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Spectrum and Bethe Ansatz Equations for the $U_q(gl(\mathcal{N}))$ Closed and Open Spin Chains in any Representation

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Dedicated to our friend Daniel Arnaudon

Abstract. We consider the N-site $U_q(gl(\mathcal{N}))$ integrable spin chain with periodic and open diagonal soliton-preserving boundary conditions. By employing analytical Bethe ansatz techniques we are able to determine the spectrum and the corresponding Bethe ansatz equations for the general case, where each site of the spin chain is associated to any representation of $U_q(gl(\mathcal{N}))$.

In the case of open spin chain, we study finite dimensional representations of the quantum reflection algebra, and prove in full generality that the pseudo-vacuum is a highest weight of the monodromy matrix.

For these two types of spin chain, we study the (generalized) "algebraic" fusion procedures, which amount to construct the quantum contraction and the Sklyanin determinant for the $U_q(\widehat{gl}(\mathcal{N}))$ and quantum reflection algebras. We also determine the symmetry algebra of these two types of spin chains, including general K and K^+ diagonal matrices for the open case.

The case of open spin chains with soliton non-preserving boundary conditions is also presented in the framework of quantum twisted Yangians. The symmetry algebra of this spin chains is studied. We also give an exhaustive classification of the invertible matricial solutions to the corresponding reflection equation.

1. Introduction

For a long time, integrable systems, in particular spin chains models, have attracted much attention. The importance of these models relies on the fact that nonperturbative expressions of physical values (eigenstates, correlation functions,...) may

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be obtained exactly. Due to this property, numerous applications have been obtained in different domains of physics (condensed matter, string theory,...) as well as mathematics (quantum groups,...). Among the different approaches used to solve integrable problems, the Bethe ansatz [1] has been historically introduced to obtain eigenstates of the XXX model proposed by Heisenberg [2]. Then, various generalizations of this ansatz have been successfully constructed and applied. In this paper, we shall use a generalization of this method, called analytical Bethe ansatz [3], to find the spectrum of the periodic $\mathcal{U}_q(gl(\mathcal{N}))$ spin chain (XXZ model) where at each site the $\mathcal{U}_q(gl(\mathcal{N}))$ representation may be different. The construction of these models follows the same lines that the one done previously for the $gl(\mathcal{N})$ spin chain [4] and is based on the finite irreducible representations of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$. Then, to find the energy spectrum, we need to know a particular eigenvector (which is simply the highest weight of the chosen representation), to determine the symmetry and to obtain the explicit form of the $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ center. The knowledge of these algebraic data is sufficient to compute the full spectrum of the so-called transfer matrix (which gathers N Hamiltonians for the N sites spin chain).

More recently, the introduction of boundaries which preserve the integrability of well-known models has been also investigated [5, 6]. In the present context, the construction of the $\mathcal{U}_q(gl(\mathcal{N}))$ spin chain with non-periodic boundaries consists in studying the representations of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ subalgebras instead of the whole algebra. In particular, we study the quantum reflection algebra and the quantum twisted Yangian which are respectively associated to the soliton preserving (SP) and soliton non-preserving (SNP) boundaries. In the SP case, we determine the symmetry algebra and the spectrum of the corresponding transfer matrix, using the analytical Bethe ansatz. This case is very similar to the closed spin chains case. In the SNP case, we present the algebraic setup and classify the matricial solutions to the corresponding reflection equation. However, the absence of diagonal solution prevents us from completing the study, for one cannot find a reference state (pseudo-vacuum).

The plan of the paper is as follows. In Section 2, we introduce the algebraic structures which will be needed for the study of spin chains models. They consist in the quantum affine algebra (Section 2.1) and the quantum reflection algebra (Section 2.2), both of them based on the $gl(\mathcal{N})$ algebra. We will also remind the irreducible finite-dimensional representations of the quantum affine algebra, and study those of the quantum reflection algebra. To our knowledge, this latter study is new. Then, in Section 3, we construct the spin chains associated to these algebraic structures: closed spin chains for quantum affine algebra (Section 3.1) and SP open spin chain for the quantum reflection algebra (Section 3.2). We compute their spectrum, through the use of the analytical Bethe ansatz. Finally, we introduce in Section 4 the framework needed for the study of SNP open spin chains. The algebraic setting consists in the quantum twisted Yangian (Section 4.1). SNP open spin chains are introduced in Section 4.2. We classify the matrices obeying the corresponding reflection equation. However, due to the absence of diagonal solution, the usual Bethe ansatz does not work: we argue on the complete treatment of these spin chains. Appendices are devoted to some properties of the R matrices involved in the study (Appendix A), as well as to the fusion procedures used for the analytical Bethe ansatz. At the algebraic level, these fusion procedures amount to construct the quantum contraction and the Sklyanin determinant for the $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ and quantum reflection algebras (Appendices B and C).

2. Algebraic structures

In this section, we describe the algebraic framework needed for the construction of the different spin chains which will be presented in the next section. Depending on the boundary conditions we will impose on the spin chain, two types of algebras will show up: the quantum affine algebra and the quantum reflection algebra. They both rely on the R matrix of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, and we have gathered the different properties of this matrix in Appendix A.

2.1. The quantum affine algebra $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$

2.1.1. Definitions. One starts from the exchange relations of finite dimensional $\mathcal{U}_q(gl(\mathcal{N}))$ algebra:

$$\begin{aligned} R_{12}\,\mathcal{L}_1^+\mathcal{L}_2^+ &= \mathcal{L}_2^+\mathcal{L}_1^+\,R_{12};\\ R_{12}\,\mathcal{L}_1^+\mathcal{L}_2^- &= \mathcal{L}_2^-\mathcal{L}_1^+R_{12};\\ R_{12}\,\mathcal{L}_1^-\mathcal{L}_2^- &= \mathcal{L}_2^-\mathcal{L}_1^-\,R_{12} \end{aligned} \tag{2.1}$$

where R_{12} is given by (A.1) and $\mathcal{L}^+$, $\mathcal{L}^-$ are upper (lower) triangular matrices defining $\mathcal{U}_q(gl(\mathcal{N}))$, i.e.,

$$\mathcal{L}^+ = \sum_{1\le a\le b\le\mathcal{N}} E_{ab}\otimes\ell^+_{ab}, \quad \mathcal{L}^- = \sum_{1\le b\le a\le\mathcal{N}} E_{ab}\otimes\ell^-_{ab} \quad \text{with } \ell^+_{ab},\ \ell^-_{ab}\in\mathcal{U}_q(gl(\mathcal{N}))\,. \tag{2.2}$$

There are supplementary relations between the diagonal elements, namely

$$\ell^+_{aa}\ell^-_{aa} = \ell^-_{aa}\ell^+_{aa} = 1\,. \tag{2.3}$$

The algebra $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ (noted for brevity $\widehat{\mathcal{U}}_q$) is defined by the following fundamental relations, known as FRT relations [7, 8]

$$R_{12}(\frac{z}{w})\ \mathcal{L}_1^\pm(z)\ \mathcal{L}_2^\pm(w) = \mathcal{L}_2^\pm(w)\ \mathcal{L}_1^\pm(z)\ R_{12}(\frac{z}{w}), \tag{2.4}$$

$$R_{12}(\frac{z}{w}\,q^c)\ \mathcal{L}_1^+(z)\ \mathcal{L}_2^-(w) = \mathcal{L}_2^-(w)\ \mathcal{L}_1^+(z)\ R_{12}(\frac{z}{w}\,q^{-c}), \tag{2.5}$$

where, as usual in auxiliary spaces formalism, $\mathcal{L}_1^\pm(z) = \mathcal{L}^\pm(z)\otimes\mathbb{I}_\mathcal{N}\in\text{End}(\mathbb{C}^\mathcal{N})\otimes\text{End}(\mathbb{C}^\mathcal{N})\otimes\widehat{\mathcal{U}}_q$, $\mathcal{L}_2^\pm(z) = \mathbb{I}_\mathcal{N}\otimes\mathcal{L}^\pm(z)\in\text{End}(\mathbb{C}^\mathcal{N})\otimes\text{End}(\mathbb{C}^\mathcal{N})\otimes\widehat{\mathcal{U}}_q$ and $R_{12}(\frac{z}{w})\in\text{End}(\mathbb{C}^\mathcal{N})\otimes\text{End}(\mathbb{C}^\mathcal{N})$ is given by (A.27). The space $\text{End}(\mathbb{C}^\mathcal{N})$ is known as the

auxiliary space and c is the central charge (which will be set to zero below).
$\mathcal{L}^{\pm}(z) \in \mathrm{End}(\mathbb{C}^{\mathcal{N}}) \otimes \widehat{\mathcal{U}}_q$ gathers the generators $L_{ab}^{(\pm n)}$ as

$$\mathcal{L}^{\pm}(z) = \sum_{n=0}^{+\infty} \sum_{a,b=1}^{\mathcal{N}} z^{\pm 2n}\, E_{ab} \otimes L_{ab}^{(\pm n)} = \sum_{a,b=1}^{\mathcal{N}} E_{ab} \otimes L_{ab}^{\pm}(z), \tag{2.6}$$

with the constraints

$$L_{ab}^{(+0)} = L_{ba}^{(-0)} = 0\,, \qquad 1 \le a < b \le \mathcal{N} \tag{2.7}$$

$$L_{aa}^{(+0)}\, L_{aa}^{(-0)} = L_{aa}^{(-0)}\, L_{aa}^{(+0)} = 1\,, \qquad a = 1, \dots, \mathcal{N}. \tag{2.8}$$

Note that, due to the convention we take for the R matrix, the $\mathcal{L}^{\pm}(z)$ operators are even in z, so that the series expansion involves even integers only.

Remark also that the "level zero" generators $L_{ab}^{(+0)}$ and $L_{ab}^{(-0)}$ span a finite dimensional algebra which is nothing but the $\mathcal{U}_q(gl(\mathcal{N}))$ algebra: in the following, we will identify $L_{ab}^{(\pm 0)}$ with $\ell_{ab}^{\mp}$.

$\widehat{\mathcal{U}}_q$ is endowed with a coproduct $\Delta\colon \widehat{\mathcal{U}}_q \to \widehat{\mathcal{U}}_q \otimes \widehat{\mathcal{U}}_q$

$$\Delta(\mathcal{L}^{\pm}(z)) = \mathcal{L}_{01}^{\pm}(zq^{\pm \frac{c_2}{2}})\, \mathcal{L}_{02}^{\pm}(zq^{\mp \frac{c_1}{2}}), \tag{2.9}$$

$$\Delta(c) = c_1 + c_2 \tag{2.10}$$

where 0 denotes the auxiliary space, and $1, 2$ label the copies of $\widehat{\mathcal{U}}_q$ in which act the operators. For instance, $\mathcal{L}_{02}^{\pm}(z)$ acts trivially in the first copy of $\widehat{\mathcal{U}}_q$, while $c_1 = c \otimes 1$ and $c_2 = 1 \otimes c$.

When $c = 0$ (which will be always the case in Section 3), the coproduct reduces to

$$\Delta(L_{ab}^{\pm}(z)) = \sum_{c=1}^{\mathcal{N}} L_{ac}^{\pm}(z) \otimes L_{cb}^{\pm}(z), \quad a, b \in \{1, \dots, \mathcal{N}\}. \tag{2.11}$$

More generally, one defines recursively for $N \ge 2$

$$\Delta^{(N+1)} = (id^{\otimes(N-1)} \otimes \Delta) \circ \Delta^{(N)}\colon \widehat{\mathcal{U}}_q \to \widehat{\mathcal{U}}_q^{\otimes(N+1)} \tag{2.12}$$

with $\Delta^{(2)} = \Delta$ and $\Delta^{(1)} = id$. The map $\Delta^{(N)}$ is also a morphism, i.e.,

$$\mathcal{T}_0^{\pm}(z) = \Delta^{(N)}\left(\mathcal{L}^{\pm}(z)\right) = \mathcal{L}_{01}^{\pm}(z)\ \mathcal{L}_{02}^{\pm}(z) \dots \mathcal{L}_{0\ N-1}^{\pm}(z)\ \mathcal{L}_{0N}^{\pm}(z) \tag{2.13}$$

also obey the relations (2.4)–(2.5) and the constraints (2.7)–(2.8). As usual, the indices $i \in \{1, \dots, N\}$, here labeling copies of $\widehat{\mathcal{U}}_q$, are suppressed from $\mathcal{T}$, and we only keep the index 0 corresponding to the 'auxiliary space'.

As we shall see below, in the context of spin chains, $\mathcal{T}^{\pm}(z)$ will be seen (after being represented) as monodromy matrices. For the moment, we remark that this construction is valid at the algebra level, i.e., before any choice of representations, a property which justifies the name of universal (or algebraic) monodromy matrices for $\mathcal{T}^{\pm}(z)$. Correspondingly, we introduce two universal transfer matrices

$$t^{\pm}(z) = \mathrm{tr}_0\, \mathcal{T}_0^{\pm}(z) \tag{2.14}$$

which are elements of $\widehat{\mathcal{U}}_q^{\otimes N}$. It can be then shown via (2.4)–(2.5) that [9]

$$\Big[t^{\varepsilon}(z),\ t^{\varepsilon'}(w)\Big] = 0\,, \qquad \varepsilon, \varepsilon' \in \{+,-\}. \tag{2.15}$$

We remark that we have *two* transfer matrices: in fact, we will see below that the construction of a spin chain needs only $\mathcal{T}^+(z)$ (hence only $t^+(z)$), because the construction based on $\mathcal{T}^-(z)$ (and $t^-(z)$) is equivalent.

2.1.2. Symmetry of the transfer matrix. The relations (2.4)–(2.5), using the form (A.28) of the R matrix, can be rewritten as (when $c = 0$):

$$\begin{aligned}&[\mathcal{T}_1(z)\,,\,\mathcal{T}_2(w)] = \\ &\Big\{\mathcal{T}_2^q(z)\mathcal{T}_1^{\bar q}(w) - \mathcal{T}_2(w)\mathcal{T}_1(z)\Big\}P_{12}^q + \frac{\mathfrak{a}(\frac{z}{w})}{\mathfrak{b}(\frac{z}{w})}\Big\{\mathcal{T}_2(w)\mathcal{T}_1(z) - \mathcal{T}_2(z)\mathcal{T}_1(w)\Big\}\mathcal{P}\end{aligned} \tag{2.16}$$

where $\mathcal{T}(x)$ stands for $\mathcal{T}^+(x)$ or $\mathcal{T}^-(x)$, $\mathcal{T}^q(z)$ for its q-deformed operator defined in relation (A.20) and $\bar q = q^{-1}$.

Taking the trace in the auxiliary space 1 leads to

$$[t(z)\,,\,\mathcal{T}(w)] = [\mathcal{T}^q(z) - \frac{\mathfrak{a}(\frac{z}{w})}{\mathfrak{b}(\frac{z}{w})}\,\mathcal{T}(z)\,,\,\mathcal{T}(w)]\,. \tag{2.17}$$

Then, using the expansions

$$\mathcal{T}^{\pm}(w) = \mathcal{L}^{\mp} + o(w^{\pm 2}) \qquad , \quad \frac{\mathfrak{a}(\frac{z}{w})}{\mathfrak{b}(\frac{z}{w})} = q^{\pm 1} + o(w^{\pm 2}) \tag{2.18}$$

one can compute the action of the $\mathcal{U}_q(gl(\mathcal{N}))$ generators on the transfer matrix:

$$[t(z)\,,\,\mathcal{L}^{\pm}] = [\mathcal{T}^q(z) - q^{\mp 1}\mathcal{T}(z)\,,\,\mathcal{L}^{\pm}]\,. \tag{2.19}$$

In particular, considering the diagonal terms, using the fact that $\mathcal{L}^{\pm}$ is triangular and the commutation relations $[\ell_{ii}^{\pm}\,,\,\mathcal{T}_{jj}(z)] = 0$, one gets the following result:

Property 2.1. *All the Cartan generators of the finite Lie algebra $\mathcal{U}_q(gl(\mathcal{N}))$ commutes with the universal transfer matrix,*

$$[\ell_{ii}^{\pm}\,,\,t(z)] = 0\,, \quad i = 1,\ldots,\mathcal{N}\,. \tag{2.20}$$

Thus, they generate a $U(1)^{\mathcal{N}}$ symmetry algebra for the closed spin chains.

2.1.3. Representations and evaluation map. Since the spin chain interpretation will be possible through the use of representations of the algebra we consider, we now describe them. The irreducible finite dimensional representations of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ are built from those of $\mathcal{U}_q(gl(\mathcal{N}))$ using the evaluation morphism.

It is easy to show that the (evaluation) map

$$\mathcal{L}^+(z) \mapsto \mathcal{L}^- - z^2\mathcal{L}^+ \tag{2.21}$$

$$\mathcal{L}^-(z) \mapsto \mathcal{L}^+ - z^{-2}\mathcal{L}^- \tag{2.22}$$

defines a homomorphism from $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ (at $c=0$) to $\mathcal{U}_q(gl(\mathcal{N}))$. Note that $\mathcal{L}^+(z)$ and $\mathcal{L}^-(z)$ are represented in the same way (up to a multiplicative function of z). In fact, when $c=0$ (which is always the case for finite dimensional representations), they play the same role and we will rather work with

$$\mathcal{L}(z) = z\mathcal{L}^+ - z^{-1}\mathcal{L}^- \,. \tag{2.23}$$

Let us remark however that both for the mathematical framework, or for the analyticity properties used in Bethe ansatz, one should work with $\mathcal{L}^+(z)$ rather than $\mathcal{L}(z)$. Unfortunately, it is this latter notation which is used in spin chains context, so that we stick to it. We will come back to this point in Section 2.2.

The construction of finite dimensional irreducible representations of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ is rooted in this evaluation morphism. Thanks to this map, each representation of $\mathcal{U}_q(gl(\mathcal{N}))$ can be lifted to a representation of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, and indeed, it can be shown that all (up to twist by sign automorphisms) finite dimensional irreducible representations of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ can be constructed in this way (when q is neither a root of unity, nor zero), see for instance [10], Section 12.2.B. Since we will use this property in the next section to build the different spin chains based on $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, we describe this construction.

We start with a highest weight finite-dimensional irreducible representation of $\mathcal{U}_q(gl(\mathcal{N}))$, $M(\boldsymbol{\varpi})$, with highest weight $\boldsymbol{\varpi} = (\varpi_1,\dots,\varpi_{\mathcal{N}})$ and associated to the highest weight vector v. This highest weight vector obeys

$$\ell^-_{ab} v = 0 \,, \qquad 1 \le b < a \le \mathcal{N} \tag{2.24}$$

$$\ell^-_{aa} v = \eta_a\, q^{-\varpi_a} v \,, \qquad 1 \le a \le \mathcal{N} \tag{2.25}$$

where (when q is generic) $\varpi_1,\dots,\varpi_{\mathcal{N}}$ are real numbers with $\varpi_a - \varpi_{a+1} \in \mathbb{Z}_+$ and $\eta_a = \pm 1, \pm i$ (see [11] and Theorem 3.1 below). When q is a root of unity, the parameters $\varpi_1,\dots,\varpi_{\mathcal{N}}$ obey more general relations. In what follows, we will formally redefine ϖ_a in such way that $\eta_a = +$, and assume that the weights ϖ_a obey the conditions for the representation to be irreducible, finite dimensional and highest weight (including the cases where q is a root of unity). Then, the repetitive action of ℓ^+_{ab}, $1 \le a < b \le \mathcal{N}$, generates the other states of the representation.
Using the evaluation homomorphism (2.23), one infers from $M(\boldsymbol{\varpi})$ an irreducible finite dimensional highest weight representation $M_z(\boldsymbol{\varpi})$ for $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$:

$$L_{ab}(z) v = 0 \,, \qquad 1 \le b < a \le \mathcal{N} \tag{2.26}$$

$$L_{aa}(z) v = \left(z q^{\varpi_a} - z^{-1} q^{-\varpi_a}\right) v \,, \qquad 1 \le a \le \mathcal{N} \,. \tag{2.27}$$

More generally, using the morphism $\Delta^{(N)}$, one constructs the tensor product of such representations $M_{zq^{a_1}}(\boldsymbol{\varpi}^{(1)}) \otimes \cdots \otimes M_{zq^{a_N}}(\boldsymbol{\varpi}^{(N)})$, with so-called 'inhomogeneity' parameters a_n and highest weights $\boldsymbol{\varpi}^{(n)} = (\varpi^{(n)}_1,\dots,\varpi^{(n)}_{\mathcal{N}})$, $n=1,\dots,N$. This tensor product infers a representation for the universal monodromy matrices

$\mathcal{T}^{\pm}(z) = \Delta^{(N)}\mathcal{L}^{\pm}(z)$:

$$T_{ab}(z)\,\omega = 0\,, \qquad 1 \le b < a \le \mathcal{N} \tag{2.28}$$

$$T_{aa}(z)\,\omega = \prod_{n=1}^{N}\Big(zq^{a_n+\varpi_a^{(n)}} - z^{-1}q^{-a_n-\varpi_a^{(n)}}\Big)\,\omega \equiv P_a(z)\,\omega \tag{2.29}$$

where $\omega = \otimes_{i=1}^{N} v^{(i)}$, $v^{(i)}$ is the highest weight vector of $M(\boldsymbol{\varpi}^{(i)})$, $i = 1,\dots,N$. The functions $P_a(z)$ are related to the Drinfel'd polynomials $D_a(z)$ characterizing the representation in the usual way, namely

$$\frac{P_a(z)}{P_{a+1}(z)} = \frac{D_a(q\,z)}{D_a(z)}. \tag{2.30}$$

Being a highest weight vector of the monodromy matrix, ω is an eigenvector of the transfer matrix:

$$t(z)\,\omega = \Lambda^0(z)\,\omega \qquad \text{with} \quad \Lambda^0(z) = \sum_{a=1}^{\mathcal{N}} P_a(z)\,. \tag{2.31}$$

In a mathematical context, the quantum contraction or quantum determinant generate central elements of the algebra $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$. In the spin chain context, they allow one to obtain constraints on the eigenvalues of the problem. The different useful formulae are gathered in Appendices B and C. For a given representation, the quantum contraction and the quantum determinant of $\mathcal{T}(z)$ take respectively the simple forms

$$\delta(\mathcal{T}(z)) = 1 \qquad \text{and} \quad \mathrm{qdet}\,\mathcal{T}(z) = \prod_{a=1}^{\mathcal{N}} P_a(zq^{\mathcal{N}-a})\,. \tag{2.32}$$

To prove these formulae, we have used relations (B.4) and (C.7) applied on the highest weight vector ω, the statement that the quantum contraction and quantum determinant are proportional to the identity matrix (because they belong to the center of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$) and the convention that the length of the identity permutation is zero.

2.2. The quantum reflection algebra

2.2.1. Definitions. The quantum reflection algebra $\mathcal{R}$ is constructed as a coideal subalgebra of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$. This general construction can be done for "any" quantum group defined by an FRT relation of type (2.4) provided the R matrix obeys relation (2.35). In a physical context, this construction reflects the possibility, starting from a periodic (closed) spin chain, to build a new chain with an (integrable) boundary (open spin chain). We focus in this subsection on (some of) the mathematical aspects of the construction, postponing the physical contents to the next section.

As a first step, one defines

$$\mathcal{B}(z) = \mathcal{L}^+(zq^{-\frac{c}{2}})\,\mathcal{L}^-(z^{-1}q^{\frac{c}{2}})^{-1}, \tag{2.33}$$

where $\mathcal{L}^{\pm}(z)$ obey the relations (2.4)–(2.5) and the existence of $\mathcal{L}^{-}(z^{-1})^{-1}$, understood as a series expansion, is guaranteed by relations (2.8). Note that the forms (2.21) and (2.22) ensure that $\mathcal{B}(z)$ is analytical. This point has to be related to the remark made in Section 2.1.3 on analyticity conditions. However, for $c = 0$, one can loosely rewrite $\mathcal{B}(z)$ as $\mathcal{L}(z)\,\mathcal{L}(z^{-1})^{-1}$, with $\mathcal{L}(z)$ defined by (2.23).

It is not difficult to show that (the generating function) $\mathcal{B}(z)$ defines a subalgebra $\mathcal{R}$, called the (quantum) reflection algebra, whose exchange relations take the form

$$R_{12}(\frac{z}{w})\,\mathcal{B}_1(z)\ R_{21}(zw)\,\mathcal{B}_2(w) = \mathcal{B}_2(w)\ R_{12}(zw)\,\mathcal{B}_1(z)\ R_{21}(\frac{z}{w}). \tag{2.34}$$

To prove (2.34) starting from (2.33), one needs the supplementary constraint on R:

$$R_{12}(z)R_{21}(w) = R_{12}(w)R_{21}(z), \tag{2.35}$$

which is indeed satisfied by the R matrix of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$.

Due to the series expansions (2.6), $\mathcal{B}(z)$ can be expanded as

$$\mathcal{B}(z) = \sum_{n=0}^{\infty}\sum_{a,b=1}^{\mathcal{N}} z^{2n}\,E_{ab}\otimes B_{ab}^{(n)} = \sum_{a,b=1}^{\mathcal{N}} E_{ab}\otimes B_{ab}(z) = \sum_{n=0}^{\infty} z^{2n}\mathcal{B}^{(n)}\,. \tag{2.36}$$

The $\mathcal{R}$ subalgebra is a left coideal (see, e.g., [12]) of the starting algebra $\widehat{\mathcal{U}}_q = \mathcal{U}_q(\widehat{gl}(\mathcal{N}))$:

$$\Delta(\mathcal{B}(z)) = \mathcal{L}^{+}_{01}(zq^{-\frac{c_1}{2}})\,\mathcal{B}_{02}(zq^{-c_1})\,\mathcal{L}^{-}_{01}(q^{\frac{c_1}{2}}z^{-1})^{-1}\ \in\ \widehat{\mathcal{U}}_q\otimes\mathcal{R} \tag{2.37}$$

where we have used the index 0 for the auxiliary space, and $1,2$ for the two copies of $\widehat{\mathcal{U}}_q$. This formula is easily generalized for $\Delta^{(N)}$:

$$\begin{aligned}\Delta^{(N)}\mathcal{B}(z) &= \mathcal{L}^{+}_{01}(zq^{-\frac{c_1}{2}})\mathcal{L}^{+}_{02}(zq^{-c_1-\frac{c_2}{2}})\ldots\mathcal{L}^{+}_{0N}(zq^{-c_1-c_2-\ldots-c_{N-1}-\frac{c_N}{2}})\\ &\times\mathcal{L}^{-}_{0N}(q^{c_1+c_2+\ldots+c_{N-1}+\frac{c_N}{2}}z^{-1})^{-1}\ldots\mathcal{L}^{-}_{02}(q^{c_1+\frac{c_2}{2}}z^{-1})^{-1}\mathcal{L}^{-}_{01}(q^{\frac{c_1}{2}}z^{-1})^{-1}\,.\end{aligned} \tag{2.38}$$

A more general solution of (2.34) is then given by [6]:

$$\mathcal{B}(z) = \mathcal{L}^{+}(zq^{-\frac{c}{2}})\ K(z)\ \mathcal{L}^{-}(q^{\frac{c}{2}}z^{-1})^{-1}, \tag{2.39}$$

where $K(z)$ is a $\mathbb{C}$-valued matricial solution of the reflection equation,

$$R_{12}(\frac{z}{w})\ K_1(z)\ R_{21}(zw)\ K_2(w) = K_2(w)\ R_{12}(zw)\ K_1(z)\ R_{21}(\frac{z}{w}). \tag{2.40}$$

The relation (2.35) ensures that $K(z) = \mathbb{I}$ is a solution to this equation.

Although not always relevant for the mathematical structure of the subalgebra, the K matrix is pertinent at the physical level, and encodes the boundary which is present in the open spin chain. Indeed, from the mathematical point of view, one can replace $K(z)$ by $U(z)K(z)U(\frac{1}{z})^{-1}$, where $U(z)$ is any matricial solution of the relation $R_{12}(\frac{z}{w})\,U_1(z)\,U_2(w) = U_2(w)\,U_1(z)\,R_{12}(\frac{z}{w})$. It amounts to a change $\mathcal{B}(z)\to U(z)\widetilde{\mathcal{B}}(z)U(\frac{1}{z})^{-1}$, where $\widetilde{\mathcal{B}}(z)$ is built on $U(z)\mathcal{L}(z)U(z)^{-1}$ instead of $\mathcal{L}(z)$. All these changes being algebra automorphisms, the mathematical structure remains unaltered while the K matrix (i.e., the boundary) has changed.

The diagonal solutions have been classified in [13], they take the form

$$K(z;\xi) = \mathrm{diag}\Big(\underbrace{\alpha,\dots,\alpha}_{\mathcal{M}},\ \underbrace{\beta,\dots,\beta}_{\mathcal{N}-\mathcal{M}}\Big), \tag{2.41}$$

$$\alpha(z;\ \xi) = z^2(z^2-\xi^2), \qquad \beta(z;\ \xi) = 1-\xi^2 z^2, \tag{2.42}$$

where ξ is a free parameter characterizing the boundary, and the normalization has been chosen to fulfill the series expansion requirement.

The modes $\mathcal{B} = \mathcal{B}(0)$ generate a finite version of the reflection algebra, whose exchange relations read

$$R_{12}\,\mathcal{B}_1\,R_{12}^{-1}\,\mathcal{B}_2 = \mathcal{B}_2\,R_{21}^{-1}\,\mathcal{B}_1\,R_{21}. \tag{2.43}$$

Remark that, due to the presence of $K(z)$, the "zero mode" $\mathcal{B}$ does not generate a full $\mathcal{U}_q(gl(\mathcal{N}))$ algebra: we will come back on this point in Section 2.2.2.

Note that one could also consider

$$\mathcal{B}^{\pm}(z) = \mathcal{L}^{\pm}(z)K(z)\mathcal{L}^{\pm}(z^{-1})^{-1}\,, \tag{2.44}$$

whose commutation relations are also of the form (2.34), but do not lead to analytical entries for $\mathcal{B}^{\pm}(z)$. However, in the spin chain context, we will be interested only with finite dimensional representations, which have $c=0$, so that we will not distinguish the three types of embedding (we remind that $\mathcal{L}^+(z)$ and $\mathcal{L}^-(z)$ are represented in the same way in evaluation representations, see Section 2.1.3). From now on, we will set $c=0$ and drop the $\pm$ superscript. Nevertheless, we want to stress that only definitions of the form (2.33) lead to a coideal subalgebra when $c \neq 0$.

The (universal) monodromy matrix related to this algebra will be defined using the morphism $\Delta^{(N)}$:

$$\begin{aligned}\mathcal{B}^N(z) &\equiv \Delta^{(N)}\mathcal{B}(z)\\ &= \mathcal{L}_{01}(z)\mathcal{L}_{02}(z)\dots\mathcal{L}_{0N}(z)K_0(z)\mathcal{L}_{0N}(z^{-1})^{-1}\dots\mathcal{L}_{02}(z^{-1})^{-1}\mathcal{L}_{01}(z^{-1})^{-1}\\ &= \mathcal{T}_0(z)\,K_0(z)\,\mathcal{T}_0(z^{-1})^{-1}\,.\end{aligned} \tag{2.45}$$

The monodromy matrix $\Delta^{(N)}\mathcal{B}(z)$ also obeys the relation (2.34). Upon representation, this monodromy matrix will correspond to an open spin chain with N sites. Finally we introduce the (universal) transfer matrix of the open spin chain [6]:

$$b(z) = \mathrm{tr}_0\Big(K_0^+(z)\,\mathcal{B}^N(z)\Big), \tag{2.46}$$

where $K^+(z)$ is a solution of the dual reflection equation:

$$\begin{aligned}&R_{12}(\frac{w}{z})\ \Big(K_1^+(z)\Big)^{t_1}\ M_1^{-1}\ R_{21}(\rho^{-2}z^{-1}w^{-1})\ M_1\ \Big(K_2^+(w)\Big)^{t_2} =\\ &\Big(K_2^+(w)\Big)^{t_2}\ M_1\ R_{12}(\rho^{-2}z^{-1}w^{-1})\ M_1^{-1}\ \Big(K_1^+(z)\Big)^{t_1}\ R_{21}(\frac{w}{z}).\end{aligned} \tag{2.47}$$

We have introduced $\rho = q^{\frac{\mathcal{N}}{2}}$ and the matrix M, defined in (A.12).

Note that M is a solution of (2.47), which ensures that this construction can be performed. In fact, from any solution $K(z)$ to the reflection equation (2.34), one can generate a solution $K^+(z)$ to the dual equation (2.47) through the combination $K^+(z) = f(z)\Big(K((\rho z)^{-1})\Big)^t M$ where $f(z)$ is an arbitrary function. Hence, starting from a diagonal solution for $K(z)$, one can deduce a diagonal solution for $K^+(z)$:

$$K^+(z) = q^{\mathcal{N}+1}\,\mathrm{diag}\,\Big(\underbrace{q^{-2}\,\widetilde{\alpha}, q^{-4}\,\widetilde{\alpha}, \dots, q^{-2\mathcal{M}_+}\,\widetilde{\alpha}}_{\mathcal{M}_+},\ \underbrace{q^{-2\mathcal{M}_+-2}\,\widetilde{\beta}, \dots, q^{-2\mathcal{N}}\,\widetilde{\beta}}_{\mathcal{N}-\mathcal{M}_+}\Big), \tag{2.48}$$

$$\text{where}\quad \widetilde{\alpha}(z;\ \xi_+) = 1-(\rho\xi_+ z)^2, \qquad \widetilde{\beta}(z;\ \xi_+) = (\rho z)^2((\rho z)^2-\xi_+^2)\ . \tag{2.49}$$

The normalization has been chosen in such a way that $K^+(z)$ has analytical entries. Let us stress that, when considering a couple of diagonal solutions given by (2.41) and (2.48), the parameters ξ, $\mathcal{M}$ and ξ_+, $\mathcal{M}_+$ are not necessarily related.

Again, it can be proved using only the reflection equations (2.34), (2.47), and the properties of the R matrix, that [6]

$$\Big[b(z),\ b(w)\Big] = 0\ . \tag{2.50}$$

It will ensure that the open spin chain derived from (2.46) is also integrable.

2.2.2. Finite dimensional subalgebras of the reflection algebra. From the reflection algebra, one deduces (in the limit $w \to 0$)[1]

$$R_{12}\,\mathcal{B}_1(z)\,R_{12}^{-1}\,\mathcal{B}_2(0) = \mathcal{B}_2(0)\,R_{21}^{-1}\,\mathcal{B}_1(z)\,R_{21} \tag{2.51}$$

where the above R matrices are the finite ones of Appendix A.1. It proves that $\mathcal{B}(0)$ generates a subalgebra and that $\mathcal{B}(z)$ is one of its representation. To identify this subalgebra, one needs the expression

$$K(0) = \mathrm{diag}(\underbrace{0,\dots,0}_{\mathcal{M}},\underbrace{1,\dots,1}_{\mathcal{N}-\mathcal{M}})\,. \tag{2.52}$$

This expression shows that $\mathcal{B}(0)$ is constructed on the generators ℓ^+_{ij} and ℓ^-_{ij} with $i,j > \mathcal{M}$. It corresponds to the (finite dimensional version) of the reflection algebra based on $\mathcal{U}_q(gl(\mathcal{N}-\mathcal{M}))$. In fact, this algebra is known to be isomorphic to the $\mathcal{U}_q(gl(\mathcal{N}-\mathcal{M}))$ algebra itself. This is mainly due to the triangular form of $\ell^\pm$, so that $\mathcal{B}(0) = \ell^+\,(\ell^-)^{-1}$ is just a (invertible) triangular decomposition[2]. Thus, we conclude that $\mathcal{B}(0)$ generates the finite dimensional $\mathcal{U}_q(gl(\mathcal{N}-\mathcal{M}))$ algebra.

[1]Note that this limit is well defined only for the form (2.39), not for the alternative expressions (2.44).

[2]However, one has to allow square roots and inverses for some generators.

One can construct from $\mathcal{B}(z)$ another subalgebra in the following way. One introduces

$$\widehat{\mathcal{B}}(z) = \frac{1}{z^2}\mathcal{B}^{-1}(z^{-1})\,, \tag{2.53}$$

which satisfies:

$$R_{12}(\frac{z}{w})\,\mathcal{B}_1(z)\,R_{21}(zw)\,\widehat{\mathcal{B}}_2(w) = \widehat{\mathcal{B}}_2(w)\,R_{12}(zw)\,\mathcal{B}_1(z)\,R_{21}(\frac{z}{w}) \tag{2.54}$$

$$R_{12}(\frac{z}{w})\,\widehat{\mathcal{B}}_1(z)\,R_{21}(zw)\,\widehat{\mathcal{B}}_2(w) = \widehat{\mathcal{B}}_2(w)\,R_{12}(zw)\,\widehat{\mathcal{B}}_1(z)\,R_{21}(\frac{z}{w})\,. \tag{2.55}$$

$\widehat{\mathcal{B}}(z)$ admits a z^{-2} series expansion, so that we get from (2.55) when $w \to \infty$:

$$R_{21}^{-1}\,\widehat{\mathcal{B}}_1(z)\,R_{21}\,\widehat{\mathcal{B}}_2(\infty) = \widehat{\mathcal{B}}_2(\infty)\,R_{12}\,\widehat{\mathcal{B}}_1(z)\,R_{12}^{-1}\,. \tag{2.56}$$

Once again, it is the form of $\widetilde{K}(z) = \frac{1}{z^2}\,K(z^{-1})^{-1}$ which determines the subalgebra generated by $\widehat{\mathcal{B}}(\infty)$:

$$\widetilde{K}(\infty) = \mathrm{diag}(\underbrace{1,\ldots,1}_{\mathcal{M}},\underbrace{0,\ldots,0}_{\mathcal{N}-\mathcal{M}})\,. \tag{2.57}$$

This proves that $\widehat{\mathcal{B}}(\infty)$ generates a $\mathcal{U}_q(gl(\mathcal{M}))$ algebra based on the generators ℓ^+_{ij} and ℓ^-_{ij} with $i,j \le \mathcal{M}$.

Finally, since $\mathcal{B}(0)$ (resp. $\widehat{\mathcal{B}}(\infty)$) depends on ℓ^+_{ij} and ℓ^-_{ij} with $i,j > \mathcal{M}$ (resp. with $i,j \le \mathcal{M}$) only, we deduce:

Lemma 2.2. *When $K(z)$ is diagonal, $\mathcal{B}(0)$ (resp. $\widehat{\mathcal{B}}(\infty)$) generate a finite dimensional $\mathcal{U}_q(gl(\mathcal{N}-\mathcal{M}))$ (resp. $\mathcal{U}_q(gl(\mathcal{M}))$) subalgebra of the quantum reflection algebra.*
These two subalgebras commute one with each other, and thus are in direct sum in the quantum reflection algebra.

2.2.3. Symmetry of the transfer matrix. To identify the symmetry algebra, we need the following lemma:

Lemma 2.3. *For any (operator valued) matrix $\mathcal{B}(z)$, one has*

$$\mathrm{tr}_1\left(M_1\,R_{12}\,\mathcal{B}_1(z)\,R_{12}^{-1}\right) = \mathrm{tr}\,(M\,\mathcal{B}(z))\,\mathbb{I} = \mathrm{tr}_1\left(M_1\,R_{21}^{-1}\,\mathcal{B}_1(z)\,R_{21}\right)\,. \tag{2.58}$$

Proof. We first prove the first equality

$$\begin{aligned}&\mathrm{tr}_1\left(M_1\,R_{12}\,\mathcal{B}_1(z)\,R_{12}^{-1}\right)\\ &\quad = \mathrm{tr}_1\left(\left(M_1\,R_{12}\,\mathcal{B}_1(z)\right)^{t_1}\left(R_{12}^{-1}\right)^{t_1}\right) = \mathrm{tr}_1\left(\mathcal{B}_1^t(z)\,R_{12}^{t_1}\,M_1\left(R_{12}^{-1}\right)^{t_1}\right) \quad (2.59)\\ &\quad = \mathrm{tr}_1\left(\mathcal{B}_1^t(z)\,M_1\right) = \mathrm{tr}_1\left(M_1\,\mathcal{B}_1(z)\right) \quad (2.60)\end{aligned}$$

where we have used (A.11).
For the second equality, one uses (A.9) to write

$$\begin{aligned}&\mathrm{tr}_1\left(M_1\,R_{21}^{-1}\,\mathcal{B}_1(z)\,R_{21}\right)\\ &\quad = \mathrm{tr}_1\left(M_1\,V_1\,V_2\,R_{12}^{-1}\,V_1\,V_2\,\mathcal{B}_1(z)\,V_1\,V_2\,R_{12}\,V_1\,V_2\right) = \quad (2.61)\end{aligned}$$

$$= \mathrm{tr}_1 \left(\left(V_1^t\, V_2\, R_{12}^{-1}\, V_1\, V_2\, \mathcal{B}_1(z) \right)^{t_1} \left(V_1\, V_2\, R_{12}\, V_1\, V_2 \right)^{t_1} \right) \tag{2.62}$$

$$= \mathrm{tr}_1 \left(\mathcal{B}_1^t(z)\, V_1^t\, V_2 \left(R_{12}^{-1} \right)^{t_1} M_1^{-1}\, R_{12}^{t_1}\, V_1^t\, V_2 \right) \tag{2.63}$$

$$= \mathrm{tr}_1 \left(\mathcal{B}_1^t(z)\, V_1^t\, V_2\, M_1^{-1}\, V_1^t\, V_2 \right) = \mathrm{tr}_1 \left(\mathcal{B}_1^t(z)\, M_1 \right) = \mathrm{tr}_1 \left(M_1\, \mathcal{B}_1(z) \right) . \tag{2.64}$$

We used (A.11) in the last steps. □

Now, from the relations (2.51) and (2.54), one deduces

$$\mathrm{tr}_1 \left(M_1 R_{12}\, \mathcal{B}_1(z)\, R_{12}^{-1} \right) \mathcal{B}_2(0) = \mathcal{B}_2(0)\, \mathrm{tr}_1 \left(M_1 R_{21}^{-1}\, \mathcal{B}_1(z)\, R_{21} \right) \tag{2.65}$$

$$\mathrm{tr}_1 \left(M_1 R_{21}^{-1}\, \mathcal{B}_1(z)\, R_{21} \right) \widehat{\mathcal{B}}_2(\infty) = \widehat{\mathcal{B}}_2(\infty)\, \mathrm{tr}_1 \left(M_1 R_{12}\, \mathcal{B}_1(z)\, R_{12}^{-1} \right) . \tag{2.66}$$

Thus, one gets the following property:

Property 2.4. *When $K^+(z) = M$ and $K(z)$ is a diagonal matrix* (2.41)*, the transfer matrix $b(z)$ defined in* (2.46) *obeys*

$$[\mathcal{B}(0)\, , \, b(z)] = [\widehat{\mathcal{B}}(\infty)\, , \, b(z)] = 0\, . \tag{2.67}$$

Hence, the open spin chain admits a finite dimensional $\mathcal{U}_q(gl(\mathcal{N} - \mathcal{M})) \oplus \mathcal{U}_q(gl(\mathcal{M}))$ symmetry algebra.

This property is valid whatever the quantum spaces are, and generalizes the results obtained in [14, 15] in the case of fundamental representations.

When $K^+(z)$ is a more general diagonal matrix, the symmetry is reduced. To study this symmetry, we need the following lemma

Lemma 2.5. *For a general diagonal matrix $K^+(z) = \mathrm{diag}(k_1(z), \ldots, k_{\mathcal{N}}(z))$, and for any* (*operator valued*) *matrix $\mathcal{B}(z) = \sum_{a,b=1}^{\mathcal{N}} B_{ab}(w)\, E_{ab}$, one has:*

$$\mathrm{tr}_+(w) \equiv \mathrm{tr}_2 \left(K_2^+(w)\, R_{12}^{-1}\, \mathcal{B}_2(w)\, R_{12} \right) = b(w)\, \mathbb{I} + \sum_{1 \le a < b \le \mathcal{N}} x_{ab}(w)\, B_{ab}(w)\, E_{ab} \tag{2.68}$$

$$\mathrm{tr}_-(w) \equiv \mathrm{tr}_2 \left(K_2^+(w)\, R_{21}^{-1}\, \mathcal{B}_2(w)\, R_{21} \right) = b(w)\, \mathbb{I} + \sum_{1 \le a < b \le \mathcal{N}} x_{ab}(w)\, B_{ba}(w)\, E_{ba} \tag{2.69}$$

$$x_{ab}(w) = (q - q^{-1}) \Big(q^{-1}\, k_a - q\, k_b - (q - q^{-1}) \sum_{a<c<b} k_c \Big) \tag{2.70}$$

where $b(z) = \mathrm{tr} \left(K^+(z)\, \mathcal{B}(z) \right)$.

Proof. Direct calculation. □

Then, one has:

Property 2.6. *Let $K(z)$* (*$K_+(z)$ resp.*) *be a diagonal solution of* (2.34) (*of* (2.47) *resp.*) *with parameters $(\xi, \mathcal{M})$* (*parameters $(\xi_+, \mathcal{M}_+)$ resp.*)*. One has the following* *commutation relations:*

$$[B_{ij}(0)\,,\,b(w)] = 0\;, \qquad i,j \le \mathcal{M}_+ \tag{2.71}$$

$$[\widehat{B}_{ij}(\infty)\,,\,b(w)] = 0\;, \qquad i,j > \mathcal{M}_+ \tag{2.72}$$

Hence, for $\mathcal{M}_{\rm ax} = \max(\mathcal{M},\mathcal{M}_+)$ *and* $\mathcal{M}_{\rm in} = \min(\mathcal{M},\mathcal{M}_+)$*, the open spin chain transfer matrix admits a* $\mathcal{U}_q(gl(\mathcal{M}_{\rm in})) \oplus \mathcal{U}_q(gl(\mathcal{M}_{\rm ax}-\mathcal{M}_{\rm in})) \oplus \mathcal{U}_q(gl(\mathcal{N}-\mathcal{M}_{\rm ax}))$ *symmetry algebra.*

Proof. Starting with the reflection equation (2.34), taking the limit $z \to 0$, then multiplying by $K_2^+(w)$ and taking the trace in the space 2, one gets

$$\mathcal{B}(0)\,\mathrm{tr}_+(w) = \mathrm{tr}_-(w)\,\mathcal{B}(0)\,,$$

where we used the notation (2.68) and (2.69). The projection of this equation on E_{ij} gives

$$[B_{ij}(0)\,,\,b(w)] = \sum_{1\le a<j} x_{aj}(w)\,B_{ia}(0)\,B_{aj}(w) - \sum_{1\le a<i} x_{ai}(w)\,B_{ia}(w)\,B_{aj}(0) \tag{2.73}$$

where $x_{ab}(w)$ is defined in (2.70).

Now, since $K^+(z)$ is a diagonal solution (2.48), its diagonal terms obey

$$k_a(z) = q^{2(b-a)}k_b(z) \quad \text{when} \quad a,b \le \mathcal{M}_+ \;\text{ or }\; a,b > \mathcal{M}_+ . \tag{2.74}$$

Using this property, one is led to $x_{ab}(z) = 0$ when $a,b \le \mathcal{M}_+$ or when $a,b > \mathcal{M}_+$. Plugging this result in eq. (2.73) for $i,j \le \mathcal{M}_+$, one gets (2.71).

In the same way, starting with (2.54), taking the limit $w \to \infty$, then exchanging the spaces 1 and 2, one obtains

$$\widehat{\mathcal{B}}(\infty)\,\mathrm{tr}_-(w) = \mathrm{tr}_+(w)\,\widehat{\mathcal{B}}(\infty)\,,$$

which leads to

$$[\widehat{B}_{ij}(\infty)\,,\,b(w)] = \sum_{i<a\le\mathcal{N}} x_{ia}(w)\,B_{ia}(w)\,\widehat{B}_{aj}(\infty) - \sum_{j<a\le\mathcal{N}} x_{ja}(w)\,\widehat{B}_{ia}(\infty)\,B_{aj}(w)\;.$$

Hence, for $i,j > \mathcal{M}_+$, one gets (2.72).

Finally, one concludes using the results of Section 2.2.2 and considering the cases $i,j \le \mathcal{M}_{\rm in}$, $\mathcal{M}_{\rm in} < i,j \le \mathcal{M}_{\rm ax}$ or $\mathcal{M}_{\rm ax} < i,j$. □

The implementation of more general non-diagonal boundaries reduces even more the symmetry leading to a finite set of conserved quantities that commute with the open transfer matrix, that is the boundary quantum algebra [16, 17].

2.2.4. Finite dimensional representations. For the study of the representations of the reflection algebra, we follow essentially the lines given in [18] for the reflection algebra based on the Yangian of $gl(\mathcal{N})$, and in [19, 20, 21] for the twisted Yangian.

We introduce the quantum comatrix, defined by

$$\widetilde{\mathcal{T}}(z)\,\mathcal{T}(zq^{\mathcal{N}-1}) = \mathrm{qdet}\,\mathcal{T}(z)\,, \tag{2.75}$$

where the quantum determinant $\mathrm{qdet}\,\mathcal{T}(z)$, which generates the center of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, is defined in equation (C.5). The quantum comatrix is essentially the inverse of $\mathcal{T}(z)$, which motivates its use in the study of quantum reflection algebras. It can be shown that it takes the form

$$\begin{aligned}\widetilde{T}_{ij}(z) = &\sum_{\sigma\in S_{\mathcal{N}-1}} (-q)^{-\ell(\sigma)+j-i}\, T_{a_{\sigma(1)},1}(z)T_{a_{\sigma(2)},2}(zq)\cdots T_{a_{\sigma(i-1)},i-1}(zq^{i-2})\\ &\times T_{a_{\sigma(i)},i+1}(zq^{i-1})\cdots T_{a_{\sigma(\mathcal{N}-1)},\mathcal{N}}(zq^{\mathcal{N}-2})\end{aligned}$$

where $(a_1,a_2,\ldots,a_{\mathcal{N}-1}) = (1,\ldots,j-1,j+1,\ldots,\mathcal{N})$. Moreover, it obeys the following lemma, proved using the steps given in [18] for the reflection algebra (based on the Yangian of $gl(\mathcal{N})$) and in [21, 20] for the twisted Yangians:

Lemma 2.7. *If ω is a highest weight vector of $\mathcal{T}(z)$, with eigenvalue $(P_1(z),\ldots,P_{\mathcal{N}}(z))$, then it is also a highest weight vector for the comatrix, with*

$$\widetilde{T}_{ii}\,\omega = P_1(zq^{\mathcal{N}-2})\cdots P_{i-1}(zq^{\mathcal{N}-i})P_{i+1}(zq^{\mathcal{N}-i-1})\cdots P_{\mathcal{N}}(z)\,\omega \equiv \widetilde{P}_i(z)\,\omega\,. \tag{2.76}$$

Since the quantum comatrix $\widetilde{T}(z)$ and the inverse matrix $T(z)^{-1}$ are related through the quantum determinant, this lemma also proves that ω is a highest weight vector for $T(z)^{-1}$, with eigenvalue

$$\begin{aligned}T^{-1}(z)_{ii}\,\omega = &\Big(\mathrm{qdet}\,T(zq^{-\mathcal{N}+1})\Big)^{-1}\widetilde{P}_i(zq^{-\mathcal{N}+1})\,\omega =\\ &\left(\prod_{a=1}^{i-1}\frac{P_a(zq^{-a})}{P_a(zq^{-a+1})}\right)\frac{1}{P_i(zq^{-i+1})}\,\omega \equiv \widehat{P}_i(z)\,\omega\,.\end{aligned} \tag{2.77}$$

This property is essential for the following theorem:

Theorem 2.8. *If ω is a highest weight vector of $\mathcal{T}(z)$, with eigenvalue $(P_1(z),\ldots,P_{\mathcal{N}}(z))$, then, when $K(z)$ is a diagonal matrix* (2.41)*, it is also a highest weight vector for $\mathcal{B}(z)$, with eigenvalues*

$$B_{ii}(z)\,\omega = \left(\Gamma_i(z)\,P_i(z)\,\widehat{P}_i(z^{-1}) + \sum_{k=1}^{i-1}\frac{q^2-1}{q^{2k}z^4-1}\,\Gamma_k(z)\,P_k(z)\,\widehat{P}_k(z^{-1})\right)\omega \tag{2.78}$$

$$\Gamma_k(z) = \frac{q^{k-1}\,\mathfrak{b}(z^2)}{\mathfrak{b}(q^{k-1}z^2)}\times\begin{cases}\alpha(z;\xi) & ;\quad k\le\mathcal{M}\\ q^{-2\mathcal{M}}\,\beta(zq^{\mathcal{M}};\xi) & ;\quad k>\mathcal{M}\end{cases} \tag{2.79}$$

where $\mathfrak{b}(z) = z - z^{-1}$, and the expression of $\widehat{P}_j(z)$ in terms of $P_k(z)$ is given in (2.77)*.*

Proof. One needs to compute $B_{ij}(z)\,\omega$ for $j\le i$. Denoting by $D_k(z)$ the diagonal elements of $K(z)$, it takes the form:

$$\begin{aligned}B_{ij}(z)\,\omega &= \sum_{1\le k\le j} T_{ik}(z)D_k(z)T^{-1}_{kj}(z^{-1})\,\omega\\ &= \delta_{ij}\,D_i(z)T_{ii}(z)T^{-1}_{ii}(z^{-1})\,\omega + \sum_{1\le k\le j} D_k(z)\Big[T_{ik}(z)\,,\,T^{-1}_{kj}(z^{-1})\Big]\,\omega\end{aligned} \tag{2.80}$$

where $T^{-1}_{kj}(w)$ stands for $\big(\mathcal{T}^{-1}(w)\big)_{kj}$. From the relations (2.4) applied to $\mathcal{T}(z)$, one deduces (for $k \le j \le i$):

$$\begin{aligned}
&\mathfrak{b}(\frac{z}{w})\,\Big[T^{-1}_{kj}(w)\,,\,T_{ik}(z)\Big]\\
&\qquad= \Big(\mathfrak{a}(\frac{z}{w})-\mathfrak{b}(\frac{z}{w})\Big)\Big\{T_{ik}(z)T^{-1}_{kj}(w)-\delta_{ij}T^{-1}_{kj}(w)T_{ik}(z)\Big\}\\
&\qquad+\,(q-q^{-1})\left\{\frac{z}{w}\Big(\sum_{1\le a<k}T_{ia}(z)T^{-1}_{aj}(w)-\delta_{ij}\sum_{1\le a<j}T^{-1}_{ka}(w)T_{ak}(z)\Big)\right.\\
&\qquad\left.+\,\frac{w}{z}\Big(\sum_{k<a\le\mathcal{N}}T_{ia}(z)T^{-1}_{aj}(w)-\delta_{ij}\sum_{j<a\le\mathcal{N}}T^{-1}_{ka}(w)T_{ak}(z)\Big)\right\}
\end{aligned}\tag{2.81}$$

with $\mathfrak{a}(x)=qx-(qx)^{-1}$ and $\mathfrak{b}(x)=x-x^{-1}$, see Appendix A.

We first consider the case $j<i$.

Applying the above relation to ω, one gets

$$\begin{aligned}
&\frac{-\mathfrak{a}(z/w)}{q-q^{-1}}\;T_{ik}(z)T^{-1}_{kj}(w)\,\omega=\\
&\qquad\qquad\frac{z}{w}\sum_{1\le a<k}T_{ia}(z)T^{-1}_{aj}(w)\,\omega+\frac{w}{z}\sum_{k<a\le\mathcal{N}}T_{ia}(z)T^{-1}_{aj}(w)\,\omega\,.
\end{aligned}\tag{2.82}$$

Considering the case $k=j$, one obtains

$$\sum_{1\le a<j}T_{ia}(z)T^{-1}_{aj}(w)\,\omega=0\,.\tag{2.83}$$

Plugging this result in the former equation, we get, through iteration

$$T_{ik}(z)T^{-1}_{kj}(w)\,\omega=0\,,\quad k\le j<i\,,\tag{2.84}$$

which proves that

$$B_{ij}(z)\,\omega=0\,,\quad j<i\,.\tag{2.85}$$

We now turn to the case $i=j$.

Applying ω to the commutator (2.81), one is led to the following equations (for $k<i$):

$$\begin{aligned}
&\frac{\mathfrak{a}(z/w)}{q-1/q}F_{ik}(z,w)+\frac{z}{w}\sum_{a<k}F_{ia}(z,w)+\frac{w}{z}\left\{\sum_{k<a<i}F_{ia}(z,w)-\sum_{a<k}G_{ka}(z,w)\right\}\\
&\qquad\qquad=-\frac{w}{z}\Big(\Psi_i(z,w)-\Psi_k(z,w)\Big)
\end{aligned}\tag{2.86}$$

$$\begin{aligned}
&\frac{\mathfrak{a}(z/w)}{q-1/q}G_{ik}(z,w)+\frac{w}{z}\sum_{a<k}G_{ia}(z,w)+\frac{z}{w}\left\{\sum_{k<a<i}G_{ia}(z,w)-\sum_{a<k}F_{ka}(z,w)\right\}\\
&\qquad\qquad=-\frac{z}{w}\Big(\Psi_i(z,w)-\Psi_k(z,w)\Big)
\end{aligned}\tag{2.87}$$

with

$$F_{ik}(z,w) = T_{ik}(z)\,T_{ki}^{-1}(w)\omega \quad ; \quad G_{ik}(z,w) = T_{ik}^{-1}(w)\,T_{ki}(z)\omega \tag{2.88}$$

$$\Psi_i(z,w) = F_{ii}(z,w) = G_{ii}(z,w) = T_{ii}(z)T_{ii}^{-1}(w)\omega = P_i(z)\widehat{P}_i(w)\,\omega\,. \tag{2.89}$$

This system in $F_{ab}(z,w)$ and $G_{ab}(z,w)$, $1 \le b < a \le \mathcal{N}$, is triangular in the first index of F and G, and invertible (modulo lower indices) in the highest first index, so that it admits a unique solution. It is then easy (but lengthy) to show that the solution is

$$F_{ik}(z,w) = (q^2-1)\,F_{ik}^{\rm red}(z,w) \tag{2.90}$$

$$G_{ik}(z,w) = (q^2-1)\,\left(\frac{z}{w}\right)^2 q^{2k-2}\,G_{ik}^{\rm red}(z,w) \tag{2.91}$$

where $F^{\rm red}$ and $G^{\rm red}$ are linear combinations of the Ψ's:

$$\begin{aligned} F_{ik}^{\rm red}(z,w) = \frac{\Psi_k(z,w)}{q^{2k}x^2-1} - \frac{q^{2i-2k-2}\,\Psi_i(z,w)}{q^{2i-2}x^2-1} \\ + (1-q^2)\sum_{a=k+1}^{i-1} \frac{q^{2(a-k-1)}\,\Psi_a(z,w)}{(q^{2a}x^2-1)(q^{2a-2}x^2-1)} \end{aligned}$$

$$\begin{aligned} G_{ik}^{\rm red}(z,w) = \frac{\Psi_k(z,w)}{q^{2k}x^2-1} - \frac{\Psi_i(z,w)}{q^{2i-2}x^2-1} \\ + (1-q^2)\sum_{a=k+1}^{i-1} \frac{q^{2a-2}\,x^2\,\Psi_a(z,w)}{(q^{2a}x^2-1)(q^{2a-2}x^2-1)} \end{aligned}$$

where $x = \frac{z}{w}$. Since $\Psi_k(z,w)$ is proportional to ω, this proves that ω is a highest weight for $\mathcal{B}(z)$. Plugging the value of $F_{ik}(z,z^{-1})$ into the equation (2.80), one gets the eigenvalues

$$\begin{aligned} B_{ii}(z)\,\omega = \Bigg\{ D_i(z)\,P_i(z)\,\widehat{P}_i(z^{-1}) + (q^2-1)\sum_{1\le k<i} D_k(z)\Bigg(\frac{P_k(z)\,\widehat{P}_k(z^{-1})}{q^{2k}z^4-1} \\ - q^{2(i-k-1)}\,\frac{P_i(z)\,\widehat{P}_i(z^{-1})}{q^{2i-2}z^4-1} + (1-q^2)\sum_{a=k+1}^{i-1} q^{2(a-k-1)} \\ \frac{P_a(z)\,\widehat{P}_a(z^{-1})}{(q^{2a}z^4-1)(q^{2a-2}z^4-1)}\Bigg)\Bigg\}\,\omega\,, \end{aligned} \tag{2.92}$$

which can be rewritten in the form

$$B_{ii}(z)\,\omega = \left\{\Gamma_i(z)\,P_i(z)\,\widehat{P}_i(z^{-1}) + \sum_{k=1}^{i-1}\frac{q^2-1}{q^{2k}z^4-1}\,\Gamma_k(z)\,P_k(z)\,\widehat{P}_k(z^{-1})\right\}\omega$$

$$\Gamma_k(z) = D_k(z) - \frac{q^2-1}{q^{2k-2}z^4-1}\sum_{a=1}^{k-1} q^{2k-2a-2}\,D_a(z)\,. \tag{2.93}$$

Implementing the diagonal form (2.41), we get (2.78)–(2.79). □

As a consequence of this theorem, we can compute the transfer matrix eigenvalue of the pseudo-vacuum ω:

Corollary 2.9. *When $K^+(z)$ is a diagonal matrix (2.48), the highest weight vector ω is an eigenvector of the transfer matrix $b(z)$ given in (2.46):*

$$b(z)\,\omega = \Lambda^0(z)\,\omega \qquad \textit{with} \quad \Lambda^0(z) = \sum_{j=1}^{\mathcal{N}} g_j(z)\,P_j(z)\widehat{P}_j(z^{-1}) \tag{2.94}$$

$$\begin{aligned} g_j(z) = \Gamma_j(z)\,q^{-j+1}\,\frac{\mathfrak{b}(q^{\mathcal{N}}\,z^2)}{\mathfrak{b}(q^j\,z^2)} & \\ \times \begin{cases} q^{2\mathcal{N}-2\mathcal{M}_+}\,\widetilde{\alpha}(zq^{\mathcal{M}_+-\mathcal{N}};\xi_+) & ; \quad j \le \mathcal{M}_+ \\ \widetilde{\beta}(z;\xi_+) & ; \quad j > \mathcal{M}_+ \end{cases} & \end{aligned} \tag{2.95}$$

where $\Gamma_j(z)$ are given in (2.79) and $\widehat{P}_j(z)$ in (2.77).

Proof. Direct calculation using the expression of $b(z)$, the diagonal form (2.48) for $K^+(z)$ and the eigenvalues (2.78). □

Although each function $g_j(z)$ possesses some poles at the points $z^2 = q^{-j}$, the residues of $\Lambda^0(z)$ vanish due to the particular forms of $\widehat{P}(z)$, $\alpha(z;\xi)$, $\beta(z;\xi)$, $\widetilde{\alpha}(z;\xi_+)$ and $\widetilde{\beta}(z;\xi_+)$.

2.2.5. Quantum contraction and Sklyanin determinant. As in the case of the Yangian, we can construct series whose coefficients give central elements. In Appendices B and C, we recall the construction of two of them: the quantum contraction of $\mathcal{B}(z)$ and the Sklyanin determinant. In the spin chain context, they allow one to obtain constraints on the eigenvalues of the problem.
If $K(z)$ is given by relation (2.41) and $K^+(z)$ by (2.48), then their quantum contractions take the following simple forms

$$\delta(K(z)) = \theta_0 q^{2\mathcal{M}-2\mathcal{N}}\;\alpha(z\rho;\xi)\;\beta(z\rho;\xi)\;\left(\frac{1}{z^2} - z^2\right) \tag{2.96}$$

and

$$\delta(K^+(z)) = \theta_0 q^{2\mathcal{M}_+}\widetilde{\alpha}\left(\frac{zq^{\mathcal{M}_+}}{\rho};\xi_+\right)\widetilde{\beta}\left(\frac{z\rho}{q^{\mathcal{M}_+}};\xi_+\right)\;\left(q^{2\mathcal{N}}z^2 - \frac{1}{q^{2\mathcal{N}}z^2}\right) \tag{2.97}$$

where we recall that $\rho = q^{\mathcal{N}/2}$. For a given representation of the reflection algebra, the quantum contraction of the generators is simply given by $\delta(\mathcal{B}(z)) = \delta(K(z))$. In the same way, the Sklyanin determinant can be computed in a particular representation and becomes

$$\mathrm{sdet}(\mathcal{B}(z)) = \mathrm{sdet}(K(z))\prod_{a=1}^{\mathcal{N}}\frac{P_a(zq^{\mathcal{N}-a})}{P_a(z^{-1}q^{1-a})}\,. \tag{2.98}$$

The Sklyanin determinant of the diagonal solution (2.41) takes the following value

$$\mathrm{sdet}(K(z)) = q^{2\mathcal{M}(\mathcal{M}-\mathcal{N})} \prod_{1\le a\le \mathcal{M}} \alpha(z_a q^{\mathcal{N}-\mathcal{M}}) \prod_{\mathcal{M}+1\le a\le \mathcal{N}} \beta(z_a) \prod_{1\le a<b\le \mathcal{N}} \mathfrak{b}\left(\frac{q}{z_a z_b}\right), \tag{2.99}$$

where we remind $\mathfrak{b}(z) = z - 1/z$ and $z_a = zq^{a-1}$. We can also compute the Sklyanin determinant of $K^+(z)$ when this latter matrix is given by relation (2.48):

$$\mathrm{sdet}(K^+(z)) = q^{2\mathcal{M}_+(\mathcal{N}-\mathcal{M}_+)} \prod_{1\le a\le \mathcal{M}_+} \widetilde{\alpha}(z_a) \prod_{\mathcal{M}_++1\le a\le \mathcal{N}} \widetilde{\beta}(z_a q^{-\mathcal{M}_+}) \prod_{1\le a<b\le \mathcal{N}} \mathfrak{b}\left(q z_a z_b \rho^2\right) . \tag{2.100}$$

The computation of the previous explicit forms of the Sklyanin determinant follows the lines of the proof made in the case of the reflection algebra associated to the Yangian [18].

3. Spin chains

Having defined the underlying algebraic structures, we now turn to the construction of spin chains. To each of the above algebras will be associated a different boundary condition: periodic (closed spin chain) or soliton preserving (SP open spin chain). In both cases, the monodromy matrix will obey the defining relations of the corresponding algebra.

3.1. The periodic spin chain

Let us first consider the algebra $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ as introduced in Section 2.1 and the corresponding algebraic monodromy matrix $\mathcal{T}(z) = \mathcal{T}^+(z)$, given in (2.13).

The transfer matrix of the system has been also defined (at the algebraic level) in (2.14), and the commutation relation (2.15) ensures the integrability of the model. We repeat that our description is purely algebraic at this stage, i.e., the entailed results are independent of the choice of representation, and thus are universal. Once we assign particular representations on each of the copies of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, which then become the so-called 'quantum spaces' of the spin chain, the algebraic construction of the monodromy and transfer matrices acquires a physical meaning. Then, one may diagonalize the transfer matrix (2.14), which is the quantity that encodes all the physical information of the system, and derive the corresponding Bethe ansatz equations [9]. This will be in fact our main objective in the subsequent sections.

Let us also remark that the Borel subalgebra generated by $\mathcal{T}^+(z)$ can be viewed as a deformation of $\mathcal{Y}_\mathcal{N}$, the Yangian of $gl(\mathcal{N})$, so that the present spin chains are 'deformation' of the spin chain models build on $\mathcal{Y}_\mathcal{N}$. Hence the present algebraic construction can be viewed as the 'deformed' counterpart of the one done for the Yangian in [4]. The study of spin chains based on the full $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$

algebra (or on the quantum double of $\mathcal{Y}_{\mathcal{N}}$) is still lacking (to our knowledge). In the same way, we have assumed that the irreducible representations are finite dimensional, to ensure the existence of the Bethe ansatz reference state. However, one can also define spin chains with infinite dimensional representations, although a Bethe ansatz is lacking in this context. These spin chains are the ones used in large N QCD: it is clear that a study of them would be of interest. Note that in the case of infinite dimensional representations, one can take a non-vanishing central charge c: the different definitions for the monodromy matrix will then become inequivalent.

3.1.1. Spectrum of the periodic spin chain. We shall now derive the spectrum of the periodic spin chain by implementing the analytical Bethe ansatz formulation. We denote by $T_0(z)$ the represented monodromy matrix.

The first step is to derive an appropriate reference state, that is a state which is an eigenvector of the transfer matrix. This is provided by the highest weight vector ω of the $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ representations, as presented in Section 2.1.3. Its eigenvalue is related to the Drinfel'd polynomials characterizing the representation, see (2.31) and (2.30).

Having determined the form of the pseudovacuum eigenvalue we may now assume the following form for the general eigenvalue

$$\Lambda(z) = \sum_{a=1}^{\mathcal{N}} P_a(z)\ A_{a-1}(z)\,, \tag{3.1}$$

where the dressing functions $A_a(z)$ may be derived by implementing a number of constraints upon the spectrum (see e.g [22, 23, 24, 4] for a more detailed description). More specifically, the constraints follow from:

1. The fusion expression (B.7) produces constraints between the dressing functions of t and those of $\widehat{t}$ (defined by relation (B.9)), while the generalized fusion (C.9) provides a relation among the dressing functions of t. Explicitly, using relation (2.32), the constraint reads as
$$\prod_{a=1}^{\mathcal{N}} A_{a-1}(zq^{N-a}) = 1\ . \tag{3.2}$$
2. Analyticity requirements, imposed on the spectrum, lead any two successive (up to relabeling) dressing functions to have common poles.
3. The fact that the R matrix and monodromy matrix in the chosen representations are written in terms of rational functions (in z) leads to the assumption that A_a should be given as products of rational functions.
4. The asymptotic behaviour of the transfer matrix provides important information about the form of the aforementioned products (see Section 3.1.2).

5. The parity of the transfer matrix[3]
$$t(-z) = t(z), \tag{3.3}$$
leads to the parity of the eigenvalues.

The following set of dressing functions satisfy all the aforementioned constraints, for $a = 0, \ldots, \mathcal{N} - 1$,
$$A_a(z) = \prod_{\ell=1}^{M^{(a)}} \frac{q^{a+2}\, z^2 - \left(z_\ell^{(a)}\right)^2}{q^{a+1}\, z^2 - q\left(z_\ell^{(a)}\right)^2} \prod_{\ell=1}^{M^{(a+1)}} \frac{q^a\, z^2 - q\left(z_\ell^{(a+1)}\right)^2}{q^{a+1}\, z^2 - \left(z_\ell^{(a+1)}\right)^2}\,, \tag{3.4}$$
where, by convention, $M^{(0)} = 0$ and $M^{(\mathcal{N})} = 0$. We show in the next section how the coefficients $M^{(a)}$ are related to the eigenvalues of the Cartan's generators. Finally, requiring analyticity of the spectrum, we obtain the following set of Bethe ansatz equations, for $a = 1, \ldots, \mathcal{N}$ and $k = 1, \ldots, M^{(a)}$:
$$\frac{P_a(q^{-\frac{a}{2}}\, z_k^{(a)})}{P_{a+1}(q^{-\frac{a}{2}}\, z_k^{(a)})} = -\prod_{\ell=1}^{M^{(a-1)}} e_{-1}\left(\frac{z_k^{(a)}}{z_\ell^{(a-1)}}\right) \prod_{\ell=1}^{M^{(a)}} e_2\left(\frac{z_k^{(a)}}{z_\ell^{(a)}}\right) \prod_{\ell=1}^{M^{(a+1)}} e_{-1}\left(\frac{z_k^{(a)}}{z_\ell^{(a+1)}}\right)\,, \tag{3.5}$$
where we defined
$$e_n(z) = \frac{z^2\, q^n - 1}{z^2 - q^n}\,. \tag{3.6}$$
Note that, due to the normalizations we have chosen, the transfer matrix is analytical everywhere but zero. Hence, the above derivation is a priori valid for $z_k^{(a)} \neq 0$. However, since we are dealing with finite dimensional representations, multiplying by an appropriate power of z, one can always cure the unphysical pole in zero.

Let us remark that the right-hand side of the BAE reflects the Lie algebra dependence (through the Cartan matrix of $gl(\mathcal{N})$), while their left-hand side shows up a representation dependence (it can be rewritten in terms of Drinfel'd polynomials solely, using the relation (2.30)). The choice of a closed spin chain model is fixed by the choice of the quantum spaces, i.e., the choice of the Drinfel'd polynomials which determine the values $P_k(\lambda)$. Once these polynomials are given, the spectrum of the transfer matrix is fixed through the resolution of the BAE.

3.1.2. Dressing and Cartan generators. We have seen that the Cartan generators of the finite dimensional $\mathcal{U}_q(gl(\mathcal{N}))$ algebra commute with the transfer matrix. It is thus natural to try to connect the dressing used for $\Lambda(z)$ to the eigenvalues of the Cartan generators. It is done in the following way.

[3]Using an additive spectral parameter this parity is nothing but the periodicity $t(\lambda + \frac{i\pi}{\mu}) = t(\lambda)$ where μ is related to the deformation parameter through $q = e^{i\mu}$ and the spectral parameters $z = e^{\mu\lambda}$.

One first take the the limit $z \to \infty$ to get

$$\Lambda(z) \underset{z\to\infty}{\sim} z^N \sum_{j=1}^{\mathcal{N}} q^{M^{(j-1)}-M^{(j)}} \prod_{n=1}^{N} q^{a_n+\varpi_j^{(n)}} \tag{3.7}$$

where by convention $M^{(0)} = 0$ and $M^{(\mathcal{N})} = 0$. On the other hand, since $\mathcal{L}(z) \underset{z\to\infty}{\sim} z\,\mathcal{L}^+$ and $\mathcal{L}^+$ is triangular, one can also compute

$$t(z) \underset{z\to\infty}{\sim} z^N \sum_{j=1}^{\mathcal{N}} \underbrace{\ell_{jj}\otimes\cdots\otimes\ell_{jj}}_{N} . \tag{3.8}$$

Now, the Cartan generators $h_j^{(1\dots N)}$, $j = 1,\dots,\mathcal{N}$, defined by

$$\underbrace{\ell_{jj}\otimes\cdots\otimes\ell_{jj}}_{N} = q^{h_{jj}^{(1)}\oplus\cdots\oplus h_{jj}^{(N)}} = q^{h_j^{(1\dots N)}} , \tag{3.9}$$

where the superscript indicates in which quantum space(s) acts the operator, have been proved to commute with $t(z)$. Then, starting from a transfer matrix eigenvector

$$t(z)\,v = \Lambda(z)\,v , \tag{3.10}$$

one can deduce its $h_j^{(1\dots N)}$ eigenvalue:

$$h_j^{(1\dots N)}\,v = \lambda_j\,v = \Big(M^{(j-1)} - M^{(j)} + \sum_{n=1}^{N}(a_n + \varpi_j^{(n)})\Big)\ v , \qquad \forall\, j = 1,\dots,\mathcal{N} \quad \text{with } M^{(0)} = 0 . \tag{3.11}$$

This result generalizes the one obtained for the usual closed spin chain (see, e.g., [15, 24]). Indeed, in this latter case, all the sites carry a fundamental representation, so that $\varpi_j^{(n)} = \delta_{j,1}$, $\forall\ n = 1,\dots,N$. Then, for the $sl(\mathcal{N})$ Cartan generators $s_j = h_{j+1}^{(1\dots N)} - h_j^{(1\dots N)}$, one gets

$$s_j\,v = \Big(2M^{(j)} - M^{(j-1)} - M^{(j+1)} - N\,\delta_{j,1}\Big)\ v , \qquad \forall\, j = 1,\dots,\mathcal{N}-1 \quad \text{with } M^{(0)} = 0 . \tag{3.12}$$

It explains the usual convention $M^{(0)} = N$ used generally for this spin chain. However, from the general result (3.11), we rather use the convention $M^{(0)} = 0$, which is more natural and allows a more condensed writing of the BAE.

3.1.3. Upper bounds on $M^{(j)}$. We consider here the case where q is not root of unity, where we have the following theorem:

Theorem 3.1 ([11])**.** *The one-dimensional irreducible representations of $\mathcal{U}_q(gl(\mathcal{N}))$ have highest weight $\boldsymbol{\eta} = (\eta_1,\dots,\eta_{\mathcal{N}})$, with $\eta_j\,, j = 1,\dots,\mathcal{N}$ taking values in $\{1, i, -i, -1\}$.*

Any finite dimensional irreducible representation of $\mathcal{U}_q(gl(\mathcal{N}))$ is described by an highest weight $\widetilde{\boldsymbol{\varpi}} = \boldsymbol{\eta}\cdot q^{\boldsymbol{\varpi}}$, where $\boldsymbol{\varpi}$ is a dominant weight of $gl(\mathcal{N})$ and $\boldsymbol{\eta}$ describes a one-dimensional representation.

Then, up to the one-dimensional representations, the study of (finite dimensional) representations of $\mathcal{U}_q(gl(\mathcal{N}))$ is equivalent to the case of classical Lie algebra. For simplicity, we will assume in this section that the representations in each sites have $\eta_j = 1$, $\forall j$. Then the tensor product of such representations closes on representations of same type.

The values (3.11) allow us to recover the upper bounds for the parameters $M^{(j)}$. Indeed, the Bethe ansatz hypothesis states that the highest weight vectors of the diagonal $gl(\mathcal{N})$ algebra, whose Cartan basis is spanned by the $h_j^{(1\dots N)}$ generators, are transfer matrix eigenvectors v, and moreover that the eigenvalues of these eigenvectors span the whole set of transfer matrix eigenvalues. Then, the different dressings are in one-to-one correspondence with the different $gl(\mathcal{N})$ representations entering the spin chain. This also implies that $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{\mathcal{N}})$ given in (3.11) must be a dominant weight for the $gl(\mathcal{N})$ algebra.

The spin chain, as a representation of the $gl(\mathcal{N})$ algebra, reads

$$\bigotimes_{n=1}^{N} V(\boldsymbol{\varpi}^{(\boldsymbol{n})}) = \bigoplus_{\boldsymbol{\lambda}\leq\boldsymbol{\varpi}} c_\lambda\, V(\boldsymbol{\lambda}) \tag{3.13}$$

where

$$\boldsymbol{\varpi} = \sum_{n=1}^{N} \boldsymbol{\varpi}^{(\boldsymbol{n})} \tag{3.14}$$

correspond to the pseudo-vacuum eigenvalue, c_λ are multiplicities and the sum is done on dominant weights smaller than $\boldsymbol{\varpi}$. We remind that the partial order on weights is defined as follows: one has $\boldsymbol{\lambda} \leq \boldsymbol{\varpi}$ if and only if $\boldsymbol{\varpi} - \boldsymbol{\lambda}$ is a positive root. Computing this quantity for the eigenvalues (3.11), one gets

$$\boldsymbol{\varpi} - \boldsymbol{\lambda} = \sum_{k=1}^{\mathcal{N}-1} M^{(k)}\,(\varepsilon_k - \varepsilon_{k+1})\,, \tag{3.15}$$

where $(\varepsilon_k - \varepsilon_{k+1})$, $k = 1, \dots, \mathcal{N}-1$, are the $sl(\mathcal{N})$ simple roots. Demanding the weights obtained from (3.11) to be indeed smaller (in the above sense) than $\boldsymbol{\varpi}$, one recovers that the parameters $M^{(j)}$ are positive integers.

From the decomposition (3.13), demanding that the weights $\boldsymbol{\lambda}$ are dominant, one can deduce upper bounds on the integers $M^{(j)}$. Since λ_j, $j = 1, \dots, \mathcal{N}$ correspond to the decomposition of $\boldsymbol{\lambda}$ on the fundamental weights ε_j, one must have $\lambda_j - \lambda_{j+1} \in \mathbb{Z}_+$, $\forall j$. Computing this quantity, one gets

$$2M^{(j)} - M^{(j-1)} - M^{(j+1)} \leq n_j \quad \text{with } n_j = \varpi_j - \varpi_{j+1}$$
$$\text{and } \varpi_j = \sum_{n=1}^{N} \varpi_j^{(n)}\,,\ j = 1, \dots, \mathcal{N}. \tag{3.16}$$

Note that $n_j \in \mathbb{Z}_+$ because we supposed as a starting point that $\boldsymbol{\varpi}$ defines an irreducible finite dimensional representation of $gl(\mathcal{N})$.

The condition (3.16) rewrites as

$$(A\,\vec{M})_j \le n_j \quad \text{with } A \text{ the } sl(\mathcal{N})\text{-Cartan matrix and } \vec{M} = \begin{pmatrix} M^{(1)} \\ \vdots \\ M^{(\mathcal{N}-1)} \end{pmatrix} . \tag{3.17}$$

Using the fact that A^{-1} has only positive entries, one deduces

$$M_j \le (A^{-1}\,\vec{n})_j \qquad \text{with} \quad \vec{n} = \begin{pmatrix} n_1 \\ \vdots \\ n_{\mathcal{N}-1} \end{pmatrix} . \tag{3.18}$$

From the explicit form $\mathcal{N}\,(A^{-1})_{ij} = \min(i,j)\,\big(\mathcal{N} - \max(i,j)\big)$, one gets

$$\mathcal{N}\,M_j \le \sum_{k=1}^{j} k(\mathcal{N}-j)\,n_k + \sum_{k=j+1}^{\mathcal{N}-1} j(\mathcal{N}-k)\,n_k\ , \ \forall j = 1,\ldots,\mathcal{N}-1\ . \tag{3.19}$$

In the particular case of fundamental representations ($\varpi_j^{(n)} = \delta_{j,1}$), one has $\varpi_k = N\delta_{k,1}$, so that one recovers the condition

$$M^{(j)} \le \frac{\mathcal{N}-j}{\mathcal{N}}\,N\ . \tag{3.20}$$

3.1.4. Example: closed spin chain with one defect. As an example, we study the case of an arbitrary defect in a fundamental spin chain. In term of representations, it means that all the representations (all the sites of the spin chain) but one are taken to be the fundamental representation of $\mathcal{U}_q(gl(\mathcal{N}))$. We suppose this particular site to be the site p, with $1 < p < N$.

The $\mathcal{U}_q(gl(\mathcal{N}))$ highest weights on each site then read $\varpi_a^{(n)} = \delta_{a,1}$ for $n \neq p$, so that the Cartan eigenvalues simplify to

$$P_a(z) = \begin{cases} (zq - z^{-1}q^{-1})^{N-1}\,(zq^{\varpi_1+\theta} - z^{-1}q^{-\varpi_1-\theta}) & \text{if} \quad a = 1 \\ (z - z^{-1})^{N-1}\,(zq^{\varpi_a+\theta} - z^{-1}q^{-\varpi_a-\theta}) & \text{if} \quad a \neq 1 \end{cases} \tag{3.21}$$

where, for simplicity, we have dropped the superscript p on the highest weight $\boldsymbol{\varpi}^{(p)}$ and set all the inhomogeneity parameters to 0, except the one of site p, that we noted θ.

The monodromy matrix reads

$$T_0(z) = R_{01}(z)\cdots R_{0,p-1}(z)\,L_{0p}(zq^{\theta})\,R_{0,p+1}(z)\cdots R_{0N}(z) \tag{3.22}$$

so that one can define a local Hamiltonian (prime denotes derivative w.r.t. z):

$$\mathcal{H} = t(1)^{-1}\, t'(1) = \mathcal{H}_{p-1,p,p+1} + \sum_{\substack{j=1 \\ j\neq p,p-1}}^{N} \mathcal{H}_{j,j+1} \tag{3.23}$$

$$\mathcal{H}_{j,j+1} = \frac{1}{q-q^{-1}}\, \mathcal{P}_{j,j+1}\, R'_{j+1,j} \quad j \neq p\,,\, p-1 \tag{3.24}$$

$$\mathcal{H}_{p-1,p,p+1} = L^{-1}_{p+1,p}(q^{\theta})\, L'_{p+1,p}(q^{\theta}) + L^{-1}_{p+1,p}(q^{\theta})\, \mathcal{P}_{p-1,p+1}\, R'_{p-1,p+1}\, L_{p+1,p}(q^{\theta}) \tag{3.25}$$

$$R'_{ab} = 2\big(\mathbb{I} - P^{q}_{ab}\big) + (q+q^{-1})\, \mathcal{P}_{ab}\,, \quad \forall\ a \neq b. \tag{3.26}$$

This formula is a direct calculation from $t(z) = \mathrm{tr}_0\, T_0(z)$, using the properties of (q-)permutations and the value of $R(1) = (q-q^{-1})\mathcal{P}$ and $R'(1)$. As usual in periodic spin chains, we have identified the site $N+1$ with the site 1.
The energies are of the form $E = \Lambda'(1)/\Lambda(1)$, where $\Lambda(z)$ is given in (3.1), with dressings (3.4), BAEs (3.5) and $P_a(z)$ as above. A straightforward calculation leads to

$$E = E_0 + 2\,(q^2-1) \sum_{\ell=1}^{M^{(1)}} \frac{\big(z^{(1)}_{\ell}\big)^2}{\Big(1 - q\,\big(z^{(1)}_{\ell}\big)^2\Big)\Big(q - \big(z^{(1)}_{\ell}\big)^2\Big)} \tag{3.27}$$

$$E_0 = (N-1)\,\frac{q+q^{-1}}{q-q^{-1}} + \frac{q^{\varpi_1+\theta} + q^{-\varpi_1-\theta}}{q^{\varpi_1+\theta} - q^{-\varpi_1-\theta}}\,. \tag{3.28}$$

Here, E_0 is the energy of the pseudo-vacuum, with normalizations as given in (3.23)–(3.25).
Of course, when the 'defect' representation is also the fundamental representation, one recovers the usual XXZ model.

3.2. Soliton preserving open spin chain

This section is devoted to the derivation of the spectrum and Bethe ansatz equations for the integrable open spin chain with soliton preserving diagonal boundaries. These boundary conditions physically describe the reflection of a soliton to itself, i.e., no multiplet change occurs under reflection. The algebraic structure associated to this kind of open spin chain is the quantum reflection algebra described in Section 2.2.

Our main aim consists in building the corresponding quantum system, i.e., the open quantum spin chain. The open spin chain may be constructed following the generalized QISM, introduced by Sklyanin [6]. It relies on tensor product realizations of the general solution (2.39), on the open spin chain monodromy matrix

$$\mathcal{B}_0(z) = \mathcal{T}_0(z)\ K^-_0(z)\ \mathcal{T}^{-1}_0(z^{-1}), \tag{3.29}$$

and on the transfer matrix $b(z) = \mathrm{tr}_0\,\big(K^+_0(z)\,\mathcal{B}_0(z)\big)$, as they were introduced in Section 2.2.1.

3.2.1. Spectrum of the open spin chain. As in the closed case, our ultimate goal is to derive the spectrum and the corresponding Bethe ansatz equations. To achieve that we shall need an appropriate reference state. Fortunately, if we restrict our attention to the case where both left and right boundaries are diagonal, there exists an obvious reference state: the highest weight vector of the $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ representation (see Section 2.2.4).

Having specified the form of the pseudo-vacuum eigenvalue (2.94), we can make the following assumption for the general eigenvalue of the open transfer matrix:

$$\Lambda(z) = \sum_{a=1}^{\mathcal{N}} g_a(z)\ P_a(z)\ \widehat{P}_a(z^{-1})\ \widetilde{A}_{a-1}(z)\,. \tag{3.30}$$

The dressing functions $\widetilde{A}(z)$ may be derived by implementing a number of constraints upon the spectrum (see, e.g., [25, 26, 4] for a more detailed description). In particular, similarly to the periodic case the constraints follow from:

1. The fusion relation (C.17). Using the explicit form of the Sklyanin determinant (2.98)–(2.100) and the value (2.95) of the function $g_a(z)$, we find the following constraint
$$\prod_{a=1}^{\mathcal{N}} \widetilde{A}_{a-1}(zq^{N-a}) = 1\ . \tag{3.31}$$
Note that this constraint is similar to the one found in the periodic case and does not depend on the chosen representations and on the boundaries.
2. Parity requirements which means that the dressing functions are only function of z^2.
3. The fact that the R matrix and monodromy matrix (in the chosen representations) are written in terms of rational functions which implies that $\widetilde{A}(z)$ are rational functions.
4. The remark that every two successive g_a have common poles, which must vanish to ensure analyticity of the eigenvalues. Namely, the vanishing of the residues at $z^2 = q^{-a}$ $(1 \le a \le \mathcal{N}-1)$ implies the following constraints
$$\widetilde{A}_{a-1}(q^{-a/2}) = \widetilde{A}_a(q^{-a/2})\ . \tag{3.32}$$

The asymptotic behaviour of the transfer matrix also provides important information about the form of the aforementioned products as explained in Section 3.2.2 (see also [24, 25]).

The following set of dressing functions satisfy all the aforementioned constraints:

$$\widetilde{A}_a(z) = \prod_{\ell=1}^{M^{(a)}} \frac{q^{a+2}\,z^2 - \left(z_\ell^{(a)}\right)^2}{q^{a+1}\,z^2 - q\left(z_\ell^{(a)}\right)^2}\ \frac{q^{a+2}\left(zz_\ell^{(a)}\right)^2 - 1}{q^{a+1}\left(zz_\ell^{(a)}\right)^2 - q} \times$$
$$\prod_{\ell=1}^{M^{(a+1)}} \frac{q^a\,z^2 - q\left(z_\ell^{(a+1)}\right)^2}{q^{a+1}\,z^2 - \left(z_\ell^{(a+1)}\right)^2}\ \frac{q^a\left(zz_\ell^{(a+1)}\right)^2 - q}{q^{a+1}\left(zz_\ell^{(a+1)}\right)^2 - 1} \qquad a = 0,\ldots,\mathcal{N}-1 \tag{3.33}$$

where we recall that $M^{(0)} = 0$ and $M^{(\mathcal{N})} = 0$. Finally by requiring the vanishing of the residues at $z = q^{-a/2} z_\ell^{(a)}$, we obtain the following set of Bethe ansatz equations, for $a = 1, \ldots, \mathcal{N}$ and $k = 1, \ldots, M^{(a)}$,

$$\frac{g_a(z_k^{(a)}\, q^{-\frac{a}{2}})}{g_{a+1}(z_k^{(a)} q^{-\frac{a}{2}})} \, \frac{\widehat{P}_a\big(q^{\frac{a}{2}}/z_k^{(a)}\big)\ P_a(z_k^{(a)}\, q^{-\frac{a}{2}})}{\widehat{P}_{a+1}\big(q^{\frac{a}{2}}/z_k^{(a)}\big)\ P_{a+1}(z_k^{(a)}\, q^{-\frac{a}{2}})}$$
$$= - \prod_{\ell=1}^{M^{(a-1)}} \widehat{e}_{-1}(z_k^{(a)};\ z_\ell^{(a-1)}) \prod_{\ell=1}^{M^{(a)}} \widehat{e}_2(z_k^{(a)};\ z_\ell^{(a)}) \prod_{\ell=1}^{M^{(a+1)}} \widehat{e}_{-1}(z_k^{(a)};\ z_\ell^{(a+1)}) \tag{3.34}$$

where we defined

$$\widehat{e}_n(z; w) = e_n(z/w)\ e_n(zw). \tag{3.35}$$

Again, the right-hand side is linked to the $gl(\mathcal{N})$ Cartan matrix, while the left-hand side is related to the chosen representations. The choice of an open spin chain model is now determined by two types of data:

1. The choice of the quantum spaces, i.e., of the Drinfel'd polynomials fixing the eigenvalues $P_k(z)$.
2. The choice of the boundary conditions, i.e., of $K(z)$ and of $K^+(z)$, which fixes $g_k(z)$.

Then, the spectrum of the transfer matrix is given by the solutions to the BAE.

3.2.2. Dressing and Cartan generators. As in the closed case, one can relate the dressing with the Cartan generators eigenvalues. Starting from a transfer matrix eigenvector v, with eigenvalue

$$b(z)\, v = \Lambda(z)\, v\ , \tag{3.36}$$

one can deduce, taking the limit $z \to \infty$, the eigenvalues for the Cartan generators of the finite dimensional symmetry algebra:

$$B_{jj}\, v = 2\Big(M^{(j-1)} - M^{(j)} + \sum_{n=1}^{N} (a_n + \varpi_j^{(n)})\Big)\ v\ ,$$
$$\forall\, j = 1, \ldots, \mathcal{N} \quad \text{with } M^{(0)} = 0 \text{ and } M^{(\mathcal{N})} = 0\,. \tag{3.37}$$

This expression is valid whatever the representations on each site are. Let us also remark that it is independent of the form of the $K(z)$ and $K^+(z)$ matrices, i.e., of the boundary condition.

The upper bounds on the allowed values for the parameters $M^{(j)}$ is deduced as in Section 3.1.3 (with the same restrictions). We get

$$\mathcal{N}\, M_j \le \sum_{k=1}^{j} k(\mathcal{N}-j)\, n_k + \sum_{k=j+1}^{\mathcal{N}-1} j(\mathcal{N}-k)\, n_k\ ,\ \forall j = 1, \ldots, \mathcal{N}-1\ . \tag{3.38}$$

In the particular case of fundamental representations, one recovers the condition

$$M^{(j)} \le \frac{\mathcal{N}-j}{\mathcal{N}}\, N\ . \tag{3.39}$$

3.2.3. Example: open spin chain with one defect. As an example, we take the same spin chain as in the closed spin chain case, Section 3.1.4, but now with open boundary conditions given by a diagonal $K(z)$ matrix (2.41) and $K^+(z) = M$. The transfer matrix (based on $T_0(z)$ as in Section 3.1.4)

$$b(z) = \mathrm{tr}_0\left(K_0^+(z)\, T_0(z)\, K_0(z)\, T_0^{-1}(\frac{1}{z})\right) \tag{3.40}$$

leads to a local Hamiltonian:

$$\mathcal{H} = \frac{1}{2\,(1-\xi^2)\,[\mathcal{N}]_q}\, b'(1) = \sum_{\substack{j=1\\ j\neq p,p-1}}^{N-1} \mathcal{H}_{j,j+1} + \mathcal{H}_{p-1,p,p+1} \tag{3.41}$$

with (we remind prime stands for z derivative)

$$\mathcal{H}_{j,j+1} = \frac{2}{q-q^{-1}}\, \mathcal{P}_{j,j+1}\, R'_{j+1,j} \qquad ,\quad j\neq p-1\,,\, p\,,\, N-1 \tag{3.42}$$

$$\mathcal{H}_{N-1,N} = \frac{1}{2(1-\xi^2)}\, K'_N(1) \tag{3.43}$$

$$\begin{aligned}\mathcal{H}_{p-1,p,p+1} = {} & L'_{p,p-1}(q^\theta)\, L^{-1}_{p,p-1}(q^\theta) + \frac{1}{q-q^{-1}}\\ & \times L_{p,p-1}(q^\theta)\, \mathcal{P}_{p-1,p+1}\, R'_{p+1,p-1}\, L^{-1}_{p,p-1}(q^\theta)\end{aligned} \tag{3.44}$$

$$R'_{ab} = 2\big(\mathbb{I} - P^q_{ab}\big) + (q+q^{-1})\, \mathcal{P}_{ab}\,,\quad \forall\ a\neq b. \tag{3.45}$$

To obtain these results, we have used

$$K(1) = (1-\xi^2)\,\mathbb{I} \qquad \text{and} \qquad \mathrm{tr}(M) = [\mathcal{N}]_q \equiv \frac{q^{\mathcal{N}} - q^{-\mathcal{N}}}{q-q^{-1}}$$

The Hamiltonian eigenvalues are of the form $\widetilde{E} = \frac{1}{2\,(1-\xi^2)\,[\mathcal{N}]_q}\,\Lambda'(1)$, where $\Lambda(z)$ is now given in (3.30), with dressings (3.33), BAEs (3.34) and

$$\pi_a(z) = zq^{\varpi_a+\theta} - z^{-1}q^{-\varpi_a-\theta} \qquad\qquad \forall a$$

$$P_a(z) = \begin{cases} (zq - z^{-1}q^{-1})^{N-1}\, \pi_1(z) & \text{if}\quad a=1\\ (z-z^{-1})^{N-1}\, \pi_a(z) & \text{if}\quad a\neq 1\end{cases}$$

$$\widehat{P}_a(z) = $$

$$\begin{cases} P_1(z)^{-1} & \text{if } a=1\\ \left(\dfrac{z-z^{-1}}{(zq-z^{-1}q^{-1})(zq^{-1}-z^{-1}q)}\right)^{N-1} \left(\displaystyle\prod_{j=1}^{a-1} \frac{\pi_j(zq^{-j})}{\pi_j(zq^{-j+1})}\right) \dfrac{1}{\pi_a(zq^{-a+1})} & \text{if } a\neq 1\end{cases}$$

Using these expressions, one gets

$$\widetilde{E} = \widetilde{E}_0 + 2\,(q^2-1) \sum_{\ell=1}^{M^{(1)}} \frac{\left(z_\ell^{(1)}\right)^2}{\left(1 - q\left(z_\ell^{(1)}\right)^2\right)\left(q - \left(z_\ell^{(1)}\right)^2\right)} \tag{3.46}$$

$$\widetilde{E}_0 = (N-2)\,\frac{q+q^{-1}}{q-q^{-1}} + \frac{2\,q^{\mathcal{N}}}{q^{\mathcal{N}} - q^{-\mathcal{N}}} + \frac{q^{\varpi_1+\theta} + q^{-\varpi_1-\theta}}{q^{\varpi_1+\theta} - q^{-\varpi_1-\theta}} + \frac{1}{1-\xi^2} \tag{3.47}$$

where $\widetilde{E}_0$ is the energy of the open spin chain pseudo-vacuum.

4. Quantum twisted Yangians and SNP spin chains

As already mentioned, the construction of the reflection algebra (hence of "soliton preserving" open spin chains) is available for any quantum group. This property is based on the (formal) existence for any algebra of the inversion antimorphism $\mathcal{L}(z) \to \mathcal{L}^{-1}(z)$. However, there are some algebras, such as the Yangian of $gl(\mathcal{N})$, or the quantum algebra $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, for which another antimorphism can be constructed, leading to other (new) coideal subalgebras, which themselves allow one to construct (new) open spin chains. These spin chains have different boundaries, known as soliton non-preserving boundaries. These physical considerations will be developed below.

4.1. The quantum twisted Yangian

We first focus on the mathematical construction, which will infer the monodromy matrix of the aforementioned spin chains. The algebra we construct is the quantum twisted Yangian, as it has been introduced in [27].

4.1.1. Definition. We start with the transposition in the auxiliary space:

$$^{t_0}\ :\ \mathcal{L}_{01}(z) = \sum_{a,b=1}^{\mathcal{N}} E_{ab} \otimes L_{ab}(z) \ \mapsto\ \mathcal{L}_{01}^{t_0}(z) = \sum_{a,b=1}^{\mathcal{N}} E_{ab} \otimes L_{ba}(z) \tag{4.1}$$

which is an antimorphism of $\widehat{\mathcal{U}}_q$. Mimicking the construction of the quantum reflection algebra, we introduce[4]:

$$\mathcal{S}(z) = \mathcal{L}^+(zq^{-\frac{c}{2}})\,G\,\mathcal{L}^-(z^{-1}q^{\frac{c}{2}})^t$$
$$\text{where } \begin{cases} G = \mathbb{I} \quad o(\mathcal{N}) \text{ case,} \\ G = \displaystyle\sum_{k=1}^{n} \Big(q\,E_{2k-1,2k} - E_{2k,2k-1}\Big) \quad sp(2n) \text{ case .} \end{cases} \tag{4.2}$$

[4] One could also take $\mathcal{S}^{\pm}(z) = \mathcal{L}^{\pm}(z) G \mathcal{L}^{\pm}(z^{-1})^t$ which obey the same exchange relations: as for quantum reflection algebra, it leads to an equivalent construction for spin chains, up to analytical properties of $\mathcal{S}(z)$.

It is easy to show that $\mathcal{S}(z)$ defines a subalgebra of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, called quantum twisted Yangian [27], with exchange relations (when $c=0$)

$$R_{12}(\frac{z}{w})\,\mathcal{S}_1(z)\,R_{12}^{t_1}(\frac{1}{zw})\,\mathcal{S}_2(w) = \mathcal{S}_2(w)\,R_{12}^{t_1}(\frac{1}{zw})\,\mathcal{S}_1(z)\,R_{12}(\frac{z}{w})\,, \tag{4.3}$$

together with the conditions

$$o(\mathcal{N})\ \text{case:} \quad \begin{cases} \mathcal{S}_{ii}(0) = 1\,,\ \forall\ i\ , \\ \mathcal{S}_{ij}(0) = 0\,,\ \forall\ i > j\ , \end{cases} \tag{4.4}$$

$$sp(\mathcal{N})\ \text{case:} \quad \begin{cases} \mathcal{S}_{2k,2k}(0)\,\mathcal{S}_{2k-1,2k-1}(0) - q^2\,\mathcal{S}_{2k,2k-1}(0)\,\mathcal{S}_{2k-1,2k}(0) = q^3\,, \forall\ k\ , \\ \mathcal{S}_{ij}(0) = 0\,,\ \forall\ i > j \text{ and } i \neq j'\ . \end{cases} \tag{4.5}$$

We have introduced the map

$$(2k)' = 2k-1\,,\ (2k-1)' = 2k\ , \quad \text{i.e.,} \quad j' = 2\left[\frac{j+1}{2}\right] + \left[\frac{j-1}{2}\right] - \left[\frac{j}{2}\right]\,. \tag{4.6}$$

Note that the above calculation uses the equality (valid for both values of G):

$$R_{12}(z)\,G_1\,R_{12}^{t_1}(w)\,G_2 = G_2\,R_{12}^{t_1}(w)\,G_1\,R_{12}(z), \tag{4.7}$$

which have to be compared with the relation (2.35) used in the case of quantum reflection algebras.

Existence of constant solutions to the relation (4.3) is ensured by the CP-invariance (A.38), which proves that any invertible diagonal matrix is a solution.

Then, the construction follows the lines of Section 2.2, see [28]. For instance one has

$$\mathcal{S}(z) = \mathcal{L}^+(zq^{-\frac{c}{2}})\,G(z)\,\mathcal{L}^-(z^{-1}q^{\frac{c}{2}})^t \tag{4.8}$$

$$\Delta(\mathcal{S}(z)) = \mathcal{L}_{01}(zq^{-\frac{c_1}{2}})\,\mathcal{S}_{02}(zq^{-c_1})\,\mathcal{L}_{01}^t(q^{\frac{c_1}{2}}\,z^{-1}) \tag{4.9}$$

where $G(z)$ is a solution of the reflection equation (4.3) such that $G(0) = G$ (up to invariance of the exchange relation).

Here, keeping in mind the spin chain interpretation, we will use an equivalent definition of the quantum twisted Yangian. The exchange relations we choose have the following form

$$R_{12}(\frac{z}{w})\ \widetilde{\mathcal{S}}_1(z)\ \bar{R}_{21}(zw)\ \widetilde{\mathcal{S}}_2(w) = \widetilde{\mathcal{S}}_2(w)\ \bar{R}_{12}(zw)\ \widetilde{\mathcal{S}}_1(z)\ R_{21}(\frac{z}{w}), \tag{4.10}$$

where the $\bar{R}$ matrices are defined in Appendix A.4. The relation between the two definitions is given by

$$\mathcal{S}(z) = \widetilde{\mathcal{S}}(z\rho^{-\frac{1}{2}})V^t. \tag{4.11}$$

The algebra generated by $\widetilde{\mathcal{S}}(z)$ can be seen as a subalgebra of $\widehat{\mathcal{U}}_q$ through the construction ($c=0$)

$$\widetilde{\mathcal{S}}(z) = \mathcal{L}^+(z\rho^{\frac{1}{2}})\,\widetilde{K}(z)\,V^t\,\mathcal{L}^-(z^{-1}\rho^{-\frac{1}{2}})^t\,V^t\,, \tag{4.12}$$

where $\widetilde{K}(z)$ is any $\mathbb{C}$-valued matricial solution of the relation (4.10). Of course, the construction (4.12) can be deduced from (4.8) using the relation (4.11). The existence of solutions is ensured by the relation (4.11), which proves that $W\,V^t$ (where W is diagonal) is indeed a solution.
In this formalism, the coproduct takes the form

$$\Delta(\widetilde{\mathcal{S}}(z)) = \mathcal{L}_{01}(z\rho^{-\frac{1}{2}}q^{-\frac{c_1}{2}})\,\widetilde{\mathcal{S}}_{02}(zq^{-c_1})\,V_0^t\mathcal{L}_{01}^t(z^{-1}\rho^{-\frac{1}{2}}q^{\frac{c_1}{2}})V_0^t\,. \tag{4.13}$$

The universal monodromy matrix will then be (c=0)

$$\begin{aligned}&\Delta^{(N)}(\widetilde{\mathcal{S}}(z)) = \\ &\quad \mathcal{L}_{01}(z\rho^{\frac{1}{2}})\cdots\mathcal{L}_{0N}(z\rho^{\frac{1}{2}})\,\widetilde{K}_0(z)\,V_0^t\,\mathcal{L}_{0,N}^{t_0}(z^{-1}\rho^{-\frac{1}{2}})\cdots\mathcal{L}_{01}^{t_0}(z^{-1}\rho^{-\frac{1}{2}})\,V_0^t\,.\end{aligned} \tag{4.14}$$

The corresponding transfer matrix is defined as

$$\widetilde{s}(z) = \mathrm{tr}_0\left\{\widetilde{K}_0^+(z)\,\Delta^{(N)}(\widetilde{\mathcal{S}}_0(z))\right\} \tag{4.15}$$

where $\widetilde{K}^+(z)$ satisfy the dual reflection equation

$$\begin{aligned}&R_{12}(\frac{w}{z})\ \widetilde{K}_1^+(z)^t\ M_1^{-1}\ \bar{R}_{21}(\rho^{-2}z^{-1}w^{-1})\ M_1\ \widetilde{K}_2^+(w)^t = \\ &\widetilde{K}_2^+(w)^t\ M_1\ \bar{R}_{12}(\rho^{-2}z^{-1}w^{-1})\ M_1^{-1}\ \widetilde{K}_1^+(z)^t\ R_{21}(\frac{w}{z}).\end{aligned} \tag{4.16}$$

Solutions to the two equations (4.10) and (4.16) are related through $\widetilde{K}^+(z) = M\,\widetilde{K}^t(z^{-1}\rho^{-\frac{1}{2}})$, so that $\widetilde{K}^+(z) = V^t$ is also a solution to (4.16).
Again, it can be proved using only the equations (4.10), (4.16), the unitarity of the R matrix, and arguments similar to the ones given in [6] (for the SP case), that

$$\Big[\widetilde{s}(z),\ \widetilde{s}(w)\Big] = 0\,. \tag{4.17}$$

It will ensure that the open spin chain derived from (4.15) is also integrable.

4.1.2. Finite dimensional subalgebras of the quantum twisted Yangian. Starting from the exchange relation (4.10), one gets

$$R_{12}\ \widetilde{\mathcal{S}}_1(z)\ \bar{R}_{21}\ \widetilde{\mathcal{S}}_2(0) = \widetilde{\mathcal{S}}_2(0)\ \bar{R}_{12}\ \widetilde{\mathcal{S}}_1(z)\ R_{21}, \tag{4.18}$$

where $\bar{R}_{12} = V_1\,R_{12}^{t_2}\,V_1$. It proves that $\widetilde{\mathcal{S}}(0)$ form a subalgebra of the twisted Yangian, and that $\widetilde{\mathcal{S}}(z)$ is a representation of it. When $\widetilde{K}(z) = G\,V^t$, this subalgebra (up to the change of basis due to (4.11)) is nothing but the the twisted quantum algebra $\mathcal{U}_q^{tw}(\mathfrak{g})$ introduced in [29], with $\mathfrak{g} = o(\mathcal{N})$ if $\theta_0 = 1$ or $\mathfrak{g} = sp(\mathcal{N})$ if $\theta_0 = -1$. It corresponds to a deformation of the classical algebra $\mathcal{U}(\mathfrak{g})$, different from the quantum algebra $\mathcal{U}_q(\mathfrak{g})$. In particular, it has no (known) proper Hopf structure, but is rather a Hopf coideal of $\mathcal{U}_q(gl(\mathcal{N}))$. Let us remark that when $\widetilde{K}$ is an antidiagonal matrix, $\widetilde{S}(0)$ is equivalent[5] to a lower triangular ($so(\mathcal{N})$ case) or block triangular ($sp(2n)$ case) matrix, in accordance with the dimension of $\mathcal{U}_q^{tw}(\mathfrak{g})$.

[5]In fact, it is $\mathcal{S}(0) = \widetilde{\mathcal{S}}(0)\,V^t$ which is triangular when G is diagonal. $\widetilde{\mathcal{S}}(0)$ is triangular with respect to the second diagonal.

Defining $\widehat{\mathcal{S}}(z) = \widetilde{\mathcal{S}}(z^{-1})^{-1}$, one gets

$$\bar{R}_{12}(\frac{z}{w})\ \widetilde{\mathcal{S}}_1(z)\ R_{21}(zw)\ \widehat{\mathcal{S}}_2(w) = \widehat{\mathcal{S}}_2(w)\ R_{12}(zw)\ \widetilde{\mathcal{S}}_1(z)\ \bar{R}_{21}(\frac{z}{w}), \tag{4.19}$$

$$R_{12}(\frac{z}{w})\ \widehat{\mathcal{S}}_1(z)\ \bar{R}_{21}(zw)\ \widehat{\mathcal{S}}_2(w) = \widehat{\mathcal{S}}_2(w)\ \bar{R}_{12}(zw)\ \widehat{\mathcal{S}}_1(z)\ R_{21}(\frac{z}{w}), \tag{4.20}$$

which shows that $\widehat{\mathcal{S}}(z)$ obeys the same relation as $\mathcal{S}(z)$. One can, as in Section 2.2.2, consider $\widehat{\mathcal{S}}(\infty)$ (if $\widetilde{K}(z)$ is a constant antidiagonal matrix), or some regularization of it (when $\widetilde{K}(z)$ depends on z). Its exchange relations are given by

$$\bar{R}_{12}\ \widetilde{\mathcal{S}}_1(z)\ R_{21}\ \widehat{\mathcal{S}}_2(\infty) = \widehat{\mathcal{S}}_2(\infty)\ R_{12}\ \widetilde{\mathcal{S}}_1(z)\ \bar{R}_{21}, \tag{4.21}$$

$$R_{21}^{-1}\ \widehat{\mathcal{S}}_1(z)\ \widetilde{R}_{21}\ \widehat{\mathcal{S}}_2(\infty) = \widehat{\mathcal{S}}_2(\infty)\ \widetilde{R}_{12}\ \widehat{\mathcal{S}}_1(z)\ R_{12}^{-1}, \tag{4.22}$$

$$\widetilde{R}_{12} = V_1 \left(R_{21}^{-1}\right)^{t_2} V_1 = V_1^t \left(R_{12}^{t_1}\right)^{-1} V_1^t. \tag{4.23}$$

Lemma 4.1. *The generators $\widetilde{\mathcal{S}}(0)$ and $\widehat{\mathcal{S}}(\infty)$ satisfy $\widetilde{\mathcal{S}}(0)\,\widehat{\mathcal{S}}(\infty) = \widehat{\mathcal{S}}(\infty)\,\widetilde{\mathcal{S}}(0) = x_0\,\mathbb{I}$ where x_0 is some constant.*
In particular, one has

$$[\widetilde{\mathcal{S}}(0)\,,\,\widehat{\mathcal{S}}(\infty)] = 0, \tag{4.24}$$

so that either $\widehat{\mathcal{S}}(\infty)$ is the inverse of $\widetilde{\mathcal{S}}(0)$, or generates a subalgebra which commutes with the one generated by $\widetilde{\mathcal{S}}(0)$.

Proof. One starts from the relation (4.19) and consider the limits $z \to 0$ then $w \to \infty$ on the one hand, and $w \to \infty$ then $z \to 0$ on the other hand. It leads to the relations

$$\bar{R}_{12}\ \widetilde{\mathcal{S}}_1(0)\ R_{12}^{-1}\ \widehat{\mathcal{S}}_2(\infty) = \widehat{\mathcal{S}}_2(\infty)\ R_{21}^{-1}\ \widetilde{\mathcal{S}}_1(0)\ \bar{R}_{21}, \tag{4.25}$$

$$\bar{R}_{12}\ \widetilde{\mathcal{S}}_1(0)\ R_{21}\ \widehat{\mathcal{S}}_2(\infty) = \widehat{\mathcal{S}}_2(\infty)\ R_{12}\ \widetilde{\mathcal{S}}_1(0)\ \bar{R}_{21}. \tag{4.26}$$

Using relation (A.4), it leads to

$$\bar{R}_{12}\ \widetilde{\mathcal{S}}_1(0)\ \widehat{\mathcal{S}}_1(\infty) = \widehat{\mathcal{S}}_2(\infty)\ \widetilde{\mathcal{S}}_2(0)\ \bar{R}_{12}, \tag{4.27}$$

which gives after some manipulations

$$V_1\ \widetilde{\mathcal{S}}_1(0)\ \widehat{\mathcal{S}}_1(\infty)\ V_1 = \left(\widehat{\mathcal{S}}_2(\infty)\ \widetilde{\mathcal{S}}_2(0)\right)^t\ , \tag{4.28}$$

whose only solution is the one given in the lemma. □

Remark that this lemma is the counterpart of Lemma 2.2.

We leave to the interested reader the complete study of the subalgebra spanned by $\widetilde{\mathcal{S}}(0)$ and $\widehat{\mathcal{S}}(\infty)$. It is related to twisted Yangians $\mathcal{U}_q^{tw}(\mathfrak{h})$, where $\mathfrak{h}$ is a subalgebra of $o(\mathcal{N})$ or $sp(\mathcal{N})$ which depends on the chosen $\widetilde{K}(z)$ matrix. We classify these latter in Section 4.2.

4.1.3. Symmetry of the universal transfer matrix. To study the symmetry of the transfer matrix, one should proceed as in the SP case. However, there is a crucial difference between the relations (2.51) and (4.18): in the first case, each side of the equality involves an R matrix and essentially its inverse, while it is essentially its transpose in the second case. Then, it is not possible to get an equivalent of Lemma 2.3 in the SNP context. Indeed, starting from (4.3), it is easy to conclude that

$$\left[\mathrm{tr}_1\left(R_{12}\,\mathcal{S}_1(z)\,R_{12}^{t_1}\right)\;,\;\mathcal{S}_2(0)\right]=0\,,\tag{4.29}$$

but $\mathrm{tr}_1\left(R_{12}\,\mathcal{S}_1(z)\,R_{12}^{t_1}\right)$ is not easily related to $\widetilde{s}(z)$, so that one cannot obtain, in this way, a symmetry of the transfer matrix. Similar conclusion is obtained when starting from (4.18) or (4.21). As a consequence, it looks as if the SNP spin chain does not possess any symmetry. In fact, the matrices $\mathcal{S}(z)$, $\widetilde{\mathcal{S}}(z)$ and $\widehat{\mathcal{S}}(z)$ are not written in a Cartan-Weyl basis, so that even the Cartan generators (which should commute with $\widetilde{s}(z)$) are difficult to identify in this presentation.

4.2. Soliton non-preserving boundaries

In this section we shall deal with the so-called soliton non-preserving boundary conditions (SNP) [30, 26, 28], which physically describe the reflection of a soliton to an anti-soliton. The corresponding equation describing such boundary conditions is given [31, 32] by (4.10). The main aim now is to derive matricial solutions of this equation.

It is convenient to consider the matrix $G(z)=\widetilde{K}(z\rho^{-\frac{1}{2}})\,V^t$ instead of $\widetilde{K}(z)$, which obeys the reflection equation (4.3).

Lemma 4.2. *If $G(z)$ is a solution of the reflection equation* (4.3)*, then $DG(z)D$ is also a solution of* (4.3)*, where D is any constant diagonal matrix.*

Proof. One multiplies (4.3) on the right and on the left by $D_1\,D_2$ and uses the property (A.38) of the matrix $R_{12}(z)$. □

This property was already noticed in [31].

Proposition 4.3. *The invertible solutions to the reflection equation* (4.10) *must be, up to a normalization factor, of the form*

$$\widetilde{K}(z)=DG(z\rho^{\frac{1}{2}})D\,V^t\tag{4.30}$$

where D is any constant invertible diagonal matrix and $G(z)$ is one of the following matrices:

- $*$ *$G(z)$ is the unit matrix,* (4.31)
- $*$ $$G(z)=\frac{q^2-z^4}{q+1}\sum_{i=1}^{\mathcal{N}}E_{ii}-z^2(z^2\pm q)\sum_{i<j}E_{ij}+(q\pm z^2)\sum_{i>j}E_{ij},\tag{4.32}$$

$$* \quad G(z) = \sum_{i=1}^{\mathcal{N}/2} (qE_{2i-1,2i} - E_{2i,2i-1}) = G \qquad \textit{when } \mathcal{N} \textit{ is even,} \tag{4.33}$$

$$* \quad G(z) = z^2 E_{1\mathcal{N}} - qz^{-2} E_{\mathcal{N}1} + \sum_{i=1}^{(\mathcal{N}-2)/2} (qE_{2i,2i+1} - E_{2i+1,2i}) \qquad \textit{when } \mathcal{N} \textit{ is even.} \tag{4.34}$$

In the case of $gl(4)$, there is one additional solution:

$$G(z) = E_{13} \pm z^{-2} E_{31} + E_{24} \pm z^{-2} E_{42} \,. \tag{4.35}$$

Note that the solution (4.34) takes an antidiagonal form in the $gl(4)$ case:

$$G(z) = z^2\, E_{14} + q\, E_{23} - \; E_{32} - qz^{-2}\, E_{41} \,. \tag{4.36}$$

Proof. The matrix $G(z) = \widetilde{K}(z\rho^{-\frac{1}{2}}) V^t = \sum_{i,j=1}^{\mathcal{N}} G_{ij}(z)\, E_{ij}$ obeys the reflection equation (4.3).
The projection of (4.3) on $E_{ii} \otimes E_{jj}$ with $i < j$ reads

$$G_{ij}(w) \left(\frac{q^2 - z^2 w^2}{qw^2(z^2 - w^2)}\, G_{ji}(z) + \frac{1}{z^2 w^2}\, G_{ij}(z) \right) = G_{ji}(w) \left(\frac{q^2 - z^2 w^2}{qz^2(z^2 - w^2)}\, G_{ij}(z) + G_{ji}(z) \right) \tag{4.37}$$

which shows that $G_{ij}(z) = 0 \Leftrightarrow G_{ji}(z) = 0$. Moreover, the relation (4.37) implies that the non-vanishing off-diagonal elements $G_{ij}(z)$ and $G_{ji}(z)$ satisfy (α_{ij} and β_{ij} are free parameters)

$$G_{ji}(z) = F_{ij}(z)\, G_{ij}(z) \quad \text{with} \quad F_{ij}(z) = z^{-2}\, \frac{\alpha_{ij} z^2 - q\beta_{ij}}{\beta_{ij} z^2 - q\alpha_{ij}} \quad , \; i < j \,. \tag{4.38}$$

Then projecting (4.3) on the $E_{ij} \otimes E_{ij}$ component (with $i \neq j$), one gets

$$G_{ii}(z)\, G_{jj}(w) = G_{ii}(w)\, G_{jj}(z) \tag{4.39}$$

which implies that $G_{ii}(z)$ can be taken as constant numbers (note that it is still true if there is only one non-vanishing diagonal element since the matrix $G(z)$ is defined up to an overall normalization factor).

► Let us first assume that all diagonal elements G_{ii} are non-zero. Then, the $E_{ii} \otimes E_{ij}$ and $E_{ii} \otimes E_{ji}$ components of (4.3) lead to (with $i < j$)

$$\frac{q^2 - w^4}{w^2}\, G_{ij}(w) - \frac{q^2 - z^2 w^2}{z^2}\, G_{ij}(z) - q(z^2 - w^2) G_{ji}(z) = 0 \tag{4.40}$$

$$\frac{q^2 - w^4}{w^2}\, G_{ji}(w) - \frac{q^2 - z^2 w^2}{w^2}\, G_{ji}(z) - q\, \frac{z^2 - w^2}{z^2 w^2}\, G_{ij}(z) = 0 \tag{4.41}$$

from which it follows

$$G_{ij}(z) = qz^2\,\frac{\beta_{ij}z^2 - q\alpha_{ij}}{z^4 - q^2} \quad \text{and} \quad G_{ji}(z) = q\,\frac{\alpha_{ji}z^2 - q\beta_{ji}}{z^4 - q^2} \qquad , \; i<j \tag{4.42}$$

where $\alpha_{ji} = \alpha_{ij}$ and $\beta_{ji} = \beta_{ij}$.
Now, one projects (4.3) on the components $E_{ij} \otimes E_{kk}$ $(i \neq j \neq k)$:

$$\begin{aligned}(1-z^2w^2)\Big(\big(z^2\,\delta_{i<k}+w^2\,\delta_{i>k}\big)G_{kj}(z)G_{ik}(w)-\big(w^2\,\delta_{j<k}+z^2\,\delta_{j>k}\big)G_{ik}(z)G_{kj}(w)\Big)\\ +(z^2-w^2)\Big(\big(\delta_{j<k}+z^2w^2\,\delta_{j>k}\big)G_{ik}(z)G_{jk}(w)\\ -\big(z^2w^2\,\delta_{i<k}+\delta_{i>k}\big)G_{kj}(z)G_{ki}(w)\Big)=0\end{aligned} \tag{4.43}$$

and on the components $E_{ij} \otimes E_{kl}$ $(i,j,k,l$ all distinct):

$$\begin{aligned}&(1-z^2w^2)\Big(\big(z^2\,\delta_{i<k}+w^2\,\delta_{i>k}\big)G_{kj}(z)G_{il}(w)-\big(w^2\,\delta_{j<l}+z^2\,\delta_{j>l}\big)G_{il}(z)G_{kj}(w)\Big)\\ &+(z^2-w^2)\Big(\big(\delta_{j<k}+z^2w^2\,\delta_{j>k}\big)G_{ik}(z)G_{jl}(w)-\big(z^2w^2\,\delta_{i<l}+\delta_{i>l}\big)G_{lj}(z)G_{ki}(w)\Big)\\ &+(q-q^{-1})z^2w^2\Big(\big(z^2\,\delta_{i<(j,k)}+w^2\,\delta_{k<i<j}+w^{-2}\,\delta_{k>i>j}+z^{-2}\,\delta_{i>(j,k)}\big)G_{ki}(z)G_{jl}(w)\\ &-\big(z^2\,\delta_{(i,l)<j}+w^2\,\delta_{i<j<l}+w^{-2}\,\delta_{i>j>l}+z^{-2}\,\delta_{(i,l)>j}\big)G_{jl}(z)G_{ki}(w)\Big)=0\end{aligned} \tag{4.44}$$

where $\delta_{m<n} = 1$ if $m<n$ and zero otherwise (and similarly $\delta_{(m,p)<n} = 1$ if $m<n$ and $p<n$ and zero otherwise, and so on).
Inserting expressions (4.42) into (4.43) and (4.44), one gets

$$\alpha_{ik}\,\beta_{jl} = \beta_{ik}\,\alpha_{jl}\,, \;\; i<(j,k)<l \quad \text{and} \quad \alpha_{ik}\,\alpha_{jl} = \beta_{ik}\,\beta_{jl}\,, \;\; (i,j)<(k,l) \tag{4.45}$$

whose solution is given by $\alpha_{ij} = \epsilon\,\beta_{ij}$ for all pair (i,j) of indices $(\epsilon = \pm 1)$, with the consistency condition

$$\beta_{ij}\,\beta_{kl} = \beta_{ik}\,\beta_{jl} = \beta_{il}\,\beta_{jk} \qquad (i<j<k<l)\,. \tag{4.46}$$

The projection on the components $E_{ij} \otimes E_{ik}$:

$$\begin{aligned}&\frac{(1-z^2w^2)(qz^2+w^2)}{q+1}\,G_{ij}(z)G_{ik}(w)-(1-z^2w^2)\big(w^2\,\delta_{j<k}+z^2\,\delta_{j>k}\big)G_{ij}(w)G_{ik}(z)\\ &+\Big((qz^2-q^{-1}w^2)\big(z^2w^2\,\delta_{i<j}+\delta_{i>j}\big)G_{jk}(w)-(z^2-w^2)\big(z^2w^2\,\delta_{i<k}+\delta_{i>k}\big)G_{kj}(z)\\ &-(q-q^{-1})z^2w^2\big(z^2\,\delta_{(i,k)<j}+w^2\,\delta_{i<j<k}+w^{-2}\,\delta_{i>j>k}+z^{-2}\,\delta_{(i,k)>j}\big)G_{jk}(z)\Big)G_{ii}=0\end{aligned} \tag{4.47}$$

then leads to the conditions (with i,j,k distinct indices)

$$(q+1)\,G_{ii}\,\beta_{jk} = q\,\beta_{ij}\,\beta_{ik}\,. \tag{4.48}$$

Since $G_{ii} \neq 0$ for all i, equation (4.48) implies that the coefficients β_{ij} (and therefore α_{ij}) are either all zero, or all non-zero. In the first case, $G(z)$ is a constant diagonal matrix (which can be brought to the unit matrix due to Lemma 4.2). In the second case, equation (4.48) is solved as

$$\beta_{ij}^2 = \frac{(q+1)^2}{q^2}\, G_{ii}\, G_{jj}\,. \tag{4.49}$$

Thanks to Lemma 4.2, one can bring the matrix $G(z)$ in the form (4.32). Finally, one can check that all remaining equations obtained from (4.3) are satisfied.
These two cases correspond to the matrices (4.31) and (4.32) exhibited in the classification (up to multiplication by a function, so that $G(z)$ is analytical).

► We now consider the case where there is at least one diagonal element G_{ii} which is zero. The projections of (4.3) on $E_{ij} \otimes E_{ik}$ and $E_{ji} \otimes E_{ki}$ reduce to

$$(qz^2+w^2)G_{ij}(z)G_{ik}(w) - (q+1)\big(w^2\,\delta_{j<k} + z^2\,\delta_{j>k}\big)G_{ij}(w)G_{ik}(z) = 0 \tag{4.50}$$

$$(qz^2+w^2)G_{ji}(z)G_{ki}(w) - (q+1)\big(z^2\,\delta_{j<k} + w^2\,\delta_{j>k}\big)G_{ji}(w)G_{ki}(z) = 0 \tag{4.51}$$

which imply $G_{ij}(z)\,G_{ik}(w) = 0$ and $G_{ji}(z)\,G_{ki}(w) = 0$ for each triple (i,j,k) of distinct indices. Since $G_{ij}(z) = 0 \Leftrightarrow G_{ji}(z) = 0$, it follows that the reflection matrix $G(z)$ has at most one non-zero element in i-th row and in i-th column. We assume that $G_{ij}(z) \neq 0$ for some $j \neq i$ (otherwise the matrix $G(z)$ is not invertible).
Considering now the projections on $E_{in} \otimes E_{mj}$ and on $E_{im} \otimes E_{mj}$, one gets for $m,n \neq i,j$

$$\big(z^2\,\delta_{i<m}+w^2\,\delta_{i>m}\big)G_{mn}(z)G_{ij}(w)-\big(z^2\,\delta_{j<n}+w^2\,\delta_{j>n}\big)G_{ij}(z)G_{mn}(w) = 0 \tag{4.52}$$

from which one can deduce that there are complex numbers μ_{mn} such that

$$G_{mn}(z) = \begin{cases} \mu_{mn}\, G_{ij}(z) & \text{for } m<i \text{ and } n<j \text{ or } m>i \text{ and } n>j\,, \\ \mu_{mn}\, z^2\, G_{ij}(z) & \text{for } m<i \text{ and } n>j\,, \\ \mu_{mn}\, z^{-2}\, G_{ij}(z) & \text{for } m>i \text{ and } n<j\,. \end{cases} \tag{4.53}$$

Now, the $E_{ji} \otimes E_{jk}$ and $E_{ij} \otimes E_{kj}$ components of (4.3) $(i,j,k$ distinct) lead to equations similar to (4.50) and (4.51), relating the entries G_{ij} and G_{kj} on the one hand and G_{ji} and G_{jk} on the other hand. They imply that $G_{jk}(z) = G_{kj}(z) = 0$ for $k \neq i,j$. Hence, the matrix $G(z)$ exhibits the following shape (taking, e.g., $i<j$):

$$G(z) = \begin{pmatrix} (M_{11}) & 0 & (M_{12}) & 0 & (M_{13}) \\ 0 & 0 & 0 & G_{ij}(z) & 0 \\ (M_{21}) & 0 & (M_{22}) & 0 & (M_{23}) \\ 0 & G_{ji}(z) & 0 & G_{jj}(z) & 0 \\ (M_{31}) & 0 & (M_{32}) & 0 & (M_{33}) \end{pmatrix} \tag{4.54}$$

where (M_{mn}) represent block submatrices whose entries are given by (4.53).
Exchanging the rôles of the indices i and j, and taking into account the relations (4.53), one obtains the following necessary conditions for the matrix $G(z)$ to be invertible:

(i) If $|j-i|>1$ with $i\neq 1$ and $j\neq \mathcal{N}$, the block submatrices (M_{13}), (M_{31}) and (M_{mm}) are identically zero and the function $F_{ij}(z)$ must be equal to $\pm z^{-2}$ $(i<j)$, see eq. (4.38).

(ii) If $|j-i|>1$, with $i=1$ or $j=\mathcal{N}$ (exclusively), the conditions are similar to the case (i), but the block submatrices (M_{1m}) for $i=1$ or (M_{m3}) for $j=\mathcal{N}$ are not present.

(iii) If $i=1$ and $j=\mathcal{N}$, only the block submatrix (M_{22}) survives and the function $F_{1\mathcal{N}}(z)$ is equal to $-qz^{-4}$, see eq. (4.38).

(iv) If $j=i+1$, the block submatrices (M_{2m}) and (M_{m2}) are not present and the function $F_{i,i+1}(z)$ is equal to $-q^{-1}$, see eq. (4.38).

• *We first treat the case* (i), where $|j-i|>1$ with $i,j\neq 1,\mathcal{N}$. Due to the vanishing of (M_{mm}), the diagonal elements G_{kk} are zero for all $k\neq j$. Hence, from the above arguments, the matrix $G(z)$ has exactly one non-zero off diagonal element in each row and in each column to be invertible. One can easily prove by recursion that there is no such matrix when $\mathcal{N}$ is odd. Since we ask for invertible solutions, there exists in (M_{12}) at least an element $G_{i'j'}(z)\neq 0$ (note that $|j'-i'|>1$ since $i'<i<j'<j$). The compatibility between the shapes (4.54) of the matrix $G(z)$ for the pairs of indices (i,j) and (i',j') implies that (1) $G_{jj}=0$, (2) $j'-i'=j-i$ and (3) the elements $G_{i'+1,j'+1},\ \ldots,\ G_{i-1,j-1}$ are non-zero. More generally, one can prove by recursion that if G_{ij} and $G_{ji}\neq 0$, the non-zero off-diagonal elements of the matrix $G(z)$ lie on two lines parallel to the diagonal of the matrix and containing the elements G_{ij} and G_{ji} respectively. The inspection of the different configurations shows that the matrix $G(z)$ is never invertible.

• *The case* (ii) (i.e., $|j-i|>1$, with $i=1$ or $j=\mathcal{N}$ exclusively) requires special attention. A careful examination of the shape (4.54) of the matrix $G(z)$ shows three possibilities:

- There exists a pair (i',j') of indices for which $G_{i'j'}\neq 0$ such that $|j'-i'|>1$ and we are back to case (i);
- There is no such (i',j') pair, and $\mathcal{N}>4$. Then, it is impossible to have exactly one non-zero off diagonal element in each row and in each column for an invertible matrix $G(z)$;
- For $\mathcal{N}=4$, an explicit calculation exhibits the solution (4.35).

• *We consider now the case* (iii), where $i=1$ and $j=\mathcal{N}$. The $E_{m1}\otimes E_{m\mathcal{N}}$ component of (4.3) gives for $m\neq 1,\mathcal{N}$ (using the fact that $G_{\mathcal{N}1}(z)=-qz^{-4}G_{1\mathcal{N}}(z)$)

$$z^2(q^2z^2-w^2)\,G_{mm}(z)\,G_{1\mathcal{N}}(w)-w^2(q^2w^2-z^2)\,G_{mm}(w)\,G_{1\mathcal{N}}(z)=0 \qquad (4.55)$$

while the $E_{m1}\otimes E_{\mathcal{N}m}$ component reads

$$z^2\,G_{mm}(z)\,G_{1\mathcal{N}}(w)-w^2\,G_{mm}(w)\,G_{1\mathcal{N}}(z)=0\,. \qquad (4.56)$$

Equations (4.55)–(4.56) imply $G_{mm}=0$ for all $m\neq 1,\mathcal{N}$. If $G_{\mathcal{N}\mathcal{N}}\neq 0$, the $E_{m\mathcal{N}}\otimes E_{\mathcal{N}n}$ component for $m,n\neq 1,\mathcal{N}$ leads to $G_{mn}(z)=G_{mn}(w)$, which implies $G_{mn}(z)=0$ when $1<(m,n)<\mathcal{N}$ due to (4.53). Therefore the matrix $G(z)$ is not invertible. Hence, we assume that all diagonal elements are zero. If there exists a pair (i', j') of indices such that $|j' - i'| > 1$, we are back to the case (i) (and the matrix $G(z)$ is not invertible in that case). Otherwise, all non-zero off diagonal elements belong to 2×2 submatrices corresponding to the indices $(m, m+1)$ of the matrix $G(z)$, denoted hereafter as 'block m'. The blocks should not overlap (i.e., a block m cannot be followed by a block $m+1$) and should be adjacent (i.e., a block m should be followed by a block $m+2$), otherwise the matrix $G(z)$ is not invertible. Since $G_{1\mathcal{N}}(z) \neq 0$, the matrix $G(z)$ contains blocks m with m even only. Using Lemma 4.2, the matrix $G(z)$ is brought to the form (4.34). Finally, all remaining equations obtained from (4.3) are satisfied.

- *We finally consider the case (iv)*, where $j = i + 1$. Suppose that $G_{mm} \neq 0$ for some $m \neq i, j$. Since $j - i = 1$, one has necessarily $m < i, j$ or $m > i, j$. The $E_{im} \otimes E_{mj}$ component of (4.3) reduces here to

$$\big(G_{ij}(z) - G_{ij}(w)\big)\, G_{mm} = 0 \tag{4.57}$$

which implies that $G_{ij}(z)$ is constant. Moreover, the projection on $E_{mi} \otimes E_{mj}$ takes here the form

$$\big(z^2 w^2\, \delta_{m<i} + \delta_{m>i}\big) G_{ij}\, G_{mm} = 0 \tag{4.58}$$

leading to $G_{ij}\, G_{mm} = 0$, which is obviously not satisfied. Therefore, one has $G_{mm} = 0$ for $m \neq i, j$. If $G_{jj} \neq 0$, $G_{ij}(z)$ is given by (4.42) with $\beta_{ij} = 0$. The projection on $E_{jn} \otimes E_{mj}$ (with $m, n \neq j, j-1$) reads

$$\Big(\big(z^2\, \delta_{j<m} + w^2\, \delta_{j>m}\big) G_{mn}(z) - \big(z^2\, \delta_{j<n} + w^2\, \delta_{j>n}\big) G_{mn}(w)\Big) G_{jj} = 0 \tag{4.59}$$

and thus leads to $G_{mn}(z) = \mu'_{mn}$ if $m, n < i$ or $m, n > j$, $G_{mn}(z) = \mu'_{mn} z^2$ if $m < i$ and $n > j$, and $G_{mn}(z) = \mu'_{mn} z^{-2}$ if $m > j$ and $n < i$. This is clearly in contradiction with (4.53), unless $G_{mn}(z) = 0$ for all $m, n \neq i, j$, giving a non-invertible matrix.

Therefore, all diagonal elements must be equal to zero. There are then three possibilities: either there exists a pair (i', j') of indices such that $|j' - i'| > 1$ corresponding to the case (i) or (ii); or $G_{1\mathcal{N}}(z) \neq 0$ and we are back to the case (iii); or $G_{1\mathcal{N}}(z) = 0$ and all non-zero off diagonal elements consist in blocks m subjected to the same constraints as in case (iii), but now with m odd. One finds then the solution (4.33) if $\mathcal{N}$ is even, while the matrix $G(z)$ is not invertible if $\mathcal{N}$ is odd. □

The solutions (4.32) were given for the first time in [31], and used in [32] in the context of sine-Gordon and affine Toda field theories on the half line. The solution (4.33) was used in [33] to define and study the quantum twisted Yangians. To our knowledge, the solutions (4.34), (4.35) and (4.36) are new. Let us stress that (4.36) is the only anti-diagonal matrix appearing in the classification. Hence, it corresponds, after the change (4.30), to the only case where the reflection equation (4.10) admits a diagonal solution. Note that with the use of non-invertible diagonal matrices, the Lemma 4.2 provides also classes of non-invertible solutions.

Of course as in the case of SP boundary conditions the ultimate goal is the derivation of the spectrum of the transfer matrix. However, in our case due to the lack of diagonal solutions of the equation (4.10), deriving the spectrum is turning to an intricate problem. This is primarily due to the fact that an obvious reference state is not available anymore. One could proceed using a generalization of the method presented in [34, 35] for the case of non-diagonal boundaries, such analysis however will not be attempted in the present work. It is under investigation.

5. Conclusion

This paper is concerned with generalizations of the XXZ model. We deal with spin chains where at each site may be associated a different representation of $\mathcal{U}_q(gl(\mathcal{N}))$, and which have various types of boundary conditions: periodic, soliton preserving or soliton non-preserving. We have computed the Bethe equations, which is the main physical results of this paper, thanks to the analytical Bethe ansatz. This method involves the classification of the representations and the knowledge of the center of the underlying algebra. Therefore, an important part of this paper is devoted to the study of the algebra $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ and its associated subalgebras: the reflection algebra and the quantum twisted Yangian. For the paper to be self-contained, we first review these well-known algebraic structures. Then, we also establish new results (concerning representations and centers) for these algebras. These results look interesting by themselves from the algebraic point of view.

Although we treat a generic case, the Bethe equations take surprisingly a very compact form, depending only on the Drinfel'd polynomials, the boundaries and the Cartan matrix of $gl(\mathcal{N})$. This simple form enlightens the deep connection between the theory of representations and the Bethe ansätze. The determination of transfer matrix eigenvectors, which is beyond the scope of analytical Bethe ansatz, would certainly be simplified by an utter understanding of this relation.

As mentioned in the paper, we do not deal with non-diagonal boundary conditions: they cannot be handled, at the moment, with the method presented here. This problem has been treated in some particular cases: the XXZ model (i.e., $\mathcal{U}_q(\widehat{gl}(2))$ with fundamental representations) has been solved in [34, 35], and some progress has been achieved for $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ with fundamental representations in [36]. However, a generic method (similar to the one used for spin chains based on Yangians [4]), which would deal with the full type of models, is still lacking.

Another open problem is worth mentioning: there exists no systematic method to compute local Hamiltonians from the transfer matrices for generic representations. It is known that, in the fundamental representation, one obtains the Hamiltonian for the closed or open XXZ model with a very simple formula. In the general framework treated in this paper, a local Hamiltonian must be found case by case, after having chosen a representation at each site. It implies in general the fusion procedure for auxiliary spaces, and no general proof for the existence of such local Hamiltonian is known when the representations at each site are different. This point is crucial, since locality is an important property in physics. The proof and the explicit construction of local Hamiltonian is thus a challenging open question. The coefficients of the transfer matrices introduced here being non-local Hamiltonians, one could be disappointed by such a result. However, the models being integrable and finite dimensional, it is reasonable to believe that the local Hamiltonian is 'hidden' in some complicated way in the series of Hamiltonians provided by the transfer matrix. In particular, it is known that the Bethe ansatz equations presented in this paper are the right ones which will parametrize the spectrum of this hypothetical local Hamiltonian.

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Appendix A. R matrices

A.1. The R matrix of $\mathcal{U}_q(gl(\mathcal{N}))$

The R matrix of the finite dimensional quantum group $\mathcal{U}_q(gl(\mathcal{N}))$ is given by

$$R = q\sum_{a=1}^{\mathcal{N}} E_{aa}\otimes E_{aa} + \sum_{1\leq a\neq b\leq \mathcal{N}} E_{aa}\otimes E_{bb} + (q-q^{-1})\sum_{1\leq a<b\leq\mathcal{N}} E_{ab}\otimes E_{ba}, \quad (A.1)$$

where E_{ab} are the elementary matrices with 1 in position (a,b) and 0 elsewhere.

This R matrix obeys the Yang-Baxter equation

$$R_{12}\ R_{13}\ R_{23} = R_{23}\ R_{13}\ R_{12}. \quad (A.2)$$

If we note $R_{12}[q]$ the matrix (A.1) with deformation parameter q, one has the properties

$$R_{12}[q]\, R_{12}[q^{-1}] = \mathbb{I}\otimes\mathbb{I}\,, \quad (A.3)$$

$$R_{12}[q] - R_{21}[q^{-1}] = (q-q^{-1})\mathcal{P}\,, \quad (A.4)$$

where $\mathcal{P}\equiv\mathcal{P}_{12}=\mathcal{P}_{21}$ is the permutation operator

$$\mathcal{P} = \sum_{a,b=1}^{\mathcal{N}} E_{ab}\otimes E_{ba}\ , \quad (A.5)$$

and

$$R_{21} = \mathcal{P}\, R_{12}\, \mathcal{P}\,. \quad (A.6)$$

It is also convenient to introduce a deformation of the permutation operator, the q-permutation operator $P^q_{12}\in \mathrm{End}(\mathbb{C}^{\mathcal{N}}\otimes\mathbb{C}^{\mathcal{N}})$ defined by

$$P^q_{12} = \sum_{i=1}^{\mathcal{N}} E_{ii}\otimes E_{ii} + q\sum_{i>j}^{\mathcal{N}} E_{ij}\otimes E_{ji} + q^{-1}\sum_{i<j}^{\mathcal{N}} E_{ij}\otimes E_{ji}\,. \quad (A.7)$$

We have gathered below some useful properties of the q-permutation. Then, the R matrix (A.1) can be rewritten as

$$R_{12} = \mathbb{I}\otimes\mathbb{I} + q\mathcal{P} - P^q_{12}\,. \tag{A.8}$$

Finally, let us note the following identities

$$R_{21} = R_{12}^{t_1t_2} = U_1U_2\,R_{12}\,U_1^{-1}U_2^{-1} \tag{A.9}$$

$$U_1\,R_{12}^{t_2}\,U_1^{-1} = U_2^t\,R_{12}^{t_1}\,(U_2^t)^{-1} \tag{A.10}$$

$$\left(R_{12}^{-1}\right)^{t_1} = M_1^{-1}\,\left(R_{12}^{t_1}\right)^{-1}\,M_1 \tag{A.11}$$

where ^{t_1} (resp. ^{t_2}) is the transposition in the first (resp. second) space, U is any antidiagonal invertible matrix $U=\sum_{a=1}^{N} u_aE_{a\bar{a}}$ and

$$M = \sum_{a=1}^{\mathcal{N}} \theta_0\,q^{\mathcal{N}-2a+1}\,E_{aa}\,, \quad \theta_0=\pm1\,. \tag{A.12}$$

Relation (A.9) implies

$$W_1\,W_2\,R_{12} \;=\; R_{12}\,W_1\,W_2 \tag{A.13}$$

where W is any invertible diagonal matrix.

A.2. Some properties of the q-permutation

The q-permutation can be rewritten in a condensed way as

$$P^q_{12} = \sum_{a,b=1}^{\mathcal{N}} q^{\mathrm{sgn}(a-b)}E_{ab}\otimes E_{ba} \tag{A.14}$$

where sgn is the sign function, with the convention $\mathrm{sgn}(0)=0$. Note that this q-permutation is not symmetric anymore,

$$P^q_{21} = \mathcal{P}\,P^q_{12}\,\mathcal{P} = P^{\bar q}_{12} \qquad \text{with} \quad \bar q = q^{-1} \tag{A.15}$$

but still obey the permutation group relations:

$$\left(P^q_{12}\right)^2 = \mathbb{I}\otimes\mathbb{I}\,, \tag{A.16}$$

$$P^q_{12}\,P^q_{23}\,P^q_{12} = P^q_{23}\,P^q_{12}\,P^q_{23}\,. \tag{A.17}$$

More generally, for q and t deformation parameters, one has:

$$P^q_{12}\,P^t_{12} = P^{q/t}_{12}\,\mathcal{P} = \mathcal{P}\,P^{t/q}_{12}\,. \tag{A.18}$$

With P^q_{12} comes the notion of *q-deformed operator*: to each operator-valued matrix

$$A = \sum_{i,j=1}^{\mathcal{N}} A_{ij}E_{ij}\,, \tag{A.19}$$

one associates its q-deformed version as

$$A^q = \sum_{i,j=1}^{\mathcal{N}} q^{\mathrm{sgn}(j-i)}A_{ij}E_{ij}\,. \tag{A.20}$$

These definitions are justified by the following relations (with $\bar{q} = q^{-1}$, $A_1 = A \otimes \mathbb{I}$, $A_2 = \mathbb{I} \otimes A$):

$$P^q_{12}\, A_1\, P^q_{12} = A^q_2 \quad \text{and} \quad P^q_{12}\, A_2\, P^q_{12} = A^{\bar{q}}_1 . \tag{A.21}$$

We will also need:

$$\mathrm{tr}_2\, A_1\, B_2\, P^q_{12} = A\, B^{\bar{q}} \quad ; \quad \mathrm{tr}_1\, A_2\, B_1\, P^q_{12} = A\, B^q \tag{A.22}$$

$$\mathrm{tr}_2\, P^q_{12}\, A_2\, B_1 = A^{\bar{q}}\, B \quad ; \quad \mathrm{tr}_1\, P^q_{12}\, A_1\, B_2 = A^q\, B \tag{A.23}$$

as well as, when $[A_1\,,\, B_2] = 0$:

$$\mathrm{tr}_1\, A_1\, B_2\, P^q_{12} = B\, A^q \quad ; \quad \mathrm{tr}_2\, A_2\, B_1\, P^q_{12} = B\, A^{\bar{q}} \tag{A.24}$$

$$\mathrm{tr}_1\, P^q_{12}\, A_2\, B_1 = B^q\, A \quad ; \quad \mathrm{tr}_2\, P^q_{12}\, A_1\, B_2 = B^{\bar{q}}\, A . \tag{A.25}$$

A.3. The R matrix of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$

We consider the $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ algebra, whose R matrix can be constructed through the Baxterization of the R matrix (A.1):

$$R_{12}(z) = z\, R_{12} - z^{-1}\, R^{-1}_{21} \tag{A.26}$$

It is given by the explicit expression

$$\begin{aligned} R(z) = \mathfrak{a}(z) \sum_{a=1}^{\mathcal{N}} E_{aa} \otimes E_{aa} + \mathfrak{b}(z) \sum_{1 \le a \ne b \le \mathcal{N}} E_{aa} \otimes E_{bb} \\ + \mathfrak{c} \sum_{1 \le a \ne b \le \mathcal{N}} z^{\mathrm{sgn}(b-a)} E_{ab} \otimes E_{ba} \end{aligned} \tag{A.27}$$

with $\mathfrak{a}(z) = qz - q^{-1}z^{-1}$, $\mathfrak{b}(z) = z - z^{-1}$ and $\mathfrak{c} = q - q^{-1} = \mathfrak{a}(1)$. It can be also written as

$$R_{12}(z) = \mathfrak{b}(z) \left(\mathbb{I} \otimes \mathbb{I} - P^q_{12} \right) + \mathfrak{a}(z)\, \mathcal{P} . \tag{A.28}$$

At $z = 1$ the R matrix reduces to the permutation operator $\mathcal{P}$ given in (A.5): $R(1) = \mathfrak{c}\, \mathcal{P}$.

In what follows we shall make use of the antidiagonal matrix $V = \sum_{a,b=1}^{\mathcal{N}} V_{ab} E_{ab}$ defined by:

$$V_{ab} = q^{\frac{a-b}{2}} \delta_{b\bar{a}} \quad \text{or} \quad V_{ab} = i(-1)^a q^{\frac{a-b}{2}} \delta_{b\bar{a}} \ . \tag{A.29}$$

where $\bar{a} = \mathcal{N}+1-a$. The second choice is forbidden for $\mathcal{N}$ odd. We will parametrize these two choices by a sign $\theta_0 = \pm 1$, the second choice being associated to $\theta_0 = -1$. With this convention, one can encompass (formally) the two choices as

$$V_{ab} = \theta_0^{a+\frac{1}{2}} q^{\frac{a-\bar{a}}{2}}\, \delta_{b\bar{a}} \equiv \theta_a\, \delta_{b\bar{a}} . \tag{A.30}$$

Note that we have in both cases $\theta_0^{\mathcal{N}} = 1$ and $V^2 = \mathbb{I}$.

The R matrix (A.27) satisfies the following properties:

Yang–Baxter equation [37, 38, 39, 40]

$$R_{12}(\frac{z}{w})\ R_{13}(z)\ R_{23}(w) = R_{23}(w)\ R_{13}(z)\ R_{12}(\frac{z}{w}) \tag{A.31}$$

$$R_{21}(z) = \mathcal{P}\, R_{12}(z)\, \mathcal{P} = R_{12}^{t_1 t_2}(z) \tag{A.32}$$

Unitarity

$$R_{12}(z)\ R_{21}(z^{-1}) = \zeta(z) \tag{A.33}$$

where

$$\zeta(z) = (qz - q^{-1}z^{-1})(qz^{-1} - q^{-1}z) \tag{A.34}$$

Crossing-unitarity

$$R_{12}^{t_1}(z)\ M_1\ R_{12}^{t_2}(z^{-1}\rho^{-2})\ M_1^{-1} = \bar{\zeta}(z\rho) \tag{A.35}$$

where $\rho = q^{\frac{\mathcal{N}}{2}}$, $M = V^t\, V$ and

$$\bar{\zeta}(z) = (z\rho - z^{-1}\rho^{-1})(z^{-1}\rho - z\rho^{-1})\,. \tag{A.36}$$

CP-invariance

$$R_{21}(z) = U_1 U_2\, R_{12}(z)\, U_1^{-1} U_2^{-1} \tag{A.37}$$

for any antidiagonal invertible matrix $U = \sum_{a=1}^{N} u_a E_{a\bar{a}}$. Examples of such matrices are given by V, V^t and also $\sum_{a=1}^{N} E_{a\bar{a}}$.
This relation implies in particular

$$\Big[W_1 W_2,\ R_{12}(z)\Big] = 0\,, \tag{A.38}$$

for any invertible diagonal constant matrix W. It can be checked by direct calculation that this property remains valid when W is not invertible. An example for an (invertible) W matrix is provided by $M = V^t\, V$, whose explicit expression is given by relation (A.12).

The R matrix can be interpreted physically as a scattering matrix [41, 40, 9] describing the interaction between two solitons that carry the fundamental representation of $gl(\mathcal{N})$.

A.4. The matrix $\bar{R}(z)$

The CP-invariance of R allows the existence, in the general case, of anti-solitons carrying the conjugate representation of $gl(\mathcal{N})$. The scattering matrix which describes the interaction between a soliton and an anti-soliton, is given by

$$R_{\bar{1}2}(z) = \bar{R}_{12}(z) = V_1 R_{12}^{t_2}(z^{-1}\rho^{-1}) V_1 \tag{A.39}$$

$$= V_2^t R_{12}^{t_1}(z^{-1}\rho^{-1}) V_2^t\ =:\ R_{1\bar{2}}(z) = \bar{R}_{21}^{t_1 t_2}(z). \tag{A.40}$$

Note that equality between the first and second lines of these relations is a consequence of the properties listed above. In the case $\mathcal{N} = 2$ and for the second choice of V ($sp(2)$ case), $\bar{R}$ is proportional to R, so that there is no genuine notion of anti-soliton. This reflects the fact that the fundamental representation of $sp(2) = sl(2)$

is self-conjugate. This does not contradict the fact that for $\mathcal{N}=2$ and for $\theta_0=+1$ ($so(2)$ case), there exists a notion of soliton and anti-soliton.

The equality between $R_{\bar{1}2}(z)$ and $R_{1\bar{2}}(z)$ in (A.39) reflects the CP invariance of R, from which one also has $R_{\bar{1}\bar{2}}=R_{12}$, i.e., the scattering matrix of two anti-solitons is equal to the scattering matrix of two solitons. The explicit form of the $\bar{R}$ matrix is

$$\bar{R}(z)=\bar{\mathfrak{a}}(z)\sum_{a=1}^{\mathcal{N}}E_{\bar{a}\bar{a}}\otimes E_{aa}+\bar{\mathfrak{b}}(z)\sum_{1\le a\neq b\le\mathcal{N}}E_{\bar{a}\bar{a}}\otimes E_{bb}+\mathfrak{c}\sum_{1\le a\neq b\le\mathcal{N}}(\theta_0 q)^{a-b}(z\rho)^{\mathrm{sgn}(b-a)}E_{\bar{b}\bar{a}}\otimes E_{ba}\qquad\text{(A.41)}$$

where we set $\bar{\mathfrak{a}}(z)=\mathfrak{a}(z^{-1}\rho^{-1})$ and $\bar{\mathfrak{b}}(z)=\mathfrak{b}(z^{-1}\rho^{-1})$.

The matrix $\bar{R}(z)$ reduces to a one dimensional projector at $z=\rho^{-1}$, i.e.,

$$\bar{R}_{12}(\rho^{-1})=(q-q^{-1})\;V_1\;\mathcal{P}_{12}^{t_2}\;V_1=(q-q^{-1})\;\mathcal{N}\;Q_{12}\,.\qquad\text{(A.42)}$$

The matrix Q_{12} is a projector (i.e., $Q^2=Q$) onto a one-dimensional space and is written as

$$Q_{12}=\frac{1}{\mathcal{N}}\sum_{a,b=1}^{\mathcal{N}}(\theta_0 q)^{a-b}E_{\bar{b}\bar{a}}\otimes E_{ba}\,.\qquad\text{(A.43)}$$

Let us remark that Q_{12} is not a symmetric operator, i.e., $Q_{12}\neq Q_{21}$. It will be important in the quantum contraction for the reflection algebra (see Section B).

The $\bar{R}$ matrix (A.40) also obeys

(i) A Yang–Baxter equation

$$\bar{R}_{12}(\frac{z}{w})\;\bar{R}_{13}(z)\;R_{23}(w)=R_{23}(w)\;\bar{R}_{13}(z)\;\bar{R}_{12}(\frac{z}{w})\qquad\text{(A.44)}$$

(ii) Unitarity

$$\bar{R}_{12}(z)\;\bar{R}_{21}(z^{-1})=\bar{\zeta}(z)\qquad\text{(A.45)}$$

(iii) Crossing-unitarity

$$\bar{R}_{12}^{t_1}(z)\;M_1\;\bar{R}_{12}^{t_2}(z^{-1}\rho^{-2})\;M_1^{-1}=\zeta(z\rho)\qquad\text{(A.46)}$$

and moreover

$$\Big[M_1M_2,\;\bar{R}_{12}(z)\Big]=0.\qquad\text{(A.47)}$$

We remind that the functions ζ and $\bar{\zeta}$ are defined in (A.34) and (A.36).

Appendix B. Review on fusion

We present in this appendix the fusion procedure for both periodic and open spin chains. Such process provides sets of constraints facilitating the derivation of the spectrum of the corresponding spin chain. The presentation will be different to the previous one made for example in [30, 25]. Here, we focus on certain algebraic aspects which may be usefull in other contexts since we give an explicit construction of central elements for $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ and the reflection algebra.

B.1. Quantum contraction for $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$

The fusion relies essentially on the fact that $\bar{R}(\rho^{-1})$ reduces to an one-dimensional projector (see (A.42)). For this purpose, let us introduce, for $\mathcal{L}^{\pm}(z)$ obeying the relations (2.4)–(2.5) at $c = 0$,

$$\widehat{\mathcal{L}}_1^{\pm}(z) = V_1 \, \Big(\big(\mathcal{L}_1^{\pm}(z\rho)\big)^{-1} \Big)^{t_1} \, V_1 \tag{B.1}$$

which satisfy

$$\bar{R}_{12}(\frac{z}{w}) \; \widehat{\mathcal{L}}_1^{\pm}(z) \; \mathcal{L}_2^{\pm}(w) = \mathcal{L}_2^{\pm}(w) \; \widehat{\mathcal{L}}_1^{\pm}(z) \; \bar{R}_{12}(\frac{z}{w}) \tag{B.2}$$

$$\bar{R}_{12}(\frac{z}{w}) \; \widehat{\mathcal{L}}_1^{-}(z) \; \mathcal{L}_2^{+}(w) = \mathcal{L}_2^{+}(w) \; \widehat{\mathcal{L}}_1^{-}(z) \; \bar{R}_{12}(\frac{z}{w}). \tag{B.3}$$

By considering $\frac{z}{w} = \rho^{-1}$ in (B.2), we conclude that

$$Q_{12}\widehat{\mathcal{L}}_1^{\pm}(z) \; \mathcal{L}_2^{\pm}(z\rho) = \mathcal{L}_2^{\pm}(z\rho)\widehat{\mathcal{L}}_1^{\pm}(z) \; Q_{12} = Q_{12}\widehat{\mathcal{L}}_1^{\pm}(z) \; \mathcal{L}_2^{\pm}(z\rho)Q_{12} \tag{B.4}$$

$$\equiv Q_{12} \; \delta(\mathcal{L}^{\pm}(z))\,. \tag{B.5}$$

The coefficients of the formal series $\delta(\mathcal{L}^{\pm}(z))$, called quantum contraction of $\mathcal{L}^{\pm}(z)$, belong to the center of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$, the proof being similar to the one used in the case of the Yangian [19].

Using the above construction of central elements for $\mathcal{T}^{\pm} = \Delta^{(\mathcal{N})}(\mathcal{L}^{\pm})$ and taking the trace over the auxiliary spaces 1 and 2 of the relation

$$\widehat{\mathcal{T}}_1^{\pm}(z)\mathcal{T}_2^{\pm}(z\rho) = Q_{12} \; \widehat{\mathcal{T}}_1^{\pm}(z)\mathcal{T}_2^{\pm}(z\rho) + (\mathbb{I}_{\mathcal{N}} \otimes \mathbb{I}_{\mathcal{N}} - Q_{12})\widehat{\mathcal{T}}_1^{\pm}(z)\mathcal{T}_2^{\pm}(z\rho)\;, \tag{B.6}$$

we obtain the fusion relation

$$\widehat{t}^{\pm}(z)t^{\pm}(z\rho) = \delta(\mathcal{T}^{\pm}(z)) + \widetilde{t}^{\pm}(z)\;. \tag{B.7}$$

We have introduced the fused transfer matrix

$$\widetilde{t}^{\pm}(z) = \mathrm{tr}_{12}\Big((\mathbb{I}_{\mathcal{N}} \otimes \mathbb{I}_{\mathcal{N}} - Q_{12})\widehat{\mathcal{T}}_1^{\pm}(z)\mathcal{T}_2^{\pm}(z\rho) \Big) \tag{B.8}$$

and the transfer matrix

$$\widehat{t}^{\pm}(z) = \mathrm{tr}_1(\widehat{\mathcal{T}}_1^{\pm}(z))\;. \tag{B.9}$$

B.2. Quantum contraction for the reflection algebra

As in the closed case we are going to exploit the fact that the R matrix reduces to a one-dimensional projector for a particular value of the spectral parameter. We need also new solutions, denoted $\bar{K}(z)$, of the same reflection equation (2.34) describing physically the reflection of an anti-soliton to an anti-soliton (see also [25, 26]). A new type of reflection equation is needed, which gives an additional constraint between $K(z)$ and $\bar{K}(z)$

$$\bar{R}_{12}(\frac{z}{w}) \; \bar{K}_1(z) \; \bar{R}_{21}(zw) \; K_2(w) = K_2(w) \; \bar{R}_{12}(zw) \; \bar{K}_1(z) \; \bar{R}_{21}(\frac{z}{w})\,. \tag{B.10}$$

Note that this equation is equivalent (exchanging spaces 1 and 2, arguments z and w, and using unitarity for $\bar{R}(z)$) to

$$\bar{R}_{12}(\frac{z}{w})\ K_1(z)\ \bar{R}_{21}(zw)\ \bar{K}_2(w) = \bar{K}_2(w)\ \bar{R}_{12}(zw)\ K_1(z)\ \bar{R}_{21}(\frac{z}{w}), \tag{B.11}$$

which shows that couple of solutions $(K(z), \bar{K}(z))$ can be consistently defined.

Lemma B.1. *Let* $K(z)$ *and* $\bar{K}(z)$ *be two diagonal solutions* (2.41) *of the reflection equation* (2.34)*, with parameters* $(\mathcal{M}, \xi)$ *and* $(\bar{\mathcal{M}}, \bar{\xi})$ *respectively. Then, they obey the additional relation* (B.10) *if and only if*

$$\mathcal{M} + \bar{\mathcal{M}} = \mathcal{N} \qquad \textit{and} \qquad \xi\, q^{\mathcal{M}/2} = \pm\, \bar{\xi}\, q^{\bar{\mathcal{M}}/2}\,. \tag{B.12}$$

Proof. Up to a multiplication by $(zw)^2$, the relation (B.10) is polynomial in z and w. Considering the $(zw)^2$ coefficient, one gets

$$\begin{aligned}\bar{\xi}^2 &\left(\sum_{a<b} q^{2b-\mathcal{N}-1}\,(k_b - k_a)\ \mathcal{E}_{ab} + (1-q^{-2}) \sum_{a<b<c} q^{2c-\mathcal{N}-1}\,(k_b - k_a)\ \mathcal{E}_{ab}\right) \\ &= (\rho\xi)^2 \left(\sum_{a<b} (\widetilde{k}_a - q^{-2}\widetilde{k}_b)\ \mathcal{E}_{ab} + (1-q^{-2}) \sum_{a<c<b} \widetilde{k}_b\ \mathcal{E}_{ab}\right)\end{aligned} \tag{B.13}$$

where $\mathcal{E}_{ab} = E_{ab} \otimes E_{ab} - E_{ba} \otimes E_{ba}$ and k_a (resp. $\widetilde{k}_a$) are the diagonal terms of $K(0)$ (resp. $V\bar{K}(0)V^t$).
Considering the case $b = a+1$, we are led to

$$q^{-1}\bar{\xi}^2(k_{a+1} - k_a) = \xi^2(\widetilde{k}_a - q^{-2}\widetilde{k}_{a+1})\,. \tag{B.14}$$

Since $k_{a+1} - k_a = 0$ if $a \neq \mathcal{M}$ and $\widetilde{k}_a - q^{-2}\widetilde{k}_{a+1} = 0$ if $a \neq \mathcal{N} - \bar{\mathcal{M}}$, this implies the first constraint. The second one is obtained setting $a = \mathcal{M}$ in (B.14).
Finally, one checks directly that these constraints are sufficient to solve the relation (B.10). □

In particular, $K(z) = \mathbb{I}$ and $\bar{K}(z) = \mathbb{I}$ are solutions to the set of constraints. Similarly, we introduce a new solution $\bar{K}^+(z)$ of the dual reflection equation (2.47) with the following consistency relation

$$\begin{aligned}&\bar{R}_{12}(\frac{w}{z})\ \Big(\bar{K}_1^+(z)\Big)^t\ M_1^{-1}\ \bar{R}_{21}(w^{-1}z^{-1}\rho^{-2})\ M_1\ \Big(K_2^+(w)\Big)^t = \\ &\Big(K_2^+(w)\Big)^t\ M_1\ \bar{R}_{12}(w^{-1}z^{-1}\rho^{-2})\ M_1^{-1}\ \Big(\bar{K}_1^+(z)\Big)^t\ \bar{R}_{21}(\frac{w}{z})\,.\end{aligned} \tag{B.15}$$

Starting from $K(z)$ and $\bar{K}(z)$, solutions to equations (2.34), (2.47) and (B.10), the matrices $K^+(z) = f(z)K(\rho^{-1}z^{-1})^tM$ and $\bar{K}^+(z) = g(z)\bar{K}(\rho^{-1}z^{-1})^tM$ are solutions to the dual reflections equations ($f(z)$ and $g(z)$ are arbitrary functions chosen in the spin chain context such that the matrices have analytical entries). Similarly to relation (2.45), we can define a new monodromy matrix

$$\bar{\mathcal{B}}_1(z) = \widehat{\mathcal{T}}_1^+(z)\ \bar{K}_1(z)\ V_1^t\Big(\mathcal{T}_1^-(z^{-1}\rho^{-1})\Big)^{t_1} V_1^t\,, \tag{B.16}$$

obeying the exchange relation (2.34) and satisfying consistency relation (B.10) with $\mathcal{B}(z)$. We can deduce from relation (B.10) (multiplying on the right by V_1V_2 and setting $\frac{w}{z}=\rho$) the following equality

$$Q_{12}\bar{\mathcal{B}}_1(z)\ \bar{R}_{21}(z^2\rho)\ \mathcal{B}_2(z\rho)V_1V_2 = \mathcal{B}_2(z\rho)\ \bar{R}_{12}(z^2\rho)\ \bar{\mathcal{B}}_1(z)\ V_1V_2\ Q_{12} \tag{B.17}$$

$$\equiv Q_{12}\delta(\mathcal{B}(z)) \tag{B.18}$$

and from relation (B.15) (transposing in both spaces 1 and 2, multiplying on the left by $V_1V_2\bar{R}_{21}(\frac{z}{w})$ and on the right by $\bar{R}_{12}(\frac{z}{w})$ and setting $\frac{z}{w}=\rho$)

$$Q_{12}V_1V_2K_2^+(z\rho)\ M_1\bar{R}_{12}(z^{-2}\rho^{-3})\ M_1^{-1}\bar{K}_1^+(z) \tag{B.19}$$

$$= V_1V_2\bar{K}_1^+(z)\ M_1^{-1}\bar{R}_{21}(z^{-2}\rho^{-3})\ M_1K_2^+(z\rho)\ \ Q_{12} \equiv Q_{12}\delta(K^+(z))\,. \tag{B.20}$$

Proposition B.2. *All the coefficients of the series $\delta(\mathcal{B}(z))$, called quantum contraction of $\mathcal{B}(z)$, belong to the center of the reflection algebra.*

Proof. Using reflection equations (2.34) and (B.10) as well as Yang-Baxter equations (A.31) and (A.44), we can show

$$\begin{aligned}&\bar{R}_{01}(\frac{w}{z})R_{02}(\frac{w}{y})\mathcal{B}_0(w)R_{20}(wy)\bar{R}_{10}(wz)\mathcal{B}_2(y)\bar{R}_{12}(yz)\bar{\mathcal{B}}_1(z)\bar{R}_{21}(\frac{z}{y}) = \\ &\qquad \mathcal{B}_2(y)\bar{R}_{12}(yz)\bar{\mathcal{B}}_1(z)\bar{R}_{21}(\frac{z}{y})\bar{R}_{01}(wz)R_{02}(wy)\mathcal{B}_0(w)R_{20}(\frac{w}{y})\bar{R}_{10}(\frac{w}{z})\,.\end{aligned} \tag{B.21}$$

Multiplying on the right by V_1V_2 and taking the particular value $y=\rho z$ in the previous relation, we obtain

$$\begin{aligned}&\bar{R}_{01}(\frac{w}{z})R_{02}(\frac{w}{\rho z})\mathcal{B}_0(w)R_{20}(\rho wz)\bar{R}_{10}(wz)Q_{12}\delta(\mathcal{B}(z)) = \\ &\qquad \delta(\mathcal{B}(z))Q_{12}V_1V_2\bar{R}_{01}(wz)R_{02}(\rho wz)\mathcal{B}_0(w)R_{20}(\frac{w}{\rho z})\bar{R}_{10}(\frac{w}{z})V_1V_2\,.\end{aligned} \tag{B.22}$$

By a direct computation, one can show the following properties

$$Q_{12}V_1V_2\bar{R}_{01}(wz)R_{02}(\rho wz)V_1V_2 = \mathfrak{a}(\rho wz)\mathfrak{a}(\frac{1}{\rho zw})Q_{12} = R_{20}(\rho wz)\bar{R}_{10}(wz)Q_{12} \tag{B.23}$$

$$Q_{12}V_1V_2R_{20}(\frac{w}{\rho z})\bar{R}_{10}(\frac{w}{z})V_1V_2 = \mathfrak{b}(\frac{\rho w}{z})\mathfrak{b}(\frac{z}{\rho w})Q_{12} = \bar{R}_{01}(\frac{w}{z})R_{02}(\frac{w}{\rho z})Q_{12}\,. \tag{B.24}$$

Using these relations, equality (B.22) can be rewritten as

$$\mathcal{B}_0(w)\delta(\mathcal{B}(z))Q_{12} = \delta(\mathcal{B}(z))\mathcal{B}_0(w)Q_{12}\,, \tag{B.25}$$

which is equivalent to the statement of the proposition. □

From the particular form (2.45) of $\mathcal{B}(z)$ and (B.16) of $\bar{\mathcal{B}}(z)$, we deduce that

$$\delta(\mathcal{B}(z)) = \delta\big(\mathcal{T}^+(z)\big)\ \delta\big(K(z)\big)\ \delta\big(\mathcal{T}^-(z^{-1})^{-1}\big) \tag{B.26}$$

where $\delta(\mathcal{T}^+(z))$, $\delta\big(\mathcal{T}^-(z^{-1})^{-1}\big)$ are defined by (B.5) and $\delta\big(K(z)\big)$ by (B.18).

Now, we can obtain the fused relation, using similar relation to (B.6),

$$\zeta(z^2\rho^2)\bar{b}(z)b(z\rho) = \delta(K^+(z))\,\delta(\mathcal{B}(z)) + \widetilde{b}(z) \tag{B.27}$$

where

$$\bar{b}(z) = \mathrm{tr}_1\left(\bar{K}_1^+(z)\bar{\mathcal{B}}_1(z)\right) \tag{B.28}$$

$$\begin{aligned}\widetilde{b}(z) = \mathrm{tr}_{12}\Big((\mathbb{I}_{\mathcal{N}}\otimes\mathbb{I}_{\mathcal{N}} - Q_{12})V_1V_2K_2^+(z\rho)\; M_1\bar{R}_{12}(z^{-2}\rho^{-3})\; M_1^{-1}\bar{K}_1^+(z)\\ \times\bar{\mathcal{B}}_1(z)\;\bar{R}_{21}(z^2\rho)\;\mathcal{B}_2(z\rho)V_1V_2\Big)\,.\end{aligned} \tag{B.29}$$

The transfer matrix $\widetilde{b}(z)$ is the so-called fused transfer matrix.

Appendix C. Generalised fusion

We describe a generalised fusion procedure for $\mathcal{U}_q(gl(\mathcal{N}))$ open and closed spin chains [42]. The procedure we use follows the lines of the construction of the quantum determinant for $\mathcal{U}_q(gl(\mathcal{N}))$, the Sklyanin determinant for quantum twisted Yangians [33] and reflection algebras. The crucial observation here is that for the general case a one-dimensional projector can be obtained by repeating the fusion procedure $\mathcal{N}$ times (recall that $\mathcal{N}^{\otimes\mathcal{N}} = 1\oplus\cdots$). The procedure described in the previous section is basically a consequence of the fact that $\mathcal{N}\otimes\bar{\mathcal{N}} = 1\,\oplus\,(\mathcal{N}^2-1)$. Let us now introduce the following necessary objects for the *generalised* fusion procedure for the closed and open spin chains (see also equations (2.13) and (2.14) in [43]).

C.1. q-antisymmetriser

We start with the q-permutation (A.7): it provides an action of the symmetric group $\mathfrak{S}_{\mathcal{N}}$ on the space $(\mathbb{C}^{\mathcal{N}})^{\otimes\mathcal{N}}$ in the following way. To each transposition $(i, i+1)\in\mathfrak{S}_{\mathcal{N}}$, we associate the q-permutation P_i^q q-permuting i-th and $(i+1)$-th spaces. Then, we set, for $\sigma\in\mathfrak{S}_{\mathcal{N}}$, $P_\sigma^q = P_{i_1}^q\ldots P_{i_l}^q$ where $\sigma = (i_1, i_1{+}1)\ldots(i_l, i_l{+}1)$ is a reduced decomposition. The decomposition does not depend on the choice of the reduced decomposition because of the braid relation satisfied by P_i^q (see equations (A.16) and (A.17)). We recall that $l = l(\sigma)$ is called length of the permutation σ. We can now introduce the q-antisymmetriser

$$\mathcal{A}^q = \frac{1}{\mathcal{N}!}\sum_{\sigma\in\mathfrak{S}_{\mathcal{N}}}(-1)^{l(\sigma)}P_\sigma^q\,. \tag{C.1}$$

This operator is the one-dimensional projector on the q-antisymmetric representation belonging to the tensor product of $\mathcal{N}$ fundamental representations of $\mathcal{U}_q(gl(\mathcal{N}))$.

The fundamental result to obtain the generalized fusion consists in writing the q-antisymmetriser in terms of a product of R-matrices

$$\mathcal{A}^q \prod_{1\le a<b\le \mathcal{N}} \left(q^{a-b} - q^{b-a}\right) = \frac{1}{\mathcal{N}!} \prod_{1\le a<b\le \mathcal{N}} R_{ab}(q^{a-b}) \tag{C.2}$$

where the product in the right-hand side is taken in the lexicographical order on the pairs (a, b).

C.2. Fusion from the quantum determinant

We define, for $a_1, \dots a_{\mathcal{N}} \in \{1, \dots, \mathcal{N}\}$,

$$\mathcal{L}^{\pm}_{\langle a_1\dots a_{\mathcal{N}}\rangle} = \mathcal{L}^{\pm}_{a_1}(z_{a_1}) \dots \mathcal{L}^{\pm}_{a_{\mathcal{N}}}(z_{a_{\mathcal{N}}}), \quad \text{with} \quad z_a = zq^{a-1}\,,\ a = 1, \dots, \mathcal{N} \tag{C.3}$$

where $\mathcal{L}^{\pm}(z)$ are solution of relations (2.4) and (2.5) with $k = 0$. Let us remark that we will make the fusion for a tensor product of auxiliary spaces (here denoted by $1, \dots, \mathcal{N}$). It is important to note the difference between these $\mathcal{N}$ auxiliary spaces and the $\mathcal{N}$ quantum spaces present, for example, in relation (2.13) or (2.38).

From relation (C.2), we can show

$$\mathcal{A}^q\, \mathcal{L}^{\pm}_{\langle 1\dots\mathcal{N}\rangle} = \mathcal{L}^{\pm}_{\langle \mathcal{N}\dots 1\rangle}\, \mathcal{A}^q = \mathcal{A}^q\, \mathcal{L}^{\pm}_{\langle 1\dots\mathcal{N}\rangle}\, \mathcal{A}^q \tag{C.4}$$

$$\equiv \mathcal{A}^q \operatorname{qdet} \mathcal{L}^{\pm}(z)\ . \tag{C.5}$$

Relation (C.5) defines formal series in terms of z whose coefficients belong to the center of $\mathcal{U}_q(\widehat{gl}(\mathcal{N}))$ (we can also show that they are algebraically independent and generate the center). Similarly, we can write

$$\operatorname{qdet} \mathcal{L}^{\pm}(z) = \sum_{\sigma\in\mathfrak{S}_N} (-q)^{-l(\sigma)} L^{\pm}_{\sigma(1),1}(z) \dots L^{\pm}_{\sigma(\mathcal{N}),\mathcal{N}}(zq^{\mathcal{N}-1}) \tag{C.6}$$

$$= \sum_{\sigma\in\mathfrak{S}_N} (-q)^{l(\sigma)} L^{\pm}_{1,\sigma(1)}(zq^{\mathcal{N}-1}) \dots L^{\pm}_{\mathcal{N},\sigma(\mathcal{N})}(z)\,. \tag{C.7}$$

Using this above construction of the quantum determinant for $\mathcal{T}^{\pm} = \Delta^{(\mathcal{N})}(\mathcal{L}^{\pm})$ and taking the trace over all the auxiliary spaces $1, \dots, \mathcal{N}$ of the following relation

$$\mathcal{T}^{\pm}_{\langle 1\dots\mathcal{N}\rangle} = \mathcal{A}^q\, \mathcal{T}^{\pm}_{\langle 1\dots\mathcal{N}\rangle} + (\mathbb{I}^{\otimes\mathcal{N}} - \mathcal{A}^q)\mathcal{T}^{\pm}_{\langle 1\dots\mathcal{N}\rangle}\ , \tag{C.8}$$

we obtain the so-called generalized fusion relation

$$\prod_{l=1}^{\mathcal{N}} t^{\pm}(z_l) = \operatorname{qdet} \mathcal{T}^{\pm}(z) + \widetilde{t}^{\pm}(z)\ . \tag{C.9}$$

We have defined the so-called fused transfer matrix $\widetilde{t}^{\pm}(z) = \operatorname{tr}_{\langle 1\dots\mathcal{N}\rangle}(\mathbb{I}^{\otimes\mathcal{N}} - \mathcal{A}^q)\mathcal{T}^{\pm}_{\langle 1\dots\mathcal{N}\rangle}$. To prove this relation, one needs to use $\operatorname{tr}_{\langle 1\dots\mathcal{N}\rangle} \mathcal{A}^q = 1$. The equation (C.9) plays a crucial role in determining the spectrum of the periodic $\mathcal{U}_q(gl(\mathcal{N}))$ spin chain.

C.3. Fusion from the Sklyanin determinant

We can use a similar construction for the reflection algebra to obtain its central elements and a fusion relation. The quantum determinant is replaced by the so-called Sklyanin determinant, defined as follows

$$\mathcal{A}^q\ \mathcal{B}_{\langle 1\dots\mathcal{N}\rangle} = \mathcal{B}_{\langle \mathcal{N}\dots 1\rangle}\ \mathcal{A}^q = \mathcal{A}^q\ \mathcal{B}_{\langle 1\dots\mathcal{N}\rangle}\ \mathcal{A}^q \tag{C.10}$$

$$\equiv \mathcal{A}^q\,\mathrm{sdet}\,\mathcal{B}(z)\ . \tag{C.11}$$

where

$$\mathcal{B}_{\langle a_1\dots a_\mathcal{N}\rangle} = \prod_{k=1,\dots,\mathcal{N}}^{\longrightarrow}\left(\mathcal{B}_{a_k}(z_{a_k})\left(\prod_{l=k+1,\dots,\mathcal{N}}^{\longrightarrow} R_{a_l a_k}(z_{a_l}z_{a_k})\right)V^t_{a_k}\right)\ . \tag{C.12}$$

In the above relations, $z_a = zq^{a-1}$ and $\mathcal{B}(z)$ is the generating function for the quantum reflection algebra defined by the relation (2.34). Relation (C.11) defines formal series in terms of z whose coefficients belong to the center of $\mathcal{R}$. To study the fusion, it is necessary to also introduce the quantity

$$K^+_{\langle a_1\dots a_\mathcal{N}\rangle} = \prod_{k=1,\dots,\mathcal{N}}^{\longleftarrow}\left(V_{a_k}\left(\prod_{l=k+1,\dots,\mathcal{N}}^{\longleftarrow} R_{a_k a_l}(z^{-1}_{a_k}z^{-1}_{a_l}\rho^{-2})\right)M^{-1}_{a_k}K^+_{a_k}(z_{a_k})\right) \tag{C.13}$$

where $K^+(z)$ is solution of relation (2.47) and we can show that

$$\mathcal{A}^q\ K^+_{\langle 1\dots\mathcal{N}\rangle} = K^+_{\langle \mathcal{N}\dots 1\rangle}\ \mathcal{A}^q = \mathcal{A}^q\ K^+_{\langle 1\dots\mathcal{N}\rangle}\ \mathcal{A}^q \tag{C.14}$$

$$\equiv \mathcal{A}^q\,\mathrm{sdet}\,K^+(z)\ . \tag{C.15}$$

Taking the trace over all the auxiliary spaces $1,\dots,\mathcal{N}$ of the following relation

$$K^+_{\langle 1\dots\mathcal{N}\rangle}\mathcal{B}_{\langle 1\dots\mathcal{N}\rangle} = \mathcal{A}^q\ K^+_{\langle 1\dots\mathcal{N}\rangle}\mathcal{B}_{\langle 1\dots\mathcal{N}\rangle} + (\mathbb{I}^{\otimes\mathcal{N}} - \mathcal{A}^q)K^+_{\langle 1\dots\mathcal{N}\rangle}\mathcal{B}_{\langle 1\dots\mathcal{N}\rangle}\ , \tag{C.16}$$

we obtain the so-called generalized fusion relation for the reflection algebra

$$\left(\prod_{a<b}\bar{\zeta}\Big(\frac{\rho^{-1}}{z_a z_b}\Big)\right)\prod_{l=1}^{\mathcal{N}} b(z_l) = \mathrm{sdet}\,K^+(z)\,\mathrm{sdet}\,\mathcal{B}(z) + \widetilde{b}(z)\ , \tag{C.17}$$

where $\bar{\zeta}(z)$ is defined in (A.36). We recall that $b(z) = \mathrm{tr}_0(K^+_0(z)\mathcal{B}_0(z))$ and we have defined the so-called fused transfer matrix $\widetilde{b}(z) = \mathrm{tr}_{\langle 1\dots\mathcal{N}\rangle}(\mathbb{I}^{\otimes\mathcal{N}} - \mathcal{A}^q)K^+_{\langle 1\dots\mathcal{N}\rangle}$ $\mathcal{B}_{\langle 1\dots\mathcal{N}\rangle}$. As discussed in [43], the matrix $K^+_{\langle 1\dots\mathcal{N}\rangle}$ is necessary so that the trace of the r.h.s. of (C.16) decouples to a product of $\mathcal{N}$ transfer matrices. Note that if we choose $\mathcal{B}(z)$ of the form (2.45), then its Sklyanin determinant can be determined in terms of the quantum determinant:

$$\mathrm{sdet}\,\mathcal{B}(z) = \mathrm{sdet}\,K(z)\ \mathrm{qdet}\,\mathcal{T}^+(z)\big(\mathrm{qdet}\,\mathcal{T}^-(z^{-1}q^{-\mathcal{N}+1})\big)^{-1}\ . \tag{C.18}$$

This factorized form of the Sklyanin determinant allows us to prove easily that its coefficients belong to the center of $\mathcal{R}$. It is also very useful to compute its explicit form in a given representation (see equation (2.98)).

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