1 + \big(E_{11} - E_{22}\big)\,(2u)^{-1}, \qquad T_{12}(u) \mapsto E_{12}u^{-1},$$
$$T_{22}(u) \mapsto 1 + \big(E_{22} - E_{11}\big)\,(2u)^{-1}, \qquad T_{21}(u) \mapsto E_{21}u^{-1}.$$

Composing this with the isomorphism $\mathfrak{sl}_2 \to \mathfrak{o}_3$ given by

$$E_{11} - E_{22} \mapsto 2F_{-1,-1}, \qquad E_{12} \mapsto \sqrt{2}\,F_{-1,0}, \qquad E_{21} \mapsto \sqrt{2}\,F_{0,-1},$$

we get another homomorphism $\mathrm{Y}(\mathfrak{gl}_2) \to \mathrm{U}(\mathfrak{o}_3)$ such that

$$T_{11}(u) \mapsto 1 + F_{-1,-1}\,u^{-1}, \qquad T_{12}(u) \mapsto \sqrt{2}\,F_{-1,0}\,u^{-1},$$
$$T_{22}(u) \mapsto 1 + F_{1,1}\,u^{-1}, \qquad T_{21}(u) \mapsto \sqrt{2}\,F_{0,-1}\,u^{-1}.$$

Finally, compose the isomorphism of Proposition 4.4 with the shift automorphism $t_{ij}(u) \mapsto t_{ij}(u-1/2)$ of $\mathrm{X}(\mathfrak{o}_3)$ and use the above formulas to get a homomorphism $\mathrm{X}(\mathfrak{o}_3) \to \mathrm{U}(\mathfrak{o}_3)$. It remains to verify that the resulting formulas for the images of $t_{ij}(u)$ agree with (4.23). This can be done by an easy straightforward calculation. For instance, for the image of $t_{0,0}(u)$ we calculate

$$\begin{aligned} t_{0,0}(u) &\mapsto T_{11}(2u-1)\,T_{22}(2u) + T_{21}(2u-1)\,T_{12}(2u) \\ &\mapsto \Big(1 + \frac{F_{-1,-1}}{2u-1}\Big)\Big(1 + \frac{F_{1,1}}{2u}\Big) + 2\cdot\frac{F_{0,-1}}{2u-1}\cdot\frac{F_{-1,0}}{2u}. \end{aligned} \tag{4.24}$$

On the other hand, formula (4.23) gives

$$\begin{aligned} t_{0,0}(u) &\mapsto 1 + \frac{2F_{0,-1}F_{-1,0} + 2F_{0,1}F_{1,0} - F_{11}^2 + F_{11} - 2\,F_{10}F_{01}}{2u\,(2u-1)} \\ &= 1 + \frac{2F_{0,-1}F_{-1,0} - F_{11}^2 - F_{11}}{2u\,(2u-1)}. \end{aligned}$$

Clearly, this agrees with (4.24). All the remaining cases are verified by a similar and even shorter calculation. Obviously, the homomorphism (4.23) is surjective. □

4.3. Extended Yangian $\mathrm{X}(\mathfrak{o}_4)$

We shall need the tensor product algebra $\mathrm{Y}(\mathfrak{gl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2)$. In order to distinguish the two copies of the algebra $\mathrm{Y}(\mathfrak{gl}_2)$, we denote the corresponding generator series respectively by $T_{ij}(u)$ and $T'_{ij}(u)$ for the first and second copies, where $i, j \in \{1, 2\}$. We also identify $T_{ij}(u) \otimes 1$ with $T_{ij}(u)$ and $1 \otimes T'_{ij}(u)$ with $T'_{ij}(u)$. As before, we combine the series $T_{ij}(u)$ and $T'_{ij}(u)$ into the matrices $T^\circ(u)$ and $T^{\circ\prime}(u)$, respectively.

The algebra $\mathrm{Y}(\mathfrak{gl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)$ is naturally equipped with an ascending filtration, where the degrees of the elements on each copy of $\mathrm{Y}(\mathfrak{gl}_2)$ are defined by (4.8).

Consider the vector space $\mathbb{C}^2$ with its canonical basis e_1, e_2 and set $V = \mathbb{C}^2 \otimes \mathbb{C}^2$. We identify V with $\mathbb{C}^4$ regarding the vectors

$$v_{-2} = e_1 \otimes e_1, \qquad v_{-1} = e_1 \otimes e_2, \qquad v_1 = e_2 \otimes e_1, \qquad v_2 = -e_2 \otimes e_2$$

as the canonical basis of V. Then $T_1^{\circ}(u)\, T_2^{\circ}{}'(u)$ may be regarded as an element of $\mathrm{End}\, V \otimes \big(\mathrm{Y}(\mathfrak{gl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2)\big)[[u^{-1}]]$. The operators P_V and Q_V in $V \otimes V$ are given by the respective formulas (2.9) and (2.10). We shall regard the matrix $T(u) = (t_{ij}(u))$ as an element of $\mathrm{End}\, V \otimes \mathrm{X}(\mathfrak{o}_4)[[u^{-1}]]$.

Proposition 4.8. *The mapping*

$$T(u) \mapsto T_1^{\circ}(u)\, T_2^{\circ}{}'(u), \tag{4.25}$$

defines an embedding $\psi\colon \mathrm{X}(\mathfrak{o}_4) \hookrightarrow \mathrm{Y}(\mathfrak{gl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2)$. *More explicitly, the images of the generators under the embedding are given by the formulas*

$$\begin{aligned}
t_{-2,-2}(u) &\mapsto T_{11}(u)\, T'_{11}(u), & t_{-2,-1}(u) &\mapsto T_{11}(u)\, T'_{12}(u),\\
t_{-2,1}(u) &\mapsto T_{12}(u)\, T'_{11}(u), & t_{-2,2}(u) &\mapsto -T_{12}(u)\, T'_{12}(u),\\
t_{-1,-2}(u) &\mapsto T_{11}(u)\, T'_{21}(u), & t_{-1,-1}(u) &\mapsto T_{11}(u)\, T'_{22}(u),\\
t_{-1,1}(u) &\mapsto T_{12}(u)\, T'_{21}(u), & t_{-1,2}(u) &\mapsto -T_{12}(u)\, T'_{22}(u),\\
t_{1,-2}(u) &\mapsto T_{21}(u)\, T'_{11}(u), & t_{1,-1}(u) &\mapsto T_{21}(u)\, T'_{12}(u),\\
t_{1,1}(u) &\mapsto T_{22}(u)\, T'_{11}(u), & t_{1,2}(u) &\mapsto -T_{22}(u)\, T'_{12}(u),\\
t_{2,-2}(u) &\mapsto -T_{21}(u)\, T'_{21}(u), & t_{2,-1}(u) &\mapsto -T_{21}(u)\, T'_{22}(u),\\
t_{2,1}(u) &\mapsto -T_{22}(u)\, T'_{21}(u), & t_{2,2}(u) &\mapsto T_{22}(u)\, T'_{22}(u).
\end{aligned}$$

Proof. We start by showing that the mapping defines an algebra homomorphism. Identifying $V \otimes V$ with the tensor product space $(\mathbb{C}^2)^{\otimes 4}$, we set

$$R_V(u) = R_{13}^{\circ}(u)\, R_{24}^{\circ}(u). \tag{4.26}$$

Lemma 4.9. *We have the equality of operators in* $V \otimes V$,

$$R_V(u) = \frac{u-1}{u} \cdot \Big(1 - \frac{P_V}{u} + \frac{Q_V}{u-1}\Big). \tag{4.27}$$

Proof. We have

$$R_V(u) = \Big(1 - \frac{P_{13}}{u}\Big)\Big(1 - \frac{P_{24}}{u}\Big) = \frac{u-1}{u}\Big(1 - \frac{P_{13}P_{24}}{u} + \frac{(1-P_{13})(1-P_{24})}{u-1}\Big).$$

It remains to note that $P_V = P_{13}P_{24}$ and $Q_V = (1-P_{13})(1-P_{24})$. This is verified by the application of the operators to all basis vectors $v_i \otimes v_j$ of $V \otimes V$. For instance, by the definition of Q_V,

$$Q_V(v_{-2} \otimes v_2) = v_{-2} \otimes v_2 + v_{-1} \otimes v_1 + v_1 \otimes v_{-1} + v_2 \otimes v_{-2},$$

while

$$\begin{aligned}
&(1-P_{13})(1-P_{24})(v_{-2} \otimes v_2) = (1-P_{13})(1-P_{24})(-e_1 \otimes e_1 \otimes e_2 \otimes e_2)\\
&= -e_1 \otimes e_1 \otimes e_2 \otimes e_2 + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_2 \otimes e_1 \otimes e_1,
\end{aligned}$$

which clearly coincides with $Q_V(v_{-2}\otimes v_2)$. The remaining relations are verified in the same way. □

By the lemma, the element $R_V(u)$ coincides with the R-matrix (2.13) for $\mathfrak{a}=\mathfrak{o}_4$, up to a scalar factor. So, in order to verify that the mapping (4.25) defines a homomorphism $\mathrm{X}(\mathfrak{o}_4)\to \mathrm{Y}(\mathfrak{gl}_2)\otimes \mathrm{Y}(\mathfrak{gl}_2)$ we need to show that the relation

$$R_V(u-v)\,T_{1'}(u)\,T_{2'}(v)=T_{2'}(v)\,T_{1'}(u)\,R_V(u-v) \tag{4.28}$$

remains valid when $T(u)$ is replaced by its image. The primed indices are used here to indicate the copies of the space V in the tensor product $V\otimes V$. The left-hand side of (4.28) reads

$$R^{\circ}_{13}(u-v)\,R^{\circ}_{24}(u-v)\,T^{\circ}_1(u)\,T^{\circ}_2{}'(u)\,T^{\circ}_3(v)\,T^{\circ}_4{}'(v).$$

Applying the RTT-relation (4.6) twice, we bring this expression to the form

$$T^{\circ}_3(v)\,T^{\circ}_4{}'(v)\,T^{\circ}_1(u)\,T^{\circ}_2{}'(u)\,R^{\circ}_{13}(u-v)\,R^{\circ}_{24}(u-v)$$

which coincides with the right-hand side of (4.28), where $T(u)$ is replaced with its image in accordance with (4.25).

The explicit images of the generators of $\mathrm{X}(\mathfrak{o}_4)$ are found by taking the matrix elements in (4.25). Indeed, the application of $T(u)$ to the basis vector v_{-2} of V gives

$$T(u)(v_{-2})=t_{-2,-2}(u)\,v_{-2}+t_{-1,-2}(u)\,v_{-1}+t_{1,-2}(u)\,v_1+t_{2,-2}(u)\,v_2,$$

while

$$T^{\circ}_1(u)\,T^{\circ}_2{}'(u)(v_{-2})=T^{\circ}_1(u)\,T^{\circ}_2{}'(u)(e_1\otimes e_1)=\sum_{a,b=1}^{2}T_{a1}(u)\,T'_{b1}(u)\,(e_a\otimes e_b)$$

$$=T_{11}(u)\,T'_{11}(u)\,v_{-2}+T_{11}(u)\,T'_{21}(u)\,v_{-1}+T_{21}(u)\,T'_{11}(u)\,v_1-T_{21}(u)\,T'_{21}(u)\,v_2.$$

This agrees with the formulas for the images of the series $t_{a,-2}(u)$ for $a=-2,-1,1,2$ given in the statement. The remaining formulas are verified in the same way.

In order to demonstrate that the homomorphism ψ is injective, observe that it preserves the respective filtrations on $\mathrm{X}(\mathfrak{o}_4)$ and $\mathrm{Y}(\mathfrak{gl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)$. Hence, we have a homomorphism of the associated graded algebras

$$\mathrm{gr}\,\mathrm{X}(\mathfrak{o}_4)\to \mathrm{gr}\,\big(\mathrm{Y}(\mathfrak{gl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)\big).$$

Identifying the graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{gl}_2)$ with $\mathrm{U}(\mathfrak{gl}_2[x])$ via the isomorphism (4.9), we get a homomorphism

$$\mathrm{gr}\,\mathrm{X}(\mathfrak{o}_4)\to \mathrm{U}(\mathfrak{gl}_2[x])\otimes\mathrm{U}(\mathfrak{gl}_2[y])$$

so that

$$\bar t^{(r)}_{-1,2}\mapsto -E_{12}\,x^{r-1},\qquad \bar t^{(r)}_{1,2}\mapsto -E_{12}\,y^{r-1},\qquad \bar t^{(r)}_{1,1}\mapsto E_{22}\,x^{r-1}+E_{11}\,y^{r-1},$$
$$\bar t^{(r)}_{2,-1}\mapsto -E_{21}\,x^{r-1},\qquad \bar t^{(r)}_{2,1}\mapsto -E_{21}\,y^{r-1},\qquad \bar t^{(r)}_{2,2}\mapsto E_{22}\,x^{r-1}+E_{22}\,y^{r-1},$$

and

$$\bar{z}_r \mapsto (E_{11} + E_{22})\, x^{r-1} + (E_{11} + E_{22})\, y^{r-1}.$$

Therefore, the injectivity of ψ follows from Corollary 3.10. □

Due to the presentation of the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$ provided by (4.13) and (4.14) we have a natural projection $\mathrm{Y}(\mathfrak{gl}_2) \to \mathrm{Y}(\mathfrak{sl}_2)$ defined by the mapping $T_{ij}(u) \mapsto \mathcal{T}_{ij}(u)$. Applying this projection to the first or second copy of $\mathrm{Y}(\mathfrak{gl}_2)$ in the tensor product algebra $\mathrm{Y}(\mathfrak{gl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2)$ and taking its composition with the embedding ψ we get homomorphisms

$$\chi^{(1)} : \mathrm{X}(\mathfrak{o}_4) \to \mathrm{Y}(\mathfrak{sl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2), \qquad \chi^{(2)} : \mathrm{X}(\mathfrak{o}_4) \to \mathrm{Y}(\mathfrak{gl}_2) \otimes \mathrm{Y}(\mathfrak{sl}_2).$$

Corollary 4.10. *The homomorphisms $\chi^{(1)}$ and $\chi^{(2)}$ are bijective.*

Proof. We only consider $\chi^{(1)}$, the proof for $\chi^{(2)}$ is similar. By the formulas of Proposition 4.8 we have

$$\chi^{(1)} : t_{-2,-2}(u)\, t_{1,1}(u-1) - t_{1,-2}(u)\, t_{-2,1}(u-1) \mapsto$$
$$\Big(\mathcal{T}_{11}(u)\, \mathcal{T}_{22}(u-1) - \mathcal{T}_{21}(u)\, \mathcal{T}_{12}(u-1)\Big)\, T'_{11}(u)\, T'_{11}(u-1) = T'_{11}(u)\, T'_{11}(u-1).$$

Therefore, all the coefficients of the series $T'_{11}(u)$ belong to the image of $\chi^{(1)}$. Hence, so do the coefficients of $\mathcal{T}_{ij}(u)$ with $i, j \in \{1, 2\}$. This implies that $\chi^{(1)}$ is surjective. To verify the injectivity of $\chi^{(1)}$ we use the same argument as in the proof of Proposition 4.8. Namely, $\chi^{(1)}$ induces a homomorphism of the associated graded algebras

$$\mathrm{gr}\, \mathrm{X}(\mathfrak{o}_4) \to \mathrm{U}(\mathfrak{sl}_2[x]) \otimes \mathrm{U}(\mathfrak{gl}_2[y])$$

and the argument is completed by the application of Corollary 3.10. □

Corollary 4.11. *The restriction of each isomorphism $\chi^{(1)}$ and $\chi^{(2)}$ to the subalgebra $\mathrm{Y}(\mathfrak{o}_4)$ induces an isomorphism $\mathrm{Y}(\mathfrak{o}_4) \to \mathrm{Y}(\mathfrak{sl}_2) \otimes \mathrm{Y}(\mathfrak{sl}_2)$.*

Proof. Again, we only consider the isomorphism $\chi^{(1)}$. The subalgebra $\mathrm{Y}(\mathfrak{o}_4)$ consists of the elements stable under all automorphisms of $\mathrm{X}(\mathfrak{o}_4)$ of the form (2.21). For any formal series $f(u)$ of the form (2.20) consider the automorphism $\widetilde{\mu}_f$ of the algebra $\mathrm{Y}(\mathfrak{sl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2)$ defined by

$$\widetilde{\mu}_f : \mathcal{T}_{ij}(u) \mapsto \mathcal{T}_{ij}(u), \qquad T'_{ij}(u) \mapsto f(u)\, T'_{ij}(u).$$

By the definition of $\chi^{(1)}$, we have the relation $\chi^{(1)} \circ \mu_f = \widetilde{\mu}_f \circ \chi^{(1)}$. This implies that if $y \in \mathrm{Y}(\mathfrak{o}_4)$ then $\chi^{(1)}(y)$ is stable under the automorphisms $\widetilde{\mu}_f$ for all series $f(u)$. Hence, the image of the subalgebra $\mathrm{Y}(\mathfrak{o}_4)$ of $\mathrm{X}(\mathfrak{o}_4)$ under the isomorphism $\chi^{(1)}$ coincides with the subalgebra $\mathrm{Y}(\mathfrak{sl}_2) \otimes \mathrm{Y}(\mathfrak{sl}_2)$ of $\mathrm{Y}(\mathfrak{sl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2)$ thus providing the desired isomorphism. □

Let us denote by c the following Casimir element for the Lie algebra $\mathfrak{o}_4$,

$$c = \frac{1}{2}\big(F_{11}^2 + F_{22}^2\big) - F_{22} + F_{21} F_{12} + F_{2,-1} F_{-1,2}.$$

In the following we use notation (2.6).

Corollary 4.12. *The mapping*

$$\mathrm{ev}\colon T(u) \mapsto 1 + \frac{F}{u} + \frac{F^2 - F - c\,1}{2u^2} \tag{4.29}$$

defines a surjective homomorphism $\mathrm{X}(\mathfrak{o}_4) \to \mathrm{U}(\mathfrak{o}_4)$.

Proof. Consider the isomorphism $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathfrak{o}_4$ given by

$$E_{11} - E_{22} \mapsto -F_{11} - F_{22}, \qquad E_{12} \mapsto F_{-2,1}, \qquad E_{21} \mapsto F_{1,-2},$$

and

$$E'_{11} - E'_{22} \mapsto F_{11} - F_{22}, \qquad E'_{12} \mapsto F_{-2,-1}, \qquad E'_{21} \mapsto F_{-1,-2},$$

where the primes indicate the basis elements of the second copy of $\mathfrak{sl}_2$. Applying the homomorphism $\mathrm{Y}(\mathfrak{gl}_2) \to \mathrm{U}(\mathfrak{sl}_2)$ used in the proof of Corollary 4.7, we get a homomorphism $\mathrm{Y}(\mathfrak{gl}_2) \otimes \mathrm{Y}(\mathfrak{gl}_2) \to \mathrm{U}(\mathfrak{o}_4)$ such that

$$\begin{aligned} T_{11}(u) &\mapsto 1 - \frac{F_{11} + F_{22}}{2}\, u^{-1}, \qquad & T_{12}(u) &\mapsto F_{-2,1}\, u^{-1}, \\ T_{22}(u) &\mapsto 1 + \frac{F_{11} + F_{22}}{2}\, u^{-1}, \qquad & T_{21}(u) &\mapsto F_{1,-2}\, u^{-1} \end{aligned}$$

and

$$\begin{aligned} T'_{11}(u) &\mapsto 1 + \frac{F_{11} - F_{22}}{2}\, u^{-1}, \qquad & T'_{12}(u) &\mapsto F_{-2,-1}\, u^{-1}, \\ T'_{22}(u) &\mapsto 1 - \frac{F_{11} - F_{22}}{2}\, u^{-1}, \qquad & T'_{21}(u) &\mapsto F_{-1,-2}\, u^{-1}. \end{aligned}$$

Using the isomorphism of Proposition 4.8 we get a homomorphism $\mathrm{X}(\mathfrak{o}_4) \to \mathrm{U}(\mathfrak{o}_4)$. It remains to verify that the resulting formulas for the images of $t_{ij}(u)$ agree with (4.29). This can be done by an easy straightforward calculation. For instance, for the image of $t_{-2,-2}(u)$ we calculate

$$\begin{aligned} t_{-2,-2}(u) \mapsto T_{11}(u)\, T'_{11}(u) &\mapsto \Big(1 - \frac{F_{11} + F_{22}}{2}\, u^{-1}\Big)\Big(1 + \frac{F_{11} - F_{22}}{2}\, u^{-1}\Big) \\ &= 1 + F_{-2,-2}\, u^{-1} + \frac{F^2_{-2,-2} - F^2_{-1,-1}}{4}\, u^{-2}. \end{aligned} \tag{4.30}$$

On the other hand, formula (4.29) gives

$$t_{-2,-2}(u) \mapsto 1 + F_{-2,-2}\, u^{-1} + \frac{F^2_{-2,-2} + F_{-2,-1}F_{-1,-2} + F_{-2,1}F_{1,-2} - F_{-2,-2} - c}{2u^2}$$

which agrees with (4.30). All the remaining cases are verified by a similar calculation. Obviously, the homomorphism (4.29) is surjective. □

Remark 4.13. The respective compositions of the evaluation homomorphisms provided by Corollaries 4.3, 4.7 and 4.12 with the shift automorphism τ_a given by (2.22) yields the homomorphisms $\mathrm{ev}_a = \mathrm{ev} \circ \tau_a$ with the evaluation parameter a. □

5. Representations of the extended Yangians

Here we introduce the highest weight representations for the extended Yangians $\mathrm{X}(\mathfrak{a})$, where as before, $\mathfrak{a} = \mathfrak{o}_{2n+1}$, $\mathfrak{sp}_{2n}$ or $\mathfrak{o}_{2n}$. We show by a standard argument that finite-dimensional irreducible representations of $\mathrm{X}(\mathfrak{a})$ are highest weight representations. Then we give necessary and sufficient conditions for the irreducible highest weight representations to be finite-dimensional. In particular, we obtain an alternative proof of Drinfeld's classification theorem for the finite-dimensional irreducible representations of the Yangians $\mathrm{Y}(\mathfrak{a})$.

5.1. Highest weight representations

A representation V of the algebra $\mathrm{X}(\mathfrak{a})$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

$$\begin{aligned} t_{ij}(u)\,\xi &= 0 \qquad \text{for} \quad -n \leqslant i < j \leqslant n, \qquad \text{and} \\ t_{ii}(u)\,\xi &= \lambda_i(u)\,\xi \qquad \text{for} \quad -n \leqslant i \leqslant n, \end{aligned} \tag{5.1}$$

for some formal series

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \cdots, \qquad \lambda_i^{(r)} \in \mathbb{C}, \tag{5.2}$$

where the value $i = 0$ only occurs in the case $\mathfrak{a} = \mathfrak{o}_{2n+1}$. The vector ξ is called the *highest vector* of V and the tuple $\lambda(u) = (\lambda_{-n}(u), \dots, \lambda_n(u))$ of the formal series is the *highest weight* of V.

Let us identify the elements $F_{ij} \in \mathfrak{a}$ with their images in $\mathrm{X}(\mathfrak{a})$ under the embedding (3.22). The defining relations (2.19) imply

$$[t_{ij}^{(1)}, t_{kl}(u)] = \delta_{kj}\, t_{il}(u) - \delta_{il}\, t_{kj}(u) - \delta_{k,-i}\,\theta_{ij}\, t_{-j,l}(u) + \delta_{l,-j}\,\theta_{ij}\, t_{k,-i}(u).$$

Also, due to (2.28) we have

$$t_{ij}^{(1)} + \theta_{ij}\, t_{-j,-i}^{(1)} = \delta_{ij}\, z_1.$$

Therefore, $F_{ij} = t_{ij}^{(1)} - \delta_{ij}\, z_1/2$. Since z_1 is central in $\mathrm{X}(\mathfrak{a})$, this gives

$$[F_{ij}, t_{kl}(u)] = \delta_{kj}\, t_{il}(u) - \delta_{il}\, t_{kj}(u) - \delta_{k,-i}\,\theta_{ij}\, t_{-j,l}(u) + \delta_{l,-j}\,\theta_{ij}\, t_{k,-i}(u). \tag{5.3}$$

Take the linear span of the elements $F_{11}, \dots, F_{nn}$ as the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{a}$ and consider the standard triangular decomposition of $\mathfrak{a}$. Then the nonzero elements F_{ij} with $i < j$ are the positive root vectors. The corresponding positive roots are

$$-\varepsilon_i - \varepsilon_j, \quad \varepsilon_i - \varepsilon_j \quad \text{with} \quad 1 \leqslant i < j \leqslant n$$

for $\mathfrak{a} = \mathfrak{o}_{2n}$,

$$-2\,\varepsilon_i \quad \text{with} \quad 1 \leqslant i \leqslant n \quad \text{and} \quad -\varepsilon_i - \varepsilon_j, \quad \varepsilon_i - \varepsilon_j \quad \text{with} \quad 1 \leqslant i < j \leqslant n$$

for $\mathfrak{a} = \mathfrak{sp}_{2n}$, and

$$-\varepsilon_i \quad \text{with} \quad 1 \leqslant i \leqslant n \quad \text{and} \quad -\varepsilon_i - \varepsilon_j, \quad \varepsilon_i - \varepsilon_j \quad \text{with} \quad 1 \leqslant i < j \leqslant n$$

for $\mathfrak{a} = \mathfrak{o}_{2n+1}$, where ε_i denotes the element of $\mathfrak{h}^*$ defined by $\varepsilon_i(F_{jj}) = \delta_{ij}$. The standard partial ordering on the set of weights of any $\mathfrak{a}$-module is now defined as follows. If α and β are two weights, then α precedes β if $\beta-\alpha$ is a $\mathbb{Z}_+$-linear combination of the positive roots.

Theorem 5.1. *Every finite-dimensional irreducible representation V of the algebra $\mathrm{X}(\mathfrak{a})$ is a highest weight representation. Moreover, V contains a unique, up to a constant factor, highest vector.*

Proof. Introduce the subspace V^0 of V by

$$V^0=\{\eta\in V \mid t_{ij}(u)\,\eta=0, \quad -n\leqslant i<j\leqslant n\}. \tag{5.4}$$

We show first that V^0 is nonzero. Consider the set of weights of V, where V is regarded as the $\mathfrak{a}$-module defined via the embedding (3.22). This set is finite and hence contains a maximal weight ν with respect to the partial ordering on the set of weights of V. The corresponding weight vector η belongs to V^0. Indeed, if $i<j$ then by (5.3) the weight of $t_{ij}(u)\,\eta$ has the form $\nu+\alpha$ for a positive root α. By the maximality of ν, we have $t_{ij}(u)\,\eta=0$.

Next, we show that all the operators $t_{kk}(u)$ preserve the subspace V^0. Consider first the case $\mathfrak{a}=\mathfrak{o}_{2n+1}$. In the following argument we write $\equiv$ for an equality of operators in V^0. Due to (5.3), it suffices to show that for any i and k we have

$$t_{i,i+1}(u)\,t_{kk}(v)\equiv 0. \tag{5.5}$$

Suppose first that $i<k$. Then (5.5) is immediate from (2.19) except for the cases $i=-k$ and $i=-k-1$. In the former case, we have $k>0$ and so (2.19) gives

$$t_{-k,-k+1}(u)\,t_{kk}(v)\equiv -\frac{1}{u-v-\kappa}\sum_{p=k}^{n} t_{-p,-k+1}(u)\,t_{pk}(v), \tag{5.6}$$

while for each $p\geqslant k$,

$$t_{-p,-k+1}(u)\,t_{pk}(v)\equiv -\frac{1}{u-v-\kappa}\sum_{q=k}^{n} t_{-q,-k+1}(u)\,t_{qk}(v).$$

Hence, $t_{-p,-k+1}(u)\,t_{pk}(v)\equiv t_{-k,-k+1}(u)\,t_{kk}(v)$. So, (5.6) implies

$$\Big(1+\frac{n-k+1}{u-v-\kappa}\Big)\,t_{-k,-k+1}(u)\,t_{kk}(v)\equiv 0$$

and thus, $t_{-k,-k+1}(u)\,t_{kk}(v)\equiv 0$ verifying (5.5).

Similarly, in the case $i=-k-1$ we have $k\geqslant 0$ and so

$$t_{-k-1,-k}(u)\,t_{kk}(v)\equiv \frac{1}{u-v-\kappa}\sum_{p=k+1}^{n} t_{kp}(v)\,t_{-k-1,-p}(u). \tag{5.7}$$

For each $p\geqslant k+1$ we have

$$\begin{aligned} t_{kp}(v)\,t_{-k-1,-p}(u)&\equiv -[t_{-k-1,-p}(u),t_{kp}(v)]\\ &\equiv -\frac{1}{u-v-\kappa}\sum_{q=k+1}^{n} t_{kq}(v)\,t_{-k-1,-q}(u).\end{aligned}$$

Therefore, $t_{kp}(v)\, t_{-k-1,-p}(u) \equiv -t_{-k-1,-k}(u)\, t_{kk}(v)$. So, (5.7) gives

$$\left(1+\frac{n-k}{u-v-\kappa}\right) t_{-k-1,-k}(u)\, t_{kk}(v) \equiv 0$$

implying (5.5) in the case under consideration.

Suppose now that $i \geqslant k$. We can write

$$t_{i,i+1}(u)\, t_{kk}(v) \equiv -[t_{kk}(v),\, t_{i,i+1}(u)].$$

Now (5.5) is immediate from (2.19) except for the cases $i=-k$ and $i=-k-1$. In the former case, we have $k \leqslant 0$ and so (2.19) gives

$$t_{-k,-k+1}(u)\, t_{kk}(v) \equiv \frac{1}{v-u-\kappa} \sum_{p=-k+1}^{n} t_{-p,k}(v)\, t_{p,-k+1}(u), \tag{5.8}$$

while for each $p \geqslant -k+1$,

$$t_{-p,k}(v)\, t_{p,-k+1}(u) \equiv -\frac{1}{v-u-\kappa} \sum_{q=-k+1}^{n} t_{-q,k}(v)\, t_{q,-k+1}(u).$$

Hence, $t_{-p,k}(v)\, t_{p,-k+1}(u) \equiv -t_{-k,-k+1}(u)\, t_{kk}(v)$. So, (5.8) implies

$$\left(1+\frac{n+k}{v-u-\kappa}\right) t_{-k,-k+1}(u)\, t_{kk}(v) \equiv 0$$

verifying (5.5).

Finally, let $i=-k-1$. Then $k<0$ and

$$\begin{aligned} t_{-k-1,-k}(u)\, t_{kk}(v) &\equiv -[t_{kk}(v),\, t_{-k-1,-k}(u)] \\ &\equiv -\frac{1}{v-u-\kappa} \sum_{p=-k}^{n} t_{-k-1,p}(u)\, t_{k,-p}(v). \end{aligned} \tag{5.9}$$

For each $p \geqslant -k$ we have

$$\begin{aligned} t_{-k-1,p}(u)\, t_{k,-p}(v) &\equiv -[t_{k,-p}(v),\, t_{-k-1,p}(u)] \\ &\equiv -\frac{1}{v-u-\kappa} \sum_{q=-k}^{n} t_{-k-1,q}(u)\, t_{k,-q}(v). \end{aligned}$$

Therefore, $t_{-k-1,p}(u)\, t_{k,-p}(v) \equiv t_{-k-1,-k}(u)\, t_{kk}(v)$. So, (5.9) gives

$$\left(1+\frac{n+k+1}{v-u-\kappa}\right) t_{-k-1,-k}(u)\, t_{kk}(v) \equiv 0$$

completing the proof of (5.5).

For the Lie algebras $\mathfrak{a}=\mathfrak{sp}_{2n}$ and $\mathfrak{o}_{2n}$ the argument is essentially the same as in the previous case. If $\mathfrak{a}=\mathfrak{sp}_{2n}$, then due to (5.3), it suffices to show that (5.5) holds for $i \in \{-n,\dots,-2,1,\dots,n-1\}$ and all k, together with the relation

$$t_{-1,1}(u)\, t_{kk}(v) \equiv 0. \tag{5.10}$$

This relation is immediate from (2.19) for $k > 1$ and $k < -1$; for the latter we apply (2.19) to the commutator $[t_{kk}(v), t_{-1,1}(u)]$. If $k = 1$ or $k = -1$ then the claim is verified by a calculation similar to the cases (5.6) and (5.8), respectively.

If $\mathfrak{a} = \mathfrak{o}_{2n}$, then it is sufficient to verify (5.5) for $i \in \{-n, \dots, -2, 1, \dots, n-1\}$ and all k, together with the relations (5.10) and $t_{-1,2}(u)\, t_{kk}(v) \equiv 0$. The calculation is again a repetition of the one for $\mathfrak{a} = \mathfrak{o}_{2n+1}$.

Now we verify that all the operators $t_{i\,i}^{(r)}$ on the space V^0 with $i \in \{-n, \dots, n\}$ and $r \geqslant 1$ comprise a commutative family. First of all, by (2.19) we have $[t_{ii}(u), t_{ii}(v)] = 0$ for any $i \neq 0$. Furthermore, for any $i < j$ such that $i + j \neq 0$ we have

$$(u-v)\,[t_{ii}(u), t_{jj}(v)] = t_{ji}(u)\, t_{ij}(v) - t_{ji}(v)\, t_{ij}(u)$$

and so, $[t_{ii}(u), t_{jj}(v)] \equiv 0$ as operators on V^0. Next, for any $0 \leqslant i \leqslant j$ set

$$A_{ij} = t_{-j,-i}(u)\, t_{ji}(v) - t_{ij}(v)\, t_{-i,-j}(u),$$

where the value $i = 0$ only occurs in the case $\mathfrak{a} = \mathfrak{o}_{2n+1}$. By (2.19), we get

$$A_{00} \equiv \frac{1}{u-v}\, A_{00} - \frac{1}{u-v-\kappa} \sum_{j=0}^{n} A_{0j}, \tag{5.11}$$

and for any $i > 0$

$$A_{ii} \equiv -\frac{1}{u-v-\kappa} \sum_{j=i}^{n} A_{ij}, \tag{5.12}$$

as operators on V^0, while for $0 \leqslant i < j$ we have

$$A_{ij} \equiv -\frac{1}{u-v-\kappa} \sum_{k=i}^{n} A_{ik} - \frac{1}{u-v-\kappa} \sum_{l=j}^{n} A_{jl}.$$

This implies

$$A_{ij} \equiv A_{ii} - A_{jj}$$

for $0 < i < j$, and

$$A_{0j} = \frac{u-v-1}{u-v}\, A_{00} - A_{jj}$$

for $j > 0$. Hence, (5.12) gives

$$\Big(1 + \frac{n-i+1}{u-v-\kappa}\Big) A_{ii} - \frac{1}{u-v-\kappa} \sum_{j=i+1}^{n} A_{jj} \equiv 0,$$

thus proving that $A_{ii} = [t_{-i,-i}(u), t_{ii}(v)] \equiv 0$ for all $i > 0$ by an obvious induction. Moreover, in the case $\mathfrak{a} = \mathfrak{o}_{2n+1}$, we derive from (5.11) that $A_{00} = [t_{00}(u), t_{00}(v)] \equiv 0$.

Since the operators $t_{i\,i}^{(r)}$ on V^0 are pairwise commuting, they have a simultaneous eigenvector $\xi \in V^0$. Then ξ satisfies the conditions (5.1). Moreover, since V is irreducible, the submodule $\mathrm{X}(\mathfrak{a})\,\xi$ must coincide with V so that V is a highest weight module over $\mathrm{X}(\mathfrak{a})$. In particular, ξ is an $\mathfrak{a}$-weight vector with a certain weight ν.

Finally, since the central elements z_r act on V as scalar operators, Corollary 3.10 implies that the vector space V is spanned by the elements

$$t^{(r_1)}_{j_1 i_1} \dots t^{(r_m)}_{j_m i_m}\, \xi, \qquad m \geqslant 0,$$

with $j_a > i_a$ and $r_a \geqslant 1$. Hence, by (5.3) the $\mathfrak{a}$-weight space V_ν is one-dimensional and spanned by the vector ξ. Moreover, if ρ is a weight of V and $\rho \neq \nu$ then ρ strictly precedes ν. This proves that the highest vector ξ of V is determined uniquely, up to a constant factor. □

Given any tuple $\lambda(u) = (\lambda_{-n}(u), \dots, \lambda_n(u))$ of formal series of the form (5.2), we define the *Verma module* $M(\lambda(u))$ as the quotient of $\mathrm{X}(\mathfrak{a})$ by the left ideal generated by all the coefficients of the series $t_{ij}(u)$ with $-n \leqslant i < j \leqslant n$, and $t_{ii}(u) - \lambda_i(u)$ for $i = -n, \dots, n$. As we shall see below, the Verma module $M(\lambda(u))$ can be trivial for some $\lambda(u)$. In the non-trivial case, the Verma module $M(\lambda(u))$ is a highest weight representation of $\mathrm{X}(\mathfrak{a})$ with the highest weight $\lambda(u)$ and the highest vector 1_λ which is the canonical image of the element $1 \in \mathrm{X}(\mathfrak{a})$. Moreover, any highest weight representation of $\mathrm{X}(\mathfrak{a})$ with the highest weight $\lambda(u)$ is isomorphic to a quotient of $M(\lambda(u))$. Regarding $M(\lambda(u))$ as an $\mathfrak{a}$-module, we obtain the weight space decomposition

$$M(\lambda(u)) = \bigoplus_{\nu} M(\lambda(u))_\nu,$$

summed over all $\mathfrak{a}$-weights $\nu = (\nu_1, \dots, \nu_n)$ of $M(\lambda(u))$, where

$$M(\lambda(u))_\nu = \{\eta \in M(\lambda(u)) \mid F_{ii}\, \eta = \nu_i\, \eta, \quad i = 1, \dots, n\}.$$

By (5.3), the set of weights of $M(\lambda(u))$ coincides with that of the $\mathfrak{a}$-Verma module with the highest weight $\lambda^{(1)} = (\lambda^{(1)}_1, \dots, \lambda^{(1)}_n)$. This set consists of all weights of the form $\lambda^{(1)} - \omega$, where ω is a $\mathbb{Z}_+$-linear combination of the positive roots.

One easily shows that any submodule K of a non-trivial Verma module $M(\lambda(u))$ admits the weight space decomposition

$$K = \bigoplus_{\nu} K_\nu, \qquad K_\nu = K \cap M(\lambda(u))_\nu.$$

This implies that the sum of all proper submodules is the unique maximal proper submodule of $M(\lambda(u))$. The *irreducible highest weight representation* $L(\lambda(u))$ of $\mathrm{X}(\mathfrak{a})$ with the highest weight $\lambda(u)$ is defined as the quotient of the Verma module $M(\lambda(u))$ by the unique maximal proper submodule.

Proposition 5.2. *Let V be a highest weight representation of $\mathrm{X}(\mathfrak{a})$ with the highest weight $\lambda(u) = (\lambda_{-n}(u), \dots, \lambda_n(u))$ with some series (5.2). Then the coefficients of the series $z(u)$ act on V as scalar operators determined by $z(u)|_V = \lambda_{-n}(u+\kappa)\, \lambda_n(u)$.*

Proof. Let ξ be the highest vector of V. Then $V = \mathrm{X}(\mathfrak{a})\, \xi$ so that $z(u)$ acts on V as a scalar function determined by its action on ξ. However, taking $k = l = n$ in

(2.28) we get

$$z(u) = \sum_{i=-n}^{n} \theta_{ni}\, t_{-i,-n}(u+\kappa)\, t_{in}(u). \tag{5.13}$$

Therefore, $z(u)\,\xi = \lambda_{-n}(u+\kappa)\,\lambda_n(u)\,\xi$. □

5.2. Representations of low rank Yangians

Using the results on representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$ (Tarasov [25, 26]; see also [7, Chapter 12], [16]), and the isomorphisms constructed in Section 4, we describe here the finite-dimensional irreducible representations of the extended Yangians $\mathrm{X}(\mathfrak{o}_3)$, $\mathrm{X}(\mathfrak{sp}_2)$ and $\mathrm{X}(\mathfrak{o}_4)$. For the sake of completeness, we also reproduce a simplified version of Tarasov's classification theorem for the representations of $\mathrm{Y}(\mathfrak{gl}_2)$; cf. [16].

We shall use the notation for the generators of $\mathrm{Y}(\mathfrak{gl}_2)$ introduced in Section 4. A representation L of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$ is called a *highest weight representation* if there exists a nonzero vector $\zeta \in L$ such that L is generated by ζ and the following relations hold

$$T_{12}(u)\,\zeta = 0 \qquad \text{and} \tag{5.14}$$

$$T_{ii}(u)\,\zeta = \mu_i(u)\,\zeta \qquad \text{for} \quad i = 1, 2. \tag{5.15}$$

for some formal series

$$\mu_i(u) = 1 + \mu_i^{(1)} u^{-1} + \mu_i^{(2)} u^{-2} + \cdots, \qquad \mu_i^{(r)} \in \mathbb{C}. \tag{5.16}$$

The vector ζ is called the *highest vector* of L, and the pair $\mu(u) = \big(\mu_1(u), \mu_2(u)\big)$ is the *highest weight* of L. A standard argument, similar to the one used in Section 5.1 (see, e.g., [16]), shows that every finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{gl}_2)$ is a highest weight representation. Given any pair of series $\mu(u) = \big(\mu_1(u), \mu_2(u)\big)$, the corresponding Verma module $M(\mu(u))$ for $\mathrm{Y}(\mathfrak{gl}_2)$ is the quotient of $\mathrm{Y}(\mathfrak{gl}_2)$ by the left ideal generated by all the coefficients of the series $T_{12}(u)$ and $T_{ii}(u) - \mu_i(u)$ for $i = 1, 2$. When the components of $\mu(u)$ satisfy the condition $\mu_1(u)\,\mu_2(u-1) = 1$ then $M(\mu(u))$ may also be regarded as a module over the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$.

The $\mathrm{Y}(\mathfrak{gl}_2)$-module $M(\mu(u))$ has a unique irreducible quotient $L(\mu(u))$. Thus, any finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{gl}_2)$ is isomorphic to $L(\mu(u))$ for a pair $\mu(u) = \big(\mu_1(u), \mu_2(u)\big)$. It remains to describe the highest weights $\mu(u)$ which correspond to finite-dimensional modules $L(\mu(u))$. This is given by the following theorem due to Tarasov [25, 26] in Drinfeld's version [12].

Theorem 5.3. *The irreducible highest weight representation $L(\mu(u))$ of $\mathrm{Y}(\mathfrak{gl}_2)$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in u such that*

$$\frac{\mu_1(u)}{\mu_2(u)} = \frac{P(u+1)}{P(u)}. \tag{5.17}$$

In this case, $P(u)$ is unique.

Proof. We shall need the following lemma.

Lemma 5.4. *If* $\dim L(\mu(u)) < \infty$ *then there exists a formal series*

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots , \qquad f_r \in \mathbb{C},$$

such that $f(u)\mu_1(u)$ *and* $f(u)\mu_2(u)$ *are polynomials in* u^{-1}.

Proof. By twisting the action of $\mathrm{Y}(\mathfrak{gl}_2)$ on $L(\mu(u))$ by the automorphism (4.11) with $g(u) = \mu_2(u)^{-1}$, we obtain a module over $\mathrm{Y}(\mathfrak{gl}_2)$ which is isomorphic to the irreducible highest weight representation $L(\nu(u), 1)$ with $\nu(u) = \mu_1(u)/\mu_2(u)$. So, we may assume without loss of generality that the highest weight of $L(\mu(u))$ has the form $\mu(u) = (\nu(u), 1)$. Let ζ denote the highest vector of the Verma module $M(\nu(u), 1)$. Since $\dim L(\nu(u), 1) < \infty$, the vectors $T_{21}^{(i)}\zeta \in M(\nu(u), 1)$ with $i \geqslant 1$ are linearly dependent modulo the maximal proper submodule K of $M(\nu(u), 1)$. Hence, $M(\nu(u), 1)$ contains a nonzero vector $\xi \in K$ of the form

$$\xi = \sum_{i=1}^{m} c_i\, T_{21}^{(i)}\zeta, \qquad c_i \in \mathbb{C}.$$

Here m is a positive integer and we may assume that $c_m \neq 0$. Then we have $T_{12}^{(r)}\xi = 0$ for all $r \geqslant 1$ because otherwise the highest vector ζ would belong to K. Write

$$\nu(u) = 1 + \nu^{(1)}u^{-1} + \nu^{(2)}u^{-2} + \cdots , \qquad \nu^{(i)} \in \mathbb{C}.$$

By the defining relations (4.2), in $M(\nu(u), 1)$ we have

$$T_{12}^{(r)}\, T_{21}^{(i)}\zeta = \sum_{a=1}^{\min(r,i)} \Big(T_{22}^{(a-1)}T_{11}^{(r+i-a)} - T_{22}^{(r+i-a)}T_{11}^{(a-1)}\Big)\zeta = \nu^{(r+i-1)}\,\zeta.$$

Hence, for all $r \geqslant 1$ we have the relations

$$\sum_{i=1}^{m} c_i\, \nu^{(r+i-1)} = 0.$$

They imply

$$\nu(u)\,(c_1 + c_2\, u + \cdots + c_m\, u^{m-1}) = (b_1 + b_2\, u + \cdots + b_m\, u^{m-1})$$

for some coefficients $b_i \in \mathbb{C}$ with $b_m = c_m$. Thus, taking now

$$f(u) = c_m^{-1} \sum_{i=1}^{m} c_i\, u^{-m+i}$$

we conclude that both $f(u)\,\nu(u)$ and $f(u)\,1$ are polynomials in u^{-1}. □

Thus, taking the composition of the representation of $\mathrm{Y}(\mathfrak{gl}_2)$ on $L(\mu(u))$ with an appropriate automorphism of the form (4.11), we can get another highest weight representation of $\mathrm{Y}(\mathfrak{gl}_2)$ where both components of the highest weight are polynomials in u^{-1}.

For any $\alpha, \beta \in \mathbb{C}$ consider the irreducible highest weight representation $L(\alpha, \beta)$ of the Lie algebra $\mathfrak{gl}_2$ and equip it with a $\mathrm{Y}(\mathfrak{gl}_2)$-module structure via the evaluation homomorphism (4.10). Let ζ denote the highest vector of $L(\alpha, \beta)$. Then

$$E_{11}\,\zeta = \alpha\,\zeta, \qquad E_{22}\,\zeta = \beta\,\zeta, \qquad E_{12}\,\zeta = 0.$$

Moreover, if $\alpha - \beta \in \mathbb{Z}_+$ then the vectors $(E_{21})^r\zeta$ with $r = 0, 1, \ldots, \alpha - \beta$ form a basis of $L(\alpha, \beta)$ so that $\dim L(\alpha, \beta) = \alpha - \beta + 1$. If $\alpha - \beta \notin \mathbb{Z}_+$ then a basis of $L(\alpha, \beta)$ is formed by the vectors $(E_{21})^r\zeta$, where r runs over all nonnegative integers.

Now let $\mu_1(u)$ and $\mu_2(u)$ be polynomials in u^{-1} of degree not more than k. Write the decompositions

$$\begin{aligned} \mu_1(u) &= (1 + \alpha_1 u^{-1}) \ldots (1 + \alpha_k u^{-1}), \\ \mu_2(u) &= (1 + \beta_1 u^{-1}) \ldots (1 + \beta_k u^{-1}), \end{aligned} \tag{5.18}$$

where the constants α_i and β_i are complex numbers (some of them are zero if the degree of the corresponding polynomial is strictly less than k).

For any $\mathrm{Y}(\mathfrak{gl}_2)$-modules L_1 and L_2, their tensor product $L_1 \otimes L_2$ is equipped with a $\mathrm{Y}(\mathfrak{gl}_2)$-module structure defined by the coproduct (4.7).

Lemma 5.5. *Re-number the parameters α_i and β_i if necessary, so that for every index $i = 1, \ldots, k-1$ the following condition holds: if the multiset $\{\alpha_p - \beta_q \mid i \leqslant p, q \leqslant k\}$ contains nonnegative integers, then $\alpha_i - \beta_i$ is minimal amongst them. Then the representation $L(\mu_1(u), \mu_2(u))$ of $\mathrm{Y}(\mathfrak{gl}_2)$ is isomorphic to the tensor product module*

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_k, \beta_k). \tag{5.19}$$

Proof. Let us denote the module (5.19) by L and let ζ_i be the highest vector of $L(\alpha_i, \beta_i)$ for $i = 1, \ldots, k$. Using the definition of the coproduct on $\mathrm{Y}(\mathfrak{gl}_2)$ we derive that the cyclic span $\mathrm{Y}(\mathfrak{gl}_2)\zeta$ of the vector $\zeta = \zeta_1 \otimes \cdots \otimes \zeta_k$ is a highest weight module with the highest weight $(\mu_1(u), \mu_2(u))$. Therefore, the proposition will follow if we prove that the module L is irreducible.

We claim that any vector $\xi \in L$ satisfying $T_{12}(u)\xi = 0$ is proportional to ζ. We shall prove this claim by induction on k. This is obvious for $k = 1$ so suppose that $k \geqslant 2$. Write any such vector ξ, which is assumed to be nonzero, in the form

$$\xi = \sum_{r=0}^{p} (E_{21})^r \zeta_1 \otimes \xi_r \quad \text{where} \quad \xi_r \in L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_k, \beta_k)$$

and p is some non-negative integer. Moreover, if $\alpha_1 - \beta_1 \in \mathbb{Z}_+$ then we may and will assume that $p \leqslant \alpha_1 - \beta_1$. We also assume that $\xi_p \neq 0$. Applying $T_{12}(u)$ to ξ, with the use of (4.7) we get

$$\sum_{r=0}^{p} \Big(T_{11}(u)(E_{21})^r \zeta_1 \otimes T_{12}(u)\xi_r + T_{12}(u)(E_{21})^r \zeta_1 \otimes T_{22}(u)\xi_r \Big) = 0. \tag{5.20}$$

Using the definition of the Yangian action on $L(\alpha_1,\beta_1)$ and commutation relations in $\mathfrak{gl}_2$, we obtain

$$T_{11}(u)(E_{21})^r\zeta_1 = (1+E_{11}u^{-1})(E_{21})^r\zeta_1 = (1+(\alpha_1-r)\,u^{-1})(E_{21})^r\zeta_1,$$

and

$$T_{12}(u)(E_{21})^r\zeta_1 = u^{-1}E_{12}\,(E_{21})^r\zeta_1 = u^{-1}\,r(\alpha_1-\beta_1-r+1)(E_{21})^{r-1}\zeta_1.$$

Hence, taking the coefficient at $(E_{21})^p\zeta_1$ in (5.20) gives

$$(1+(\alpha_1-p)\,u^{-1})\,T_{12}(u)\,\xi_p = 0,$$

implying the relation $T_{12}(u)\,\xi_p = 0$. By the induction hypothesis, applied to the $\mathrm{Y}(\mathfrak{gl}_2)$-module $L(\alpha_2,\beta_2)\otimes\cdots\otimes L(\alpha_k,\beta_k)$, the vector ξ_p must be proportional to $\zeta_2\otimes\cdots\otimes\zeta_k$. Therefore, using (4.7) we get

$$T_{22}(u)\,\xi_p = (1+\beta_2u^{-1})\dots(1+\beta_ku^{-1})\,\xi_p. \tag{5.21}$$

In order to complete the proof of the claim it now suffices to show that p must be equal to zero. Suppose by way of contradiction that $p\geqslant 1$. Then taking the coefficient at $(E_{21})^{p-1}\zeta_1$ in (5.20) we derive

$$(1+(\alpha_1-p+1)u^{-1})\,T_{12}(u)\,\xi_{p-1} + u^{-1}\,p(\alpha_1-\beta_1-p+1)\,T_{22}(u)\,\xi_p = 0.$$

Hence, multiplying by u^k and taking into account (5.21) we get

$$(u+\alpha_1-p+1)u^{k-1}T_{12}(u)\,\xi_{p-1} + \,p(\alpha_1-\beta_1-p+1)(u+\beta_2)\dots(u+\beta_k)\,\xi_p = 0.$$

Now observe that the vector $u^{k-1}T_{12}(u)\,\xi_{p-1}$ depends on u polynomially. This follows by an easy induction with the use of (4.7). So, taking the value $u = -\alpha_1+p-1$ we obtain the relation

$$p\,(\alpha_1-\beta_1-p+1)(\alpha_1-\beta_2-p+1)\dots(\alpha_1-\beta_k-p+1) = 0.$$

But this is impossible due to the conditions on the parameters α_i and β_i. Thus, p must be zero and the claim follows.

Suppose now that M is a nonzero submodule of L. Then M must contain a nonzero vector ξ such that $T_{12}(u)\,\xi = 0$. Indeed, this follows from the fact that the set of $\mathfrak{gl}_2$-weights of L has an upper boundary. The above argument thus shows that M contains the vector ζ. It remains to prove that the cyclic span $K = \mathrm{Y}(\mathfrak{gl}_2)\,\zeta$ coincides with L.

Denote by $\varkappa$ the anti-automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_2)$, defined by

$$\varkappa\colon t_{ij}(u)\mapsto t_{3-i,3-j}(-u). \tag{5.22}$$

Consider the vector space L^* dual to L. That is, L^* is spanned by all linear maps $\sigma\colon L\to\mathbb{C}$ satisfying the condition that the linear span of the vectors $\eta\in L$ such that $\sigma(\eta)\neq 0$, is finite-dimensional. Equip L^* with a $\mathrm{Y}(\mathfrak{gl}_2)$-module structure by setting

$$(y\,\sigma)(\eta) = \sigma(\varkappa(y)\,\eta) \quad \text{for} \quad y\in\mathrm{Y}(\mathfrak{gl}_2) \quad \text{and} \quad \sigma\in L^*,\ \eta\in L.$$

It is easy to see that the dual module $L(\alpha,\beta)^*$ to the evaluation module $L(\alpha,\beta)$ is isomorphic to $L(-\beta,-\alpha)$. Moreover, the $\mathrm{Y}(\mathfrak{gl}_2)$-module L^* is isomorphic to the tensor product module

$$L(-\beta_1,-\alpha_1)\otimes\cdots\otimes L(-\beta_k,-\alpha_k).$$

This is deduced from the fact that the anti-automorphism $\varkappa$ commutes with the coproduct Δ, where $\varkappa$ is extended to $\mathrm{Y}(\mathfrak{gl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)$ by $\varkappa(x\otimes y)=\varkappa(x)\otimes\varkappa(y)$ for $x,y\in\mathrm{Y}(\mathfrak{gl}_2)$. Furthermore, the highest vector ζ_i^* of the module $L(-\beta_i,-\alpha_i)\cong L(\alpha_i,\beta_i)^*$ can be identified with the element of $L(\alpha_i,\beta_i)^*$ such that $\zeta_i^*(\zeta_i)=1$ and $\zeta_i^*(\eta_i)=0$ for all weight vectors $\eta_i\in L(\alpha_i,\beta_i)$ whose weights are different from (α_i,β_i).

Suppose now that the submodule K of L is proper and consider its annihilator

$$\operatorname{Ann} K=\{\xi^*\in L^*\mid \xi^*(\eta)=0\quad\text{for all}\quad \eta\in K\}.$$

Then $\operatorname{Ann} K$ is a nonzero submodule of L^*, which does not contain the vector $\zeta_1^*\otimes\cdots\otimes\zeta_k^*$. However, this contradicts the claim verified in the first part of the proof, because the condition on the parameters α_i and β_i remain satisfied after we replace each α_i by $-\beta_i$ and each β_i by $-\alpha_i$. □

By this lemma, all differences $\alpha_i-\beta_i$ must be nonnegative integers because the representation $L(\lambda_1(u),\lambda_2(u))$ is finite-dimensional. Then the polynomial

$$P(u)=\prod_{i=1}^{k}(u+\beta_i)(u+\beta_i+1)\dots(u+\alpha_i-1) \tag{5.23}$$

obviously satisfies (5.17).

Conversely, suppose (5.17) holds for a polynomial $P(u)=(u+\gamma_1)\dots(u+\gamma_p)$. Set

$$\nu_1(u)=(1+(\gamma_1+1)u^{-1})\dots(1+(\gamma_p+1)u^{-1}),$$
$$\nu_2(u)=(1+\gamma_1u^{-1})\dots(1+\gamma_pu^{-1}),$$

and consider the tensor product module

$$L=L(\gamma_1+1,\gamma_1)\otimes L(\gamma_2+1,\gamma_2)\otimes\cdots\otimes L(\gamma_p+1,\gamma_p)$$

of $\mathrm{Y}(\mathfrak{gl}_2)$. Obviously, this module is finite-dimensional. The cyclic $\mathrm{Y}(\mathfrak{gl}_2)$-span of the tensor product of the highest vectors of $L(\gamma_i+1,\gamma_i)$ is a highest weight module with the highest weight $(\nu_1(u),\nu_2(u))$. Since this submodule is finite-dimensional, then so is its irreducible quotient $L(\nu_1(u),\nu_2(u))$. Since

$$\frac{\nu_1(u)}{\nu_2(u)}=\frac{\mu_1(u)}{\mu_2(u)},$$

there exists an automorphism of $\mathrm{Y}(\mathfrak{gl}_2)$ of the form (4.11) such that its composition with the representation $L(\nu_1(u),\nu_2(u))$ is isomorphic to $L(\mu_1(u),\mu_2(u))$. Thus, the latter is also finite-dimensional.

Finally, suppose that $Q(u)$ is another monic polynomial in u and

$$\frac{P(u+1)}{P(u)} = \frac{Q(u+1)}{Q(u)}.$$

This means that the ratio $P(u)/Q(u)$ is periodic in u which is only possible for $P(u) = Q(u)$. □

The polynomial $P(u)$ is called the *Drinfeld polynomial* of the representation $L(\mu(u))$.

We now apply Theorem 5.3 to the low rank extended Yangians.

Corollary 5.6. *The Verma module $M(\lambda(u))$ over $\mathrm{X}(\mathfrak{sp}_2)$ is non-trivial for any highest weight $\lambda(u) = (\lambda_{-1}(u), \lambda_1(u))$. Moreover, the $\mathrm{X}(\mathfrak{sp}_2)$-module $L(\lambda(u))$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in u such that*

$$\frac{\lambda_{-1}(u)}{\lambda_1(u)} = \frac{P(u+2)}{P(u)}. \tag{5.24}$$

In this case, $P(u)$ is unique.

Proof. This is immediate from Proposition 4.1 and Theorem 5.3. □

The evaluation homomorphism provided by Corollary 4.3 allows one to regard any irreducible $\mathfrak{sp}_2$-module $V(\mu)$ as an $\mathrm{X}(\mathfrak{sp}_2)$-module. The corresponding evaluation module is immediately identified with an irreducible highest weight module.

Proposition 5.7. *The evaluation module $V(\mu)$ over $\mathrm{X}(\mathfrak{sp}_2)$ is isomorphic to $L(\lambda(u))$ with*

$$\lambda_{-1}(u) = 1 - \mu_1\, u^{-1} \qquad \textit{and} \qquad \lambda_1(u) = 1 + \mu_1\, u^{-1}. \qquad \square$$

Corollary 5.8. *The Verma module $M(\lambda(u))$ over $\mathrm{X}(\mathfrak{o}_3)$ is non-trivial if and only if the highest weight $\lambda(u) = \big(\lambda_{-1}(u), \lambda_0(u), \lambda_1(u)\big)$ satisfies the condition*

$$\lambda_{-1}(u-1/2)\,\lambda_1(u) = \lambda_0(u-1/2)\,\lambda_0(u). \tag{5.25}$$

Moreover, if this condition holds then the $\mathrm{X}(\mathfrak{o}_3)$-module $L(\lambda(u))$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in u such that

$$\frac{\lambda_0(u)}{\lambda_1(u)} = \frac{P(u+1/2)}{P(u)}. \tag{5.26}$$

In this case, $P(u)$ is unique.

Proof. Let the Verma module $M(\lambda(u))$ be non-trivial. By Proposition 4.4, we may regard $M(\lambda(u))$ as a $\mathrm{Y}(\mathfrak{gl}_2)$-module. In particular, we have

$$T_{11}(2u)\,T_{11}(2u+1)\,1_\lambda = \lambda_{-1}(u)\,1_\lambda,$$

where 1_λ is the highest vector of $M(\lambda(u))$. This implies that 1_λ is an eigenvector for $T_{11}(u)$, that is, $T_{11}(u)\,1_\lambda = \mu_1(u)\,1_\lambda$ for a certain series $\mu_1(u)$. Moreover, this series satisfies

$$\mu_1(2u)\,\mu_1(2u+1) = \lambda_{-1}(u). \tag{5.27}$$

Similarly, $T_{22}(u)\, 1_\lambda = \mu_2(u)\, 1_\lambda$ for a series $\mu_2(u)$ satisfying

$$\mu_2(2u)\, \mu_2(2u+1) = \lambda_1(u). \tag{5.28}$$

Furthermore, by the defining relations (4.3) we have

$$T_{12}(2u)\, T_{22}(2u+1) + T_{22}(2u)\, T_{12}(2u+1) = 2\, T_{12}(2u+1)\, T_{22}(2u).$$

Since $t_{0,1}(u)\, 1_\lambda = 0$ we derive that $T_{12}(u)\, 1_\lambda = 0$. Hence, using the action of $t_{0,0}(u)$ on 1_λ we also get

$$\mu_1(2u)\, \mu_2(2u+1) = \lambda_0(u). \tag{5.29}$$

This gives the condition (5.25).

Conversely, if the condition (5.25) holds for a highest weight $\lambda(u)$ then there exist series $\mu_1(u)$ and $\mu_2(u)$ satisfying (5.27), (5.28) and (5.29). Consider the Verma module $M(\mu_1(u), \mu_2(u))$ over $\mathrm{Y}(\mathfrak{gl}_2)$. Using the formulas of Proposition 4.4, we find that the highest vector $1_\mu \in M(\mu_1(u), \mu_2(u))$ satisfies the conditions (5.1) for the action of $\mathrm{X}(\mathfrak{o}_3)$.

The argument of the first part of the proof shows that, regarded as a $\mathrm{Y}(\mathfrak{gl}_2)$-module, the module $L(\lambda(u))$ is isomorphic to $L(\mu_1(u), \mu_2(u))$ with $\mu_1(u)$ and $\mu_2(u)$ satisfying (5.27), (5.28) and (5.29). So writing the relation of Theorem 5.3 in terms of the series $\lambda_i(u)$, we get the desired condition. □

The evaluation homomorphism provided by Corollary 4.7 allows one to regard any irreducible $\mathfrak{o}_3$-module $V(\mu)$ as an $\mathrm{X}(\mathfrak{o}_3)$-module.

Proposition 5.9. *The evaluation module $V(\mu)$ over $\mathrm{X}(\mathfrak{o}_3)$ is isomorphic to $L(\lambda(u))$ with*

$$\lambda_{-1}(u) = \frac{(2u-\mu_1)(2u-\mu_1-1)}{2u\,(2u-1)},$$
$$\lambda_0(u) = \frac{(2u+\mu_1)(2u-\mu_1-1)}{2u\,(2u-1)},$$
$$\lambda_1(u) = \frac{(2u+\mu_1)(2u+\mu_1-1)}{2u\,(2u-1)}.$$

Proof. This is immediate from Corollary 4.7, as the Casimir element c acts on $V(\mu)$ as multiplication by the scalar $(\mu_1^2-\mu_1)/2$. □

Corollary 5.10. *The Verma module $M(\lambda(u))$ over $\mathrm{X}(\mathfrak{o}_4)$ is non-trivial if and only if the highest weight $\lambda(u) = \big(\lambda_{-2}(u), \lambda_{-1}(u), \lambda_1(u), \lambda_2(u)\big)$ satisfies the condition*

$$\lambda_{-2}(u)\, \lambda_2(u) = \lambda_{-1}(u)\, \lambda_1(u). \tag{5.30}$$

Moreover, if this condition holds then the $\mathrm{X}(\mathfrak{o}_4)$-module $L(\lambda(u))$ is finite-dimensional if and only if there exist monic polynomials $P(u)$ and $Q(u)$ in u such that

$$\frac{\lambda_{-1}(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)} \qquad \text{and} \qquad \frac{\lambda_1(u)}{\lambda_2(u)} = \frac{Q(u+1)}{Q(u)}. \tag{5.31}$$

In this case, $P(u)$ and $Q(u)$ are determined uniquely.

Proof. Suppose that the Verma module $M(\lambda(u))$ over $\mathrm{X}(\mathfrak{o}_4)$ is non-trivial. Using the isomorphism $\chi^{(1)}$ provided by Corollary 4.10, we shall regard $M(\lambda(u))$ as a module over the algebra $\mathrm{Y}(\mathfrak{sl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)$. As was seen in the proof of Corollary 4.10,

$$\chi^{(1)}\colon t_{-2,-2}(u)\, t_{1,1}(u-1) - t_{1,-2}(u)\, t_{-2,1}(u-1) \mapsto T'_{11}(u)\, T'_{11}(u-1).$$

This implies that 1_λ is an eigenvector for $T'_{11}(u)$, that is, $T'_{11}(u)\, 1_\lambda = \mu'_1(u)\, 1_\lambda$ for a certain series $\mu'_1(u)$. Similarly, $T'_{22}(u)\, 1_\lambda = \mu'_2(u)\, 1_\lambda$ for a series $\mu'_2(u)$. Then, by the formulas of Proposition 4.8, we also have

$$\mathcal{T}_{11}(u)\, 1_\lambda = \mu_1(u)\, 1_\lambda \qquad \text{and} \qquad \mathcal{T}_{22}(u)\, 1_\lambda = \mu_2(u)\, 1_\lambda$$

for some series $\mu_1(u)$ and $\mu_2(u)$. Moreover, we have the relations

$$\begin{aligned} \lambda_{-2}(u) &= \mu_1(u)\, \mu'_1(u), & \lambda_{-1}(u) &= \mu_1(u)\, \mu'_2(u), \\ \lambda_1(u) &= \mu_2(u)\, \mu'_1(u), & \lambda_2(u) &= \mu_2(u)\, \mu'_2(u), \end{aligned} \tag{5.32}$$

which imply (5.30). Conversely, if (5.30) holds for some series $\lambda_i(u)$, then there exist series $\mu_i(u)$ and $\mu'_i(u)$ satisfying (5.32) together with the condition $\mu_1(u)\, \mu_2(u-1) = 1$. Consider the $\mathrm{Y}(\mathfrak{sl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)$-module $M(\mu_1(u), \mu_2(u)) \otimes M(\mu'_1(u), \mu'_2(u))$. The vector $1_\mu \otimes 1_{\mu'}$ satisfies the conditions (5.1) for the action of the series $t_{ij}(u)$ thus proving that the $X(\mathfrak{o}_4)$-module $M(\lambda(u))$ is non-trivial.

Finally, the argument of the first part of the proof shows that, regarded as a $\mathrm{Y}(\mathfrak{sl}_2)\otimes\mathrm{Y}(\mathfrak{gl}_2)$-module, the module $L(\lambda(u))$ is isomorphic to $L(\mu_1(u), \mu_2(u)) \otimes L(\mu'_1(u), \mu'_2(u))$ with the $\mu_i(u)$ and $\mu'_i(u)$ satisfying (5.32). By Theorem 5.3, the module $L(\mu_1(u), \mu_2(u)) \otimes L(\mu'_1(u), \mu'_2(u))$ is finite-dimensional if and only if there exist monic polynomials $P(u)$ and $Q(u)$ in u such that

$$\frac{\mu_1(u)}{\mu_2(u)} = \frac{P(u+1)}{P(u)} \qquad \text{and} \qquad \frac{\mu'_1(u)}{\mu'_2(u)} = \frac{Q(u+1)}{Q(u)}.$$

Writing these formulas in terms of the $\lambda_i(u)$ we get the desired conditions. □

The evaluation homomorphism provided by Corollary 4.12 allows one to regard any irreducible $\mathfrak{o}_4$-module $V(\mu)$ as an $\mathrm{X}(\mathfrak{o}_4)$-module.

Proposition 5.11. *The evaluation module $V(\mu)$ over $\mathrm{X}(\mathfrak{o}_4)$ is isomorphic to $L(\lambda(u))$ with*

$$\begin{aligned} \lambda_{-2}(u) &= \frac{(2u-\mu_1-\mu_2)(2u+\mu_1-\mu_2)}{4u^2}, \\ \lambda_{-1}(u) &= \frac{(2u-\mu_1-\mu_2)(2u-\mu_1+\mu_2)}{4u^2}, \\ \lambda_1(u) &= \frac{(2u+\mu_1-\mu_2)(2u+\mu_1+\mu_2)}{4u^2}, \\ \lambda_2(u) &= \frac{(2u-\mu_1+\mu_2)(2u+\mu_1+\mu_2)}{4u^2}. \end{aligned}$$

Proof. This follows from Corollary 4.12, as the Casimir element c acts on $V(\mu)$ as multiplication by the scalar $(\mu_1^2+\mu_2^2)/2-\mu_2$. □

Remark 5.12. More general evaluation modules $V(\mu)_a$ with $a \in \mathbb{C}$ over $\mathrm{X}(\mathfrak{a})$ for $\mathfrak{a} = \mathfrak{sp}_2$, $\mathfrak{o}_3$ and $\mathfrak{o}_4$ can be obtained by using the respective evaluation homomorphisms $\mathrm{ev}_a : \mathrm{X}(\mathfrak{a}) \to \mathrm{U}(\mathfrak{a})$ instead of ev; see Remark 4.13. Then $V(\mu)_a$ will be isomorphic to the irreducible highest weight module $L(\lambda(u))$, where the components $\lambda_i(u)$ are found from the formulas of Propositions 5.7, 5.9 or 5.11 by replacing u with $u-a$.

5.3. Classification theorems

Our goal here is to prove classification theorems for the finite-dimensional irreducible representations of the extended Yangians $\mathrm{X}(\mathfrak{a})$ for $\mathfrak{a} = \mathfrak{o}_{2n+1}$, $\mathfrak{sp}_{2n}$, and $\mathfrak{o}_{2n}$. The corresponding theorem for the Yangian $\mathrm{Y}(\mathfrak{sl}_N)$ implies that every finite-dimensional irreducible representation of $\mathrm{Y}(\mathfrak{sl}_N)$ is isomorphic to a subquotient of the tensor product of the fundamental representations [12], [7, Chapter 12]. We shall use the following version of the well-known construction of the fundamental representations of $\mathrm{Y}(\mathfrak{sl}_N)$. They are obtained by restriction from the corresponding representation of $\mathrm{Y}(\mathfrak{gl}_N)$ which is obtained by a simple particular case of the fusion procedure; see, e.g., [9], [20]. The vector space $\mathbb{C}^N$ carries an irreducible representation of $\mathrm{Y}(\mathfrak{gl}_N)$ with the action of the generators given by

$$T_{ij}(u) \mapsto \delta_{ij} + e_{ij}\, u^{-1}, \qquad i,j \in \{1,\dots,N\},$$

where the e_{ij} denote the standard matrix units. So,

$$T_{ij}(u)\, e_k = \delta_{ij}\, e_k + \delta_{jk}\, e_i\, u^{-1},$$

where $e_1,\dots,e_N$ denote the canonical basis of $\mathbb{C}^N$. Since for any $b \in \mathbb{C}$ the mapping $T_{ij}(u) \mapsto T_{ij}(u-b)$ defines an automorphism of $\mathrm{Y}(\mathfrak{gl}_N)$, using the coproduct (4.7), we can equip the tensor product $(\mathbb{C}^N)^{\otimes m}$ with the action of $\mathrm{Y}(\mathfrak{gl}_N)$ by the rule

$$T_{ij}(u)\,(e_{i_1} \otimes \cdots \otimes e_{i_m}) = \sum_{a_1,\dots,a_{m-1}=1}^{N} T_{ia_1}(u)\, e_{i_1} \otimes T_{a_1a_2}(u+1)\, e_{i_2} \otimes \cdots \otimes T_{a_{m-1}j}(u+m-1)\, e_{i_m}. \qquad (5.33)$$

For any $1 \leqslant m < N$ set

$$\xi_m = \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}\sigma \cdot e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(m)} \in (\mathbb{C}^N)^{\otimes m}.$$

Then ξ_m has the properties

$$T_{ij}(u)\,\xi_m = 0 \qquad \text{for all} \quad 1 \leqslant i < j \leqslant N \qquad (5.34)$$

and

$$T_{ii}(u)\,\xi_m = \begin{cases} \dfrac{u+m}{u+m-1}\,\xi_m & \text{if} \quad 1 \leqslant i \leqslant m, \\ \xi_m & \text{if} \quad m+1 \leqslant i \leqslant N. \end{cases}$$

Thus, the vector ξ_m generates a highest weight module over $\mathrm{Y}(\mathfrak{gl}_N)$ whose irreducible quotient is isomorphic to a fundamental module; see [7, Chapter 12], [16].

Consider the extended Yangian $\mathrm{X}(\mathfrak{a}')$ for the subalgebra $\mathfrak{a}'$ of $\mathfrak{a}$ of rank $n-1$. That is,

$$\mathfrak{a}' = \mathfrak{o}_{2n-1},\ \mathfrak{sp}_{2n-2},\ \mathfrak{o}_{2n-2} \quad \text{respectively for} \quad \mathfrak{a} = \mathfrak{o}_{2n+1},\ \mathfrak{sp}_{2n},\ \mathfrak{o}_{2n}.$$

Note that $\mathrm{X}(\mathfrak{a}')$ is not a natural subalgebra of $\mathrm{X}(\mathfrak{a})$. Let V be an $\mathrm{X}(\mathfrak{a})$-module. Set

$$\begin{aligned} V^+ = \{\eta \in V \mid t_{k,n}(u)\,\eta = 0 \qquad & \text{for} \quad k < n \qquad \text{and} \\ t_{-n,k}(u)\,\eta = 0 \qquad & \text{for} \quad k > -n\}. \end{aligned}$$

Lemma 5.13. *The subspace V^+ is stable under all operators $t_{ij}(u)$ with the condition $-n+1 \leqslant i,j \leqslant n-1$. Moreover, these operators form a representation of the algebra $\mathrm{X}(\mathfrak{a}')$ on V^+, where each operator $t_{ij}(u)$ is the image of the generator series of $\mathrm{X}(\mathfrak{a}')$ with the same name.*

Proof. For any $\eta \in V^+$ we have the following relations modulo elements of V^+ which are implied by (2.19): if $k < n$ and $-n+1 \leqslant i,j \leqslant n-1$ then

$$t_{kn}(v)\,t_{ij}(u)\,\eta \equiv -[t_{ij}(u), t_{kn}(v)]\,\eta \equiv \frac{\delta_{k,-i}}{u-v-\kappa}\,\theta_{i,-n} t_{-n,j}(u)\,t_{nn}(v)\,\eta.$$

However, applying again (2.19), we find that

$$t_{-n,j}(u)\,t_{nn}(v)\,\eta \equiv -\frac{1}{u-v-\kappa}\,t_{-n,j}(u)\,t_{nn}(v)\,\eta.$$

Therefore, $t_{-n,j}(u)\,t_{nn}(v)\,\eta \equiv 0$ implying $t_{kn}(v)\,t_{ij}(u)\,\eta \equiv 0$. A similar calculation shows that for any $k > -n$ and $-n+1 \leqslant i,j \leqslant n-1$ we also have $t_{-n,k}(v)\,t_{ij}(u)\,\eta \equiv 0$ proving the first part of the lemma.

In order to prove the second part, suppose that the indices i,j,k,l satisfy the condition $-n+1 \leqslant i,j,k,l \leqslant n-1$. Then by (2.19) for any $\eta \in V^+$ we have

$$\begin{aligned} [t_{ij}(u), t_{kl}(v)]\,\eta &= \frac{1}{u-v}\Big(t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u)\Big)\,\eta \\ &- \frac{1}{u-v-\kappa}\Big(\delta_{k,-i}\sum_{p=-n}^{n}\theta_{ip}\,t_{pj}(u)\,t_{-p,l}(v) - \delta_{l,-j}\sum_{p=-n}^{n}\theta_{jp}\,t_{k,-p}(v)\,t_{ip}(u)\Big)\,\eta. \end{aligned}$$

Writing the right-hand side modulo V^+, we get

$$\begin{aligned} &\frac{1}{u-v}\Big(t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u)\Big)\,\eta \\ &- \frac{1}{u-v-\kappa}\Big(\delta_{k,-i}\sum_{p=-n+1}^{n-1}\theta_{ip}\,t_{pj}(u)\,t_{-p,l}(v) - \delta_{l,-j}\sum_{p=-n+1}^{n-1}\theta_{jp}\,t_{k,-p}(v)\,t_{ip}(u)\Big)\,\eta \\ &- \frac{1}{u-v-\kappa}\Big(\delta_{k,-i}\,\theta_{i,-n}\,t_{-n,j}(u)\,t_{n,l}(v) - \delta_{l,-j}\,\theta_{j,-n}\,t_{k,n}(v)\,t_{i,-n}(u)\Big)\,\eta. \end{aligned}$$

Applying again (2.19), we obtain

$$t_{-n,j}(u)\,t_{n,l}(v)\,\eta \equiv -\frac{1}{u-v-\kappa}\sum_{p=-n+1}^{n-1}\theta_{-n,p}\,t_{pj}(u)\,t_{-p,l}(v)\,\eta$$
$$-\frac{1}{u-v-\kappa}\Big(t_{-n,j}(u)\,t_{n,l}(v)-\delta_{l,-j}\,\theta_{j,-n}\,t_{n,n}(v)\,t_{-n,-n}(u)\Big)\,\eta.$$

Hence,

$$t_{-n,j}(u)\,t_{n,l}(v)\,\eta \equiv -\frac{1}{u-v-\kappa+1}\sum_{p=-n+1}^{n-1}\theta_{-n,p}\,t_{pj}(u)\,t_{-p,l}(v)\,\eta$$
$$+\frac{1}{u-v-\kappa+1}\,\delta_{l,-j}\,\theta_{j,-n}\,t_{n,n}(v)\,t_{-n,-n}(u)\,\eta.$$

Similarly, $t_{k,n}(v)\,t_{i,-n}(u)\,\eta \equiv -[t_{i,-n}(u),t_{k,n}(v)]\,\eta$ and

$$[t_{i,-n}(u),t_{k,n}(v)]\,\eta \equiv -\frac{1}{u-v-\kappa}\,\delta_{k,-i}\,\theta_{i,-n}\,t_{-n,-n}(u)\,t_{nn}(v)\,\eta$$
$$+\frac{1}{u-v-\kappa}\Big(\sum_{p=-n+1}^{n-1}\theta_{-n,p}\,t_{k,-p}(v)\,t_{ip}(u)+t_{k,n}(v)\,t_{i,-n}(u)\Big)\,\eta$$

which gives

$$t_{k,n}(v)\,t_{i,-n}(u)\,\eta \equiv \frac{1}{u-v-\kappa+1}\,\delta_{k,-i}\,\theta_{i,-n}\,t_{-n,-n}(u)\,t_{nn}(v)\,\eta$$
$$-\frac{1}{u-v-\kappa+1}\sum_{p=-n+1}^{n-1}\theta_{-n,p}\,t_{k,-p}(v)\,t_{ip}(u)\,\eta.$$

Combining these expressions, we come to the following relation

$$[t_{ij}(u),t_{kl}(v)]\,\eta \equiv \frac{1}{u-v}\Big(t_{kj}(u)\,t_{il}(v)-t_{kj}(v)\,t_{il}(u)\Big)\,\eta$$
$$-\frac{1}{u-v-\kappa+1}\Big(\delta_{k,-i}\sum_{p=-n+1}^{n-1}\theta_{ip}\,t_{pj}(u)\,t_{-p,l}(v)-\delta_{l,-j}$$
$$\sum_{p=-n+1}^{n-1}\theta_{jp}\,t_{k,-p}(v)\,t_{ip}(u)\Big)\,\eta$$
$$+\frac{1}{(u-v-\kappa)(u-v-\kappa+1)}\,\delta_{k,-i}\,\delta_{l,-j}\,\theta_{ij}\,[t_{-n,-n}(u),t_{nn}(v)]\,\eta.$$

Finally, by (2.19),

$$[t_{-n,-n}(u),t_{nn}(v)]\,\eta \equiv -\frac{1}{u-v-\kappa}\,[t_{-n,-n}(u),t_{nn}(v)]\,\eta,$$

so that $[t_{-n,-n}(u),t_{nn}(v)]\,\eta \equiv 0$. This yields the desired relations between the operators $t_{ij}(u)$ on V^+ since $\kappa-1=\kappa'$ coincides with the value of the parameter κ for the Lie algebra $\mathfrak{a}'$. □

Proposition 5.14. *The Verma module $M(\lambda(u))$ over $\mathrm{X}(\mathfrak{a})$ is non-trivial if and only if the components of the highest weight $\lambda(u)$ satisfy the conditions*

$$\frac{\lambda_{-n+i-1}(u+\kappa-i)}{\lambda_{-n+i}(u+\kappa-i)}=\frac{\lambda_{n-i}(u)}{\lambda_{n-i+1}(u)} \tag{5.35}$$

for $i=1,\dots,n-1$ if $\mathfrak{a}=\mathfrak{o}_{2n}$ or $\mathfrak{sp}_{2n}$, and for $i=1,\dots,n$ if $\mathfrak{a}=\mathfrak{o}_{2n+1}$.

Proof. Suppose first that $M(\lambda(u))$ is non-trivial. We use induction on n taking Corollaries 5.6, 5.8 and 5.10 as the induction base. Let us apply $t_{-n,-n+1}(u)\, t_{n,n-1}(v)$ to the highest vector 1_λ of $M(\lambda(u))$. By (2.19) we have

$$t_{-n,-n+1}(u)\, t_{n,n-1}(v)\, 1_\lambda = \\ -\frac{1}{u-v-\kappa}\Big(t_{-n,-n+1}(u)\, t_{n,n-1}(v)+\lambda_{-n+1}(u)\,\lambda_{n-1}(v)-\lambda_{-n}(u)\,\lambda_n(v)\Big)\, 1_\lambda,$$

which implies

$$(u-v-\kappa+1)\, t_{-n,-n+1}(u)\, t_{n,n-1}(v)\, 1_\lambda=\lambda_{-n}(u)\,\lambda_n(v)\, 1_\lambda-\lambda_{-n+1}(u)\,\lambda_{n-1}(v)\, 1_\lambda.$$

Putting $u=v+\kappa-1$ and replacing v by u we obtain (5.35) for $i=1$. Furthermore, by Lemma 5.13, the subspace $M(\lambda(u))^+$ of $M(\lambda(u))$ is a module over $\mathrm{X}(\mathfrak{a}')$. The highest vector 1_λ belongs to $M(\lambda(u))^+$ and generates a highest weight $\mathrm{X}(\mathfrak{a}')$-module with the highest weight $(\lambda_{-n+1}(u),\dots,\lambda_{n-1}(u))$. So, the remaining conditions hold by the induction hypothesis.

Conversely, suppose that $\lambda(u)$ satisfies the conditions. Consider the left ideal I of the algebra $\mathrm{X}(\mathfrak{a})$ generated by the coefficients of the series $t_{ij}(u)$ with $i<j$ where $i+j>0$ or $i+j\geqslant 0$ for the orthogonal or symplectic case, respectively; and by the coefficients of the series $t_{ii}(u)-\lambda_i(u)$ for $i=1,\dots,n$ and $z(u)-\lambda_{-n}(u+\kappa)\,\lambda_n(u)$. By Corollary 3.10, the quotient $\widetilde{M}(\lambda(u))=\mathrm{X}(\mathfrak{a})/I$ is non-trivial. Let 1_λ be the image of $1\in\mathrm{X}(\mathfrak{a})$ in the quotient. It suffices to verify that the vector 1_λ satisfies all the conditions (5.1). Now we use Corollary 3.10 again. Let us choose the total ordering on the elements $t_{ij}^{(r)}$ and z_r with the conditions on the indices as in the statement of the corollary, in such a way that any element $t_{ij}^{(r)}$ with $i>j$ precedes any element $t_{kk}^{(s)}$ while the latter precedes any element of the form $t_{ij}^{(r)}$ with $i<j$. We shall regard $\mathrm{X}(\mathfrak{a})$ as the adjoint $\mathfrak{a}$-module with the action defined on the generators by (5.3). For any pair $k<l$ and any $r\geqslant 1$ write the element $t_{kl}^{(r)}$ as a linear combination of the ordered monomials. The $\mathfrak{a}$-weight of each of the monomials coincides with the $\mathfrak{a}$-weight of $t_{kl}^{(r)}$. Then the relation $t_{kl}^{(r)}\, 1_\lambda=0$ follows because the vector 1_λ is annihilated by any monomial occurring in the combination. The same argument shows that 1_λ is an eigenvector for the action of any element $t_{kk}^{(s)}$. Thus, the $\mathrm{X}(\mathfrak{a})$-module $\widetilde{M}(\lambda(u))$ is a Verma module $M(\widetilde{\lambda}(u))$. It remains to verify that its highest weight $\widetilde{\lambda}(u)$ coincides with $\lambda(u)$. This holds for the components of $\widetilde{\lambda}(u)$ with positive subscripts by the definition of $\widetilde{M}(\lambda(u))$. Furthermore, since $z(u)\, 1_\lambda=\lambda_{-n}(u+\kappa)\,\lambda_n(u)\, 1_\lambda$, (5.13) implies that

$t_{-n,-n}(u)\,1_\lambda = \lambda_{-n}(u)\,1_\lambda$. So, $\widetilde{\lambda}_{-n}(u) = \lambda_{-n}(u)$. By the first part of the proof, since the Verma module $M(\widetilde{\lambda}(u))$ is non-trivial, the conditions (5.35) must hold for the components of $\widetilde{\lambda}(u)$. This shows that $\widetilde{\lambda}(u) = \lambda(u)$, and thus $M(\lambda(u))$ is non-trivial. □

Corollary 5.15. *The irreducible highest weight module* $L(\lambda(u))$ *over* $\mathrm{X}(\mathfrak{a})$ *exists if and only if the conditions* (5.35) *hold.*

Proof. If $L(\lambda(u))$ exists then the conditions (5.35) are derived by repeating the argument of the first part of the proof of Proposition 5.14. Conversely, if the conditions hold then the Verma module $M(\lambda(u))$ is non-trivial by Proposition 5.14. Therefore, the irreducible quotient $L(\lambda(u))$ of $M(\lambda(u))$ exists. □

We are now in a position to prove the classification theorem for finite-dimensional irreducible representations of the extended Yangian $\mathrm{X}(\mathfrak{a})$.

Theorem 5.16. *Every finite-dimensional irreducible* $\mathrm{X}(\mathfrak{a})$*-module is isomorphic to* $L(\lambda(u))$ *where* $\lambda(u)$ *satisfies the conditions* (5.35) *and there exist monic polynomials* $P_1(u), \dots, P_n(u)$ *in* u *such that*

$$\frac{\lambda_{i-1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)}, \qquad \text{for} \quad i = 2, \dots, n \tag{5.36}$$

and also

$$\frac{\lambda_0(u)}{\lambda_1(u)} = \frac{P_1(u+1/2)}{P_1(u)}, \qquad \text{if} \quad \mathfrak{a} = \mathfrak{o}_{2n+1},$$

$$\frac{\lambda_{-1}(u)}{\lambda_1(u)} = \frac{P_1(u+2)}{P_1(u)}, \qquad \text{if} \quad \mathfrak{a} = \mathfrak{sp}_{2n},$$

$$\frac{\lambda_{-1}(u)}{\lambda_2(u)} = \frac{P_1(u+1)}{P_1(u)}, \qquad \text{if} \quad \mathfrak{a} = \mathfrak{o}_{2n}.$$

Conversely, if (5.35) *and the above conditions on the highest weight* $\lambda(u)$ *are satisfied then* $L(\lambda(u))$ *exists and has finite dimension.*

The polynomials $P_1(u), \dots, P_n(u)$ are called the *Drinfeld polynomials* corresponding to the finite-dimensional representation $L(\lambda(u))$.

Proof. Due to Theorem 5.1, every finite-dimensional irreducible $\mathrm{X}(\mathfrak{a})$-module is isomorphic to $L(\lambda(u))$ for some highest weight $\lambda(u)$. Then $\lambda(u)$ must satisfy (5.35) by Corollary 5.15 since $L(\lambda(u))$ exists. Now we argue by induction on n taking Corollaries 5.6, 5.8 and 5.10 as the induction base. Observe that if $n \geqslant 2$ then by (2.19), the mapping

$$T_{ij}(u) \mapsto t_{i+n-2,j+n-2}(u), \qquad i,j \in \{1,2\}$$

defines a homomorphism $\mathrm{Y}(\mathfrak{gl}_2) \to \mathrm{X}(\mathfrak{a})$. So, $L(\lambda(u))$ can be regarded as a $\mathrm{Y}(\mathfrak{gl}_2)$-module. The highest vector $1_\lambda \in L(\lambda(u))$ then satisfies

$$T_{11}(u)\,1_\lambda = \lambda_{n-1}(u)\,1_\lambda, \qquad T_{22}(u)\,1_\lambda = \lambda_n(u)\,1_\lambda, \qquad T_{12}(u)\,1_\lambda = 0.$$

Since the cyclic span $\mathrm{Y}(\mathfrak{gl}_2)\, 1_\lambda$ is finite-dimensional, we derive from Theorem 5.3 that there exists a monic polynomial $P_n(u)$ such that (5.36) holds for $i = n$. Furthermore, by Lemma 5.13, the subspace $L(\lambda(u))^+$ is a module over $\mathrm{X}(\mathfrak{a}')$. The highest vector 1_λ belongs to $L(\lambda(u))^+$ and generates a highest weight $\mathrm{X}(\mathfrak{a}')$-module with the highest weight $(\lambda_{-n+1}(u), \dots, \lambda_{n-1}(u))$. Since the cyclic span $\mathrm{X}(\mathfrak{a}')\, 1_\lambda$ is finite-dimensional, the remaining conditions on the $\lambda_i(u)$ hold by the induction hypothesis.

Suppose now that the highest weight $\lambda(u)$ satisfies the given conditions. Then $L(\lambda(u))$ exists by Corollary 5.15. We need to show that $\dim L(\lambda(u)) < \infty$. Observe that the n-tuple of Drinfeld polynomials corresponding to an $\mathrm{X}(\mathfrak{a})$-module $L(\lambda(u))$ determines the highest weight $\lambda(u)$ up to a simultaneous multiplication of all components $\lambda_i(u)$ by a series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. On the other hand, the composition of the action of $\mathrm{X}(\mathfrak{a})$ on $L(\lambda(u))$ with the automorphism (2.21) yields a representation of $\mathrm{X}(\mathfrak{a})$ isomorphic to $L(\lambda'(u))$, where the components of $\lambda'(u)$ are given by $\lambda'_i(u) = f(u)\, \lambda_i(u)$. Therefore, it suffices to prove that a particular module $L(\lambda(u))$ corresponding to an arbitrary n-tuple of Drinfeld polynomials is finite-dimensional.

We shall use the coproduct (2.24) to equip the tensor product of two $\mathrm{X}(\mathfrak{a})$-modules with an $\mathrm{X}(\mathfrak{a})$-module structure.

Lemma 5.17. *Let $L(\lambda(u))$ and $L(\mu(u))$ be two irreducible highest weight modules over $\mathrm{X}(\mathfrak{a})$ with*

$$\lambda(u) = (\lambda_{-n}(u), \dots, \lambda_n(u)) \qquad \text{and} \qquad \mu(u) = (\mu_{-n}(u), \dots, \mu_n(u)).$$

Then the tensor product $1_\lambda \otimes 1_\mu$ of the highest vectors of $L(\lambda(u))$ and $L(\mu(u))$ generates a highest weight submodule V over $\mathrm{X}(\mathfrak{a})$ in $L(\lambda(u)) \otimes L(\mu(u))$ with the highest weight

$$(\lambda_{-n}(u)\, \mu_{-n}(u), \dots, \lambda_n(u)\, \mu_n(u)). \tag{5.37}$$

Moreover, if the modules $L(\lambda(u))$ and $L(\mu(u))$ are finite-dimensional with the corresponding n-tuples of Drinfeld polynomials $(P_1(u), \dots, P_n(u))$ and $(Q_1(u), \dots, Q_n(u))$, respectively, then the n-tuple of Drinfeld polynomials corresponding to the irreducible quotient of V is $(P_1(u)\, Q_1(u), \dots, P_n(u)\, Q_n(u))$.

Proof. It follows easily from (2.24) that the vector $\xi = 1_\lambda \otimes 1_\mu$ satisfies (5.1) with the highest weight (5.37). The second statement now follows from the relations defining the Drinfeld polynomials. □

By the lemma, we only need to show that if an irreducible highest weight module $L(\lambda(u))$ corresponds to an n-tuple of Drinfeld polynomials of the form $P_j(u) = 1$ for all $j \neq i$ and $P_i(u) = u - b$ for certain $i \in \{1, \dots, n\}$ and $b \in \mathbb{C}$, then $\dim L(\lambda(u)) < \infty$. Furthermore, the composition of the action of $\mathrm{X}(\mathfrak{a})$ on $L(\lambda(u))$ with an automorphism of the form (2.22) yields a representation of $\mathrm{X}(\mathfrak{a})$ whose n-tuple of Drinfeld polynomials is $P_j(u) = 1$ for all $j \neq i$ and $P_i(u) = u - a - b$. Thus, it suffices to prove the claim for all values of the index i and a certain particular value of $b \in \mathbb{C}$.

Consider the representation of $\mathrm{X}(\mathfrak{a})$ on $\mathbb{C}^N$ defined in (3.19) with $c = 0$ so that

$$t_{ij}(u) \mapsto \delta_{ij} + e_{ij}\, u^{-1} - \theta_{ij}\, e_{-j,-i}\,(u+\kappa)^{-1}.$$

Equip the tensor product $(\mathbb{C}^N)^{\otimes m}$ with an $\mathrm{X}(\mathfrak{a})$-action by

$$t_{ij}(u)\,(e_{i_1}\otimes\cdots\otimes e_{i_m}) = \sum_{a_1,\dots,a_{m-1}=-n}^{n} t_{ia_1}(u)\, e_{i_1}\otimes t_{a_1a_2}(u+1)\, e_{i_2}\otimes\cdots\otimes t_{a_{m-1}j}(u+m-1)\, e_{i_m}, \quad (5.38)$$

where we use the coproduct (2.24) on $\mathrm{X}(\mathfrak{a})$ and the automorphism (2.22). For any $1 \leqslant m \leqslant n$ set

$$\xi_m = \sum_{\sigma\in\mathfrak{S}_m} \operatorname{sgn}\sigma\cdot e_{-n-1+\sigma(1)}\otimes\cdots\otimes e_{-n-1+\sigma(m)} \in (\mathbb{C}^N)^{\otimes m}.$$

We claim that ξ_m satisfies

$$t_{ij}(u)\,\xi_m = 0 \qquad \text{for all} \quad -n \leqslant i < j \leqslant n \quad (5.39)$$

and

$$t_{ii}(u)\,\xi_m = \begin{cases} \dfrac{u+m}{u+m-1}\,\xi_m & \text{if} \quad -n \leqslant i \leqslant -n+m-1,\\ \xi_m & \text{if} \quad -n+m \leqslant i \leqslant n-m,\\ \dfrac{u+\kappa-1}{u+\kappa}\,\xi_m & \text{if} \quad n-m+1 \leqslant i \leqslant n. \end{cases} \quad (5.40)$$

Denote by $P^{(m)}$ the operator in $(\mathbb{C}^N)^{\otimes m}$ which acts on the basis vectors by

$$P^{(m)}\,(e_{i_1}\otimes\cdots\otimes e_{i_m}) = e_{i_m}\otimes\cdots\otimes e_{i_1}.$$

We have $P^{(m)}(\xi_m) = \alpha\,\xi_m$, where $\alpha = 1$ or -1. The definition (5.38) implies the following relation for the action of $\mathrm{X}(\mathfrak{a})$ on $(\mathbb{C}^N)^{\otimes m}$,

$$\theta_{ij}\, t_{-j,-i}(u) = P^{(m)}\, t_{ij}(-u-\kappa-m+1)\, P^{(m)}. \quad (5.41)$$

Due to (5.3), in order to verify (5.39) in the case $\mathfrak{a} = \mathfrak{o}_{2n+1}$, it therefore suffices to consider the values $j = i+1$ with $-n \leqslant i \leqslant -1$. Since the expression for the vector ξ_m only involves the tensor products $e_{i_1}\otimes\cdots\otimes e_{i_m}$ with negative subscripts i_k, we may assume that the summation indices $a_1,\dots,a_{m-1}$ in (5.38) are all negative. Indeed, $t_{ia_1}(u)\, e_{i_1} = 0$ unless $a_1 < 0$ implying $t_{a_1a_2}(u+1)\, e_{i_2} = 0$ unless $a_2 < 0$ etc. However, in this case the formula (5.38) takes the same form as its $\mathrm{Y}(\mathfrak{gl}_N)$-counterpart (5.33) if we take into account the convention on the basis vector indices. Therefore, the relations $t_{i,i+1}(u)\,\xi_m = 0$ and, hence (5.39), are implied by the corresponding property (5.34) of the vector ξ_m in the case of $\mathrm{Y}(\mathfrak{gl}_N)$. Moreover, this argument also proves (5.40) for the non-positive values of i. The application of (5.41) completes the proof of (5.40).

The same argument applies to the cases $\mathfrak{a} = \mathfrak{sp}_{2n}$ and $\mathfrak{a} = \mathfrak{o}_{2n}$ which also shows that $t_{-1,1}(u)\,\xi_m = 0$ together with $t_{-1,2}(u)\,\xi_m = 0$ for $\mathfrak{a} = \mathfrak{o}_{2n}$.

Thus, in the case $\mathfrak{a} = \mathfrak{sp}_{2n}$ for any $m \in \{1, \dots, n-1\}$ the vector ξ_m generates a highest weight submodule of $(\mathbb{C}^N)^{\otimes m}$ whose n-tuple of Drinfeld polynomials is $P_j(u) = 1$ for all $j \neq m$ and $P_m(u) = u + \kappa - 1$, while ξ_n generates a highest weight submodule of $(\mathbb{C}^N)^{\otimes n}$ whose n-tuple of Drinfeld polynomials is $P_1(u) = u + n - 1$ and $P_j(u) = 1$ for $j \neq 1$. This completes the proof of the theorem in the symplectic case, as the irreducible highest weight modules over $\mathrm{X}(\mathfrak{a})$ with such n-tuples of Drinfeld polynomials are finite-dimensional.

Similarly, the proof is also complete in the case $\mathfrak{a} = \mathfrak{o}_{2n}$ and the values $m \in \{1, \dots, n-2\}$, as well as in the case $\mathfrak{a} = \mathfrak{o}_{2n+1}$ for the values $m \in \{1, \dots, n-1\}$. In order to complete the proof in the remaining cases, we shall use the spinor representations of the orthogonal Lie algebras. The spinor representation $V(-1/2, \dots, -1/2)$ of the Lie algebra $\mathfrak{o}_{2n+1}$ can be realized in the 2^n-dimensional space Λ_n of polynomials in n anti-commuting variables $\xi_1, \dots, \xi_n$,

$$\Lambda_n = \text{span of } \{\xi_{i_1} \dots \xi_{i_k} \mid 1 \leqslant i_1 < \dots < i_k \leqslant n, \quad 0 \leqslant k \leqslant n\}.$$

The generators of $\mathfrak{o}_{2n+1}$ act on this space as the operators

$$\begin{aligned} F_{ij} = \xi_i \, \partial_j - \frac{1}{2} \, \delta_{ij}, \qquad F_{-j,i} = \partial_i \, \partial_j, \qquad F_{j,-i} = \xi_i \, \xi_j, \\ F_{0,i} = \frac{1}{\sqrt{2}} \, \partial_i, \qquad F_{i,0} = \frac{1}{\sqrt{2}} \, \xi_i, \end{aligned} \tag{5.42}$$

where $i, j \in \{1, \dots, n\}$ and ∂_i is the left derivative over ξ_i. The restriction of Λ_n to the subalgebra $\mathfrak{o}_{2n} \subset \mathfrak{o}_{2n+1}$ (spanned by the elements F_{ij} with $i, j \neq 0$) splits into the direct sum of two irreducible submodules, $\Lambda_n = \Lambda_n^+ \oplus \Lambda_n^-$, where Λ_n^+ (respectively, Λ_n^-) is the subspace of Λ_n spanned by the even (respectively, odd) monomials in the generators ξ_i. We have the isomorphisms

$$\Lambda_n^+ \cong V(-1/2, \dots, -1/2) \qquad \text{and} \qquad \Lambda_n^- \cong V(1/2, -1/2, \dots, -1/2). \tag{5.43}$$

The highest weight vectors of the $\mathfrak{o}_{2n}$-modules Λ_n^+ and Λ_n^- are, respectively, the vectors 1 and ξ_1.

Lemma 5.18. *Each spinor representation of $\mathfrak{o}_N$ can be extended to a representation of the algebra $\mathrm{X}(\mathfrak{o}_N)$ by the rule*

$$t_{ij}(u) \mapsto \delta_{ij} + F_{ij} \, u^{-1}, \qquad i, j \in \{-n, \dots, n\}.$$

Proof. The claim follows by a direct verification that the images of $t_{ij}(u)$ satisfy the defining relations (2.19) with the use of the following identity of operators in each spinor representation:

$$(F^2)_{ij} = \Big(\frac{\kappa}{2} + \frac{1}{4}\Big) \, \delta_{ij} + \kappa \, F_{ij}, \tag{5.44}$$

where F is defined in (2.6). Indeed, in the particular case $i = j = n$, the identity is verified by a straightforward calculation. The general case then follows by commuting both sides of this particular identity with appropriate generators F_{ij}. □

The lemma implies that the spinor representation $V(-1/2,\dots,-1/2)$ of $\mathfrak{o}_N$ becomes an irreducible highest weight representation of $\mathrm{X}(\mathfrak{o}_N)$ with the highest weight $\lambda(u)$, where

$$\lambda_i(u) = 1 + \frac{1}{2}u^{-1} \quad \text{for} \quad i \leqslant -1, \qquad \lambda_i(u) = 1 - \frac{1}{2}u^{-1} \quad \text{for} \quad i \geqslant 1$$

and $\lambda_0(u) = 1$ (the latter only occurs for $N = 2n+1$). The corresponding n-tuple of Drinfeld polynomials is $(u - 1/2, 1, \dots, 1)$ in both cases $N = 2n$ and $N = 2n+1$. Finally, the spinor representation $V(1/2, -1/2, \dots, -1/2)$ of $\mathfrak{o}_{2n}$ becomes an irreducible highest weight representation of $\mathrm{X}(\mathfrak{o}_{2n})$ with the highest weight $\lambda(u)$, where

$$\lambda_i(u) = 1 + \frac{1}{2}u^{-1} \quad \text{for} \quad i \leqslant -2 \quad \text{and} \quad i = 1,$$
$$\lambda_i(u) = 1 - \frac{1}{2}u^{-1} \quad \text{for} \quad i \geqslant 2 \quad \text{and} \quad i = -1.$$

The corresponding n-tuple of Drinfeld polynomials is $(1, u - 1/2, 1, \dots, 1)$. □

Theorem 5.16 allows us to get another proof of Drinfeld's classifications theorem for the Yangian modules [12]; cf. [7, Chapter 12].

Corollary 5.19. *Any finite-dimensional irreducible representation of the Yangian* $\mathrm{Y}(\mathfrak{a})$ *is isomorphic to the restriction of an* $\mathrm{X}(\mathfrak{a})$*-module* $L(\lambda(u))$ *to the subalgebra* $\mathrm{Y}(\mathfrak{a})$*, where the components of* $\lambda(u)$ *satisfy the conditions of Theorem* 5.16. *In particular, such representations of* $\mathrm{Y}(\mathfrak{a})$ *are parameterized by the tuples* $(P_1(u), \dots, P_n(u))$ *of monic polynomials in* u.

Proof. By Theorem 3.1, any finite-dimensional irreducible representation V of $\mathrm{Y}(\mathfrak{a})$ can be extended to a representation of $\mathrm{X}(\mathfrak{a})$ where the elements of the center $\mathrm{ZX}(\mathfrak{a})$ act as scalar operators. By Theorem 5.16, the $\mathrm{X}(\mathfrak{a})$-module V is isomorphic to $L(\lambda(u))$ for an appropriate highest weight $\lambda(u)$. This allows one to attach a tuple of polynomials $(P_1(u), \dots, P_n(u))$ to the $\mathrm{Y}(\mathfrak{a})$-module V.

Conversely, given any n-tuple of polynomials $(P_1(u), \dots, P_n(u))$, there exists a highest weight $\lambda(u)$ such that the conditions of Theorem 3.1 hold. Moreover, the components of $\lambda(u)$ are uniquely determined up to simultaneous multiplication by a formal series in u^{-1}. This implies that the corresponding $\mathrm{X}(\mathfrak{a})$-module $L(\lambda(u))$ is determined up to twisting by an appropriate automorphism (2.21). However, the subalgebra $\mathrm{Y}(\mathfrak{a})$ consists of the elements stable under all such automorphisms. This yields the desired parametrization of the representations of $\mathrm{Y}(\mathfrak{a})$. □

The finite-dimensional irreducible representations $L(\lambda(u))$ corresponding to the n-tuples of Drinfeld polynomials of the form $(1, \dots, u-a, 1, \dots, 1)$, where $a \in \mathbb{C}$ and $u-a$ is on the i-th position, are called the *fundamental representations* of $\mathrm{X}(\mathfrak{a})$ or $\mathrm{Y}(\mathfrak{a})$. The following corollary was established in the proof of Theorem 5.16.

Corollary 5.20. *Every finite-dimensional irreducible representation of* $\mathrm{Y}(\mathfrak{a})$ *is isomorphic to a subquotient of a tensor product of the fundamental representations.*

□

5.4. Fundamental representations

In this section we give a more explicit description of the fundamental representations of the algebras $\mathrm{X}(\mathfrak{a})$ and $\mathrm{Y}(\mathfrak{a})$. We shall follow the general approach of the paper by Chari and Pressley [6]. However, contrary to [6], we avoid using the theorem describing the singularities of R-matrices.

We start with the orthogonal case $\mathfrak{a}=\mathfrak{o}_N$. The fundamental representations with the n-tuples of Drinfeld polynomials $(u-1/2,1,\dots,1)$ and $(1,u-1/2,1,\dots,1)$ (the latter for $N=2n$ only), were constructed in the proof of Theorem 5.16.

Now let $N=2n+1$. The tensor square of the spinor representation Λ_n of $\mathfrak{o}_{2n+1}$ has the following decomposition into irreducibles:

$$\Lambda_n\otimes\Lambda_n\cong\bigoplus_{p=0}^{n}V(\mu^{(p)}), \tag{5.45}$$

where $\mu^{(p)}=(0,\dots,0,-1,\dots,-1)$ with p zeros. Note that $V(\mu^{(p)})$ is a fundamental representation of $\mathfrak{o}_{2n+1}$ for any $1\leqslant p\leqslant n-1$. It corresponds to the fundamental weight ω_{n-p} in a more standard notation. The highest weight vector v_p of $V(\mu^{(p)})$ is given in an explicit form by

$$v_p=\sum(-1)^{j_1+\dots+j_l}\,\xi_{i_1}\cdots\xi_{i_k}\otimes\xi_{j_1}\cdots\xi_{j_l}, \tag{5.46}$$

summed over all partitions of the set $\{1,\dots,p\}$ into the disjoint union of two subsets $\{i_1,\dots,i_k\}$ and $\{j_1,\dots,j_l\}$ so that $p=k+l$ with $k,l\geqslant 0$ while $i_1<\cdots<i_k$ and $j_1<\cdots<j_l$.

By Lemma 5.18, we may regard Λ_n as an $\mathrm{X}(\mathfrak{o}_{2n+1})$-module. Furthermore, using the coproduct (2.24) and the automorphism (2.22), we can equip $\Lambda_n\otimes\Lambda_n$ with an $\mathrm{X}(\mathfrak{o}_{2n+1})$-action by

$$t_{ij}(u)(\eta\otimes\zeta)=\sum_{k=-n}^{n}\Big(\delta_{ik}+F_{ik}\,(u-a)^{-1}\Big)\,\eta\otimes\Big(\delta_{kj}+F_{kj}\,u^{-1}\Big)\,\zeta, \tag{5.47}$$

where $\eta,\zeta\in\Lambda_n$ and $a\in\mathbb{C}$ is a fixed constant.

Proposition 5.21. *If $a=p-1/2$ then the vector $v_p\in\Lambda_n\otimes\Lambda_n$ has the properties*

$$t_{ij}(u)\,v_p=0\qquad\text{for}\quad -n\leqslant i<j\leqslant n \tag{5.48}$$

and

$$t_{ii}(u)\,v_p=\begin{cases}\dfrac{(u-p)(u+1/2)}{u\,(u-p+1/2)}\,v_p & \text{for}\quad 0\leqslant i\leqslant p,\\[2mm] \dfrac{(u-p)(u-1/2)}{u\,(u-p+1/2)}\,v_p & \text{for}\quad p+1\leqslant i\leqslant n.\end{cases} \tag{5.49}$$

Proof. By the definition (5.47), we have

$$t^{(1)}_{ij}\,(\eta\otimes\zeta)=F_{ij}\,\eta\otimes\zeta+\eta\otimes F_{ij}\,\zeta$$

and

$$t_{ij}^{(r)}\,(\eta\otimes\zeta)=a^{r-2}\sum_{k=-n}^{n}F_{ik}\,\eta\otimes F_{kj}\zeta+a^{r-1}\,F_{ij}\,\eta\otimes\zeta \tag{5.50}$$

for $r\geqslant 2$. In particular,

$$t_{ij}^{(r+1)}\,(\eta\otimes\zeta)=a\,t_{ij}^{(r)}\,(\eta\otimes\zeta) \tag{5.51}$$

for any $r\geqslant 2$. Since v_p is the highest weight vector in the $\mathfrak{o}_{2n+1}$-module $V(\mu^{(p)})$, we have the relations $t_{ij}^{(1)}\,v_p=0$ for $-n\leqslant i<j\leqslant n$ and

$$t_{ii}^{(1)}\,v_p=\begin{cases}0 & \text{for}\quad 0\leqslant i\leqslant p,\\ -v_p & \text{for}\quad p+1\leqslant i\leqslant n.\end{cases}$$

Now, (5.3) implies that

$$[F_{i-1,i},t_{i,i}^{(2)}]=t_{i-1,i}^{(2)},\qquad i=1,\dots,n.$$

Furthermore, taking the $(i-1,i)$ entry in (2.26) and comparing the coefficients at u^{-2} we get

$$t_{i-1,i}^{(2)}-\sum_{k=-n}^{n}t_{i-1,k}^{(1)}\,t_{k,i}^{(1)}+t_{-i,-i+1}^{(2)}-\kappa\,t_{-i,-i+1}^{(1)}=0.$$

Hence, (5.48) will follow if we prove that v_p is an eigenvector for all the operators $t_{ii}^{(2)}$ with $i=1,\dots,n$. By (5.50), we have the following equality of operators in $\Lambda_n\otimes\Lambda_n$,

$$t_{ii}^{(2)}=\sum_{k=-n}^{n}(F_{ik}\otimes 1)(F_{ki}\otimes 1+1\otimes F_{ki})-(F^2)_{ii}\otimes 1+a\,F_{ii}\otimes 1.$$

Note that each element $F_{ki}\in\mathfrak{o}_{2n+1}$ acts on $\Lambda_n\otimes\Lambda_n$ as the operator

$$\Delta(F_{ki})=F_{ki}\otimes 1+1\otimes F_{ki}.$$

Due to (5.44), in the spinor representation Λ_n we have $(F^2)_{ii}=n/2+(n-1/2)\,F_{ii}$. Moreover, we have $\Delta(F_{ki})\,v_p=0$ for $k<i$ and for $1\leqslant i<k\leqslant p$. The latter follows from the fact that each vector $\Delta(F_{k,k-1})\,v_p$ with $k\in\{2,\dots,p\}$ is annihilated by all operators $\Delta(F_{j,j+1})$ and hence must be zero, as the $\mathfrak{o}_{2n+1}$-module $V(\mu^{(p)})$ is irreducible. Recalling that $a=p-1/2$ we thus get for any $i\in\{1,\dots,p\}$,

$$t_{ii}^{(2)}\,v_p=\sum_{k=p+1}^{n}(F_{ik}\otimes 1)\,\Delta(F_{ki})\,v_p+(p-n)\,(F_{ii}\otimes 1)\,v_p-n/2\,v_p.$$

Using the expression (5.46) for v_p and the formulas (5.42) it is now easy to derive the relation $t_{ii}^{(2)}\,v_p=-p/2\cdot v_p$. If $i\in\{p+1,\dots,n\}$ then

$$t_{ii}^{(2)}\,v_p=\sum_{k=i}^{n}(F_{ik}\otimes 1)\,\Delta(F_{ki})\,v_p+(p-n)\,(F_{ii}\otimes 1)\,v_p-n/2\,v_p.$$

Using again (5.46) and (5.42), we find that $\Delta(F_{ki})\, v_p = 0$ for $k > i$ which gives $t^{(2)}_{ii}\, v_p = (-p/2 + 1/2)\, v_p$. Thus, (5.48) is proved. For any $i > 0$ the relation (5.49) is now implied by (5.51) with $j = i$. Finally, we have $t^{(2)}_{00}\, v_p = -p/2 \cdot v_p$ which is verified by a similar calculation. This implies (5.49) for $i = 0$. □

Due to Proposition 5.21, the cyclic span $W_p = \mathrm{X}(\mathfrak{o}_{2n+1})\, v_p$ of the highest vector $v_p \in \Lambda_n \otimes \Lambda_n$ is a highest weight module over $\mathrm{X}(\mathfrak{o}_{2n+1})$. By the following theorem, W_p is irreducible. This module is finite-dimensional, and if $1 \leqslant p \leqslant n-1$ then the corresponding n-tuple of Drinfeld polynomials is $(1, \dots, u-1/2, 1, \dots, 1)$ with $u - 1/2$ on the $(p+1)$-th position; see Theorem 5.16. So, this yields a construction of the fundamental representations of $\mathrm{X}(\mathfrak{o}_{2n+1})$ alternative to the one used in the proof of Theorem 5.16. The following is a version of a result of Chari and Pressley [6, Theorem 6.2] and earlier results of Ogievetsky, Reshetikhin and Wiegmann [22]. We assume that $1 \leqslant p \leqslant n-1$ and $a = p - 1/2$.

Theorem 5.22. *The* $\mathrm{X}(\mathfrak{o}_{2n+1})$*-module* W_p *is irreducible. Its restriction to the universal enveloping algebra* $\mathrm{U}(\mathfrak{o}_{2n+1})$ *is given by*

$$W_p|_{\mathrm{U}(\mathfrak{o}_{2n+1})} \cong \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}).$$

Proof. By Corollary 3.10 and Proposition 5.21 the vector space W_p is spanned by the elements

$$t^{(r_1)}_{j_1 i_1} \dots t^{(r_m)}_{j_m i_m}\, v_p, \qquad m \geqslant 0,$$

with $j_a > i_a$ and $r_a \geqslant 1$. By (5.3), the $\mathfrak{o}_{2n+1}$-weights of W_p have the form $\mu^{(p)} - \omega$, where ω is a $\mathbb{Z}_+$-linear combination of the positive roots; see their description in the beginning of Section 5.1. However, any $\mathbb{Z}_+$-linear combination of the positive roots has the form $k_1\,\varepsilon_1 + \dots + k_n\,\varepsilon_n$, where the k_i are integers and the sum $k_1 + \dots + k_n$ is a non-positive integer. Since $\mu^{(p)} - \mu^{(l)} = \varepsilon_{l+1} + \dots + \varepsilon_p$ for $l < p$, we conclude that, as an $\mathfrak{o}_{2n+1}$-module,

$$W_p \subseteq \bigoplus_{s=p}^{n} V(\mu^{(s)}). \tag{5.52}$$

We shall now demonstrate that none of the irreducible $\mathfrak{o}_{2n+1}$-modules of the form $V(\mu^{(s)})$ with $s = p+1, p+3, \dots$ can occur in the irreducible decomposition of W_p. We need the following lemma which holds for any value of the parameter a.

Lemma 5.23. *For any* $s \in \{2, \dots, n\}$ *in the* $\mathrm{X}(\mathfrak{o}_{2n+1})$*-module* $\Lambda_n \otimes \Lambda_n$ *we have*

$$t^{(2)}_{-s+1,s}\, v_s = (a - s + 1/2)\, v_{s-2}.$$

Proof. By (5.50), we have

$$t^{(2)}_{-s+1,s} = \sum_{k=-n}^{n} (F_{-s+1,k} \otimes 1)\, \Delta(F_{ks}) - (F^2)_{-s+1,s} \otimes 1 + a\, F_{-s+1,s} \otimes 1.$$

Furthermore, (5.44) implies $(F^2)_{-s+1,s} = (n-1/2)\,F_{-s+1,s}$. Moreover, in the $\mathfrak{o}_{2n+1}$-submodule $V(\mu^{(s)})$ of $\Lambda_n \otimes \Lambda_n$ we have $\Delta(F_{ks})\,v_s = 0$ for $k \leqslant s$. Hence, applying (5.42) we obtain

$$t^{(2)}_{-s+1,s}\,v_s = \sum_{k=s+1}^{n} (\partial_k\,\partial_s \otimes 1)(\xi_k\,\partial_s \otimes 1 + 1 \otimes \xi_k\,\partial_s)\,v_s + (a-n+1/2)\,(\partial_s\,\partial_{s-1} \otimes 1)\,v_s.$$

Finally, using the formula (5.46) for v_s we come to

$$t^{(2)}_{-s+1,s}\,v_s = (a-s+1/2)\,(\partial_s\,\partial_{s-1} \otimes 1)\,v_s = (a-s+1/2)\,v_{s-2}. \qquad \square$$

Now, if the irreducible module $V(\mu^{(s)})$ with $s = p+2i-1$ for some $i \geqslant 1$ occurs in the irreducible decomposition of W_p then W_p would also contain $V(\mu^{(p-1)})$ by Lemma 5.23. But this contradicts (5.52). Thus, as an $\mathfrak{o}_{2n+1}$-module,

$$W_p \subseteq \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}). \tag{5.53}$$

We now need the following counterpart of Lemma 5.23.

Lemma 5.24. *Let $s \in \{2,\dots,n\}$. If $a \neq -s+1/2$ then the projection of the vector $t^{(2)}_{s,-s+1}\,v_{s-2} \in \Lambda_n \otimes \Lambda_n$ onto the component $V(\mu^{(s)})$ in the decomposition* (5.45) *is nonzero.*

Proof. Let us introduce a bilinear form on the vector space Λ_n by

$$\langle \xi_{i_1} \cdots \xi_{i_k}, \xi_{j_1} \cdots \xi_{j_l} \rangle = \delta_{IJ},$$

where $I = \{i_1,\dots,i_k\}$ and $J = \{j_1,\dots,j_l\}$ are subsets of $\{1,\dots,n\}$ such that $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$, with $\delta_{IJ} = 1$ if $I = J$, and 0 otherwise. The form possesses the covariance property with respect to the action of $\mathfrak{o}_{2n+1}$,

$$\langle F_{ij}\,\eta,\,\zeta \rangle = \langle \eta,\,F_{ji}\,\zeta \rangle, \qquad \eta,\zeta \in \Lambda_n.$$

Extend the form $\langle\ ,\ \rangle$ to a bilinear form on the tensor product space $\Lambda_n \otimes \Lambda_n$ by

$$\langle \eta_1 \otimes \eta_2, \zeta_1 \otimes \zeta_2 \rangle = \langle \eta_1, \zeta_2 \rangle \langle \eta_2, \zeta_1 \rangle.$$

One easily verifies that this form inherits the covariance property. In particular, the irreducible components $V(\mu^{(s)})$ in the decomposition (5.45) are pairwise orthogonal. So the lemma will follow if we prove that $\langle t^{(2)}_{s,-s+1}\,v_{s-2}, v_s \rangle \neq 0$. However, a direct calculation with the use of (5.50) shows that for any $\eta,\zeta \in \Lambda_n \otimes \Lambda_n$ we have

$$\langle t^{(2)}_{ij}\,\eta,\,\zeta \rangle = \langle \eta,\,\big(t^{(2)}_{ji} + a\,(1 \otimes F_{ji} - F_{ji} \otimes 1)\big)\,\zeta \rangle.$$

Hence, using Lemma 5.23 and the formulas (5.42) we find that

$$\begin{aligned} \langle t^{(2)}_{s,-s+1}\,v_{s-2}, v_s \rangle &= \langle v_{s-2},\,\big(t^{(2)}_{-s+1,s} + a\,(1 \otimes F_{-s+1,s} - F_{-s+1,s} \otimes 1)\big)\,v_s \rangle \\ &= (-a-s+1/2)\,\langle v_{s-2},\,v_{s-2} \rangle \neq 0, \end{aligned}$$

completing the proof of the lemma. $\square$

If $a = p - 1/2$ then the condition of Lemma 5.24 is satisfied for any $s \in \{2, \dots, n\}$. Thus, Lemmas 5.23 and 5.24 imply that the $\mathrm{X}(\mathfrak{o}_{2n+1})$-module W_p is irreducible and its $\mathfrak{o}_{2n+1}$-irreducible decomposition coincides with the right-hand side of (5.53). □

Consider now the case $\mathfrak{a} = \mathfrak{o}_{2n}$. As we mentioned in the previous section, the restriction of the $\mathfrak{o}_{2n+1}$-module Λ_n to the subalgebra $\mathfrak{o}_{2n}$ splits into the direct sum of two irreducible submodules, $\Lambda_n = \Lambda_n^+ \oplus \Lambda_n^-$, and we have the isomorphisms (5.43). We have the following tensor product decompositions of the $\mathfrak{o}_{2n}$-modules:

$$\Lambda_n^+ \otimes \Lambda_n^+ \cong \bigoplus_{r=0}^{[n/2]} V(\mu^{(2r)}), \tag{5.54}$$

$$\Lambda_n^+ \otimes \Lambda_n^- \cong \bigoplus_{r=0}^{[(n-1)/2]} V(\mu^{(2r+1)}), \tag{5.55}$$

where $\mu^{(p)} = (0, \dots, 0, -1, \dots, -1)$ with p zeros. Note that $V(\mu^{(p)})$ is a fundamental representation of $\mathfrak{o}_{2n}$ for any $2 \leqslant p \leqslant n - 1$. The highest weight vector v_p of $V(\mu^{(p)})$ in the decompositions (5.54) and (5.55) is given by (5.46) with the following additional restrictions: both k and l are even for (5.54) with $p = 2r$, while k is even and l is odd for (5.55) with $p = 2r + 1$.

By Lemma 5.18, we may regard Λ_n^+ and Λ_n^- as $\mathrm{X}(\mathfrak{o}_{2n})$-modules. As in the previous case, we equip the tensor products $\Lambda_n^+ \otimes \Lambda_n^+$ and $\Lambda_n^+ \otimes \Lambda_n^-$ with an $\mathrm{X}(\mathfrak{o}_{2n})$-action by

$$t_{ij}(u)(\eta \otimes \zeta) = \sum_{k=-n}^{n} \Big(\delta_{ik} + F_{ik}\,(u-a)^{-1}\Big)\,\eta \otimes \Big(\delta_{kj} + F_{kj}\,u^{-1}\Big)\,\zeta, \tag{5.56}$$

where $a \in \mathbb{C}$ is a fixed constant. In the following proposition we consider the cases of even and odd p simultaneously. If $p = 2r$ then $v_p \in \Lambda_n^+ \otimes \Lambda_n^+$ and if $p = 2r + 1$ then $v_p \in \Lambda_n^+ \otimes \Lambda_n^-$.

Proposition 5.25. *If $a = p - 1$ then the vector v_p has the properties*

$$t_{ij}(u)\, v_p = 0 \qquad \text{for} \quad -n \leqslant i < j \leqslant n \tag{5.57}$$

and

$$t_{ii}(u)\, v_p = \begin{cases} \dfrac{(u-p+1/2)(u+1/2)}{u\,(u-p+1)}\, v_p & \text{for} \quad -1 \leqslant i \leqslant p, \\[2ex] \dfrac{(u-p+1/2)(u-1/2)}{u\,(u-p+1)}\, v_p & \text{for} \quad p+1 \leqslant i \leqslant n. \end{cases} \tag{5.58}$$

Proof. The proof is essentially the same as for Proposition 5.21 with the use of the relation (5.44). The calculation of the eigenvalues of the operators $t_{ii}^{(2)}$ on v_p

gives

$$t_{ii}^{(2)}\, v_p = \begin{cases} (1/4 - p/2)\, v_p & \text{for} \quad -1 \leqslant i \leqslant p, \\ (3/4 - p/2)\, v_p & \text{for} \quad p+1 \leqslant i \leqslant n. \end{cases}$$

These imply the desired properties. □

The cyclic span $W_p = \mathrm{X}(\mathfrak{o}_{2n})\, v_p$ of the vector v_p is a highest weight module over $\mathrm{X}(\mathfrak{o}_{2n})$. By the following theorem, W_p is irreducible. This module is finite-dimensional, and if $2 \leqslant p \leqslant n-1$ then the corresponding n-tuple of Drinfeld polynomials is $(1, \ldots, u-1/2, 1, \ldots, 1)$ with $u-1/2$ on the $(p+1)$-th position; see Theorem 5.16. So, W_p is a fundamental module over $\mathrm{X}(\mathfrak{o}_{2n})$. The following is the $\mathfrak{o}_{2n}$-counterpart of Theorem 5.22. We assume that $2 \leqslant p \leqslant n-1$ and $a = p-1$.

Theorem 5.26. *The* $\mathrm{X}(\mathfrak{o}_{2n})$*-module* W_p *is irreducible. Its restriction to the universal enveloping algebra* $\mathrm{U}(\mathfrak{o}_{2n})$ *is given by*

$$W_p|_{\mathrm{U}(\mathfrak{o}_{2n})} \cong \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}).$$

Proof. Considering the $\mathfrak{o}_{2n}$-weights of W_p and using Corollary 3.10, we conclude that, as an $\mathfrak{o}_{2n}$-module,

$$W_p \subseteq \bigoplus_{i=0}^{[(n-p)/2]} V(\mu^{(p+2i)}). \tag{5.59}$$

The equality in (5.59) and irreducibility of the $\mathrm{X}(\mathfrak{o}_{2n})$-module W_p is implied by the following two lemmas which are verified in the same way as their $\mathfrak{o}_{2n+1}$-counterparts.

Lemma 5.27. *For any* $s \in \{2, \ldots, n\}$ *in the* $\mathrm{X}(\mathfrak{o}_{2n})$*-module* $\Lambda_n^+ \otimes \Lambda_n^+$ *or* $\Lambda_n^+ \otimes \Lambda_n^-$ *for even or odd* s*, respectively, we have*

$$t_{-s+1,s}^{(2)}\, v_s = (a - s + 1)\, v_{s-2}.$$ □

Lemma 5.28. *Let* $s \in \{2, \ldots, n\}$*. If* $a \neq -s+1$ *then the projection of the vector* $t_{s,-s+1}^{(2)}\, v_{s-2}$ *onto the component* $V(\mu^{(s)})$ *in the decomposition* (5.54) *or* (5.55)*, respectively, is nonzero.* □

In particular, if $a = p-1$ then the condition of Lemma 5.28 is satisfied for any $s \in \{2, \ldots, n\}$. This completes the proof of the theorem. □

We conclude by showing that each fundamental representation of the Lie algebra $\mathfrak{sp}_{2n}$ can be extended to the algebra $\mathrm{X}(\mathfrak{sp}_{2n})$ providing a fundamental representation of the latter. Due to Theorem 3.1, it suffices to prove the corresponding statement for the Yangian $\mathrm{Y}(\mathfrak{sp}_{2n})$. We follow the argument of [6] adopting it to the presentation of $\mathrm{Y}(\mathfrak{sp}_{2n})$ provided by Corollary 3.2. For any indices $k, l \in \{-n, \ldots, n\}$ introduce the elements $J_{kl} \in \mathrm{Y}(\mathfrak{sp}_{2n})$ by

$$J_{kl} = \tau_{kl}^{(2)} - \frac{1}{2} \sum_{i=-n}^{n} \tau_{ki}^{(1)} \tau_{il}^{(1)}.$$

We shall identify the universal enveloping algebra $\mathrm{U}(\mathfrak{sp}_{2n})$ with a subalgebra of $\mathrm{Y}(\mathfrak{sp}_{2n})$ via the embedding (3.21). Denote by $\mathcal{J}$ the subspace of $\mathrm{Y}(\mathfrak{sp}_{2n})$ spanned by all elements J_{kl}.

Lemma 5.29. *The subspace $\mathcal{J}$ is stable under the adjoint action of the Lie algebra $\mathfrak{sp}_{2n}$. Moreover, the $\mathfrak{sp}_{2n}$-module $\mathcal{J}$ is isomorphic to the adjoint representation.*

Proof. We easily derive from (5.3) that

$$[F_{ij}, J_{kl}] = \delta_{kj}\, J_{il} - \delta_{il}\, J_{kj} - \delta_{k,-i}\, \theta_{ij}\, J_{-j,l} + \delta_{l,-j}\, \theta_{ij}\, J_{k,-i}.$$

This proves the first claim. For the proof of the second, take the coefficient at u^{-2} in the relation (3.6). This gives

$$\tau_{kl}^{(2)} + \theta_{kl}\, \tau_{-l,-k}^{(2)} + \kappa\, \tau_{kl}^{(1)} - \sum_{i=-n}^{n} \tau_{ki}^{(1)} \tau_{il}^{(1)} = 0, \tag{5.60}$$

where we have used the relation $\tau_{kl}^{(1)} + \theta_{kl}\, \tau_{-l,-k}^{(1)} = 0$. Replacing k and l respectively by $-l$ and $-k$ in (5.60), then multiplying it by θ_{kl} and adding the result to (5.60) yields $J_{kl} + \theta_{kl}\, J_{-l,-k} = 0$. The argument is completed by observing that $\dim \mathcal{J} = \dim \mathfrak{sp}_{2n}$ by Corollary 3.7. □

The following lemma is straightforward from the defining relations of $\mathrm{Y}(\mathfrak{sp}_{2n})$ given in Corollary 3.2.

Lemma 5.30. *The algebra $\mathrm{Y}(\mathfrak{sp}_{2n})$ is generated by the elements F_{kl} and J_{kl} with $k, l \in \{-n, \ldots, n\}$.* □

The fundamental representations of $\mathfrak{sp}_{2n}$ are the modules $V(\mu^{(p)})$ where the highest weights have the form $\mu^{(p)} = (0, \ldots, 0, -1, \ldots, -1)$ with p zeros, for the values $p = 0, 1, \ldots, n-1$. In a more common notation, $V(\mu^{(p)})$ corresponds to the fundamental weight ω_{n-p}. Denote by $W_p(a)$ the fundamental representation of $\mathrm{Y}(\mathfrak{sp}_{2n})$ corresponding to the n-tuple of Drinfeld polynomials $(1, \ldots, u-a, 1, \ldots, 1)$ with $a \in \mathbb{C}$ and $u - a$ on the $(p+1)$-th position. By Theorem 5.16, the $\mathrm{Y}(\mathfrak{sp}_{2n})$-module $W_p(a)$ is isomorphic to the restriction of the $\mathrm{X}(\mathfrak{sp}_{2n})$-module $L(\lambda(u))$ to the subalgebra $\mathrm{Y}(\mathfrak{sp}_{2n})$, where the components of $\lambda(u)$ are given by

$$\lambda_i(u) = \begin{cases} \dfrac{u-a-p}{u-a-p-1} & \text{if} \quad -n \leqslant i \leqslant -p-1, \\ 1 & \text{if} \quad -p \leqslant i \leqslant p, \\ \dfrac{u-a}{u-a+1} & \text{if} \quad p+1 \leqslant i \leqslant n \end{cases}$$

for $p = 1, \ldots, n-1$, and

$$\lambda_i(u) = \begin{cases} \dfrac{u-a+1}{u-a} & \text{if} \quad -n \leqslant i \leqslant -1, \\ \dfrac{u-a+1}{u-a+2} & \text{if} \quad 1 \leqslant i \leqslant n \end{cases}$$

for $p = 0$. So, $W_p(a)$ may also be regarded as an $\mathrm{X}(\mathfrak{sp}_{2n})$-module. Recall that the universal enveloping algebra $\mathrm{U}(\mathfrak{sp}_{2n})$ is embedded into $\mathrm{X}(\mathfrak{sp}_{2n})$ via (3.22).

The following is essentially a reformulation of a particular case of [6, Theorem 6.1].

Theorem 5.31. *The restriction of $W_p(a)$ to $\mathrm{U}(\mathfrak{sp}_{2n})$ is isomorphic to the fundamental module $V(\mu^{(p)})$. Moreover, the action of $\mathrm{Y}(\mathfrak{sp}_{2n})$ on $V(\mu^{(p)})$ is determined by the assignment $J_{kl} \mapsto b\, F_{kl}$ with $b = -(n-p+1)/2 + a$.*

Proof. By Theorem 5.1, the $\mathrm{X}(\mathfrak{sp}_{2n})$-module $W_p(a)$ contains a unique, up to a constant factor, highest vector ξ. By the Poincaré–Birkhoff–Witt theorem for $\mathrm{X}(\mathfrak{sp}_{2n})$ and the relations (5.3), ξ is a unique weight vector of the weight $\mu^{(p)}$ in the $\mathfrak{sp}_{2n}$-module $W_p(a)$. Furthermore, the irreducible decomposition of this module takes the form

$$W_p(a) = V(\mu^{(p)}) \oplus \bigoplus_{\nu} c(\nu)\, V(\nu), \tag{5.61}$$

summed over the weights ν strictly preceding $\mu^{(p)}$ with respect to the standard partial ordering on the set of $\mathfrak{sp}_{2n}$-weights, where the $c(\nu)$ are some multiplicities. Consider the $\mathfrak{sp}_{2n}$-module homomorphism

$$\psi\colon \mathcal{J} \otimes V(\mu^{(p)}) \to W_p(a) \tag{5.62}$$

defined by

$$\psi\colon J_{kl} \otimes v \mapsto J_{kl}\, v, \qquad v \in V(\mu^{(p)}).$$

By Lemma 5.29, the $\mathfrak{sp}_{2n}$-module $\mathcal{J}$ is isomorphic to $V(\rho)$ with $\rho = (0, \dots, 0, -2)$. It is well known that the irreducible decomposition of $V(\rho) \otimes V(\mu^{(p)})$ contains $V(\mu^{(p)})$ with multiplicity one, and does not contain any modules $V(\nu)$ with ν strictly preceding $\mu^{(p)}$; see, e.g., [13]. Therefore, the homomorphism ψ must be multiplication by a scalar on the component $V(\mu^{(p)})$ and zero on the other irreducible constituencies of $V(\rho) \otimes V(\mu^{(p)})$. Then by Lemma 5.30, the subspace $V(\mu^{(p)})$ of $W_p(a)$ is stable under the action of $\mathrm{Y}(\mathfrak{sp}_{2n})$ and thus $W_p(a) = V(\mu^{(p)})$ since W_p is an irreducible $\mathrm{Y}(\mathfrak{sp}_{2n})$-module. This proves the first part of the theorem and shows that the action of the elements J_{kl} on $V(\mu^{(p)})$ is given by $J_{kl} \mapsto b\, F_{kl}$ for some $b \in \mathbb{C}$. By Lemma 5.30, this determines the action of $\mathrm{Y}(\mathfrak{sp}_{2n})$ on $V(\mu^{(p)})$. Finally, the exact value of b is found by calculating the eigenvalue of the operator J_{nn} on the highest vector ξ of $L(\lambda(u)) \cong W_p(a)$. This eigenvalue remains unchanged if we multiply all components of $\lambda(u)$ by the formal series $f(u) \in 1 + \mathbb{C}[[u^{-1}]]\, u^{-1}$ defined from the relation

$$f(u)\, f(u+\kappa)\, \lambda_{-n}(u+\kappa)\, \lambda_n(u) = 1.$$

In the case $1 \leqslant p \leqslant n-1$ we obtain

$$f(u) = 1 + (n-p)\, u^{-2} + \cdots .$$

By Proposition 5.2, we have $z(u) = 1$ in the $\mathrm{X}(\mathfrak{sp}_{2n})$-module $L\big(f(u)\lambda(u)\big)$ so that the eigenvalue of $\tau_{nn}(u)$ on the highest vector of $L\big(f(u)\lambda(u)\big)$ is $f(u)\,\lambda_n(u)$. This allows one to find the eigenvalue of $\tau_{nn}^{(2)}$ which turns out to be $(n-p)/2 - a + 1$. Since the eigenvalue of $\tau_{nn}^{(1)} = F_{nn}$ on the highest vector is -1, the eigenvalue of J_{nn} is $(n-p+1)/2 - a$ proving the claim for the case under consideration. In the case $p = 0$ the value of b is found by the same calculation. □

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