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An Extension Principle for the Einstein-Vlasov System in Spherical Symmetry

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Abstract. We prove that "first singularities" in the non-trapped region of the maximal development of spherically symmetric asymptotically flat data for the Einstein-Vlasov system must necessarily emanate from the center. The notion of "first" depends only on the causal structure and can be described in the language of terminal indecomposable pasts (TIPs). This result suggests a local approach to proving weak cosmic censorship for this system. It can also be used to give the first proof of the formation of black holes by the collapse of collisionless matter from regular initial configurations.

1 Introduction

A fundamental problem in mathematical relativity is to resolve the so-called *weak cosmic censorship conjecture*, the statement that for "reasonable" Einstein-matter systems, generic asymptotically flat data do not lead to singularities visible from infinity.

The notion of "reasonable" above is of course not a precise one, and depends very much on the context one has in mind. A natural matter source for models is provided by kinetic theory. The simplest example is then a self-gravitating collisionless gas. The study of the equations describing such a gas, the Einstein-Vlasov system, was initiated by Choquet-Bruhat in [1], where the existence of a unique maximal development was proven for the Cauchy problem.

The problem of weak cosmic censorship concerns the global behaviour of the maximal development for asymptotically flat initial data. Given the current state of the art in nonlinear evolution equations, symmetry must be imposed on initial data for there to be any hope of making progress. The global study of the initial value problem for the Einstein-Vlasov equations for spherically symmetric asymptotically flat initial data was begun in [7], where, in particular, it was proven that for sufficiently small initial data, the maximal development was future causally geodesically complete. The analysis took place in so-called Schwarzschild coordinates. In [8], an extension principle was proven, again in these coordinates, saying in particular that if the solution stopped existing after finite coordinate time t, there was necessarily a singularity at the center. These results were meant to provide a first step for a *global* existence theorem in Schwarzschild coordinates. If this coordinate system could then be shown to cover the domain of outer communications, and if null infinity could moreover be shown to be complete, this would then imply a proof of weak cosmic censorship for this system.

There is another approach to the problem of weak cosmic censorship, due to Christodoulou [3], for the problem of a self-gravitating spherically symmetric scalar field. Christodoulou showed that initial data leading to a naked singularity was codimension 1 in the space of all initial data. This was shown by embedding such exceptional data in a one-dimensional subset of the space of initial data, such that all other initial data in this subset evolved to a spacetime with the following property, which can be expressed in the language of causal sets [6]. Given a terminal indecomposable past (TIP) with compact intersection with the Cauchy surface, then the domain of dependence of any open set containing this intersection contains a trapped surface. The statement that this latter property is true for generic initial data can be termed the *trapped surface conjecture*. From this property, the completeness of null infinity was then inferred, proving weak cosmic censorship.

It turns out that the relation between the existence of trapped surfaces and the completeness of null infinity is quite general. Specifically, in [12], it was proven that a weaker version of the trapped surface conjecture is sufficient to prove weak cosmic censorship for a wide variety of matter in spherical symmetry. In particular, the completeness of null infinity follows from the existence of a *single* trapped or marginally trapped surface in the maximal development. The only really restrictive hypothesis on the matter is that “first” singularities necessarily emanate from the center. Here, the notion of “first” is tied to the causal structure and can be formulated in terms of TIPs.

The goal of this paper is to prove that the above mentioned hypothesis of [12] is indeed satisfied by the Einstein-Vlasov system. As noted before, extension principles similar in spirit to this one have been proven before (cf. [8, 10]). These earlier results, however, concern the portion of the development of the Einstein-Vlasov system covered by particular coordinate systems. Thus, these previous results, as far as they concern the maximal development itself, are weaker than the results presented here, and in particular, are not sufficient to deduce the assumptions of [12].[1]

Finally, we make the following remark: In view of [9], there do exist spherically symmetric asymptotically flat initial data for the Einstein-Vlasov system possessing a trapped surface. Thus, the results of this paper provide in particular the first proof of the existence of solutions for collisionless matter representing the formation of a black hole.

2 Initial data

Initial data in this paper are always given as follows:

1. We have a C^∞ Riemannian manifold $(\Sigma, \bar{g})$, together with an additional symmetric 2-tensor K_{ab}, such that there do not exist closed antitrapped surfaces

[1] Of course, the results of [8, 10] also say something about the behaviour of the coordinate system to which they apply, something not addressed here.

in the data, and a compactly supported function f_0 defined on the tangent bundle of Σ, such that these satisfy

$$\begin{aligned} \bar{R} - K_{ab}K^{ab} + (\mathrm{tr}K)^2 &= 16\pi \int f_0(p^a)p^a p_a/(1+p^a p_a)^{1/2}\sqrt{\bar{g}}dp^1dp^2dp^3 \ , \\ \nabla_a K^a{}_b - \nabla_b(\mathrm{tr}K) &= 8\pi \int f_0(p^a)p_a\sqrt{\bar{g}}dp^1dp^2dp^3 \ . \end{aligned}$$

Here the metric $\bar{g}$ is used to move indices and to define the trace and covariant derivative. $\bar{R}$ is the scalar curvature of $\bar{g}$ and $\sqrt{\bar{g}}$ the square root of its determinant.

2. A smooth $SO(3)$ action on Σ such that $\bar{g}$, K_{ab}, f_0 are preserved, and such that $\Sigma/SO(3)$ inherits naturally the structure of a 1-dimensional manifold.

Here and throughout this paper physical units are chosen so that the gravitational constant has the numerical value unity. We recall the definition of a closed antitrapped surface. Let S be a surface in Σ which is closed, i.e., compact without boundary. Suppose that there is a preferred choice n^a of an outward normal to this surface and let σ_{ab} be the second fundamental form of S in Σ corresponding to the outward normal. Then S is said to be antitrapped if $\mathrm{tr}\sigma < -\mathrm{tr}K + K_{ab}n^a n^b$.

3 The maximal development

The theorem of Choquet-Bruhat [1], applied to the data considered here, together with a standard argument on preservation of symmetry, yields

Proposition 1. *There exists a unique C^∞ collection $(\mathcal{M}, g, f)$ such that*

1. *g and f satisfy the Einstein-Vlasov equations*
2. *$(\mathcal{M}, g)$ is globally hyperbolic,*
3. *$(\mathcal{M}, g, f)$ induces the initial data $(\Sigma, \bar{g}, K, f_0)$ and Σ is a Cauchy surface*
4. *Any other collection $(\mathcal{M}, g, f)$ with these properties 1–3 can be embedded in the given one.*

Moreover, $SO(3)$ acts smoothly by isometry on $\mathcal{M}$ and preserves f, and $\mathcal{Q} = \mathcal{M}/SO(3)$ inherits the structure of a time-oriented 2-dimensional Lorentzian manifold, with timelike boundary Γ, the center.

Let $\pi : \mathcal{M} \to \mathcal{Q}$ denote the natural projection. On $\mathcal{Q}$ we can define the so-called area-radius function

$$r(p) = \sqrt{\mathrm{Area}(\pi^{-1}(p))/4\pi} \ .$$

We have $r(p) = 0$ iff $p \in \Gamma$. We can always choose global future directed null coordinates on $\mathcal{Q}$, i.e., such that the metric takes the from $-\Omega^2 dudv$. The metric of $\mathcal{M}$ then takes the form:

$$-\Omega^2 dudv + r^2\gamma \tag{1}$$

where $\gamma = \gamma_{AB} dx^A dx^B$ is the standard metric on S^2 and x^A, $A = 2, 3$, are local coordinates on S^2. Let u and v be chosen so that $\frac{\partial}{\partial u}$ points "inwards" and $\frac{\partial}{\partial v}$ "outwards". Such definitions are meaningful in view of the assumption of asymptotic flatness. We define

$$\nu = \partial_u r \ ,$$

$$\lambda = \partial_v r \ .$$

The assumption of no antitrapped surfaces initially means by definition that

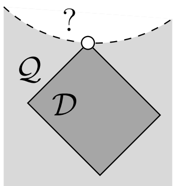
$$\nu < 0 \tag{2}$$

holds on the initial hypersurface. It follows that it holds throughout $\mathcal{Q}$ as a consequence of the Einstein equations and the dominant energy condition [2].

We shall call the region where $\lambda > 0$, the *regular region*, and denote it $\mathcal{R}$. We call the region where $\lambda = 0$ the *marginally trapped region*, and denote it by $\mathcal{A}$, and finally, we shall can the region where $\lambda < 0$ the *trapped region*, and denote it by $\mathcal{T}$.

4 The extension theorem

The extension principle proven in this paper will apply to a region $\mathcal{D} \subset \mathcal{Q}$ with Penrose diagram:



(i.e., a subset $\mathcal{D} = [u_1, u_2] \times [v_1, v_2] \setminus (u_2, v_2))$ such that

$$\mathcal{D} \subset \mathcal{R} \cup \mathcal{A} \ .$$

Let $\mathcal{C}_{\text{in}}$ and $\mathcal{C}_{\text{out}}$ be the parts of the boundary of $\mathcal{D}$ defined by $v = v_1$ and $u = u_1$ respectively. One can think of $\mathcal{D}$ as the "top" of a non-trapped non-central indecomposable past (IP) corresponding to a candidate "first" singularity. In this language, the result of this paper is that such an IP cannot be a TIP, i.e.,

Theorem 1. *If $\mathcal{D} \subset \mathcal{Q}$, then $\mathcal{D} \subset J^-(q)$ for a $q \in \mathcal{Q}$.*

The theorem thus says that there is no singularity of this form after all!

As one might expect, the proof of Theorem 1 proceeds by obtaining *a priori* estimates in $\mathcal{D}$ and then applying an appropriate local existence result. The *a priori* estimates make use of a certain energy flux along null hypersurfaces. This fact, together with the fact that regular null coordinates can always be chosen, makes it natural to stick to these. We give the form of the equations in local null coordinates in the next two sections. Then, in Section 7, we formulate a local existence theorem (Proposition 2) for a double characteristic initial value problem. The "time" of existence, in the sense of null coordinates, will depend only on the C^2 norm of the metric and the C^1 norm (and the support) of f. We obtain energy estimates in Section 8, and use these, together with the structure of the Vlasov equation, to derive in Sections 9–10 *a priori* estimates for the norm of Proposition 2. The proof of Theorem 1 will follow immediately in Section 11. Finally, in Section 12, we state two applications of our results, discussed already in the Introduction.

The above theorem depends on having a well-behaved matter model and the analogous result must be expected to fail for dust. This is illustrated by the Penrose diagram Fig. 1 in [13].

5 The Einstein equations in null coordinates

The reader should consult [2] for general facts about the initial value problem in spherical symmetry. When specialized to this case, the Einstein equations are:

$$\partial_u\partial_v r = -\frac{\Omega^2}{4r} - \frac{1}{r}\lambda\nu + 4\pi r T_{uv} \ , \tag{3}$$

$$\partial_u\partial_v \log\Omega = -4\pi T_{uv} + \frac{\Omega^2}{4r^2} + \frac{1}{r^2}\lambda\nu - \frac{\pi\Omega^2}{r^2}\gamma^{AB}T_{AB} \ , \tag{4}$$

$$\partial_v(\Omega^{-2}\partial_v r) = -4\pi r T_{vv}\Omega^{-2} \ , \tag{5}$$

$$\partial_u(\Omega^{-2}\partial_u r) = -4\pi r T_{uu}\Omega^{-2} \ . \tag{6}$$

The former two equations can be viewed as wave equations for r and Ω, while the latter two equations can be viewed as constraint equations on null hypersurfaces. A specific choice of matter model, such as a collisionless gas, leads to expressions for the components of the energy-momentum tensor.

6 The Vlasov equation

To describe the Vlasov equation in local coordinates, we need a coordinate system on $T\mathcal{M}$. Let p^u, p^v, and p^A denote the functions on $T\mathcal{M}$, defined by writing an arbitrary $\mathbf{X} \in T\mathcal{M}$ as

$$\mathbf{X} = p^u\frac{\partial}{\partial u} + p^v\frac{\partial}{\partial v} + p^A\frac{\partial}{\partial x^A} \ .$$

Together with the pull-back of the coordinates on spacetime these functions define a local coordinate system on $T\mathcal{M}$.

Let $P \subset T\mathcal{M}$ be defined by

$$P = \{g(\mathbf{X},\mathbf{X}) = -1\} \ ,$$

where $\mathbf{X}$ ranges over future-pointing vectors. We call P the mass shell. It follows that

$$-\Omega^2 p^u p^v + r^2\gamma_{AB}p^A p^B = -1 \ . \tag{7}$$

We use p^u, p^A and the pull-back of the coordinates on spacetime to define coordinates on P and p^v is regarded as a function of these coordinates defined by the relation (7). The Vlasov equation is an equation for a non-negative function

$$f : P \to \mathbf{R}$$

which, in the case that f is spherically symmetric, is given by

$$\begin{aligned} p^u\frac{\partial f}{\partial u} + p^v\frac{\partial f}{\partial v} &= (\partial_u(\log\Omega^2)(p^u)^2 + 2\Omega^{-2}r\lambda\gamma_{AB}p^A p^B)\frac{\partial f}{\partial p^u} \\ &\quad + 2r^{-1}(\nu p^u + \lambda p^v)p^A\frac{\partial f}{\partial p^A} \ . \end{aligned} \tag{8}$$

In deriving this we have used the expressions for the Christoffel symbols given in Appendix A and the fact that a spherically symmetric function f on the mass shell is a function of the variables u, v, p^u, and $\gamma_{AB}p^A p^B$. This implies the identity

$$p^A\frac{\partial f}{\partial x^A} = \Gamma^A_{BC}p^B p^C\frac{\partial f}{\partial p^A}$$

which has been used to simplify the Vlasov equation. Note that both the expressions $\gamma_{AB}p^A p^B$ and $p^A\frac{\partial}{\partial p^A}$ have a meaning independent of the particular choice of coordinates x^A on S^2.

Finally, to close the system, we must define the energy-momentum tensor. We first note that for any point $q \in \mathcal{M}$, it follows that P_q, as a spacelike hypersurface in $T_q\mathcal{M}$, inherits a volume form from the Lorentzian metric. In local coordinates this volume form can be written $r^2(p^u)^{-1}dp^u\sqrt{\gamma}dp^A dp^B$ or alternatively $r^2(p^v)^{-1}dp^v\sqrt{\gamma}dp^A dp^B$, where $\sqrt{\gamma}$ is the square root of the determinant of γ_{AB}. We then have

$$T_{ab} = \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty r^2 p_a p_b f(p^u)^{-1}\sqrt{\gamma}dp^u dp^A dp^B \ , \tag{9}$$

where $p_a = g_{ab}p^b$. It follows immediately that this matter model satisfies the energy conditions:

$$T_{uv} \geq 0, T_{vv} \geq 0, T_{uu} \geq 0 \ . \tag{10}$$

7 A local existence theorem

To prove our extension theorem, we will certainly need to appeal to some sort of local existence theorem. In particular, it is the norm in this theorem that will tell us what quantities we must bound *a priori* in $\mathcal{D}$. In principle, one could try to prove estimates so as to apply the local existence result of [1]. For various reasons, however, the following local existence theorem for a characteristic initial value problem will be more convenient:

Proposition 2. *Let $k \geq 2$. Let Ω, r be positive C^k-functions defined on $[0,d]\times\{0\}\cup\{0\}\times[0,d]$, and let f be a non-negative C^{k-1} function defined on the part of the mass shell over $[0,d]\times\{0\}\cup\{0\}\times[0,d]$. Suppose that equations* (5), (6) *hold on $\{0\}\times[0,d]$ and $[0,d]\times\{0\}$ respectively, where T_{uu} and T_{vv} are defined by* (9), *and suppose in addition that the C^k compatibility condition holds at $(0,0)$. Define the norm:*

$$\begin{aligned} N_u &= \sup_{[0,d]\times\{0\}} \{|\Omega|, |\Omega^{-1}|, |\partial_u\Omega|, |\partial_u^2\Omega|, |r|, |r|^{-1}, |\partial_u r|, |\partial_u^2 r|\ , \\ &\quad S, |f|, |\partial_u f|, |\partial_{p^u} f|, |\partial_{p^A} f|_\gamma\}\ , \\ N_v &= \sup_{\{0\}\times[0,d]} \{|\Omega|, |\Omega^{-1}|, |\partial_v\Omega|, |\partial_v^2\Omega|, |r|, |r|^{-1}, |\partial_v r|, |\partial_v^2 r|\ , \\ &\quad S, |f|, |\partial_v f|, |\partial_{p^u} f|, |\partial_{p^A} f|_\gamma\}\ , \end{aligned}$$

$$N = \sup\{N_u, N_v\}\ ,$$

were S denotes the supremum of $(p^u)^2 + (p^v)^2 + \gamma_{AB}p^Ap^B$ on the support of f and $|v_A|_\gamma = (\gamma^{AB}v_Av_B)^{1/2}$. Then there exists a δ, depending only on N, and C^k functions (unique among C^2 functions) r, Ω and a C^{k-1} function (unique among C^1 functions) f, satisfying equations (3), (4), (5), (6), (8) *in $[0,\delta^*]\times[0,\delta^*]$, where $\delta^* = \min\{d,\delta\}$, such that the restriction of these functions to $[0,d]\times\{0\}\cup\{0\}\times[0,d]$ is as prescribed.*

Proof. See Appendix B.

The compatibility conditions referred to in the statement of the proposition are as follows. The data includes the values of the function f on the part of the mass shell over $[0,d]\times\{0\}$. All derivatives of f tangential to this manifold can be calculated by direct differentiation. By using the field equations transverse derivatives (and thus all derivatives) of f can be computed up to order $k-1$. In a similar way, all derivatives up to order $k-1$ can be computed on $\{0\}\times[0,d]$. The condition that derivatives determined in these two different ways agree at $(0,0)$ is what is referred to above as the C^k compatibility condition.

Let us add the remark that, defining g on $\mathcal{M}$ by (1), the above gives rise to a solution of the Einstein-Vlasov equations upstairs, with the obvious relation to characteristic data, interpreted upstairs.

8 Energy estimates

A fundamental fact about the analysis of spherically symmetric Einstein matter systems in the non-trapped region is the existence of energy estimates.

To describe these, let us first settle for a particular null-coordinate description of the set $\mathcal{D}$. We normalize our u-coordinate such that $\nu = -1$ along $\mathcal{C}_{\text{in}}$. For the

v coordinate, we first define the quantity

$$\kappa = -\frac{1}{4}\Omega^2\nu^{-1} \ .$$

and then define v such that $\kappa = 1$ along $\mathcal{C}_{\text{out}}$. $\mathcal{D}$ is thus given by $[0,U] \times [0,V] \setminus \{(U,V)\}$.

The concept of energy in spherical symmetry is given by the so-called *Hawking mass*, given by:

$$m = \frac{r}{2}(1 - \partial^a r \partial_a r) = \frac{r}{2}(1 - 2g^{uv}\partial_u r \partial_v r) = \frac{r}{2}(1 + 4\Omega^{-2}\lambda\nu) \ .$$

We will also introduce the so-called *mass-aspect* function

$$\mu = \frac{2m}{r} \ .$$

Note that

$$\kappa(1-\mu) = \lambda \ . \tag{11}$$

From (3)–(6), we compute the identities:

$$\begin{aligned} \partial_u m &= 8\pi r^2 \Omega^{-2}(T_{uv}\nu - T_{uu}\lambda) \\ &= -2\pi\kappa^{-1} r^2 T_{uv} + 2\pi \frac{1-\mu}{\nu} r^2 T_{uu} \ , \end{aligned} \tag{12}$$

$$\begin{aligned} \partial_v m &= 8\pi r^2 \Omega^{-2}(T_{uv}\lambda - T_{vv}\nu) \\ &= -2\pi \frac{1-\mu}{\nu} r^2 T_{uv} + 2\pi\kappa^{-1} r^2 T_{vv} \ . \end{aligned} \tag{13}$$

The first point to note is that the signs of (12) and (13), together with the signs of λ and ν, give *a priori* bounds for both r and m. Indeed, set

$$\begin{aligned} m_0 &= m(U,0) \geq 0 \ , & r_0 &= r(U,0) > 0 \ , \\ M &= m(0,V) \ , & R &= r(0,V) \ . \end{aligned}$$

By (2) and the fact that $\mathcal{D} \subset \mathcal{R} \cup \mathcal{A}$, we have that

$$r_0 \leq r \leq R \tag{14}$$

throughout $\mathcal{D}$. On the other hand, (12), (13) and (10) give $\partial_u m \leq 0$, $\partial_v m \geq 0$, and thus

$$m_0 \leq m \leq M \ . \tag{15}$$

Now we make a trivial observation. In view of the fact that we have the *a priori* bounds (15), if we reexamine the equations (12), (13), keeping in mind that both terms on the right-hand side have the same sign, we obtain the bounds:

$$\int_{v_1}^{v_2} \frac{2\pi(1-\mu)}{-\nu} r^2 T_{uv}(u,v)dv \leq M - m_0 \ , \tag{16}$$

$$\int_{v_1}^{v_2} 2\pi\kappa^{-1}r^2 T_{vv}(u,v)dv \leq M - m_0 \ , \tag{17}$$

$$\int_{u_1}^{u_2} 2\pi\kappa^{-1}r^2 T_{uv}(u,v)du \leq M - m_0 \ , \tag{18}$$

$$\int_{u_1}^{u_2} \frac{2\pi(1-\mu)}{-\nu} r^2 T_{uu}(u,v)du \leq M - m_0 \ . \tag{19}$$

These will be our *energy estimates.*

As we shall see, our use of the above estimates will not quite be symmetric for u and v. The reason is this: The "constraint" equation (6) can be seen to be equivalent to the following equation for κ:

$$\partial_u \kappa = 4\pi r \nu^{-1} T_{uu}\kappa \ . \tag{20}$$

From (2), (20) and (10), we see immediately

$$0 < \kappa \leq 1 \tag{21}$$

throughout $\mathcal{D}$, i.e., $\kappa^{-1} \geq 1$. This means that *a priori* we control $\int T_{vv}dv$, but not $\int T_{uu}du$.

Finally, note that we can rewrite equation (3) as

$$\partial_v \nu = 2r^{-2}\kappa\nu m + 4\pi r T_{uv} \ , \tag{22}$$

or alternatively

$$\partial_u \lambda = 2r^{-2}\kappa\nu m + 4\pi r T_{uv} \ . \tag{23}$$

Thus, integrating (22), in view of (21), (15), (14), and (10), we have that

$$\nu \geq -e^{2r_0^{-2}MV} = -\tilde{N} \ . \tag{24}$$

9 C^1 estimates for the metric

So far, we have not used the Vlasov equation, only the energy condition (10). Indeed, all estimates obtained so far are familiar from the results of [12]. To go further, we must use the Vlasov equation itself and the special structure of the energy-momentum tensor. In this section, we shall estimate the support of f and show C^1 estimates for the metric.

Before proceeding, let us give names to bounds on certain quantities on the initial segments $\mathcal{C}_{\text{in}} \cup \mathcal{C}_{\text{out}}$. Define

$$G = \max\left\{ \sup_{[0,U]\times\{0\}} |\partial_u \log\Omega^2|, \sup_{\{0\}\times[0,V]} |\partial_v \log\Omega^2| \right\}$$

$$F = \sup_{\pi_1^{-1}(\{0\}\times[0,V]\cup[0,U]\times\{0\})} f \ ,$$

where π_1 denotes the projection from the mass shell, define Σ to be supremum of the radius of support of f in the p^v and p^u directions along $\pi_1^{-1}(\{0\} \times [0,V] \cup [0,U] \times \{0\})$, and define X be the supremum of $r^4\gamma_{AB}p^Ap^B$ over the support of f.

Let us note first two easy bounds. Clearly,

$$0 \leq f \leq F$$

throughout the mass shell over $\mathcal{D}$. Moreover, by (14) and conservation of angular momentum applied to geodesics, it follows that

$$r4\gamma_{AB}p^Ap^B \leq X \tag{25}$$

for any $x \in P$ in the support of f over $\mathcal{D}$. In particular, in the expressions defining energy-momentum, we can thus always replace an integral over the variables p^A by the integral over the ball of radius X about the origin.

We have the following:

Lemma 1. *The inequality*

$$-g_{uv}g^{AB}T_{AB} \leq 2T_{uv}$$

holds throughout $\mathcal{D}$.

Proof. The inequality is equivalent to the statement that the trace of the energy-momentum tensor is non-positive. This holds for collisionless matter independently of symmetry assumptions. It is proved straightforwardly by taking a trace in the formula defining the energy-momentum tensor in general coordinates with the spacetime metric. □

We can rewrite (4) as

$$\partial_u(\partial_v \log\Omega^2) = -8\pi T_{uv} - 4\kappa m r^{-3}\nu + 8\pi\kappa\nu r^{-2}\gamma^{AB}T_{AB}\ . \tag{26}$$

Integrating (26), applying the above lemma, the energy estimate (18), and the bounds (14), (15), (21), we estimate $\partial_v \log\Omega^2$:

$$\begin{aligned}
|\partial_v \log\Omega^2| &\leq G + \left|\int 8\pi T_{uv}du - \int 8\pi\kappa\nu r^{-2}\gamma^{AB}T_{AB}du\right| - \int 4\kappa m r^{-3}\nu du \\
&\leq G + \int 8\kappa r^{-2}\left(2\pi r^2\kappa^{-1}T_{uv}\right)du - \int 4\kappa m r^{-3}\nu du \\
&\leq G + 8r_0^{-2}\int 2\pi r^2\kappa^{-1}T_{uv}du - \int 4\kappa m r^{-3}\nu du \\
&\leq G + 8r_0^{-2}\int 2\pi r^2\kappa^{-1}T_{uv}du - \int 4\kappa m r^{-3}\nu du \\
&\leq G + 8r_0^{-2}(M - m_0) + 2(r_0^{-2} - R^{-2})M \\
&= G'\ .
\end{aligned} \tag{27}$$

Integrating now (41), using (27), we obtain

$$|\log\Omega^2(u,v)| \leq |\log\Omega^2(u,0)| + G'V \ ,$$

and thus, since $|\log\Omega^2(u,0)| \leq C$ for some C, we have,

$$0 < c \leq \Omega^2(u,v) \leq D \ . \tag{28}$$

Now, we turn to estimate the projection to the p^v-axis of the support of f. We proceed by considering the geodesic equation. Let $\gamma(s)$ be a geodesic crossing $\{0\}\times[0,V]\cup[0,U]\times\{0\}$ at $s=0$, such that $\gamma'(0)$ is in the support of f. Let $p^v(s)$ denote the $\frac{\partial}{\partial v}$ component of the tangent vector of γ. We have

$$(p^v)'(s) = -\Gamma^v_{vv}(p^v)^2 - \Gamma^v_{AB}p^Ap^B \ . \tag{29}$$

using the Christoffel symbols in Appendix A. Integrating (29), we have now by (37)

$$\begin{aligned} p^v(s) &= p^v(0)e^{-\int_0^s\Gamma^v_{vv}(p^v)d\tilde{s}} - \int_0^s \Gamma^v_{AB}p^A(\tilde{s})p^B(\tilde{s})e^{-\int_{\tilde{s}}^s\Gamma^v_{vv}(p^v)d\bar{s}}d\tilde{s} \\ &= p^v(0)e^{-\int_{v(0)}^{v(s)}\Gamma^v_{vv}dv} - \int_0^s \Gamma^v_{AB}p^Ap^Be^{-\int_{v(\tilde{s})}^{v(s)}\Gamma^v_{vv}dv}d\tilde{s} \\ &= p^v(0)e^{-\int_{v(0)}^{v(s)}\Gamma^v_{vv}dv} + \int_{v(0)}^{v(s)} 2(-\nu)\Omega^{-2}r\gamma_{AB}p^Ap^Be^{-\int_{v(\tilde{s})}^{v(s)}\Gamma^v_{vv}dv}(p^v)^{-1}dv \ . \end{aligned}$$

Thus, for $s' < s$, (replacing 0 with s') we have, by (28) and (24), the inequality

$$p^v(s) \leq p^v(s')e^{-\int_{v(s')}^{v(s)}\Gamma^v_{vv}dv} + \int_{v(s')}^{v(s)} 2\tilde{N}c^{-1}r\gamma_{AB}p^Ap^Be^{-\int_{v(\tilde{s})}^{v(s)}\Gamma^v_{vv}dv}(p^v)^{-1}dv \ . \tag{30}$$

Suppose $p^v(s) > 2\Sigma$ for some $0 \leq v(s) \leq V$, and let s' be the last previous time $s > s' > 0$ such that $p^v(s') \geq 2\Sigma$, i.e., we have $p^v(s^*) \geq 2\Sigma$ on $[s^*, s]$. By (30), (27), the angular momentum bound (25), and (41), we have

$$p^v(s) \leq 2\Sigma e^{VG'} + r_0^{-3}X\tilde{N}c^{-1}e^{VG'}V(2\Sigma)^{-1} \ ,$$

i.e.,

$$p^v(s) \leq \tilde{C} \ . \tag{31}$$

We can now easily estimate T_{uu} pointwise:

$$\begin{aligned} T_{uu} &= \int_0^\infty \int_{|\gamma_{AB}p^Ap^B|\leq Xr^{-4}} r^2(p_u)^2 f\frac{dp^v}{p^v}\sqrt{\gamma}dp^Adp^B \\ &= (g_{uv})^2\int_0^\infty \int_{|\gamma_{AB}p^Ap^B|\leq Xr^{-4}} r^2(p^v)^2 f\frac{dp^v}{p^v}\sqrt{\gamma}dp^Adp^B \\ &= 4\nu^2\kappa^2\int_0^\infty \int_{|\gamma_{AB}p^Ap^B|\leq Xr^{-4}} r^2(p^v)^2 f\frac{dp^v}{p^v}\sqrt{\gamma}dp^Adp^B \end{aligned}$$

$$
\begin{aligned}
&= 4\nu^2 F\kappa^2 \int_0^{\tilde{C}} \int_{|\gamma_{AB}p^A p^B| \le X r^{-4}} r^2 (p^v) dp^v \sqrt{\gamma} dp^A dp^B \\
&\le 16\pi\, r_0^{-2} \nu^2 F \tilde{C}^2 X^2 = \nu^2 E \\
&\le \tilde{N}^2 E \ ,
\end{aligned}
$$

in view of (31), (24), (21), (14) and the angular momentum bound (25). (Note that $T_{uu}\nu^{-2} \le E$ is a coordinate invariant[2] bound.) Integrating (20), we obtain now

$$\kappa \ge e^{-\int 4\pi r \frac{T_{uu}}{\nu^2} \nu du} \ge e^{-4\pi R E \tilde{N} U} \ .$$

(Actually, we have in fact already estimated κ from below since $\kappa^{-1} = 4(-\nu)\Omega^{-2}$.)

From the inequality

$$p_u p_v \le \frac{1}{2}(p_u^2 + p_v^2) \ , \qquad \text{we have} \qquad T_{uv} \le \frac{1}{2}(T_{uu} + T_{vv}) \ .$$

This allows us to estimate $\partial_u \log \Omega^2 = \Gamma^u_{uu}$:

$$
\begin{aligned}
|\Gamma^u_{uu}| &\le G + \left| \int 8\pi T_{uv} dv - \int 8\pi\kappa\nu r^{-2} \gamma^{AB} T_{AB} dv \right| - \int 4\kappa m r^{-3} \nu dv \\
&\le G + 8\pi \tilde{N}^2 E V + \int 8\pi T_{vv} dv - \int 4\kappa m r^{-3} \nu dv \quad \le \quad \bar{C} \ .
\end{aligned}
$$

We can easily obtain an estimate now for T_{vv}. λ can be bounded by integrating (3).

10 C^2 estimates for the metric

In this section, we derive C^2 estimates on the metric and C^1 estimates for f. The ideas of this section originate in [7].

It has already been shown that the following quantities are bounded: r, r^{-1}, m, m^{-1}, κ, κ^{-1}, ν, ν^{-1}, λ, Ω, Ω^{-1}, all first order derivatives of Ω, all components of the energy-momentum tensor, and all Christoffel symbols in (36)–(41). From these estimates and (22) and (23), it follows that $\partial_v \nu$ and $\partial_u \lambda$ are bounded, from (12) and (13) it follows that $\partial_u m$ and $\partial_v m$ are bounded, and from (26), it follows that $\partial_u \partial_v \Omega$ is bounded. Writing $\nu = -\frac{1}{4}\Omega^2 \kappa^{-1}$ and differentiating in u, we see from (20) that $\partial_u \nu$ is bounded, while writing $\kappa = -\frac{1}{4}\Omega^2 \nu^{-1}$ and differentiating in v, we see that $\partial_v \kappa$ is bounded, and thus, from (11), we see that $\partial_v \lambda$ is bounded. These estimates and the formulas (36)–(41) allow us to control all first order derivatives of the Christoffel symbols, except $\partial_u \Gamma^u_{uu}$ and $\partial_v \Gamma^v_{vv}$.

Since the components of the curvature tensor can be expressed in terms of those derivatives of the Christoffel symbols which have already been estimated, we obtain bounds for all components of the curvature tensor in our coordinates. The above estimates allow us to estimate the first derivatives of the exponential map

[2] i.e., it does not depend on the normalization of u

on the tangent bundle. This, in turn allows one to estimate the derivatives of f in terms of initial data.

We can, however, argue more directly as follows. Let us abbreviate the Vlasov equation (8) by $X(f) = 0$ where X is the Vlasov operator written in these coordinates. Note that p^v is to be thought of as expressed in terms of p^u and p^A via the mass shell condition (7).

Define $f_1 = \partial_u f - p^u \partial_u \log \Omega^2 \partial_{p^u} f$. Differentiating the Vlasov equation with respect to v, p^u and p^A gives the following equations:

$$\begin{aligned} X(\partial_v f) &= -(\partial_v p^v)\partial_v f + (\partial_u \partial_v \log \Omega^2 (p^u)^2 + \partial_v(-2\Omega^{-2} r\lambda)\gamma_{AB} p^A p^B)\partial_{p^u} f \\ &\quad + 2(\partial_v(\nu r^{-1})p^u + \partial_v(\lambda r^{-1})p^v + \lambda r^{-1}\partial_v p^v)p^A \partial_{p^A} f \ , \qquad (32) \\ X(\partial_{p^u} f) &= -\partial_u f - (\partial_{p^u} p^v)\partial_v f + 2\partial_u \log \Omega^2 p^u \partial_{p^u} f \\ &\quad + 2(\nu r^{-1} + \lambda r^{-1}\partial_{p^u} p^v)p^A \partial_{p^A} f \ , \qquad (33) \\ X(p^D \partial_{p^D} f) &= -p^D(\partial_{p^D} p^v)\partial_v f - 4\Omega^{-2} r\lambda \gamma_{AB} p^A p^B \partial_{p^u} f \\ &\quad + 2r^{-1}\lambda p^D \partial_{p^D} p^v p^A \partial_{p^A} f \ . \qquad (34) \end{aligned}$$

Differentiating the Vlasov equation with respect to u gives the following equation for f_1:

$$\begin{aligned} X(f_1) \;=\; & -p^u \partial_u(\log \Omega^2) X(\partial_{p^u} f) - \partial_u p^v \partial_v f \\ + \; & (-p^u p^v \partial_u \partial_v \log \Omega^2 - \partial_u \log \Omega^2 (\partial_u \log \Omega^2 (p^u)^2 + 2\Omega^{-2} r\lambda \gamma_{AB} p^A p^B) \\ - \; & 2\partial_u(\Omega^{-2} r\lambda)\gamma_{AB} p^A p^B)\partial_{p^u} f \\ + \; & 2(\partial_u(\nu r^{-1})p^u + \partial_u(\lambda r^{-1})p^v + \partial_u p^v \lambda r^{-1})p^A \partial_{p^A} f \ . \qquad (35) \end{aligned}$$

The quantity $X(\partial_{p^u} f)$ can be substituted for by one of the previous equations and $\partial_u f$ may be eliminated from the equations in favour of f_1. The result is a linear system of equations for the evolution of $(f_1, \partial_v f, \partial_{p^u} f, p^A \partial_{p^A} f)$ along the characteristics of the Vlasov equation. The coefficients are known to be bounded and so we can conclude that $\partial_u f$, $\partial_v f$, $\partial_{p^u} f$ and $p^A \partial_{p^A} f$ are also bounded. (Note that since p^u and p^v are bounded the derivative with respect to X is uniformly equivalent to a derivative along the characteristic with respect to u or v as parameter.)

From this, we immediately estimate $\partial_u T_{ab}$ and $\partial_v T_{ab}$ pointwise. We now estimate $\partial_u \Gamma^u_{uu}$ by differentiating (26) in u and integrating in v, and similarly, $\partial_v \Gamma^v_{vv}$ by differentiating in v and integrating in u. Note that $|\partial_{p^A}|_\gamma$ can also be bounded. This can be seen by passing from polar to Cartesian coordinates and noting that the resulting metric components are C^2. As a consequence f is C^1.

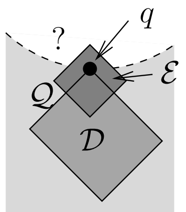
11 The Proof of Theorem 1

Let $N/2$ denote the sup of the norm defined in Proposition 2, where the sup is taken now in all of $\mathcal{D}$. By the estimates of the previous section, we have that $N/2 < \infty$. Let δ be the constant of Proposition 2 corresponding to N. Consider the point $(U - \delta/2, V - \delta/2)$. Translate the coordinates so that this point is $(0,0)$. Since $\mathcal{Q}$ is by definition open, by continuity, there exists a $\delta > \delta^* > \delta/2$ such that

$$\{0\} \times [0, \delta^*] \cup [0, \delta^*] \times \{0\} \subset \mathcal{Q}$$

and the assumptions of Proposition 2 hold on $\{0\} \times [0, \delta^*] \cup [0, \delta^*] \times \{0\}$, with N and δ^* as already defined. It follows that there exists a unique solution of in

$$\mathcal{E} = [0, \delta^*] \times [0, \delta^*] \ .$$



Thus the solution coincides in $\mathcal{E} \cap \mathcal{Q}$ by uniqueness. One sees that $\mathcal{E} \cup \mathcal{Q}$ is clearly the quotient of a development of initial data. By maximality of $\mathcal{M}$, we must have $\mathcal{E} \cup \mathcal{Q} \subset \mathcal{Q}$. Thus, in particular, in the old coordinates we have $(U, V) \in \mathcal{Q}$, and the theorem holds with $q = (U, V)$.

12 Applications

We will say that a spherically symmetric maximal development has a *black hole*, if $\mathcal{I}^+$ is *complete* in the sense of [4],[3] and if $J^-(\mathcal{I}^+)$ has a non-empty complement.

We have shown that the results of [12] apply to our matter model. In particular, the fact that the complement of $J^-(\mathcal{I}^+)$ is non-empty implies the completeness of null infinity. That this set is non-empty can be inferred in turn from the existence of a single trapped or marginally trapped surface. Asymptotically flat spherically symmetric solutions of the Einstein-Vlasov system possesing a trapped surface were constructed in [9]. Thus we have

Corollary 1. *There exist solutions of the Einstein-Vlasov system which develop from regular initial data and contain black holes.*

The fundamental open question in gravitational collapse is to show that generically, either the solution is future geodesically complete or a black hole forms. In view of [12] and the results of this paper we have

Corollary 2. *Suppose that for generic initial data, the maximal development either contains a trapped surface or marginally trapped surface, or is future causally geodesically complete. Then weak cosmic censorship is true.*

Thus, weak cosmic censorship can be reduced to a slightly weaker version of Christodoulou's *trapped surfaces conjecture.* As remarked in the Introduction, this suggests a local approach to its proof (cf. [3]).

[3]See [12] for a definition of $\mathcal{I}^+$ in this context.

A The Christoffel symbols

Note:

$$g_{uv} = -\frac{1}{2}\Omega^2 \ ,$$
$$g^{uv} = -2\Omega^{-2} \ ,$$
$$\Omega^2 = -4\kappa\nu \ .$$

The nonvanishing Christoffel symbols are given by:

$$\Gamma^u_{AB} = -g^{uv} r\lambda\gamma_{AB} \ , \tag{36}$$
$$\Gamma^v_{AB} = -g^{uv} r\nu\gamma_{AB} \ , \tag{37}$$
$$\Gamma^A_{Bv} = \lambda r^{-1}\delta^A_B \ , \tag{38}$$
$$\Gamma^A_{Bu} = \nu r^{-1}\delta^A_B \ , \tag{39}$$
$$\Gamma^u_{uu} = \partial_u \log\Omega^2 \ , \tag{40}$$
$$\Gamma^v_{vv} = \partial_v \log\Omega^2 \ . \tag{41}$$

In fact the Christoffel symbols Γ^C_{AB}, which depend on a choice of coordinates on the spheres of symmetry need not vanish but the expressions for them are not needed in this paper.

B Proof of Proposition 2

The proof of local existence follows from simpler considerations than the proof of the estimates of Sections 8–10. In particular, one does not need to consider energy estimates, for one can recover naive pointwise estimates using the smallness parameter. As in Section 10, the idea of [7] again makes its appearance, to show C^1 bounds on f directly from C^0 bounds on the curvature, before bounding the C^2 norm of the metric. Since all these methods have appeared before, we will only sketch the details here.

Let initial data be fixed. Define the space

$$A \subset C^2([0,\delta]\times[0,\delta]) \times C^1([0,\delta]\times[0,\delta]) \ ,$$

for δ to be determined later, consisting of all twice continuously differentiable nonnegative functions r, continuously differentiable nonnegative functions Ω, extending the prescribed values, such that

$$N^{-1}/2 \le r \le 2N \ , \tag{42}$$
$$N^{-1}/2 \le \Omega \le 2N \ , \tag{43}$$
$$\sup\{|\partial_u r|, |\partial_v r|, |\partial_u^2 r|, |\partial_v^2 r|\} \le 2N \ , \tag{44}$$
$$\sup\{|\partial_u \Omega|, |\partial_v \Omega|\} \le 2N \ . \tag{45}$$

Consider the subset $B \subset A$, consisting of those (r, Ω) for which Ω is C^2, and for which

$$\sup\{|\partial_u^2 \Omega|, |\partial_v^2 \Omega|, |\partial_u \partial_v \Omega|\} \leq 2N \ . \tag{46}$$

Note that the closure of B in A, denoted $\overline{B}$, consists of (r, Ω) such that $\partial_u \Omega$, $\partial_v \Omega$, are Lipschitz, with Lipschitz constants given by the above.

We shall define in the next few paragraphs a continuous map $\Phi : \overline{B} \to A$ taking (r, Ω) to $(\tilde{r}, \tilde{\Omega})$.

Given r, Ω, first, let f be defined to solve the Vlasov equations on the metric defined by r and Ω, with given initial conditions. Note that since the Christoffel symbols of this metric are Lipschitz, it follows that geodesics can be defined, and thus f can be defined by the requirement that it is preserved by geodesic motion. It follows immediately that

$$0 \leq f \leq N \ , \tag{47}$$

and, after appropriately restricting to sufficiently small δ, it follows easily by integration of the geodesic equations that

$$S \leq 2N \ . \tag{48}$$

In the case where $(r, \Omega) \in B$, we have that f is in fact C^1, since the exponential map is differentiable. If δ is chosen sufficiently small, it is clear from (42)–(46) that, in this case, we can arrange for

$$\sup\{|\partial_v f|, |\partial_u f|, |\partial_{p_u} f|, |\partial_{p^A} f|_\gamma\} \leq 2N \ . \tag{49}$$

Given now f, we can define T^{uv}, T^{vv}, T^{uu} in the standard way. In view of (42)–(45), (47), and (48), these terms can be estimated. Now, set $\nu = \partial_u r$, $\lambda = \partial_v r$. We define $\tilde{r}$ by

$$\tilde{r}(u,v) = r(u,0) + r(0,v) - r(0,0) + \int_0^u \int_0^v -\frac{1}{4} r^{-2}\Omega^2 - \frac{1}{r}\lambda\nu + 4\pi r \Omega^4 T^{uv} du dv \ . \tag{50}$$

By appropriate differentiation of (50), it is clear from our bounds thus far that we can define and estimate $\tilde{\nu} = \partial_u \tilde{r}$, $\tilde{\lambda} = \partial_v \tilde{r}$, and $\partial_u \partial_v \tilde{r}$. We can retrieve the bound (42) for $\tilde{r}$ by integration of the $\tilde{\nu}$, after restricting to small δ. For $(r, \Omega) \in B \subset \bar{B}$, it is clear we can also define and estimate $\partial_u^2 \tilde{r}$, $\partial_v^2 \tilde{r}$, by differentiating (50) twice in u or twice in v, in view of the fact that all other derivatives, including $\partial_u T^{uv}$, $\partial_u \nu$, etc., are clearly defined and bounded, in view of (49), and since these derivatives are defined initially. By appropriate choice of δ, we can clearly arrange–for $(r, \Omega) \in B$– so as to retrieve the bound (44).

Define now $\tilde{\Omega} > 0$ by the relation

$$\begin{aligned} \log \tilde{\Omega}^2 \quad &= \quad \log \Omega^2(u,0) + \log \Omega^2(0,v) - \log \Omega^2(0,0) \\ &\quad + \int_0^u \int_0^v (-8\pi T_{uv} + \frac{1}{2}\Omega^2 \tilde{r}^{-2} + 2\tilde{r}^{-2}\tilde{\lambda}\tilde{\nu} - 2\pi \Omega^2 \tilde{r}^{-2} \gamma^{AB} T_{AB}) du dv \ . \end{aligned} \tag{51}$$

Again, for small enough δ, it is clear that one can arrange for $\tilde{\Omega}$ to satisfy (43).

Differentiating (51) appropriately, in view of the initial conditions for $\tilde{\Omega}$, it follows that, for $(r,\Omega) \in \overline{B}$, $\tilde{\Omega}$ is C^1, and for δ small enough satsfies (45), while for $(r,\Omega) \in B$, $\tilde{\Omega}$ is C^2, and for δ small enough, satisfies (46).

Thus, we have shown that after judicious choice of δ, Φ maps B to itself. By continuity, it maps $\overline{B}$ to itself.

The map Φ can easily be shown to be a contraction in B for the norm of A, i.e., we can show that

$$d_A((\tilde{r}_1,\tilde{\Omega}_1),(\tilde{r}_2,\tilde{\Omega}_2)) \leq \epsilon d_A((r_1,\Omega_1),(r_2,\Omega_2)) \ , \tag{52}$$

for an $\epsilon < 1$ and all $(r_i,\Omega_i) \in B$. To see this, define first f_i, corresponding to (r_i,Ω_i). Let Γ_i denote an arbitrary Christoffel symbol for (r_i,Ω_i). We clearly have

$$|\Gamma_1 - \Gamma_2| \leq C d_A((r_1,\Omega_1),(r_2,\Omega_2)) \ .$$

We easily obtain

$$|f_1 - f_2| \leq C\delta \sup_{\Gamma} |\Gamma_1 - \Gamma_2| \sup_{i=1,2} (|\partial f_i| + |f_i|) \ .$$

Clearly we can also bound $\sup|T_1^{uv} - T_2^{uv}| \leq C \sup|f_1 - f_2|$. One bounds $(\nu_1 - \nu_2)$ by expressing $\partial_v(\tilde{\nu}_1 - \tilde{\nu}_2)$ as a linear combination of $\Omega_1 - \Omega_2$, $r_1 - r_2$, $\nu_1 - \nu_2$, $\lambda_1 - \lambda_2$ and $(T_1^{uv} - T_2^{uv})$ with bounded coefficients. One immediately obtains a similar bound for $\sup|\tilde{r}_1 - \tilde{r}_2|$. The terms $\sup|\partial_u\tilde{r}_1 - \partial_u\tilde{r}_2|$, $\sup|\partial_v\tilde{r}_1 - \partial_v\tilde{r}_2|$, and $\sup|\partial_u\partial_v\tilde{r}_1 - \partial_u\partial_v\tilde{r}_2|$, can be handled in the same way. One then obtains a bound of the above form for $\sup|\partial_v \log\tilde{\Omega}_1^2 - \partial_v \log\tilde{\Omega}_2^2|$, and similarly for $\sup|\partial_u \log\tilde{\Omega}_1^2 - \partial_u \log\tilde{\Omega}_2^2|$. Either of these bounds of course implies a bound for $\sup|\tilde{\Omega}_1^2 - \tilde{\Omega}_2^2|$.

To bound $\sup|\partial_u^2\tilde{r}_1 - \partial_u^2\tilde{r}_2|$, we compute

$$\begin{aligned}
\partial_u^2\tilde{r} &= \partial_u^2\tilde{r}|_{v=0} + \int_0^v \partial_u\left(-\frac{1}{4}r^{-2}\Omega^2 - r^{-1}\lambda\nu\right) \\
&\quad + 4\pi\partial_u(r\Omega^4)T^{uv} + 4\pi r\Omega^4\partial_u T^{uv} dv \qquad (53)\\
&= \partial_u^2\tilde{r}|_{v=0} + \int_0^v \partial_u\left(-\frac{1}{4}r^{-2}\Omega^2 - r^{-1}\lambda\nu\right) + 4\pi\partial_u(r\Omega^4)T^{uv} \\
&\quad - 4\pi r\Omega^4\partial_v T^{vv} + 4\pi r\Omega^4(\sum T\cdot\Gamma)dv \\
&= \partial_u^2\tilde{r}|_{v=0} - 4\pi r\Omega^4 T^{vv}(u,v) + 4\pi r\Omega^4 T^{uv}(u,0) \\
&\quad + \int_0^v \partial_u\left(-\frac{1}{4}r^{-2}\Omega^2 - r^{-1}\lambda\nu\right) + 4\pi\partial_u(r\Omega^4)T^{uv} \\
&\quad + 4\pi\partial_v(r\Omega^4)T^{vv} + 4\pi r\Omega^4(\sum T\cdot\Gamma)dv \ . \qquad (54)
\end{aligned}$$

Here we have used the equation $\nabla_a T^{ab} = 0$, which follows from the Vlasov equation, and we have integrated by parts. It is now clear that estimates for differences follow as before. We argue in an entirely analogous way for $\sup|\partial_v^2\tilde{r}_1 - \partial_v^2\tilde{r}_2|$.

After restricting to sufficiently small δ, all constants in the above bounds can be made small. We thus have indeed shown (52). It follows by continuity that Φ is also a contraction on $\overline{B} \subset A$, and thus, since $\overline{B}$ is closed, has a fixed point in $\overline{B}$.

Given such a fixed point (r, Ω), define f as before. To show that (r, Ω, f) corresponds to a solution of the equations, we have basically only to show that f and $\partial_u\Omega$, $\partial_v\Omega$, which *a priori* are Lipschitz, are in fact C^1. (In particular, from this it will follow that the constraint equations (5)–(6) are also satisfied.) But, in view of the fact that f is initially C^1, it follows that f is C^1 if the exponential map is C^1. (The C^2 compatibility condition is used at the point.) But this latter fact follows from the continuity of the curvature, as shown in Exercise 6.2 of Chapter V of [5] [4]. That the curvature is continuous follows by computation, since r is C^2, Ω is C^1 and $\partial_u\partial_v\Omega$ is C^0, and $\partial_u^2\Omega$ and $\partial_v^2\Omega$ do not appear in the expressions for curvature. From the C^1 property of f, the C^2 property of Ω follows immediately. Similarly, higher regularity follows immediately if it is assumed. $\Box$

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[4]If the reader does want to apply to this fact, then one can argue as follows: in view of the computations above, in the space B, we have that curvature is in fact C^1 with estimates; since derivatives of the exponential map are computed by integrating curvature on geodesics, and geodesics certainly depend C^1 on their initial conditions, in view of the fact that the Christoffel symbols are C^1 with bounds in B, it follows that we have C^2 estimates for the exponential map in B, and thus by an easy compactness argument, the exponential map of the fixed point must be C^1. There is only one catch with this argument: r and Ω^2 have to be assumed to be initially C^3 to differentiate (51) and (53) three times.

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