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Quantum Incompressibility and Razumov Stroganov Type Conjectures

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Abstract. We establish a correspondence between polynomial representations of the Temperley and Lieb algebra and certain deformations of the Quantum Hall Effect wave functions. When the deformation parameter is a third root of unity, the representation degenerates and the wave functions coincide with the domain wall boundary condition partition function appearing in the conjecture of A.V. Razumov and Y.G. Stroganov. In particular, this gives a proof of the identification of the sum of the entries of the $O(n)$ transfer matrix and a six vertex-model partition function, alternative to that of P. Di Francesco and P. Zinn-Justin.

1 Introduction

This paper is aimed at establishing a correspondence between the deformation of certain wave functions of the Hall effect and polynomial representations of the Temperley and Lieb (T.L.) algebra.

This work originates from an attempt to understand the conjecture of A.V. Razumov and Y.G. Stroganov [1][2][3], and some partial results towards its proof by P. Di Francesco and P. Zinn-Justin [4].

We consider the analogue of spin singlet wave functions of the Hall effect when one deforms the permutations into the braid group. This amounts to analyze some simple representations of the T.L. algebra on a space of polynomials in N_e variables where N_e is the number of electrons. The relation with the Hall effect arises when we require certain incompressibility properties.

One of the wave functions we consider here is the Halperin wave function [5] for a system of spin one half electrons at filling factor two. When the deformation parameter q is a third root of unity, the braid group representation degenerates into a trivial representation. In this way, we obtain a proof alternative to, and apparently simpler than, that given in [4] of the equality between the sum of the components of the transfer matrix eigenvector and the six vertex model partition function with domain wall boundary conditions [2][6].

Another wave function we consider is the Haldane Rezayi wave function [7][1] describing a system of electrons of spin one half at filling factor one. This wave function is a permanent, and its deformation is described in terms of Gaudin's determinants [8]. When q is a third root of unity, it degenerates to the square of the six vertex model partition function.

[1]More precisely a minor modification of it considered in [15].

In a separate publication [9], we shall consider the Moore Read wave function describing spinless bosons at filling factor one [10]. Its deformation involves an extension of the braid group known as the Birman-Wenzl algebra [11] which can be represented on a polynomial space similarly to the cases presented here. In some appropriate limit, the representation degenerates and the wave function coincides with the transfer matrix eigenvector considered in [12] related to the conjecture of J. De Gier and B. Nienhuis [13].

In general, when a Quantum Hall Effect wave function is discovered, it is soon after observed experimentally. We argue here, that as a bonus, Quantum Hall Effect wave functions and their deformations yield nice mathematical objects. Moreover, all these objects seem to be in relation with striking conjectures emanating from the six vertex model.

Since the permutation group relevant in the quantum Hall effect is technically simpler than the braid group case, let us for pedagogical reasons explain why finding a wave function turns out to be a useful tool to obtain a polynomial representations of the permutation algebra. Essentially, the rest of the paper extends the idea presented here to the braid group case.

We consider electrons in a strong magnetic field projected in the lowest Landau level. In a specific gauge the orbital wave functions are given by:

$$\psi_n(z) = \frac{z^n}{\sqrt{n!}}\, e^{-\frac{z\bar{z}}{4l^2}}, \tag{1}$$

where $z = x+iy$ is the coordinate of the electron, and l the magnetic length defines the length scale related to the strength of the magnetic field. These orbitals are shells of radius $\sqrt{2n}\,l$ occupying an area $2\pi l^2$. Each orbital n is represented by a monomial z^n.

The quantum Hall effect [14] ground state Ψ is obtained by combining these individual orbitals into a manybody wave function. A monomial $z_1^{\lambda_1} \dots z_{N_e}^{\lambda_{N_e}}$ describes a configuration where the electron j occupies the orbital λ_j. The wave function is a linear combinations of such monomials. The effect of the interactions is to impose some vanishing properties when electrons are in contact: $\Psi \sim (z_i - z_j)^m$ with m an integer when $z_i - z_j \to 0$.

The physical properties are mainly characterized by the filling factor ν which is the number of electrons per unit cell of area $2\pi l^2$. When the filling factor is equal to ν, the accessible orbitals and thus the maximal degree in each variable is bounded by $\nu^{-1} N_e$. On the other hand, the effect of the interactions (m) is to force the electrons to occupy more space, thus to occupy higher orbitals and and has the effect of increasing the degree. The problem is thus to obtain wave functions with the maximal possible filling factor (equivalently the lowest degree in each variable) compatible with the vanishing properties imposed by the interactions.

Once such a wave function is obtained, it is the nondegenerate lowest energy state of a Hamiltonian invariant under the permutations, thus we know that it is left invariant under the permutations. By disentangling the coordinate part from the spin part, we obtain an irreducible representation of the permutation algebra acting on polynomials.

Let us illustrate this point in the case of the Halperin wave function [5] which describes a system of spin one half electrons at filling factor two. Due to the Pauli principle, or to a δ potential interaction between electrons of the same spin, the wave function must vanish when two electrons of the same spin come into contact. Each independent orbital can be occupied with two electrons of opposite spin, which is why the maximal filling factor is equal to two.

An equivalent way to impose the constraint is to require that any linear combination of the spin components of the wave function vanishes when three electrons come into contact. The reason for this is that two of the electrons involved will necessary have the same spin. When this constraint is taken into account with the minimal degree hypotheses, one obtains a space of polynomials which can be recombined with the spin components into a wave function changing sign under the permutations. Thus we know a priori that the spatial part of the wave function carries an irreducible representation of the permutation algebra dual to that of the spins. This is precisely by generalizing this argument to the braid group case that we obtain the representations of the T.L. algebra mentioned above.

In the permutation group case case, the components have the simple structure of a product of two Slater determinants grouping together the electrons with the same spin and one does not require to recourse to this machinery.

Let us now briefly indicate why the Halperin wave function may have something to do with the eigenvector of a transfer matrix in the link pattern formulation [16]. The wave function is a spin singlet, and the spin components can best be described in a resonating valance bond (RVB) picture as follows: The labels of the electrons are disposed cyclically around a circle and are connected by a link when two electrons form a spin singlet. Links are not allowed to cross in order to avoid overcounting states. These RVB states coincide with the link patterns of [16]. Thus, the Halperin wave function as the eigenvector of the transfer matrix develops on a basis of link patterns. By deforming the permutation action on link patterns into a T.L. algebra action, one is forced to deform accordingly the polynomial representation so as to insure the invariance of the total wave function. When q is a third root of unity, this property is shared by the transfer matrix eigenvector and allows to identify the two.

In the braid group case, the situation is technically more involved than for the permutations. Nevertheless, the minimal degree hypothesis combined with some annulation constraint satisfied by linear combination of the spin components yields a wave function with the correct invariance properties. A major difference with the Hall effect is that the cancellation no longer occurs at coincident points, but at points shifted proportionally to the deformation parameter q. Typically, we require that for three arbitrary electron labels $i < j < k$ ordered cyclically, the wave function vanishes when the corresponding coordinates take the values z, q^2z, q^4z.

One is also led to study the affine extension in order to impose cyclic invariance properties which are tautologically satisfied with the permutations. While defined in a natural way on the link patterns, the cyclic properties require to introduce a shift parameter s when we identify the coordinate $i + N_e$ with the coordinate i: $z_{i+N_e} = sz_i$. When this shift parameter is related in a specific way to the braid group deformation parameter, the generalized statistics properties can be established coherently. Here, $z_{i+N_e} = q^6 z_i$, but the same annulation property can also be satisfied with s not related to q, and this can be achieved at the price of doubling the degree and enlarging the algebra [9].

In the Haldane Rezayi case, [7], the interactions are such that the wave function must vanish as the square of the distance when electrons of the same spin come into contact. For the same reason as before, this amounts to impose that any linear combination of its spin components vanishes as the square of the distance when three electrons come into contact. This wave function is a permanent, and its deformation is described in terms of Gaudin's determinants [8]. It degenerates to the square of the six vertex model partition function when the deformation parameter is a third root of unity.

The paper is organized as follows. In Section 2, we recall some properties about Hecke algebras and their polynomial representations. Section 3 introduces the T.L. algebra representation used here. Section 4 is the core of the paper where we work out the deformed Hall effect wave functions.

We have attempted to be self contained, but in order not to overload the text with technicalities, we have relegated most of the proofs to appendices to which we refer when it is useful.

2 Hecke Algebra

In this section, we recall some known facts about the Hecke and Temperley and Lieb algebras [17][18].

The Braid group algebra is generated by the braid group generators $t_1, t_2, \ldots, t_{n-1}$, obeying the braid relations:

$$\begin{aligned} t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, \\ t_i t_j &= t_j t_i, \quad \text{if } |i-j| > 1 \end{aligned} \tag{2}$$

for $1 \le i \le n-1$. It can be convenient to use the notation t_{ii+1} instead of t_i, and we will use it when necessary. The Hecke algebra is the quotient of the Braid group algebra by the relations:

$$(t_i - q)(t_i + \frac{1}{q}) = 0, \tag{3}$$

It can also be defined using the projectors $e_i = t_i - q$ obeying the relations:

$$\begin{aligned} e_i^2 &= \tau e_i, \\ e_i e_j &= e_j e_i, \quad \text{if } |i-j| > 1, \\ e_i e_{i+1} e_i - e_i, &= e_{i+1} e_i e_{i+1} - e_{i+1}, \end{aligned} \tag{4}$$

where we set $\tau = -(q+q^{-1})$. The Temperley-Lieb (T.L.) algebra $\mathcal{A}_n$ is the quotient of Hecke algebra by the relations:

$$e_i e_{i+1} e_i - e_i = e_{i+1} e_i e_{i+1} - e_{i+1} = 0. \tag{5}$$

In $\mathcal{A}_n$, a trace can be defined [18] as:

$$\operatorname{tr}(x e_p) = \tau^{-1} \operatorname{tr}(x), \quad \forall x \in \mathcal{A}_p. \tag{6}$$

The affine Hecke algebra, [19][20][21], is an extension of the Hecke algebra (3) by generators y_i, $1 \le i \le n$ obeying the following relations:

$$\begin{aligned} &\text{a)} \quad y_i y_j = y_j y_i, \\ &\text{b)} \quad t_i y_j = y_j t_i, \quad \text{if } j \neq i, i+1, \\ &\text{c)} \quad t_i y_{i+1} = y_i t_i^{-1}, \quad \text{if } i \le n-1. \end{aligned} \tag{7}$$

In Appendix D.2, we indicate why (7c) is natural from the Yang-Baxter algebra point of view.

This algebra can be endowed with two possible involutions: $e_i^* = e_i$, $y_i^* = y_i^{\pm 1}$, $q^* = q^{\pm 1}$.

The symmetric polynomials in the y_i are central elements.

We define the affine T.L. algebra $\mathcal{A}'_n$ as the extension of the T.L. algebra (5) by the generators y_i.

2.1 Yang's realization of the Affine relations

The commutation relations of the affine generators y_i become simpler to understand if we assume that we have a representation of the permutations k_{ij} acting in the natural way on the indices. Let us introduce the operators $x_{ij} = t_{ij} k_{ij}$ for $i < j$ and $x_{ji} = x_{ij}^{-1}$. These operators obey the Yang's relations:

$$\begin{aligned} x_{ij} x_{ji} &= 1, \\ x_{ij} x_{kl} &= x_{kl} x_{ij} \quad \text{if } i \neq j \neq k \neq l, \\ x_{ij} x_{ik} x_{jk} &= x_{jk} x_{ik} x_{ij} \quad \text{if } i < j < k. \end{aligned} \tag{8}$$

We also assume that we have commuting operators s_i such that $s_i s_j x_{ij} = x_{ij} s_i s_j$.

Using (8), one verifies that the operators introduced in [19], (see also [21][22]):

$$\begin{aligned} y_1 &= x_{12} x_{13} \dots x_{1n} s_1, \\ y_2 &= x_{23} x_{24} \dots x_{2n} s_2 x_{21}, \\ y_n &= s_n x_{n1} x_{n2} \dots x_{nn-1} \end{aligned} \tag{9}$$

commute. Indeed, they coincide with the scattering matrices of Yang [23]. (7b) follows directly from (8) once we substitute $t_i = x_{ii+1} k_{ii+1}$. (7c) is a direct consequence of the definition (9) of y_i.

Gathering the permutation operators k_{ij} together, we can obtain another presentation of the y_i. Let us introduce the cyclic operator:

$$\sigma = k_{n-1n} \dots k_{23} k_{12} s_1. \tag{10}$$

Then we have:

$$\begin{aligned} y_1 &= t_1 t_2 \dots t_{n-1} \sigma, \\ y_2 &= t_1^{-1} y_1 t_1^{-1}, \\ y_n &= t_{n-1}^{-1} y_{n-1} t_{n-1}^{-1}. \end{aligned} \tag{11}$$

We can define an additional generator to the t_i: $t_n = \sigma t_1 \sigma^{-1}$, which makes the relations (2) become cyclic. One has:

$$\sigma t_i = t_{i-1} \sigma. \tag{12}$$

So that the affine Hecke (or T.L.) algebra is generated by the generators t_i and the cyclic operator σ obeying (12) and does not require a representation of the permutations. σ^n is a central element which can be set equal to one, and $\sigma^* = \sigma^{-1}$ if we take $t^* = t^{-1}$.

Given the Hecke algebra, there is a simple realization of the affine Hecke algebra which consists in taking $y_1 = 1$. Then, σ is defined as:

$$\sigma = t_{n-1}^{-1} \dots t_1^{-1}. \tag{13}$$

Using the braid relations, one sees that $\sigma t_i = t_{i-1} \sigma$ for $i > 1$, and one can define t_n by $t_n = \sigma t_1 \sigma^{-1}$. Using the braid relations again, one gets $\sigma t_n = t_{n-1} \sigma$. This defines an operator σ which allows to construct the affine generators with (11).

2.2 Polynomial representations

Consider polynomials in $z_1, z_2, \dots, z_n$, a basis of which is given by the monomials: $z^\mu = z_1^{\mu_1} z_2^{\mu_2} \dots z_n^{\mu_n}$. We restrict ourselves to a fixed total degree $|\mu| = \sum \mu_i$. There is a natural action of the permutations and of the operators s_i on this space defined by:

$$\begin{aligned} \bar{\psi}(z_1, \dots z_i \dots z_j \dots, z_n) k_{ij} &= \bar{\psi}(z_1, \dots z_j \dots z_i \dots, z_n), \\ \bar{\psi}(z_1, \dots, z_i \dots, z_n) s_i &= c \bar{\psi}(z_1 \dots, s z_i \dots, z_n). \end{aligned} \tag{14}$$

It is convenient to consider the polynomials in an infinite set of variables z_i, $i \in \mathcal{Z}$, with the identification: $z_{i+n} = s z_i$. The operator $\bar{\sigma}$ (10) takes the form:

$$\bar{\psi} \bar{\sigma}(z_i) = c \bar{\psi}(z_{i+1}). \tag{15}$$

The condition $\sigma^n = 1$ imposes the relation $c^n s^{|\mu|} = 1$.

As shown in the Appendix D.1, it is straightforward to derive the following representation of the Hecke relations (2), (3):

$$\bar{t}_{ij} = -q^{-1} + (1 - k_{ij})\frac{qz_i - q^{-1}z_j}{z_i - z_j}. \tag{16}$$

In this way we obtain a representation of the affine Hecke algebra acting on homogenous polynomials of a given total degree.

The operators x_{ij} take the form:

$$x_{ij} = -q^{-1} + (q - q^{-1})(1 - k_{ij})\frac{z_j}{z_i - z_j}. \tag{17}$$

In the Appendix D.3, we show that there is a natural order on the monomial basis, z^μ, for which the operators x_{ij}, and hence the y_i are realized as lower triangular matrices.

The operator $y = y_1 + \cdots + y_n$ can be seen to commute with the Hecke generators. It is therefore equal to a constant in an irreducible representation. Its eigenvalue evaluated on the highest weight polynomial P_λ thus characterizes the representation. It is given by:

$$y_\lambda = c(-q)^{1-n}(s^{\lambda_1} + s^{\lambda_2}q^2 + \cdots + s^{\lambda_n}q^{2(n-1)}). \tag{18}$$

If λ' is a permutation of the partition λ, we say that $z^{\lambda'}$ is of degree λ. In this paper, we are mainly concerned with the monomials $z^{\lambda'}$, of degree:

$$\lambda = (\frac{n}{2} - 1, \frac{n}{2} - 1, \frac{n}{2} - 2, \frac{n}{2} - 2, \ldots, 0, 0), \tag{19}$$

and of total degree $|\lambda| = \frac{n}{2}(\frac{n}{2} - 1)$.

We will consider the subclass λ_π of permutations of λ indexed by the standard Young tableaus with two columns of $\frac{n}{2}$ boxes:

$$z^{\lambda_\pi} = (z_{\mu_1}z_{\nu_1})^{\frac{n}{2}}(z_{\mu_2}z_{\nu_2})^{\frac{n}{2}-1}\ldots(z_{\mu_{\frac{n}{2}}}z_{\nu_{\frac{n}{2}}})^0, \tag{20}$$

with $\mu_1 > \mu_2 > \cdots > \mu_{\frac{n}{2}}$, $\nu_1 > \nu_2 > \cdots > \nu_{\frac{n}{2}}$, and $\mu_i > \nu_i$. To simplify notations, we denote these monomials by z^π instead of z^{λ_π}.

We identify the standard Young tableaus with the paths $\pi = [h_i]$ introduced in the Appendix A: $h_0 = h_n = 0$, $h_i \geq 0$ and $h_{i+1} - h_i = \pm 1$. These paths are obtained using the rule: $h_i - h_{i-1} = 1$ if $i \in \{\mu_j\}$, and $h_i - h_{i-1} = -1$ if $i \in \{\nu_j\}$. For the paths, we use the order $\pi \geq \pi'$, if $[h_i] \geq [h'_i]$ $\forall i$, which coincides with the reverse order for the monomials: $z^\pi \leq z^{\pi'}$.

This identification is illustrated in Fig. 1, following the Appendix A.1.

3 Representation of the affine T.L. algebra on words

For n even, there is a simple representation $(\mathcal{H}_n)$ of the T.L. algebra $\mathcal{A}_n$ obtained as follows. One considers the left action of $\mathcal{A}_n$ on the space $\mathcal{A}_n\alpha$ where α is the minimal projector $\alpha = e_1e_3\dots e_{n-1}$. A basis of this space is given by reduced monomial words in the e_i. The elements of this basis can be put into correspondence with paths or link patterns. In the Appendix A we exhibit a basis of reduced words and we define an order relation on the reduced words.

A scalar product can be defined as:

$$\pi^*\pi' = \langle\pi|\pi'\rangle\alpha, \tag{21}$$

where $e_i^* = e_i$ and the involution reverses the order of the letters. In the link-pattern representation, this scalar product is given by: τ^l where l is the number of loops one gets by concatenating the link patterns of π and π'. If $\tau = -(q+q^{-1})$ with q not a root of one, this scalar product is positively definite [18]. For this scalar product the T.L. generators e_i are by construction hermitian.

To obtain the affine algebra representation, let us define as in (13) the cyclic operator:

$$\sigma = -q^{\frac{n}{2}-2}t_{n-1}^{-1}\dots t_1^{-1}, \tag{22}$$

where the normalization is such that in the link-pattern representation, σ acts by cyclicly permuting the indices $i \to i-1$ (see Appendix A.4). One can define an additional generator, $e_{n1} = \sigma e_{12}\sigma^{-1}$, which acts in the same way as e_{ii+1} with the two indices $1, n$. The affine generators are constructed using (11) with $y_1 = -q^{\frac{n}{2}-2}$.

In the Appendix A.3, we show that the operators y_i are realized as triangular matrices in $\mathcal{H}_n$, they are hermitian for the choice $q = q^*$. Their sum $y = \sum_i y_i$ is constant with a value given by:

$$y = -(q+q^{-1})\frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q-q^{-1}}. \tag{23}$$

There is an imbedding of $\mathcal{H}_{n-2}$ into $\mathcal{H}_n$ given by $\pi \to \pi e_1$ and a projection E from $\mathcal{H}_n$ to $\mathcal{H}_{n-2}$ given by:

$$e_1\pi = \tau E(\pi)e_1. \tag{24}$$

This projection is both triangular and hermitian.

In the Appendix A.2, we identify $\mathcal{H}_n$ with $\mathcal{A}_{\frac{n}{2}}$. This allows us to interpret the projection E as a conditional expectation value of $\mathcal{A}_{\frac{n}{2}} \to \mathcal{A}_{\frac{n}{2}-1}$ [18].

4 q-deformed Quantum Hall Effect wave functions

4.1 Statement of the Problem

Let us consider a vector Ψ:

$$\Psi = \sum_\pi \pi F_\pi(z_i), \tag{25}$$

constructed in the following way. The vectors π are the basis vector $\mathcal{H}_n$ on which the T.L. algebra acts to the left. F_π are homogeneous polynomials in the variables $z_1, z_2, \ldots, z_n$ (n is even). The polynomial coefficients of Ψ carry a representation of the affine Hecke algebra generated by the operators $\bar{t}_i$ and $\bar{\sigma}$ acting to the right. The problem is to determine the coefficients F_π in such a way that both actions give the same result on the vector Ψ:

$$\begin{aligned}\Psi\bar{t}_i &= t_i\Psi \\ \Psi\bar{\sigma} &= \sigma\Psi,\end{aligned} \tag{26}$$

The first of these relations is equivalent to the more familiar relation (63) derived in Appendix D.1.

Said differently, we look for a dual action of the affine T.L. algebra acting on polynomials. Unless we specify it, we address this problem for a generic value of the parameter q, not a root of unity, for which the T.L. algebra is semisimple [18].

4.2 Module $\mathcal{M}_n$

The dual representation of $\mathcal{H}_n$ is obtained by acting with the T.L. generators on the dual F_ω of the highest vector $\omega \in \mathcal{H}_n$. ω is given by the sequence ($a_{2p+1} = p+1$) in the characterization of words we use in the Appendix A and is fully characterized by the property that it can be written $\omega = e_i\pi$ only for $i = \frac{n}{2}$. The dual vector F_ω must therefore be annihilated by all the e_i with $i \neq n/2$. We realize the module $\mathcal{H}_n$ upon acting on F_ω with the generators e_i for $1 \leq i \leq n-1$. We define:

$$\mathcal{M}_n = \mathbf{Vec}\{\bar{\psi} = F_\omega\psi\}, \tag{27}$$

where we denote with a bar $\bar{\psi}$ the result of the action of ψ, a monomial in e_i, on F_ω. Thus we have $\bar{1} = F_\omega$. In the Appendix C.1, we show that $\mathcal{M}_n$ defined in this way is a module over the T.L. algebra as long as the e_i obey the Hecke relations (4). In other words, the projectors $U^-_{i,i+1} = e_ie_{i+1}e_i - e_i$ are null in $\mathcal{M}_n$. This formal module is however not isomorphic to $\mathcal{H}_n$ unless F_ω obeys some supplementary condition (56). Here, we construct a representation of the T.L. algebra by identifying a state F_ω dual to ω and satisfying the condition (56).

The expression of the T.L. generators $e_i = t_i - q$ for $1 \leq i \leq n-1$ follows from (16):

$$\begin{aligned} e_i &= -\frac{qz_{i+1} - q^{-1}z_i}{z_{i+1} - z_i}(1 + k_{ii+1}) \\ e_i - \tau &= (1 - k_{ii+1})\frac{qz_i - q^{-1}z_{i+1}}{z_i - z_{i+1}}. \end{aligned} \tag{28}$$

The effect of e_i and $\tau - e_i$ is to split a polynomial $\bar{\psi}$ into two polynomials belonging to $\mathcal{M}_n$, $\bar{\psi} = S_1 + (qz_i - q^{-1}z_{i+1})S_2$, where both S_1 and S_2 are symmetrical under the exchange of z_i and z_{i+1}. This decomposition is unique and characterizes the projector e_i.

It can be convenient to distinguish the representation on $\mathcal{H}_n$ from its dual on $\mathcal{M}_n$. When this is the case, we denote $\bar{e}_i$ the dual projectors which act on polynomials.

One verifies that:

$$\Delta_p(z_1,\ldots,z_p) = \prod_{1\le i<j\le p} (qz_i - q^{-1}z_j) \tag{29}$$

is annihilated by all the $\bar{e}_i,\ 1\le i\le p-1$, and this defines Δ_p up to a product by a symmetric polynomial in $z_1,\ldots,z_p$. Therefore, the minimal degree polynomial candidate for F_ω is:

$$F_\omega = \Delta_{\frac{n}{2}}(z_1,\ldots,z_{\frac{n}{2}})\Delta_{\frac{n}{2}}(z_{\frac{n}{2}+1},\ldots,z_n). \tag{30}$$

This polynomial cannot be q-antisymmetrized over $\frac{n}{2}+1$ variables, since the result would have a degree at least $\frac{n}{2}$ in z_1, and this is the content of the condition (56). Thus, $\mathcal{M}_n$ is a simple module which can be identified with $\mathcal{H}_n$. This representation is characterized by its Young diagram $(2^{\frac{n}{2}})$ having two columns of length $\frac{n}{2}$. Its dimension is given by the Catalan number $C_n = \binom{n}{\frac{n}{2}} - \binom{n}{\frac{n}{2}-1}$.

If we denote $\pi|_\omega$ the coefficient of ω in the reduced expression of π, we can identify the polynomials $\bar\psi\in\mathcal{M}_n$ with the dual of $\mathcal{H}_n$ through the relation: $\bar\psi(\pi) = \psi\pi|_\omega$.

We can also introduce the dual basis F_π defined by its action on reduced words:

$$F_\pi(\pi') = \delta_{\pi,\pi'}. \tag{31}$$

Let π_ψ be the complementary word of ψ (defined in Appendix C) such that one can write $\psi\pi_\psi = \omega$ without reducing the expression. One has $\psi\pi_\psi|_\omega = 1$ and $\psi\pi|_\omega = 0$ if $\pi < \pi_\psi$. Expanding $\bar\psi$ on the basis F_π, we get: $\bar\psi = \sum_{\pi\ge\pi_\psi}\psi\pi|_\omega F_\pi$, and by inverting the triangular system, we can obtain the expression of F_π.

Let us verify that the highest monomial of $\bar\psi$, and thus of F_{π_ψ} as well, is proportional to z^{π_ψ}. We show this by recursion. It is true for $\psi = 1$: $\bar{1} = F_\omega$, $\pi_1 = \omega$ and z^ω is the highest monomial of F_ω. We assume that the property is true for $\psi' < \psi$. If $\psi\ne 1$, one can write $\psi = \psi' e_i$ with $\psi' < \psi$, and we have $\pi_{\psi'} = e_i\pi_\psi$ with $\pi_\psi < \pi_{\psi'}$.

Then, according to the recursion hypothesis, the highest monomial of $\bar\psi'$ is $z^{\pi_{\psi'}}$ which contains the factor $z_i^m z_{i+1}^n$ with $n > m$. Since $z_i^m z_{i+1}^n e_i = -qz_i^n z_{i+1}^m +$ lower monomials, the highest monomial of $\bar\psi = \bar\psi' e_i$ is z^{π_ψ}.

We also obtain the normalization coefficient of z^π up to a global factor: $F_\pi = c_\pi z^\pi +$ lower monomials, with $c_\pi = (-\frac{1}{q})^{l_\pi}$, and l_π is the number of letters e_i entering the reduced expression of π.

4.3 Module $\mathcal{M}'_n$

We now consider a larger module $\mathcal{M}'_n \supset \mathcal{M}_n$ by letting the operator $\bar\sigma$ defined in (15) act on the polynomials. We will put some constraint on the parameter s

(which characterizes $\bar{\sigma}$) to have $\mathcal{M}'_n = \mathcal{M}_n$. We consider the simple case $n = 4$ in the Appendix B and we obtain $s = q^6$ which is the general case as we show here.

Let us assume that $\mathcal{M}'_n = \mathcal{M}_n$ and see what constraints s must satisfy to identify $\bar{\sigma}$ defined by its action on polynomials (15) with σ defined in terms of generators (13).

We observe that $\sigma^{-1}\omega_n = e_1\omega_{n-2}$, where ω_{n-2} is the highest state in $\mathcal{H}_{n-2}$. This can easily be verified in the link pattern representation. Thus, we must have:

$$\sigma E(\pi)e_1|_{\omega_n} = E(\pi)e_1|_{\sigma^{-1}\omega_n} = E(\pi)|_{\omega_{n-2}}. \tag{32}$$

Let us consider the dual to the projection E, E' from $\mathcal{M}_n \to \mathcal{M}_{n-2}$ defined as $\bar{\psi}e_1 = \tau E'(\bar{\psi})$. E' needs to satisfy the conditions:

$$\begin{aligned} &\text{a)} \quad E'(\bar{\psi}e_1) = \tau E'(\bar{\psi}) \\ &\text{b)} \quad E'(\bar{\psi}e_i) = E'(\bar{\psi})e_i \ \forall i > 2 \\ &\text{c)} \quad E'(\bar{\psi}e_1) = 0 \Rightarrow \bar{\psi}e_1 = 0. \end{aligned} \tag{33}$$

From (32), in order to identify $\bar{\sigma}$ with σ, we see that the projection E' must satisfy:

$$E'(F_{\omega_n}\bar{\sigma}) = F_{\omega_{n-2}}. \tag{34}$$

E' can be realized as:

$$E'(\bar{\psi}) = c'\frac{1}{\phi(z, z_i)}\bar{\psi}(z_1 = z, z_2 = q^2 z, z_i), \tag{35}$$

where $\phi(z, z_i) = \prod_{i=3}^{n}(z_i - q^4 z)$ and c' is a normalization constant. E' verifies (33a,b) by construction as can be seen from the expression (28) of $e_1 - \tau$.

Using the explicit expression (15) of σ, we have:

$$\begin{aligned} E'(F_{\omega_n}\sigma) = c' s^{-\frac{n}{4}-\frac{1}{2}} \frac{1}{\phi(z, z_i)} \prod_{3}^{\frac{n}{2}+1} (q^3 z - q^{-1} z_i) \\ \times \prod_{\frac{n}{2}+2}^{n} (q z_i - q^{-1} s z) F_{\omega_{n-2}}(z_3, \ldots, z_n). \end{aligned} \tag{36}$$

which imposes $s = q^6$ for the polynomial in the numerator to be proportional to $\phi(z, z_i)$ and (34) to be satisfied.

To identify $\mathcal{M}_n$ and $\mathcal{M}'_n$, we give a more convenient characterization of $\mathcal{M}'_n$. Consider the space $\mathcal{M}''_n$ of homogenous polynomials in n variables, and of the minimal total degree, obeying the property:

$$\text{(P)}: \ \bar{\psi}(z_i = z, z_j = q^2 z, z_k = q^4 z) = 0, \quad \text{if } i, j, k, \text{ are cyclically ordered.} \tag{37}$$

This property is obviously compatible with the cyclic identification $z_{i+n} = q^6 z_i$, it is thus preserved by $\bar{\sigma}$ (15). By applying (P) to the triplets $(1, 2, j)$, we see that the projection (35) is well defined from $\mathcal{M}''_n$ to $\mathcal{M}''_{n-2}$.

We show that $\mathcal{M}''_n = \mathcal{M}_n$. For this, we first show that $\mathcal{M}''_n$ is a module over the T.L. algebra $\mathcal{A}_n$ and that it contains $\mathcal{M}_n$, then we show that $\mathcal{M}''_n$ is irreducible over $\mathcal{A}_n$.

To show that $\mathcal{M}''_n$ is a module over $\mathcal{A}_n$, we verify that the generators e_i preserve the property (P). Assuming that the polynomial $\bar{\psi}$ verifies (P) we verify that $\bar{\psi} e_i$ obeys (P) for a cyclically ordered triplet k, l, m. If $\{i, i+1\} \cap \{k, l, m\} = \emptyset$, it is obvious. If $i + 1 = k$, it results from the fact that $\bar{\psi}$ obeys (P) for the triplets i, l, m and $i+1, l, m$. The same type of argument applies if $i = m$. If $\{i, i+1\} \subset \{k, l, m\}$, $\bar{\psi}(e_i - \tau)$ is proportional to $(q z_i - q^{-1} z_{i+1})$ and therefore obeys (P).

Let us show that (33c) is satisfied in $\mathcal{M}''_n$. If $E'(\bar{\psi} e_1) = 0$, $\bar{\psi} e_1$ vanishes when $z_2 = q^2 z_1$, and from the definition (28) of e_1, it is symmetric in z_1, z_2. It is therefore divisible by $(z_1 - q^2 z_2)(z_2 - q^2 z_1)$. Hence, $\bar{\psi} e_1/(z_1 - q^2 z_2)$ satisfies (P) and has a total degree reduced by one. It is thus equal to zero according to our minimal degree hypothesis.

It is clear that F_ω satisfies the property (P). To show that $\mathcal{M}_n \subset \mathcal{M}''_n$, we need to show that the degree of the polynomials in $\mathcal{M}''_n$ is the degree $\frac{n}{2}(\frac{n}{2} - 1)$ of F_ω. We proceed by recursion on n and for the moment, we exclude the case where e_1 is represented as zero in $\mathcal{M}''_n$. Due to (33c) there are polynomials $\bar{\psi}$ in $\mathcal{M}''_n$ such that $E'(\bar{\psi}) \neq 0$. This implies that $\bar{\psi}$ has a degree at least $n - 2$ in z_1, z_2. We can apply the recursion hypothesis to $E'(\bar{\psi}) \in \mathcal{M}''_{n-2}$ to conclude that the minimal degree is $\frac{n}{2}(\frac{n}{2} - 1)$.[2]

To show that $\mathcal{M}''_n$ is irreducible as a T.L. module, we use the recursion hypothesis that $\mathcal{M}''_{n-2} = \mathcal{M}_{n-2}$. Due to (33c), E' is injective from $\mathcal{M}''_n e_1$ to $E'(\mathcal{M}''_n) \subset \mathcal{M}_{n-2}$. Since $\mathcal{M}_n e_1 = \mathcal{M}_{n-2} \subset \mathcal{M}''_n e_1$, we have $\mathcal{M}''_n e_1 = \mathcal{M}_n e_1$. Thus, if $\mathcal{M}''_n$ contains an irreducible submodule $R \neq \mathcal{M}_n$, $R e_1 = 0$. If $R e_1 = 0$, from (5) we see that all the e_i are represented as 0 in R, and therefore, the polynomials in R are proportional to Δ_n defined in (29) times a symmetric polynomial. Since the total degree of Δ_n is larger than $\frac{n}{2}(\frac{n}{2} - 1)$, $R = 0$. We conclude that $\mathcal{M}''_n = \mathcal{M}_n$ as a T.L. module.

Finally, to identify $\mathcal{M}''_n$ and $\mathcal{M}_n$ as affine modules, we observe that $y_1 = \sigma^{-1}\bar{\sigma}$ commutes with $\mathcal{A}_{n-1}$ generated by $e_2, \ldots, e_n$. Since $\mathcal{M}_n$ is irreducible over $\mathcal{A}_{n-1}$ [18], y_1 is proportional to the identity, thus σ and $\bar{\sigma}$ can be identified.

[2] The same argument shows that the maximal degree of the polynomials in $\mathcal{M}''_n$ is $\geq \lambda$ for the order defined in the Appendix D.3.

4.3.1 Relation with the Macdonald Polynomials and the work of Di Francesco and Zinn-Justin

As a check of consistency, we must verify that the two expressions of the eigenvalue of the central operators y (18), (23) are the same when $s = q^6$. This is indeed the case if we substitute in (18) the degree λ (19) of the highest polynomial in $\mathcal{M}_n$ and $c = q^{3(1-\frac{n}{2})}$.

For a generic s, the operator y (18) can be diagonalized on the basis of symmetric polynomials and its eigenvectors define the Macdonald polynomials [24]. We have seen that when $s = q^6$, the polynomial representation is reducible. As a counterpart, some diagonal elements $y_{\lambda'}$ of y become degenerate with y_λ, for example, $\lambda'_2 = \lambda_2 - 1$, $\lambda'_5 = \lambda_5 + 1$. Thus, y cannot be diagonalized. We must use another operator such as $\frac{dy}{ds}$ to define the analogous symmetric polynomial.

In the non semisimple case $q^2 + q + 1 = 0$, $(\tau = 1)$, the T.L. representation admits a sub-representation given by $\mathbf{Vec}\{\sum x_\pi \pi,\ \text{with}\ \sum x_\pi = 0\}$. The trivial representation Ω is obtained by equating to zero these vectors. The dual polynomial $F_\Omega = \sum_\pi F_\pi$ is therefore symmetrical of degree λ, and obeys the property (P) (37). This completely determines it to be proportional to the Schur function s_λ with λ given by (19). Indeed, s_λ has a degree λ and satisfies (P) since three columns of the determinant which defines it become linearly dependant when we make the substitution (P). By the same argument as used in 4.3, the degree of a symmetric polynomial satisfying (P) must be at least λ (relatively to the order of partitions which follows from Appendix D.3) which prove its unicity.

Following [4], in this limit, the F_π can be identified with the components of a transfer matrix eigenvector fully characterized by the relation (63). Thus, the sum of these components is s_λ.

It would be interesting to see if in this limit, F_Ω can be recovered as the eigenvector of some operator such as $\frac{dy}{ds}$.

4.4 Representation on Gaudin's determinants

It is well known that the Bethe scalar products [8] can be expressed using a quotient of two determinants. Here, we construct a representation of the T.L. algebra acting on these quotients. We split the variables z_i into $A = \{z_1, \dots, z_{\frac{n}{2}}\}$ and $B = \{z_{\frac{n}{2}+1}, \dots, z_n\}$. We also introduce p a square root of q, $p^2 = q$. We define the polynomial F'_ω:

$$F'_\omega(z_1, \dots, z_n) = \frac{\left|(p^2 z_i - p^{-2} z_j)^{-1}(p z_i - p^{-1} z_j)^{-1}\right|}{\left|(p z_i - p^{-1} z_j)^{-1}\right|} \Delta_n(z_1, \dots, z_n), \quad \text{with } i \in A,\ j \in B. \tag{38}$$

The first factor is the ratio of the Gaudin determinant with the Cauchy determinant [8]. It is also related to the domain wall boundary condition partition function [6] of a six vertex model with weights: $a = qx - q^{-1}y$, $b = px - p^{-1}y$, $c = \sqrt{xy}(p - p^{-1})$ [3].

The second factor Δ_n (29) insures that $F'_\omega(z_i)$ is a polynomial. This factor has an innocuous effect on the T.L. algebra since:

$$\Delta_n(z_1, \ldots, z_n)t_i = \tilde{t}_i \Delta_n(z_1, \ldots, z_n), \tag{39}$$

where $\tilde{t}_i$ is obtained from t_i (16) by the substitution $q \to -1/q$ which preserves the relations (2), (3), but exchanges e_i with $e_i - \tau$.

The ratio of the two determinants being symmetrical in the two sets of variables A and B, F'_ω is annihilated by all the e_i with $i \neq \frac{n}{2}$.

To show that the action of the T.L. algebra (28) on F'_ω produces an irreducible module, we proceed as in 4.3. Consider the space $\mathcal{M}_n$ of homogenous polynomials in n variables, and of the minimal total degree, obeying the property:

$$\begin{aligned}
&(\mathrm{P}'):\\
&\bar{\psi}(z_{i_1} = q^{a_1}z, z_{i_2} = q^{a_2}z, z_{i_3} = q^{a_3}z) = 0, \quad \text{if } i_1, i_2, i_3, \text{ are cyclically ordered,}\\
&\qquad\qquad \text{and for: } (a_1, a_2, a_3) = (-1, 0, 1), (-1, 1, 0), (-2, 0, 2), (0, -1, 1).
\end{aligned} \tag{40}$$

Note that these triplets are stable under the cyclic permutation, $(a_1, a_2, a_3) \to (a_3 - 2, a_1 + 1, a_2 + 1)$, and the transpositions, $(a_i, a_{i+1}) \to (a_{i+1}, a_i)$, whenever $|a_{i+1} - a_i| = 1$.

From the cyclic invariance, we deduce that this space is preserved under the action of σ (15) if we take $s = q^3$.

By applying the property (P′) to z_1, z_2, z_i with $(a_1, a_2, a_3) = (-1, 1, 0)$ and $(-2, 0, 2)$, we can define a projection (35) from $\mathcal{M}_n \to \mathcal{M}_{n-2}$. The polynomial $\phi(z, z_i)$ is now a product of two factors $\phi(z, z_i) = \prod_{i=3}^{n}(qz - z_i)(q^4 z - z_i)$. Arguing as in 4.3, we see that this projection satisfies the properties (33).

This space is stable under the action of the generators e_i. The proof is similar to the one given in 4.3 and requires the stability of the triplets (a_1, a_2, a_3) under the transpositions. The minimal degree is now $n(\frac{n}{2} - 1) = 2|\lambda|$ with $|\lambda|$ given by (19).

Let us show that F'_ω (38) satisfies the property (P′) (40). We consider (i_1, i_2, i_3) and (a_1, a_2, a_3). If the variables z_l, z_m with $l < m$, corresponding to two a_i which differ by 2, belong to the same set A or B, $F'_\omega(z_m = q^2 z_l) = 0$ due to the factor Δ_n. Otherwise, two variables $z_l = z \in A$ and $z_m = q^2 z \in B$ differ by a factor q^2. By isolating the contribution of the pole $(p^2 z_l - p^{-2} z_m)$ in the Gaudin determinant, we factorize a term $\prod_i(qz - z_i)$ coming from the Cauchy denominator, and this enables to conclude that $F'_\omega(z_{i_1} = q^{a_1}z, z_{i_2} = q^{a_2}z, z_{i_3} = q^{a_3}z) = 0$ in all the other cases.

[3] Notice that for this six vertex model, $\Delta = \frac{a^2+b^2-c^2}{ab} = p + \frac{1}{p} \neq \tau$.

Arguing as in 4.3 we conclude that $\mathcal{M}_n$ is an irreducible module over the affine T.L. algebra and that it coincides with the module obtained upon acting with the generators on F'_ω.

We verify again that the eigenvalue of the central operators y (18) is given by (23). Now, $s = q^3$ instead of q^6 in 4.3, but the degree 2λ (19) of the highest polynomial in $\mathcal{M}_n$ is doubled and c keeps the same value $c = q^{3(1-\frac{n}{2})}$.

In the nonsemisimple case $q^2 + q + 1 = 0$, using the result of [25] we see that the components F'_π are given by the product of the components F_π of the last section with the Schur function s_λ: $F'_\pi = s_\lambda F_\pi$, and therefore, $\sum_\pi F'_\pi = s_\lambda^2$.

5 Conclusion

Let us conclude with a few comments and questions.

On the mathematical side, this work provides a unification ground around the conjectures relating the eigenvector components of a loop model transfer matrix, the six vertex model domain wall boundary condition partition function and other mathematical objects. It opens the possibility to deform the polynomials underlying these conjectures by presenting them from the algebra representation point of view. We believe that these conjectures are related to incompressibility, and we hope to return to this point in a future publication.

From a technical point of view, it would be interesting to repeat the Jones construction of Appendix A.2 on the polynomials directly. This would allow to recover in a direct way the product structure which they carry since they are associated to elements of the T.L. algebra.

The precise correspondence between the polynomial obtained here and the Macdonald polynomials needs to be clarified.

Finally, do the deformed wave functions considered here have anything to do with physics? At this moment, we have no answer to this question. A step towards a physical interpretation would be to identify a scalar product and a Hermitian Hamiltonian for which these wave functions are the ground states. This could also be useful to access to the excited states (polynomials of higher degree obeying the constraint (P)) which play an important role in the Quantum Hall Effect.

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A Word representation

A.1 Reduced words

The module $\mathcal{H}_n$ is obtained by acting with the T.L. generators of $\mathcal{A}_n$ on the lowest state $\alpha = e_1 e_3 \dots e_{n-1}$. Using the relations (5), we obtain a basis of $\mathcal{H}_n$ given by reduced words π:

$$\pi = (e_{a_{n-1}} e_{a_{n-1}+1} \dots e_{n-1}) \dots (e_{a_{2p+1}} e_{a_{2p+1}+1} \dots e_{2p+1}) \dots (e_{a_3} e_{a_3+1} \dots e_3) e_1, \tag{41}$$

with, $a_{2p+1} \leq 2p+1$, and $1 < a_3 < \cdots < a_{2p+1} < \cdots < a_{n-1}$. So, a word is fully characterized by the sequence (a_{2p+1}).

On reduced words there is a natural order relation: $\pi > \pi'$ if π is written $b\pi'$ with b a monomial. One has $\pi \geq \pi'$ if $a_{2p+1} \leq a'_{2p+1}$ for all p.

Another way to represent a reduced word is in terms of paths. Let m_i be the number of times the generator e_i appears in the reduced expression of π. One has $m_{2i} = m_{2i-1}$ or $m_{2i-1} - 1$ and $m_{2i+1} = m_{2i}$ or $m_{2i} + 1$. We define $h_{2i} = 2m_{2i} - 1$, $h_{2i-1} = 2m_{2i-1} - 2$ and $h_0 = h_n = 0$ by convention. We can describe the words π by the paths $\pi = [h_i]$ where $h_0 = h_n = 0$, $h_i \geq 0$ and $h_{i+1} - h_i = \pm 1$. Using the path representation, one has $\pi \geq \pi'$, if $[h_i] \geq [h'_i]\ \forall i$.

If π' is a non reduced word, by reducing it, one decreases the number of times the generator e_i appears in its expression. We thus see that the order relation can be presented in a weaker form valid for non reduced words: If π' is a word, not necessarily reduced and π is a reduced word, $\pi > \pi'$ if π' can be obtained by erasing letters e_k from the (reduced) expression of π.

Finally, there is way to characterize this representation in terms of link patterns. It is convenient to dispose the n points cyclically around a circle. A link pattern is obtained by pairing all the points in the set $\{1, 2, \dots, n\}$: $\pi = \{[i_1, i_2], [i_3, i_4], \dots, [i_{n-1}, i_n]\}$, in such a way that two links never cross. In practice, if $[i, j]$ is a link, then the other links $[k, l]$ are either inside, or outside the interval $[i, j]$. The action of $e_{i,i+1}$ is given by: $e_{i,i+1}\{[i, i+1], \dots, [i_{n-1}, i_n]\} = \tau\{[i, i+1], \dots, [i_{n-1}, i_n]\}$, and $e_{i,i+1}\{[i, j], [i+1, k] \dots, [i_{n-1}, i_n]\} = \{[i, i+1], [j, k], \dots\}$. In this representation, $\alpha = \{[1, 2], [3, 4], \dots, [n-1, n]\}$, and $\omega = \{[1, n], [2, n-1], \dots, [\frac{n}{2} - 1, \frac{n}{2} + 1]\}$.

These representations are illustrated in Fig. 1.

A.2 Identifying $\mathcal{H}_n$ with $\mathcal{A}_{\frac{n}{2}}$

The link pattern representation allows to identify in a natural way $\mathcal{H}_n$ with $\mathcal{A}_{\frac{n}{2}}$. If we split $\{1, 2, \dots, n\}$ into two subsets: $\{1, 2, \dots, \frac{n}{2}\}$ and $\{\frac{n}{2}, \dots, n\}$, the product $\pi * \pi'$ is defined on the link patterns by identifying the last $\frac{n}{2}$ points of π with the first $\frac{n}{2}$ points of π' through $i \equiv n + 1 - i$ and concatenating the links obtained in this way. The link pattern $\pi * \pi'$ is obtained by removing the loops which appear in this concatenating operation by giving them a weight τ.

Another identification can be achieved on paths by folding a path of length n into a loop of length $\frac{n}{2}$. In this way, we realize $\mathcal{A}_{\frac{n}{2}}$ as the algebra of double paths acting on Bratteli diagrams [26][17][18].

In this identification, $\mathcal{A}_{\frac{n}{2}}$ is a bimodule over itself. The first $\frac{n}{2} - 1$ generators $e_i \in \mathcal{A}_n$ are identified with the generators of $\mathcal{A}_{\frac{n}{2}}$ acting to the left, while the last $\frac{n}{2} - 1$ generators are identified with e_{n+1-i} acting to the right.

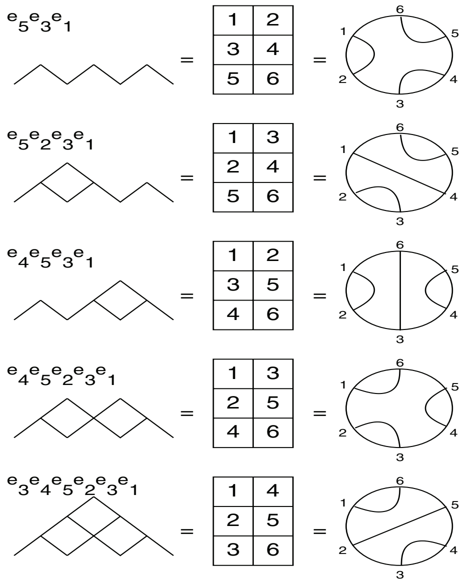

Figure 1. The three different ways to represent a word illustrated in the case of $\mathcal{H}_6$.

The state ω is the identity in $\mathcal{A}_{\frac{n}{2}}$, and the trace in $\mathcal{A}_{\frac{n}{2}}$ coincides with the scalar product with ω in $\mathcal{A}_n$:

$$\mathrm{tr}(x) = \tau^{-\frac{n}{2}} \langle \omega | x \rangle. \tag{42}$$

The projection: $E_{\frac{n}{2}} = \sigma^{-\frac{n}{2}+1} E \sigma^{\frac{n}{2}-1}$, with E given by (24) can be reinterpreted as a conditional expectation value [18], $E_{\frac{n}{2}} : \mathcal{A}_{\frac{n}{2}} \to \mathcal{A}_{\frac{n}{2}-1}$. Jones construction enables then to construct $e_{\frac{n}{2}} \in \mathcal{A}_{\frac{n}{2}+1}$ algebraically from the knowledge of $E_{\frac{n}{2}}$.

A.3 Triangularity of y_m

Let us show that the affine generators $y_{m+1} = t_m^{-1} t_{m-1}^{-1} \dots t_1^{-2} \dots t_m^{-1}$ are triangular in the word representation. It is obvious for $y_1 = 1$ and $y_2 = t_1^{-2}$ since e_1 is triangular. We proceed by recursion and assume that y_k are triangular for $k < m+1$. Using these hypotheses, we show that y_{m+1} is also triangular.

First we show that y_{m+1} acts diagonally on α. To study the action of y_{m+1} on α, we distinguish the two cases m odd or even. If m is odd, then:

$$y_{m+1}\alpha = t_m^{-1} y_m t_m^{-1} e_m \cdots = -q t_m^{-1} y_m e_m \cdots = -\lambda_m q t_m^{-1} e_m \cdots = q^2 \lambda_m \alpha, \tag{43}$$

where λ_m is the eigenvalue of y_m on α. If m is even, we make use of the fact that $t_{m-1}^{-1} t_m^{-1} e_{m-1} = \frac{1}{q} e_m e_{m-1}$ and the same relation with the indices m and $m-1$ exchanged to obtain:

$$
\begin{aligned}
y_{m+1}\alpha &= t_m^{-1}t_{m-1}^{-1}y_{m-1}t_{m-1}^{-1}t_m^{-1}e_m\cdots = \frac{1}{q}t_m^{-1}y_{m-1}e_me_{m-1}\ldots \\
&= \frac{1}{q}\lambda_{m-1}t_m^{-1}t_{m-1}^{-1}e_me_{m-1}\cdots = \frac{1}{q^2}\lambda_{m-1}\alpha.
\end{aligned}
\tag{44}
$$

We deduce that α is an eigenstate of y_m with the eigenvalue λ_m obeying the recursion relations $\lambda_{2m} = q^2\lambda_{2m-1}$, $\lambda_{2m+1} = \frac{1}{q^2}\lambda_{2m-1}$. Together with the fact that $\lambda_1 = 1$, we deduce (18).

To show that y_{m+1} is triangular on words $\neq \alpha$. We proceed by recursion and assume that y_{m+1} acts in a triangular way on words $< \pi$ and show that the property is also true for π.

Let us consider the action of y_{m+1} on a reduced word $\pi \neq \alpha$. This word can be put under the form $\pi = e_i\pi'$ where $\pi' < \pi$. We consider the three cases, $i \neq m, m+1$, $i = m$, $i = m+1$. In the third case, either the word can be written in the form $e_{m+1}e_m\pi'$ with π' reduced, or it can be written $e_p\pi'$ with $p < m$. The second possibility reduces to the first case and we need only consider the first possibility.

We observe that y_{m+1} commutes with e_i: $y_{m+1}e_i = e_iy_{m+1}$ if $i > m+1$ or if $i < m$. It is obvious if $i > m+1$ and follows from the braid relations if $i < m$. In the three cases we can thus write:

$$
\begin{aligned}
y_{m+1}e_i\pi' &= e_i(y_{m+1}\pi') \quad \text{for } i \neq m, m+1, \\
y_{m+1}e_m\pi' &= t_m^{-1}y_mt_m^{-1}e_m\pi' = -qt_m^{-1}(y_me_m\pi'), \\
y_{m+1}e_{m+1}e_m\pi' &= t_m^{-1}y_mt_m^{-1}e_{m+1}e_m\pi' = t_m^{-1}(y_me_m\pi' + \frac{1}{q}y_me_{m+1}e_m\pi').
\end{aligned}
\tag{45}
$$

It follows from the hypothesis that the terms in brackets are less than π. In the first case because $\pi' < \pi$, and in the two others because y_m is assumed to be triangular.

To conclude that y_{m+1} is triangular, we must show that the action of e_i in the first case and e_m in the two other cases preserves the triangularity: If $e_i\pi$ is a reduced word and $\pi' \leq \pi$, then, $e_i\pi' \leq e_i\pi$. If $e_m\pi$ is a reduced word and $\pi' \leq e_m\pi$, then $e_m\pi' \leq e_m\pi$. Finally, if $e_{m+1}e_m\pi$ is a reduced word and $\pi' \leq e_{m+1}e_m\pi$, then $e_m\pi' \leq e_{m+1}e_m\pi$. These properties follow from the weak form of the order relation. This concludes the proof of triangularity of y_{m+1}.

A.4 Action of σ on words

The action of $\sigma = -q^{\frac{n}{2}-2}t_{n-1}^{-1}\ldots t_1^{-1}$ on words can be computed similarly. First, using the braid relation (2), one sees that $\sigma e_i = e_{i-1}\sigma$ for $i > 1$. To fully characterize its action, we must compute $(\sigma\alpha)$. Using $t_1^{-1}e_1 = -qe_1$ and $t_{m+1}^{-1}t_m^{-1}e_{m+1} = \frac{1}{q}e_me_{m+1}$ we obtain:

$$
\sigma\alpha = \prod_{i=1}^{\frac{n}{2}-1} e_{2i}\alpha. \tag{46}
$$

Thus, $(\sigma\alpha)$ can be characterized by the property:

$$e_{2i}(\sigma\alpha) = \tau(\sigma\alpha), \tag{47}$$

for $1 \le i \le \frac{n}{2}$. $(\sigma\alpha)$ can then be used as a lowest state to construct a canonical basis by acting on it with the generators $e_2, \dots, e_n$.

B Explicit construction of $\mathcal{M}_4$

Let us construct $\mathcal{M}_4$ the dual of $\mathcal{H}_4$. The basis of $\mathcal{H}_4$ is given by the words $e_1 e_3 = \alpha$, $e_2 e_1 e_3 = \omega$. So we search for a vector Ψ of the form:

$$\Psi = F_\alpha(z_1, \dots, z_4)\alpha + F_\omega(z_1, \dots, z_4)\omega, \tag{48}$$

where F_α, F_ω are polynomials of degree $(1,1)$ in the variables z_i. The action of the T.L. affine algebra is given by the matrices:

$$e_1 = e_3 = \begin{pmatrix} \tau & 1 \\ 0 & 0 \end{pmatrix},\ e_2 = e_4 = \begin{pmatrix} 0 & 0 \\ 1 & \tau \end{pmatrix},\ \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{49}$$

We can obtain the dual representation by acting with the generators on $F_\omega \equiv (0,1)$ annihilated by e_1, e_3. The minimum degree polynomial annihilated by e_1, e_3 is given by:

$$F_\omega = (qz_1 - q^{-1}z_2)(qz_3 - q^{-1}z_4). \tag{50}$$

Let us take σ of the form:

$$F(z_1, z_2, z_3, z_4)\sigma = cF(z_2, z_3, z_4, sz_1). \tag{51}$$

We obtain two different expression for $F_\alpha \equiv (1,0)$ which we must equate. One results from the cyclic property: $F_\alpha = F_\omega \bar{\sigma}$, the other given by: $F_\alpha = F_\omega(\bar{e}_2 - \tau)$.

We get the equation:

$$\frac{(qz_1 - q^{-1}z_2)(qz_3 - q^{-1}z_4) - (qz_1 - q^{-1}z_3)(qz_2 - q^{-1}z_4)}{z_2 - z_3} = c(qz_4 - q^{-1}sz_1), \tag{52}$$

which determines $s = q^6$, $c = q^{-3}$, and:

$$F_\alpha = (qz_2 - q^{-1}z_3)(q^{-2}z_4 - q^2 z_1). \tag{53}$$

C Module F_ω

Let us define a T.L. module M defined in terms of a state F_ω satisfying $F_\omega e_i = 0$ for $i \neq \frac{n}{2}$. The module is obtained by acting with the T.L. generators and reducing words using the T.L. relations (5). In this module, a canonical basis is:

$$\bar{\psi} = F_\omega(e_{\frac{n}{2}} e_{\frac{n}{2}-1} \dots e_{a_{\frac{n}{2}}+1} e_{a_{\frac{n}{2}}}) \dots (e_p e_{p-1} \dots e_{a_p}) \dots (e_{n-1} \dots e_{a_{n-1}}), \tag{54}$$

where the p take the all the values between $\frac{n}{2}$ and $n-1$ and the a_p are restricted by the conditions: $a_p \leq p+1$, $a_{\frac{n}{2}} < a_{\frac{n}{2}+1} \cdots < a_p < \cdots < a_{n-1}$. The convention is that if $a_p = p+1$, the sequence $(e_p \dots e_{a_p})$ is empty. A word $\bar{\psi}$ is fully characterized by the sequence (a_p). The word can also be associated to the Young diagram $[\mu_{p+1-\frac{n}{2}}] = [p - a_p + 1]$.

There is a reflection symmetry, $i \to n-i$, and an alternative description of the module in terms of reflected words:

$$\bar{\psi} = F_\omega(e_{\frac{n}{2}} \dots e_{b_{\frac{n}{2}}-1} e_{b_{\frac{n}{2}}}) \dots (e_p \dots e_{b_p - 1} e_{b_p}) \dots (e_1 \dots e_{b_1}), \tag{55}$$

$1 \leq p \leq \frac{n}{2}$, $b_p \geq p-1$, $b_{\frac{n}{2}} > \cdots > b_1$. It is associated to the dual Young diagram $[\mu'_{\frac{n}{2}-p+1}] = [b_p - p + 1]$.

A similar order relation as defined earlier holds for reduced words, $\bar{\psi}' < \bar{\psi}$ if $\bar{\psi}$ can be written $\bar{\psi} = \bar{\psi}' a$. For non reduced words $\bar{\psi}'$, it is sufficient that $\bar{\psi}'$ can be obtained by erasing letters e_k from the (reduced) expression of $\bar{\psi}$.

In general, the module F_ω is reducible, it will be irreducible if F_ω satisfies the Fock condition:

$$F_\omega(1 + \sum_{m=0}^{\frac{n}{2}-1} q^{m+1} t_{\frac{n}{2}} \dots t_{\frac{n}{2}-m}) = 0. \tag{56}$$

In this case, the only allowed words $\bar{\psi}$ (54) can be associated to their complementary π_ψ in such a way that one can write without reducing the expression:

$$\psi \pi_\psi = \omega. \tag{57}$$

Thus, we get the supplementary constraint $a_p > 2p + 1 - n$, $b_p < 2p - 1$.

C.1 Reducing the Hecke Module to its T.L. form

Let us consider a module M' over the Hecke algebra defined by acting with the Hecke algebra generators satisfying (4) on the state F_ω satisfying $F_\omega e_i = 0$ for $i \neq \frac{n}{2}$. We want to show that the Hecke algebra acts as a T.L. algebra on this module. For this, we first show that the Hecke relations (4) are sufficient to reduce the word basis of M' to be of the T.L. form (54). Thus, M' and M can be identified as vector spaces. From this, we will deduce that $M' = M$ as modules. In other words, the projectors $U^-_{i,i+1} = e_i e_{i+1} e_i - e_i$ are null in M'.

Let us assume that it is not true. Since all the basis elements of M' are obtained upon acting on F_ω with letters e_k, there is a basis element $\bar{\psi} e_i$ which cannot be expressed as a linear combination of words of the form (54) although $\bar{\psi}$ is of the form (54). Among all the $\bar{\psi}$ which verify this property, we can take the smallest possible for the order relation, so that that $\bar{\psi}' e_i$ is of the form (54) when $\bar{\psi}' < \bar{\psi}$. We show that this leads to a contradiction.

Let us consider the word $\bar{\psi} e_i$. It is a word of the form (54) in the three following cases. When $\bar{\psi} e_i$ is a reduced word $> \bar{\psi}$, for $i = a_p - 1$ if $a_p - 1 > a_{p-1}$.

When $\bar{\psi}e_i = \tau\bar{\psi}$ when $i = a_p$ and $a_p > a_{p+1} - 1$. When $\bar{\psi}e_i = 0$ if $i < a_{\frac{n}{2}} - 1$ or $i > b_{\frac{n}{2}} + 1$.

The two remaining cases to consider are: First, when $a_p < i < a_{p+1} - 1$ for some p. Second, when $a_p < i \leq a_p + k$ if $a_{p+k} = a_p + k$ with $k \geq 1$. The second case can be studied similarly to the first one using the reflection symmetry $i \to n - i$ and corresponds to $b_{p'} > i > b_{p'-1} + 1$.

In the first case, $\bar{\psi}e_i = \bar{\psi}'(e_p \dots e_{a_p+1}e_{a_p})e_i(e_{p+1} \dots e_{a_{p+1}}) \dots$, and using the relation (4), we see that:

$$e_p \dots e_{a_p+1}e_{a_p}e_i = e_{i-1}e_p \dots e_{a_p+1}e_{a_p} + e_p \dots e_{i+1}(e_i - e_{i-1})e_{i-2} \dots e_{a_p+1}e_{a_p}. \tag{58}$$

The second term is $< \bar{\psi}$ and therefore of the T.L. form by the recursion hypothesis. The first term can be eliminated by repeating this relation $p - \frac{n}{2}$ times to push e_i and then $e_{i-1}, \dots, e_{i+\frac{n}{2}-p}$ to the left of the word. The last application of the relation gives a term $F_\omega e_{i+\frac{n}{2}-p-1} = 0$ since $i + \frac{n}{2} - p - 1 < \frac{n}{2}$.

This exhaust all the possibilities and $\bar{\psi}e_i$ can always be expressed as a linear combination of reduced T.L. words (54) in contradiction with the hypothesis. Therefore, the word basis of M' coincides with the word basis (54).

To conclude that $M' = M$, let us consider the projectors $U^-_{i,i+1} = e_ie_{i+1}e_i - e_i$, and the space $M'' \subset M'$ annihilated by all the $U^-_{i,i+1}$. The space M'' defines a module for the T.L. algebra. Since $F_\omega \in M''$, this module can be identified with M. Therefore, M is a subspace of M' with the same dimension, and thus, $M = M'$.

D Yang-Baxter Equation and Polynomials

D.1 Polynomial representation of the Hecke generators

In this section, we derive the expression of the Hecke generators $\bar{t}_i$ (16) from the Yang-Baxter equation.

The Yang-Baxter algebra [8] (also called $RLL = LLR$ relation) can be expressed as:

$$R_{12}(z_1, z_2)L_1(z_1)L_2(z_2) = L_2(z_2)L_1(z_1)R_{12}(z_1, z_2), \tag{59}$$

where $R_{12}(z_1, z_2)$ is a solution of the Yang-Baxter equation:

$$R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2). \tag{60}$$

If we assume that $R_{12}(z_1, z_2) = Y_{12}(z_1, z_2)P_{12}$ where P_{12} acts in the natural way on the spin indices, $(P_{12}t_{13} = t_{23}P_{12})$, but commutes with z_i, (59) rewrites as:

$$Y_{12}(z_1, z_2)L_2(z_1)L_1(z_2) = L_2(z_2)L_1(z_1)Y_{12}(z_1, z_2) = L_2(z_1)L_1(z_2)k_{12}, \tag{61}$$

where k_{12} acts to the left by permuting the variables z_1, z_2. The normalization of $Y(z_1, z_2)$ is such that:

$$Y_{12}(z_1, z_2)Y_{12}(z_2, z_1) = 1. \tag{62}$$

It is therefore consistent to demand that the Y_{ii+1} act as a representation of the permutation algebra on some wave function Ψ:

$$Y_{12}(z_1, z_2)\Psi(z_1, z_2) = \Psi(z_2, z_1) = \Psi(z_1, z_2)k_{12}. \tag{63}$$

The Y_{ij} are called Yang's operators in [8].

A well-known solution of (60) in terms of the Hecke algebra (3) is:

$$Y_{12}(z) = \frac{t_{12} - zt_{12}^{-1}}{zq - q^{-1}}, \tag{64}$$

where $z = \frac{z_1}{z_2}$.

Substituting (64) in (63), we can also rewrite this relation as:

$$t_{12}\Psi(z_1, z_2) = \Psi(z_1, z_2)\bar{t}_{12}, \tag{65}$$

where $\bar{t}_{12}$ takes the form:

$$\bar{t}_{12} = -q^{-1} + (1 - k_{12})\frac{qz_1 - q^{-1}z_2}{z_1 - z_2}, \tag{66}$$

and this coincides with (16).

D.2 Commutation relations of the affine generators y_i

We motivate the commutation relation (7c) from the Yang-Baxter algebra (59) point of view. This also reveals a complete symmetry between the spectral parameters z_i and the generators y_i.

Let us substitute the spectral parameters z_i with the affine generators y_i in $L(z_i)$, and we require that the relation (65) are preserved under the action of the algebra L_i on Ψ:

$$t_{12}L_1(y_1)L_2(y_2)\Psi = L_1(y_1)L_2(y_2)\Psi\bar{t}_{12}, \tag{67}$$

assuming that (65) holds for Ψ.

To avoid cumbersome expressions, we use here the transposed notation $\bar{a}X$ for $X\bar{a}$. We must then transpose back the final algebraic relations we deduce. In the transposed notations (67) is equivalent to:

$$(t_{12} - \bar{t}_{12})L_1(y_1)L_2(y_2) = 0, \tag{68}$$

under the hypothesis that $t_{12} = \bar{t}_{12}$ to the right of any expression. Let us for the moment assume that $\bar{t}_{12}$ commutes with the symmetrical expressions in y_1, y_2.

After substituting the expression of $L_i(y_i)$ deduced from (64):

$$L_1(y_1) = (yt_{10} - y_1 t_{10}^{-1})P_{01}, \tag{69}$$

the term proportional to y^0 requires that $\bar{t}_{12}$ commutes with $y_1 y_2$, while the term proportional to y imposes that:

$$(t_{12} - \bar{t}_{12})(y_2 t_{01} t_{12}^{-1} + y_1 t_{01}^{-1} t_{12}) = 0, \tag{70}$$

under the hypothesis that $t_{01} = \bar{t}_{12}$ to the right of any expression. This gives:

$$\begin{aligned} y_2\bar{t}_{12} + (q - q^{-1})y_1 - \bar{t}_{12}y_1 &= 0, \\ y_1 - \bar{t}_{12}y_2\bar{t}_{12} &= 0, \end{aligned} \tag{71}$$

which is equivalent to $y_2\bar{t}_{12} = \bar{t}_{12}^{-1}y_1$ and implies in particular that $\bar{t}_{12}$ commutes with the symmetrical expressions in y_1, y_2. After transposition, it yields (7c) back.

Alternatively, we can substitute z_i for y_i in (7c) and verify that the relation is obeyed when we use the expression (66) of $\bar{t}_i$.

D.3 Eigenvalues of the y_j in the polynomial case

We show that the operators y_j defined with the polynomial representation 2.2 are triangular matrices. Let us recall the expression of y_i:

$$y_i = x_{ii+1}x_{ii+2}\dots x_{in}s_i x_{i1}\dots x_{ii-1} \tag{72}$$

where the operator $x_{i,j}$ takes the form for $i < j$:

$$x_{ij} = -q^{-1} + (q - q^{-1})(1 - k_{ij})\frac{z_j}{z_i - z_j}, \tag{73}$$

and the operators s_i act as:

$$P(z_1, \dots, z_i \dots, z_n)s_i = cP(z_1 \dots, sz_i, \dots, z_n). \tag{74}$$

x_{12} commutes with $z_1 z_2$ and with z_k for $k \neq 1, 2$. It acts triangularly on the monomials $z_1^m,\ z_2^m$ as follows:

$$\begin{aligned} z_1^m x_{12} &= -q^{-1}z_1^m + (q - q^{-1})(z_1^{m-1}z_2 + z_1^{m-2}z_2^2 + \dots + z_2^m) \quad m \geq 0 \\ z_2^m x_{12} &= -qz_2^m - (q - q^{-1})(z_1^{m-1}z_2 + z_1^{m-2}z_2^2 + \dots + z_1 z_2^{m-1}) \quad m > 0. \end{aligned} \tag{75}$$

From these expressions, we determine which new monomials $z^{\lambda'}$ can appear when one acts with x_{12} on the monomial z^{λ}. First, all the λ'_l for $l \neq 1, 2$ are equal to λ_l. Then, if $\{\lambda'_i\lambda'_j\} \neq \{\lambda_i\lambda_j\}$ with $\{i, j\} = \{1, 2\}$ and $\lambda'_j \leq \lambda'_i$, we must have $\{\lambda'_i, \lambda'_j\} = \{\lambda_i - p, \lambda_j + p\}$ for some integer p. Finally, if $\{\lambda'_1\lambda'_2\} = \{\lambda_1\lambda_2\}$, the only possibility is that: $(\lambda'_1, \lambda'_2) = (\lambda_2, \lambda_1)$ with $\lambda_1 > \lambda_2$.

Let us define an order on the monomials by saying that z^{λ} is larger than $z^{\lambda'}$ if either λ' is obtained from λ by a sequence of squeezing operations $\{\lambda_i, \lambda_j\} \to \{\lambda_i - 1, \lambda_j + 1\}$ with $\lambda_i > \lambda_j + 1$, or λ' is a permutation of λ and can be obtained from λ by a sequence of permutations $(\lambda_i, \lambda_{i+1}) \to (\lambda_{i+1}, \lambda_i)$ with $\lambda_i > \lambda_{i+1}$. It follows from the above analysis that the action of y_j on a monomial produces only monomials which are smaller with respect to this order. Thus the eigenvalues of the operators y_j are given by the diagonal elements in the monomial basis.

It follows from this that, given the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, the eigenvalues corresponding to the monomials associated to it are all obtained by permutations of the multiplet:

$$(y_j) = c(-q)^{1-n}(t^{\lambda_j} q^{2(j-1)}). \tag{76}$$

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