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Spectral Shift Function in the Large Coupling Constant Limit

Mouez Dimassi

Abstract. In the large-coupling constant limit we obtain an asymptotic expansion in powers of $\mu^{-\frac{1}{\delta}}$ of the derivative of the spectral shift function corresponding to $(-\Delta+\mu W(x),-\Delta)$. Here the potential $W(x)$ is positive and $W(x)\sim\omega_0(\frac{x}{|x|})|x|^{-\delta}$ near infinity for some $\delta>n$ and $\omega_0\in C^\infty(S^{n-1})$.

1 Introduction

In the present paper we study the asymptotic behavior of the spectral shift function (SSF) of the Schrödinger operator in the large coupling constant limit. Consider the Schrödinger operator

$$P_1=-\Delta+\mu\,W(x),\quad \mu\gg 1,$$

in the n-dimensional space $\mathbb{R}^n$. We assume that W is a real C^∞-smooth function which satisfies

$$|\partial_x^\alpha W(x)|\le C_\alpha(1+|x|)^{-\delta},\quad \delta>n. \tag{1}$$

Under this assumption, P_1 admits a unique self-adjoint realization in $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$ which is the Sobolev space of order 2. We still denote by P_1 this realization.

The assumption (1) implies that the spectral shift function $\xi(\lambda,\mu)$ related to P_1 and $P_0=-\Delta$ is well defined in the sense of distribution:

$$\operatorname{tr}(f(P_1)-f(P_0))=\int_{\mathbb{R}}f'(\lambda)\xi(\lambda,\mu)\,d\lambda=-\langle\xi'(\cdot,\mu),f\rangle,\quad f\in C_0^\infty(\mathbb{R}).$$

The asymptotic behavior of the SSF of the Schrödinger operator has been intensively studied in the last twenty years in different aspects (see [4,7,12,13,20,21,23] and the references given there).

In the semi-classical regime (i.e., $H(h)=-h^2\Delta+W(x)$, $(h\searrow 0)$) a Weyl type asymptotics of the SSF with sharp remainder estimates has been obtained (see [20,21,23,24]). On the other hand, if an energy $\lambda>0$ is non-trapping for the classical Hamiltonian $p(x,\xi)=|\xi|^2+W(x)$ (i.e., for all $\rho_0\in p^{-1}(\{\lambda\})$, $|\exp(tH_p)(\rho_0)|\to\infty$, when $t\to\pm\infty$) a complete asymptotic expansion in powers of h for the derivative of the SSF has been obtained (see [20,21,23,24]). Similar results are well known for the SSF at high energies (see [3,4,13,14,19]).

There are only a few works treating the SSF in the large coupling constant limit. In this case, the asymptotic behavior of the SSF depends both on the sign of the perturbation and on its decay properties at infinity.

For the case of non-positive perturbation $W \leq 0$ satisfying (1) the following formula has been obtained for almost every $\lambda \in \mathbb{R}$ (see [15]),

$$\xi(\lambda, \mu) = -\mu^{n/2}(c_0 + o(1)), \quad (\mu \to +\infty)$$
$$c_0 = (2\pi)^{-n}\kappa_0 \int_{\mathbb{R}^n} |W(x)|^{\frac{n}{2}} dx,$$

where $\kappa_0 = \mathrm{vol}(\{x \in \mathbb{R}^n;\ |x| < 1\})$.

For the case of non-negative perturbation $W \geq 0$ satisfying $W(x) \sim_{|x|\to\infty} \omega_0(\frac{x}{|x|})|x|^{-\delta}$ with $\delta > n$ it has been proved in [16] (see also [17]) that

$$\xi(\lambda, \mu) = \mu^{\frac{n}{\delta}}(b_0 + o(1)),\ (\mu \to +\infty), \tag{2}$$
$$b_0 = (2\pi)^{-n}\kappa_0 \int_{\mathbb{R}^n} \Big((\lambda)_+^{\frac{n}{2}} - (\lambda - \omega_0(\frac{x}{|x|})|x|^{-\delta})_+^{\frac{n}{2}}\Big) dx,$$

where $(\lambda)_+ = \max(\lambda, 0)$.

In [16,17], the proof of (2) is based on a representation for the SSF in terms of the counting function of the spectrum of some family of compact operators. By this method one can find the main term in the asymptotic of the SSF with a weaker assumption on the perturbation. However, it seems quite difficult to establish with these techniques an asymptotic formula involving sharp remainder estimates.

Nevertheless, for $\omega_0 > 0$, the potential is repulsive and one expect to get even a complete asymptotic expansion in powers of $h = \mu^{-\frac{1}{\delta}}$ as in the semi-classical case.

Our main goal in this paper is to use the semi-classical analysis to prove sharp results. Assuming that $W > 0$ and $W(x) \sim_{|x|\to\infty} \omega_0(\frac{x}{|x|})|x|^{-\delta}$ with $\delta > n$, we will show that the contribution of the domain $\Omega_\mu = \{x \in \mathbb{R}^n;\ \mu W(x) \gg 1\}$ to the remainder estimates is $\mathcal{O}(\mu^{-s})$ for every s. On the other hand, on $\mathbb{R}^n \setminus \Omega_\mu$, we have $\mu\, W(x) \sim \Phi_0(hx)$ where $\Phi_0(x) = \omega_0(\frac{x}{|x|})|x|^{-\delta}$ and $h = \mu^{-\frac{1}{\delta}}$. Thus, modulo an error of order $\mathcal{O}(\mu^{-\infty})$, we will reduce the study of the SSF corresponding to (P_1, P_0) to the one of $(-h^2\Delta + \varphi(x,h), -h^2\Delta)$ where $\varphi(x,h) = \Phi_0(x) + h\Phi_1(x) + \ldots$ has an asymptotic expansion in powers of h (see Section 3). By applying the results of [24,21,20], we will obtain a complete asymptotic expansions in powers of $\mu^{-\frac{1}{\delta}}$ for $\xi'(\lambda, \mu)$, and we will give explicitly their leading terms.

2 Main results

In this section we state the assumptions and we formulate our results as Theorem 1–2.

Let $W \in C^\infty(\mathbb{R}^n;]0,+\infty[)$. We suppose that there exists a sequence $(w_j)_{j\geq 0} \in C^\infty(S^{n-1},\mathbb{R})$ such that for every integer $N \geq 0$, there is $R_N \in C^\infty(\mathbb{R}^n;\mathbb{R})$ such that:

$$W(x) = \sum_{j=0}^{N} w_j(\frac{x}{|x|})|x|^{-\delta-j} + R_N(x) \quad \text{for } |x| \gg 1, \tag{3}$$
$$\forall\, \beta,\ \exists\, C_\beta,\ |\partial_x^\beta R_N(x)| \leq C_\beta (1+|x|)^{-|\beta|-\delta-N-1} \quad \forall x,$$

with $\delta > n$ and

$$\omega_0(y) > 0, \quad \text{for all } y \in S^{n-1}. \tag{4}$$

Our first Theorem concerns the weak asymptotic of $\xi'(\lambda,\mu)$.

Theorem 1 *Assume that the assumptions* (3) *and* (4) *are satisfied. Let* $I =]a,b[$ *with* $a > 0$ *be an open bounded interval. For* $f \in C_0^\infty(I)$*, the following asymptotic expansion holds:*

$$\operatorname{tr}\Big(f(P_1) - f(P_0)\Big) \sim \mu^{\frac{n}{\delta}} \sum_{j=0}^{\infty} a_j(f)\mu^{-\frac{j}{\delta}}, \quad \mu \to \infty, \tag{5}$$

with

$$a_0(f) = (2\pi)^{-n} \int\int_{\mathbb{R}^{2n}} \Big(f(|\xi|^2 + \omega_0(\frac{x}{|x|})|x|^{-\delta}) - f(|\xi|^2)\Big) dx d\xi, \tag{6}$$
$$a_1(f) = (2\pi)^{-n} \int\int_{\mathbb{R}^{2n}} f'\Big(|\xi|^2 + \omega_0(\frac{x}{|x|})|x|^{-\delta}\Big)\, \omega_1(\frac{x}{|x|})|x|^{-\delta-1} dx d\xi.$$

The coefficients $a_j(f)$ *are distribution depending on* f*. Moreover, if* $\overline{I} \subset]0,+\infty[$*, we have*

$$a_j(f) = -\langle \gamma_j(\cdot), f\rangle,\ \ \forall f \in C_0^\infty(I).$$

Here $\gamma_j(\lambda)$ *are smooth functions of* $\lambda \in I$*. In particular,*

$$\gamma_0(\lambda) = \frac{n}{2}(2\pi)^{-n}\kappa_0 \int_{\mathbb{R}^n_x} \Big(\lambda^{\frac{n}{2}-1} - (\lambda - \omega_0(\frac{x}{|x|})|x|^{-\delta})_+^{\frac{n}{2}-1}\Big) dx,$$
$$\gamma_1(\lambda) = \frac{n(n-2)}{4\delta}(2\pi)^{-n}\kappa_0 \int_{S^{n-1}} \omega_1(\theta)\omega_0(\theta)^{\frac{n-1}{\delta}-1} d\theta \int_0^\lambda (\lambda-u)^{\frac{n}{2}-2} u^{\frac{1-n}{\delta}} du \quad \textit{for } n \neq 2,$$
$$\gamma_1(\lambda) = (4\pi\delta)^{-1} \int_0^{2\pi} \omega_1(\theta)\omega_0(\theta)^{\frac{1}{\delta}-1} d\theta\, \lambda^{-\frac{1}{\delta}}, \quad \textit{for } n = 2.$$

The proof of Theorem 1 is contained in Section 4. Our main result concerning the derivative of the spectral shift function is the following.

Theorem 2 *Assume that the assumptions* (3) *and* (4) *are satisfied, and let* $[a,b] \subset \mathbb{R}^+ \setminus \{0\}$ *be a compact interval. Then the following full asymptotic expansion holds*

$$\xi'(\lambda,\mu) \sim \mu^{\frac{n}{\delta}} \sum_{j=0}^{\infty} \gamma_j(\lambda)\mu^{-\frac{j}{\delta}}, \quad \textit{as } \mu \to \infty, \tag{7}$$

uniformly for $\lambda \in [a,b]$*. The coefficients* $\gamma_j(\lambda)$ *are given in Theorem* 1*. Furthermore, this expansion has derivate in* λ *to any order.*

Remark. We can compute explicitly all the coefficients $a_j(f)$ and $\gamma_j(\lambda)$. In fact, this is a simple consequence of Proposition 1 (see Section 3) and of the functional calculus of h-pseudodifferential operators (see [5, Chapter 8]).

3 Preliminaries

Throughout this section, we fix $I =]a,b[$ with $a > 0$ and we assume the assumptions (3) and (4) are fulfilled. Set $\mu = h^{-\delta}$. By $B(x_0, r)$ we denote the ball of center x_0 and radius r. For $M > 0$, we define:

$$\Omega_M(h) = \{x \in \mathbb{R}^n;\ h^{-\delta}W(x) > M\}.$$

Using the fact that $W > 0$ and $\omega_0 > 0$, we conclude that there exists small enough $h_0 > 0$ such that

$$B(0, \beta_1 M^{-\frac{1}{\delta}} h^{-1}) \subset \Omega_M(h) \subset B(0, \beta_2 M^{-\frac{1}{\delta}} h^{-1}),$$

for all $h \in]0, h_0]$. Here β_1 and β_2 are constants satisfying $\beta_1 < \Big(\inf_{x\in S^{n-1}}\omega_0(x)\Big)^{\frac{1}{\delta}}$ and $\beta_2 > \Big(\sup_{x\in S^{n-1}}\omega_0(x)\Big)^{\frac{1}{\delta}}$.

Let $\chi \in C_0^\infty(B(0, \beta_1 M^{-\frac{1}{\delta}}); [0,1])$, with $\chi = 1$ near zero. Set

$$\varphi(x,h) := (1-\chi(x))h^{-\delta}W(\frac{x}{h}) + M\chi(x),$$
$$V_h(x) := h^{-\delta}W(x) - \varphi(hx,h).$$

From the expressions for φ_h, V_h and Ω_h, we easily get:

$$\operatorname{supp} V_h \subset \Omega_M(h), \tag{8}$$

$$\varphi(hx,h) \geq \frac{M}{2}, \quad \text{for all } x \in \Omega_{\frac{M}{2}}(h), \tag{9}$$

$$|\partial_x^\alpha \varphi(x,h)| \leq C_\alpha, \quad \text{uniformly for } h \in]0,h_0]\,. \tag{10}$$

On the other hand, (3) shows that, for any $N \in \mathbb{N}$ there exist $\phi_0, \dots, \phi_N$, $K_N(\cdot,h)$ $\in C^\infty(\mathbb{R}^n;\mathbb{R})$, uniformly bounded with respect to $h \in]0,h_0]$ together with their derivatives such that:

$$\varphi(x,h)=\sum_{j=0}^{N}\phi_j(x)\,h^j+h^{N+1}K_N(x,h), \tag{11}$$

with

$$\phi_0(x)=(1-\chi(x))\omega_0(\frac{x}{|x|})|x|^{-\delta}+M\chi(x).$$

Notice that,

$$\phi_0(x)<M\Longrightarrow\phi_0(x)=\omega_0(\frac{x}{|x|})|x|^{-\delta}. \tag{12}$$

In fact, for $x\in\operatorname{supp}\chi$ we have $\omega_0(\frac{x}{|x|})|x|^{-\delta}>\beta_1^{\delta}|x|^{-\delta}>M$, which implies that $\phi_0(x)>M$ for $x\in\operatorname{supp}\chi$. From this we deduce (12). Consequently, for $M>b$ we have

$$p(x,\xi):=|\xi|^2+\phi_0(x)=|\xi|^2+\omega_0(\frac{x}{|x|})|x|^{-\delta}\quad\text{near } p^{-1}([a,b]). \tag{13}$$

Let $t\to\theta(t)\in C^\infty(\mathbb{R};\mathbb{R})$ be equal to t for $t\geq\frac{M}{2}$ and $\theta(t)\geq\frac{M}{3}$ for all t. We define:

$$F_1(x,h)=\theta(h^{-\delta}W(x)),$$
$$F_2(x,h)=\theta(\varphi(hx,h)).$$

Let Ω be a small complex neighborhood of I. From now on, we choose $M>b$ large enough so that

$$F_i(x,h)-\Re z\geq\frac{M}{4},\quad i=1,2, \tag{14}$$

uniformly for $z\in\Omega$. Set

$$P_{F_i}=-\Delta+F_i(x,h).$$

It follows from (10) that

$$\partial_x^\alpha F_i(x,h)=\mathcal{O}_\alpha(h^{-\delta}). \tag{15}$$

On the other hand, the inequality (14) implies that

$$\Omega\ni z\to(z-P_{F_i})^{-1}\in\mathcal{L}(L^2(\mathbb{R}^n);\;L^2(\mathbb{R}^n))\text{ is holomorphic.} \tag{16}$$

Finally (9) shows that

$$\mathrm{d}\Big(\operatorname{supp}V_h,\operatorname{supp}(h^{-\delta}W(x)-F_1(x,h))\Big)\geq\frac{a_1(M)}{h}, \tag{17}$$
$$\mathrm{d}\Big(\operatorname{supp}V_h,\operatorname{supp}(\varphi(hx,h)-F_2(x,h))\Big)\geq\frac{a_2(M)}{h},$$

with constants $a_1(M),a_2(M)>0$ independent of h.

For $z \in \Omega$ with $\Im z \neq 0$, let us introduce the operator

$$G(z) := (z - P_1)^{-1} - (z - P_2)^{-1} - (z - P_{F_1})^{-1} V_h (z - P_{F_2})^{-1}, \qquad (18)$$

where

$$P_2 = -\Delta + \varphi(hx, h).$$

Lemma 1 *The operator $G(z)$ is a trace class one and the following estimate holds uniformly for $z \in \Omega$ with $\Im z \neq 0$:*

$$\|G(z)\|_{\mathrm{tr}} = \mathcal{O}(h^\infty |\Im z|^{-2}). \qquad (19)$$

Here, by $a(x) = \mathcal{O}(\langle x \rangle^{-\infty})$ we mean $a(x) = \mathcal{O}_q(\langle x \rangle^{-q})$ for all $q \in \mathbb{N}$.

Proof. From the resolvent equation, we have:

$$(z - P_1)^{-1} - (z - P_2)^{-1} = (z - P_1)^{-1} V_h (z - P_2)^{-1}, \qquad (20)$$

$$(z - P_1)^{-1} = (z - P_{F_1})^{-1} + (z - P_1)^{-1} (h^{-\delta} W - F_1)(z - P_{F_1})^{-1}, \qquad (21)$$

$$(z - P_2)^{-1} = (z - P_{F_2})^{-1} + (z - P_{F_2})^{-1} (\varphi - F_2)(z - P_2)^{-1}. \qquad (22)$$

By inserting (21) and (22) in the right-hand side of (20), we obtain

$$\begin{aligned} G(z) &= (z - P_{F_1})^{-1} V_h (z - P_{F_2})^{-1} (\varphi - F_2)(z - P_2)^{-1} \qquad (23) \\ &+ (z - P_1)^{-1} (h^{-\delta} W - F_1)(z - P_{F_1})^{-1} V_h (z - P_{F_2})^{-1} \\ &+ (z - P_1)^{-1} (h^{-\delta} W - F_1)(z - P_{F_1})^{-1} V_h (z - P_{F_2})^{-1} (\varphi - F_2)(z - P_2)^{-1} \\ &=: A(z) + B(z) + C(z). \end{aligned}$$

Let $K_i(x, y, z, h)$ be the integral kernel of

$$\mathcal{K}_i = V_h(x)(z - P_{F_i})^{-1} \begin{cases} \varphi - F_2, & i = 1, \\ h^{-\delta} W - F_1, & i = 2. \end{cases}$$

By a classical result (see [18]), the integral kernel $R(x, y, z)$ of $(z - P_{F_i})^{-1}$ satisfies:

$$\forall \alpha, \beta,\ \partial_x^\alpha \partial_y^\beta R(x, y, z, h) = \mathcal{O}_{\alpha,\beta}(h^{-\delta}) e^{-c|x-y|},\ \ |x - y| > 1,$$

where c and $\mathcal{O}_{\alpha,\beta}$ are independent of $(z, h) \in \Omega \times]0, h_0]$. Combining this with (17), we get

$$\partial_x^\alpha \partial_y^\beta K_i(x, y, z, h) = \mathcal{O}_{\alpha,\beta,L}(h^\infty) \exp\Big(-\frac{1}{C} \left[\mathrm{d}(x, \operatorname{supp} V_h) + (y, \operatorname{supp} V_h)\right]\Big). \quad (24)$$

where C is independent of $(z, h) \in \Omega \times]0, h_0]$. A complete proof of (24) is given in [6].

According to the calculus of pseudo-differential operator (see for instance [5, Chapter 9]) the trace norm of $\mathcal{K}_i$ can be estimated by

$$\|\mathcal{K}_i\|_{\rm tr} \leq C_n \sum_{|\alpha|+|\beta|\leq 2n+1} \|\partial_x^\alpha \partial_y^\beta K_i\|_{L^1(\mathbb{R}^{2n})}, \tag{25}$$

with a constant C_n depending only on n.

Taking together (23), (24), (25) and using the fact that (24) remains true for the kernels of the adjoint operator as well as the fact that $\|(z-P_2)^{-1}\| = \mathcal{O}(|\Im z|^{-1})$, $\|(z-P_1)^{-1}\| = \mathcal{O}(|\Im z|^{-1})$, we get the lemma.

Now we are ready to state the main results of this section.

Proposition 1 *Let $f \in C_0^\infty(I;\mathbb{R})$. We have*

$$\operatorname{tr}\Big(f(P_1)-f(P_0)\Big) = \operatorname{tr}\Big(f(\tilde{P}_2(h))-f(\tilde{P}_0(h))\Big) + \mathcal{O}(h^\infty), \tag{26}$$

$$\operatorname{tr}\Big(f(P_1)-f(P_2)\Big) = \lim_{\epsilon\searrow 0}\frac{i}{2\pi}\int f(\lambda)\Big[\operatorname{tr}(G(\lambda+i\epsilon)) - \operatorname{tr}(G(\lambda-i\epsilon))\Big]d\lambda, \tag{27}$$

where the limit is taken in the sense of distributions. Here

$$\tilde{P}_2(h) = -h^2\Delta + \varphi(x,h), \quad \tilde{P}_0(h) = -h^2\Delta.$$

Proof. Let $f \in C_0^\infty(\mathbb{R},\mathbb{R}^+)$ and let $\tilde{f}(z) \in C_0^\infty(\Omega)$ be an almost analytic extension of f, such that

$$\widetilde{f} = f, \quad \text{on } \mathbb{R}, \quad \overline{\partial}\widetilde{f}(z) = \mathcal{O}(|\Im z|^2).$$

The functional calculus due to Helffer-Sjöstrand (see for instance [5, Chapter 8]) yields

$$f(P_1)-f(P_2) = -\frac{1}{\pi}\int \overline{\partial}\widetilde{f}(z)\Big((z-P_1)^{-1}-(z-P_2)^{-1}\Big)\, L(dz),$$

Here $L(dz) = dxdy$ is the Lebesgue measure on the complex plane $\mathbb{C}\sim\mathbb{R}^2_{x,y}$.

Notice that, if $\Omega \ni z \to K(z)$ is holomorphic, then $\int \overline{\partial}\tilde{f}(z)K(z)L(dz) = 0$. Taking into account (16), we obtain

$$f(P_1)-f(P_2) = -\frac{1}{\pi}\int \overline{\partial}\widetilde{f}(z)G(z)\, L(dz). \tag{28}$$

Combining this with Lemma 1 and using (19) and that $\overline{\partial}\widetilde{f}(z) = \mathcal{O}(|\Im z|^2)$, we get

$$\|f(P_1)-f(P_2)\|_{\rm tr} = \mathcal{O}(h^\infty),$$

which yields

$$\operatorname{tr}\Big(f(P_1)-f(P_0)\Big) = \operatorname{tr}\Big(f(P_2)-f(P_0)\Big) + \mathcal{O}(h^\infty).$$

Now, by the change of variable $y = hx$, (26) follows from the property of cyclic invariance of the trace.

To establish (27), we write the right-hand side of (28) in the form

$$\lim_{\epsilon\searrow 0} -\frac{1}{\pi}\Big(\int_{\Im z>0}\bar\partial_z \tilde f(z)\,\mathrm{tr}(G(z+i\epsilon))L(dz)+\int_{\Im z<0}\bar\partial_z \tilde f(z)\,\mathrm{tr}(G(z-i\epsilon))L(dz)\Big).$$

Using that $\mathrm{tr}(G(z+i\epsilon))$ (resp. $\mathrm{tr}(G(z-i\epsilon))$ is holomorphic on $\{z\in\Omega : \Im z>0\}$ (resp. $\{z\in\Omega : \Im z<0\}$) and applying the Green formula, we obtain (27).

4 Proof of Theorem 1

Let $p(x,\xi) = |\xi|^2+\phi_0(x)$ be the principal symbol of $\tilde P_2(h) = -h^2\Delta+\varphi(x,h)$. We recall from (11) that $\varphi(x,h)$ has an asymptotic expansion in powers of h. According to Theorem 1.1 in [21], we have

$$\mathrm{tr}\Big(f(\tilde P_2(h))-f(\tilde P_0(h))\Big)\sim\sum_{j=0}^{\infty}a_j(f)h^{j-n},\quad (h\searrow 0) \tag{29}$$

with

$$\begin{aligned} a_0(f) &= (2\pi)^{-n}\int\int_{\mathbb{R}^{2n}}\Big(f(|\xi|^2+\phi_0(x))-f(|\xi|^2)\Big)\,dxd\xi & (30)\\ &= (2\pi)^{-n}\int\int_{\mathbb{R}^{2n}}\Big(f(|\xi|^2+\omega_0\frac{x}{|x|})|x|^{-\delta})-f(|\xi|^2)\Big)\,dxd\xi, \\ a_1(f) &= (2\pi)^{-n}\int\int_{\mathbb{R}^{2n}}f'(|\xi|^2+\omega_0\frac{x}{|x|})|x|^{-\delta})\,\omega_1\frac{x}{|x|})|x|^{-\delta-1}\,dxd\xi, & (30)\end{aligned}$$

where we have used (13).

The coefficient a_j is a finite sum of term of the form $\int\int c_l(x,\xi)f^{(l)}(p(x,\xi))$ $dxd\xi$, where c_l depends on ϕ_i and their derivatives (see [5, Chapter 8]). Clearly, if

$$dp(x,\xi)\neq 0,\quad \text{for all } (x,\xi)\in p^{-1}([a,b]) \tag{31}$$

then $a_j(f) = -\langle\gamma_j(\cdot),f\rangle$, $\forall f\in C_0^\infty(I)$. In view of (26) and (29), to complete the proof of Theorem 1 we only have to prove (31).

By observing that $x\cdot\nabla_x(\omega_0(\frac{x}{|x|}))=0$, we get

$$x\cdot\nabla_x(\omega_0(\frac{x}{|x|})|x|^{-\delta}) = -\delta\omega_0(\frac{x}{|x|})|x|^{-\delta}. \tag{32}$$

which together with (4) and (13) yields (31). The expressions of $\gamma_0(\lambda)$ and $\gamma_1(\lambda)$ follow from those of $a_0(f)$ and $a_1(f)$ by a simple computation.

5 Proof of Theorem 2

Let $I = [a, b]$ be a compact interval in $]0, +\infty[$, and let Ω be a bounded complex neighborhood of I with $\Omega \subset \{\Re z > 0\}$. Set $\Omega_\pm = \{z \in \Omega;\ \pm\Im z > 0\}$. In this section, we freely use the notations of the previous sections.

Lemma 2 *For all $s > k - \frac{1}{2}$ with $k \in \mathbb{N}$, we have*

$$\|\langle hx\rangle^{-s}(z - P_i)^{-k}\langle hx\rangle^{-s}\| = \mathcal{O}(h^{-k}), \tag{33}$$

uniformly for $z \in \Omega_\pm$.

Proof. By the change of variable $y = hx$, we reduce the problem to showing that

$$\|\langle x\rangle^{-s}(z - \tilde{P}_i)^{-k}\langle x\rangle^{-s}\| = \mathcal{O}(h^{-k}), \qquad (34)_i$$

where $\tilde{P}_1 = -h^2\Delta + \mu W(\frac{x}{h})$, $\tilde{P}_2 := \tilde{P}_2(h) = -h^2\Delta + \varphi(x, h)$.

Let $\{a, b\} = \frac{\partial a}{\partial \xi}\frac{\partial b}{\partial x} - \frac{\partial a}{\partial x}\frac{\partial b}{\partial \xi}$ be the Poisson bracket. Making use of the fact that $p(x, \xi) = |\xi|^2 + \omega_0(\frac{x}{|x|})|x|^{-\delta}$ near $p^{-1}([a, b])$, and using (32), we get

$$\{x \cdot \xi, p(x, \xi)\} = 2|\xi|^2 + \delta\phi_0(x) \geq \inf(2, \delta)\, p(x, \xi) \geq a \inf(2, \delta), \tag{35}$$

in $p^{-1}([a, b])$. This shows that every energy $E \in [a, b]$ is non-trapping for the classical Hamiltonian $p(x, \xi)$ (see for instance [8, Proposition 21.3]). Now, applying Lemma 3.5 in [24] to the operator $\tilde{P}_2$ we obtain $(34)_2$.

The proof of $(34)_1$ is similar, since in the classical allowed region (i.e., $\{(x, \xi) \in \mathbb{R}^{2n};\ |\xi|^2 + \mu W(x) \in [a, b]\}$) the symbols of P_1 and P_2 coincide. For the reader convenience, let us point out the main change in the proof of $(34)_1$.

Let $g \in C_0^\infty(\Omega \cap \mathbb{R})$ and let $f \in C_0^\infty(\Omega \cap \mathbb{R})$ be equal to 1 on $\operatorname{supp} g$. An immediate consequence of (19) and (28) is that

$$\|f(P_1) - f(P_2)\| = \mathcal{O}(h^\infty). \tag{36}$$

By using Helffer-Sjöstrand formula and (22), we obtain

$$f(P_2) = -\frac{1}{\pi}\int \overline{\partial}\widetilde{f}(z)(z - P_{F_2})^{-1}(\varphi - F_2)(z - P_2)^{-1}\, L(dz). \tag{37}$$

We recall that $\Omega \ni z \to (z - P_{F_2})^{-1}$ is holomorphic.

Let $\Psi_h \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\Psi_h \subset \operatorname{supp} V_h$ and $|\partial_x^\alpha \Psi_h(x)| = \mathcal{O}(h^{m-|\alpha|})$ for some constant m.

Using (37) and the exponential decay properties of the integral kernel of $(z - P_{F_2})^{-1}$ (see the proof of Lemma 1), we deduce that

$$\|\Psi_h f(P_2)\| = \mathcal{O}(h^\infty), \tag{38}$$

which together with (36) yields

$$\|\Psi_h f(P_1)\| = \mathcal{O}(h^\infty). \tag{39}$$

Define $A = \frac{1}{2}(hD_x \cdot x + x \cdot hD_x)$. We decompose

$$f(\tilde{P}_2)[\tilde{P}_2, A]f(\tilde{P}_2) - f(\tilde{P}_1)[\tilde{P}_1, A]f(\tilde{P}_1) = \Big(f(\tilde{P}_2) - f(\tilde{P}_1)\Big)[\tilde{P}_2, A]f(\tilde{P}_2) + f(\tilde{P}_1)[\tilde{P}_2 - \tilde{P}_1, A]f(\tilde{P}_2) + f(\tilde{P}_1)[\tilde{P}_1, A]\Big(f(\tilde{P}_2) - f(\tilde{P}_1)\Big).$$

Since $\tilde{P}_i$ is unitarily equivalent to P_i and $P_1 - P_2 = V_h$, it follows from (36), (38), (39) and the last equality that

$$f(\tilde{P}_2)[\tilde{P}_2, A]f(\tilde{P}_2) - f(\tilde{P}_1)[\tilde{P}_1, A]f(\tilde{P}_1) = \mathcal{O}(h^\infty). \tag{40}$$

On the other hand, (35) implies (see [20])

$$f(\tilde{P}_2)[\tilde{P}_2, A]f(\tilde{P}_2) \geq Chf^2(\tilde{P}_2). \tag{41}$$

Since $g(\tilde{P}_1) = f(\tilde{P}_1)g(\tilde{P}_1)$, it follows from (40) and (41)

$$g(\tilde{P}_1)[\tilde{P}_1, A]g(\tilde{P}_1) = g(\tilde{P}_1)\Big(f(\tilde{P}_1)[\tilde{P}_1, A]f(\tilde{P}_1)\Big)g(\tilde{P}_1) \geq Chg(\tilde{P}_1)f^2(\tilde{P}_2)g(\tilde{P}_1) + \mathcal{O}(h^\infty)g^2(\tilde{P}_1) \geq (\tilde{C}h + \mathcal{O}(h^\infty))g^2(\tilde{P}_1).$$

In the last inequality we have used the fact that $f(\tilde{P}_2) = f(\tilde{P}_1) + \mathcal{O}(h^\infty)$ as well as the fact that $f(\tilde{P}_1)g(\tilde{P}_1) = g(\tilde{P}_1)$.

Thus, for h small enough

$$g(\tilde{P}_1)[\tilde{P}_1, A]g(\tilde{P}_1) \geq C_0 hg^2(\tilde{P}_1). \tag{42}$$

Now, making use of (42) and repeating the argument in [20] (see also [9], [23]) we prove $(34)_1$.

The proof of Theorem 2, will be a simple consequence of (27) and the following lemma.

Lemma 3 *We have*

$$\mathrm{tr}\Big[G(z)\Big] = \mathcal{O}(h^\infty),$$

uniformly for $z \in \Omega_\pm$.

Proof. Consider the equation (23). We will prove that $\mathrm{tr}\Big[C(z)\Big] = \mathcal{O}(h^\infty)$ uniformly for $z \in \Omega_\pm$. The terms $A(z)$ and $B(z)$ can be treated more easily. Using the resolvent identity

$$(z - P_2)^{-1} = (z - P_1)^{-1} - (z - P_2)^{-1}V_h(z - P_1)^{-1},$$

we decompose $C(z)$ as a sum of two terms $I_1 + I_2$ as follows

$$C(z) = I_1 + I_2 =$$
$$\operatorname{tr}\Big((z-P_1)^{-1}(h^{-\delta}W - F_1)(z-P_{F_1})^{-1}V_h(z-P_{F_1})^{-1}(\varphi - F_2)(z-P_1)^{-1}\Big) -$$
$$\operatorname{tr}\Big((z-P_1)^{-1}(h^{-\delta}W - F_1)(z-P_{F_1})^{-1}V_h(z-P_{F_2})^{-1}$$
$$(\varphi - F_2)(z-P_2)^{-1}V_h(z-P_1)^{-1}\Big).$$

To treat I_2 we use the cyclicity of the trace and write

$$I_2 = -\operatorname{tr}(\langle x\rangle^{-2\delta}(z-P_1)^{-2}\langle x\rangle^{-2\delta}(h^{-\delta}W - F_1)\langle x\rangle^{2\delta}(z-P_{F_1})^{-1}V_h$$
$$(z-P_{F_2})^{-1}\langle x\rangle^{\delta}(\varphi - F_2)\langle x\rangle^{-\delta}(z-P_2)^{-1}\langle x\rangle^{-\delta}V_h\langle x\rangle^{3\delta}).$$

From (24), we deduce that

$$\|(h^{-\delta}W - F_1)\langle x\rangle^{2\delta}(z-P_{F_2})^{-1}V_h(z-P_{F_2})^{-1}\langle x\rangle^{\delta}(\varphi - F_2)\|_{\mathrm{tr}} = \mathcal{O}(h^{\infty}).$$

Note that the definition of V_h and (5) yield $\|V_h\langle x\rangle^{3\delta}\|_\infty = \mathcal{O}(h^{-4\delta})$. Combining this with Lemma 2, we get $I_2 = \mathcal{O}(h^\infty)$ uniformly for $z \in \Omega_\pm$.

Applying the same arguments for I_1 and using that

$$\|\langle x\rangle^{\delta}(z-P_{F_i})^{-1}\langle x\rangle^{-\delta}\| = \mathcal{O}(h^{-\delta}),$$

we obtain $I_1 = \mathcal{O}(h^\infty)$, uniformly for $z \in \Omega_\pm$. This completes the proof of the lemma.

End of the proof of Theorem 2.

Let $\xi_1(\lambda,\mu)$ (resp. $\xi_2(\lambda,\mu)$) be the SSF corresponding to (P_1, P_0) (resp. $(\tilde{P}_2(h), \tilde{P}_0(h))$). An immediate consequence of Lemma 3 and (27) is that

$$\xi_2'(\lambda,\mu) = \xi_1'(\lambda,\mu) + \mathcal{O}(h^\infty),$$

which together with Theorem 6.1 in [24] implies (7).

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Mouez Dimassi
Département de Mathématiques
Université Paris 13
F-93430 Villetaneuse, France
email: dimassi@math.univ-paris13.fr

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