Ann. Henri Poincaré 6 (2005) 991 – 1023

1424-0637/05/060991-33, *Published online* 15.11.2005
DOI 10.1007/s00023-005-0232-x

Annales Henri Poincaré

Heat-Kernel Approach to UV/IR Mixing on Isospectral Deformation Manifolds

Victor Gayral

Abstract. We work out the general features of perturbative field theory on noncommutative manifolds defined by isospectral deformation. These (in general curved) 'quantum spaces', generalizing Moyal planes and noncommutative tori, are constructed using Rieffel's theory of deformation quantization by actions of $\mathbb{R}^l$. Our framework, incorporating background field methods and tools of QFT in curved spaces, allows to deal both with compact and non-compact spaces, as well as with periodic and non-periodic deformations, essentially in the same way. We compute the quantum effective action up to one loop for a scalar theory, showing the different UV/IR mixing phenomena for different kinds of isospectral deformations. The presence and behavior of the non-planar parts of the Green functions is understood simply in terms of off-diagonal heat kernel contributions. For periodic deformations, a Diophantine condition on the noncommutivity parameters is found to play a role in the analytical nature of the non-planar part of the one-loop reduced effective action. Existence of fixed points for the action may give rise to a new kind of UV/IR mixing.

1 Introduction

Noncommutative geometry (NCG), specially in Connes' algebraic and operatorial formulation [4], is an attempt to free oneself from the classical differential structure framework in modeling and understanding space-time, while keeping in algebraic form geometry's tools such as metric and spin structures, vector bundles and connection theory. The NCG framework is well adapted to deal with quantum field theory over 'quantum' space-time (NCQFT) [34]. However, there is a lack of computable examples crucially needed to progress in this direction. Here we present a large class of models, the isospectral deformation manifolds, in which we show the intrinsic nature of UV/IR mixing through the analysis of a scalar theory.

In [6, 7] Connes, Landi and Dubois-Violette gave a method to generate noncommutative spaces based on the noncommutative torus paradigm. For any closed Riemannian spin (this last condition could be relaxed for our purpose) manifold with isometry group of rank $l \geq 2$, one can build a family of noncommutative spaces, called isospectral deformations by the authors. The terminology comes from the fact that the underlying spectral triple, that is, the dual object $(C^\infty(M_\Theta), L^2(M,S), D\!\!\!/)$ encoding all the topological, differential, metric and spin structures of the original manifold, and so defining the 'quantum Riemannian' space [5], has the same space of spinors and the same Dirac operator as the undeformed one $(C^\infty(M), L^2(M,S), D\!\!\!/)$; only the algebra is modified.

More precisely, the noncommutative algebra $C^\infty(M_\Theta)$ can be defined as a fixed point algebra under a group action [7]:

$$C^\infty(M_\Theta) := \Big(C^\infty(M)\widehat{\otimes}\mathbb{T}^l_\Theta\Big)^{\alpha\widehat{\otimes}\tau^{-1}}, \tag{1.1}$$

where $\mathbb{T}^l_\Theta$ is a l-dimensional NC torus(-algebra) with deformation matrix $\Theta \in M_l(\mathbb{R}), \Theta^t = -\Theta$; α is the action of $\mathbb{T}^l$ on M given by an Abelian part of its isometry group, τ is the standard action of $\mathbb{T}^l$ on $\mathbb{T}^l_\Theta$ and $\widehat{\otimes}$ is a suitable tensor product completion. By the Myers-Steenrod Theorem [26], which asserts that $\mathrm{Isom}(M,g) \subset SO(n)$ for any n-dimensional compact Riemannian manifold (M,g), one can see that the class of such manifolds whose isometry group has rank greater or equal to two is far from small.

Várilly [33] and Sitarz [31] independently remarked that this construction fits into Rieffel's theory of deformation quantization for actions of $\mathbb{R}^l$ [28]. Given a Fréchet algebra A with seminorms $\{p_i\}_{i\in I}$ and a strongly continuous isometric (with respect to each seminorms) action of $\mathbb{R}^l$, one can deform the product of the subalgebra A^∞, consisting of smooth elements of A with respect to the generators X^k, $k \in \{1,\dots,l\}$ of the action α. The algebra A^∞ can be canonically endowed with a new set of seminorms $\{\tilde{p}_{i,m}\}_{i\in I, m\in\mathbb{N}}$ given by $\tilde{p}_{i,m}(.) := \sup_{j\leq i}\sum_{|\beta|\leq m} p_j(X^\beta.)$, $\beta\in\mathbb{N}^l$. Those seminorms have the property of being compatible with the deformed product defined by the A^∞-valued oscillatory integral:

$$a\star_\Theta b := (2\pi)^{-l}\int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\ \alpha_{\frac{1}{2}\Theta y}(a)\alpha_{-z}(b),\ a,b\in A^\infty.$$

Here Θ is the (real, skewsymmetric) deformation $l\times l$ matrix, $< y,z >= \sum_{i=1}^l y^i\, z^i$, and if we denote by A^∞_Θ the algebra $(A^\infty, \star_\Theta)$, the deformation process verifies $(A^\infty_\Theta)^\infty_{\Theta'} = A^\infty_{\Theta+\Theta'}$, and hence is reversible. In [16], we investigate the equivalent of (1.1) in the non-periodic case and extend the construction of isospectral non-periodic deformations (called also θ-deformations to distinguish them from q-deformations) to non-compact manifolds within Rieffel's framework, whose paradigms are now the Moyal planes [12].

Although we will not use directly the fixed point characterization (1.1), we want to insist on its crucial importance to understand the situation. Indeed, such a characterization means that we are transferring the noncommutative structure of the NC torus or of the Moyal plane inside the commutative algebra of smooth functions, in a way compatible with the Riemannian structure.

The first studied examples of NCQFT were the NC tori and the Moyal planes, in pioneer works like [3, 11, 21, 23, 24, 34] (see also [10] and [32] for reviews). In those flat space situations, the main novelty in regard to renormalization aspects is that two kinds of Feynman diagrams coexist, respectively called planar and non-planar. The first one yields ordinary UV divergences, while the non-planar graphs, characterized by vertices which depend on external momenta through a phase, are finite except for some values of the incoming momenta. That happens in particular for the zero mode in $\lambda\varphi^{\star_\Theta 4}$ theory on the NC torus and in the limit $p^\mu \to 0$ for the same theory on the Moyal plane. This is the famous UV/IR entanglement phenomenon, which gives rise to difficulties for any renormalization scheme.

In this paper, we show that for any (in general non-flat) isospectral deformation, UV/IR mixing in (Euclidean) NCQFT exists as in the (flat) paradigmatic examples of the NC torus and the Moyal planes.

In the next section, isospectral deformations are constructed and their basic NCG properties are reviewed. The third section is devoted to the study of the $\lambda\varphi^{\star_\Theta 4}$ theory. One derives a field expansion from a (modified) heat kernel asymptotics to compute the effective action up to one loop. This construction gives a simple algebraic meaning to the presence and behavior of planar and non-planar sectors in those theories. In sections 4 and 5, using off-diagonal heat-kernel estimates, we prove the inherent generic character of the divergent structures for all kinds of isospectral deformations. Fixed points for the $\mathbb{R}^l$ action potentially yield a new kind of UV/IR mixing.

2 Isospectral deformations

As explained in the Introduction, isospectral deformations are curved noncommutative spaces generalizing Moyal planes and noncommutative torus. To construct those NC Riemannian spaces (spectral triples), we use an approach developed in [16]. Advantages of this twisted product approach *à la* Rieffel are that it allows to treat on the same footing compact and non-compact cases (unital and non-unital algebras) as well as periodic and non-periodic deformations, and that it is well adapted for Hilbertian analysis.

Let (M, g) be a locally compact, complete, connected, oriented Riemannian n-dimensional manifold without boundary, and let α be a smooth isometric action of $\mathbb{R}^l$, $2 \leq l \leq n$

$$\alpha : \mathbb{R}^l \longrightarrow \mathrm{Isom}(M, g) \subset \mathrm{Diff}(M),$$

where l is less or equal to the rank of the isometry group of (M, g). We can then define a deformed or twisted product. The isometric action α yields a group of automorphisms on $C^\infty(M)$ that we will again denote by α: for all $z \in \mathbb{R}^l$

$$\alpha_z f(p) := f(\alpha_{-z}(p)).$$

For brevity we will often write $z.p \equiv \alpha_z(p)$ to designate the action of a group element on a point of the manifold. Obviously, the group action property reads

$$z_1.(z_2.p) = (z_1 + z_2).p \quad \text{and} \quad 0.p = p.$$

The infinitesimal generators of this action

$$X_j(.) := \frac{\partial}{\partial z^j}\alpha_z(.)\Big|_{z=0}, \quad j = 1, \dots, l,$$

are ordinary smooth vector fields, so they leave $C_c^\infty(M)$ invariant. Hence, given a real skewsymmetric $l \times l$ matrix Θ, one defines the deformed product of any $f, h \in C_c^\infty(M)$ as a bilinear product on $C_c^\infty(M)$ with values in $C^\infty(M) \cap L^\infty(M, \mu_g)$ by the oscillatory integral

$$f \star_\Theta h := (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y \, d^l z \; e^{-i<y,z>} \; \alpha_{\frac{1}{2}\Theta y}(f) \alpha_{-z}(h), \tag{2.1}$$

where $< y, z >:= \sum_{j=i}^{l} y^j z^j$ can be viewed as the pairing between $\mathbb{R}^l$ and its dual group. In spite of appearances this formula is symmetric, even with a degenerate Θ matrix (see the discussion near the end of this section), as one can rewrite the deformed product:

$$f \star_\Theta h := (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y \, d^l z \; e^{i<y,z>} \; \alpha_{-y}(f) \alpha_{\frac{1}{2}\Theta z}(h).$$

The non-locality of this product generates a non-preservation of supports. In particular, the twisted product of two functions with disjoint support turns out to be non-zero a priori. Whereas in the periodic case ($\ker \alpha \simeq \mathbb{Z}^l$) the fixed point characterization gives rise to a reasonable locally convex topology on the invariant sub-algebra of the algebraic tensor product $\left(C^\infty(M) \otimes \mathbb{T}^l_\Theta \right)^{\alpha \otimes \tau^{-1}}$ or $\left(C_c^\infty(M) \otimes \mathbb{T}^l_\Theta \right)^{\alpha \otimes \tau^{-1}}$ depending whether M is compact or not, to obtain a smooth algebra structure in the non-periodic case one has to complete $C_c^\infty(M)$ to a Fréchet algebra with seminorms defined through the measure associated to the Riemannian volume form, so that the action becomes strongly continuous and isometric with respect to each seminorm. This feature is investigated in [16]. In the sequel, as we mainly work at the linear level, $C_c^\infty(M)$ will be deemed "large enough".

The associativity of the product (2.1) can be easily checked. The ordinary integral with Riemannian volume form μ_g is a trace (a proof is provided in [16]):

$$\int_M \mu_g \; f \star_\Theta h = \int_M \mu_g \; f \, h = \int_M \mu_g \; h \star_\Theta f; \tag{2.2}$$

α is still an automorphism for the deformed product:

$$\alpha_z(f) \star_\Theta \alpha_z(h) = \alpha_z(f \star_\Theta h); \tag{2.3}$$

the complex conjugation is an involution:

$$(f \star_\Theta h)^* = h^* \star_\Theta f^*; \tag{2.4}$$

and the Leibniz rule is satisfied for the generators of the action

$$X^k(f \star_\Theta h) = X^k(f) \star_\Theta h + f \star_\Theta X^k(h), \; k = 1, \ldots, l. \tag{2.5}$$

In fact, the Leibniz rule is satisfied for any order one differential operator which commutes with the action α, thus for the Dirac operator when the manifold has a spin structure.

We have basically two distinct situations. When the group action is effective ($\ker\alpha = \{0\}$), i.e., for a non-periodic deformation, it is seen that the good topological assumption on α in order to avoid serious difficulties is properness. That is, we assume the map

$$(z,p) \in \mathbb{R}^l \times M \mapsto (p, \alpha_z(p)) \in M \times M$$

to be proper. Recall that a map between topological spaces is proper if the preimage of any compact set is compact as well. On the other hand, for periodic deformations the action factors through a torus action $\tilde{\alpha} : \mathbb{R}^l/\mathbb{Z}^l \to \mathrm{Isom}(M,g)$, and the factorized action $\tilde{\alpha}$ is automatically proper.

When M is compact, α must be periodic to be proper, while in the non-compact case both situations appear. We point out that the (non-compact) non-periodic case is the most difficult one. First, when the manifold is not compact, the essential spectrum of the Laplacian is non-empty, so its negative powers are no longer compact operators. Furthermore, for periodic deformations (of compact manifolds or not) we have a spectral subspace decomposition, indexed by the dual group of $\mathbb{T}^l$, which does simplify proofs and computations.
We do not explicitly treat the mixed case $\alpha : \mathbb{R}^d \times \mathbb{T}^{l-d} \to \mathrm{Isom}(M,g)$, but its general features will be clear from what follows.

The hypothesis of geodesically completeness of M guarantees selfadjointness of the (closure of the) Laplace-Beltrami operator Δ restricted to (the dense subset $C_c^\infty(M)$ of) $L^2(M,\mu_g)$, the separable Hilbert space of squared integrable functions with respect to the measure space (M,μ_g). In our convention, $\Delta = (d+\delta)^2$ is positive, and reduced to 0-forms $\Delta = \delta d = *_H d *_H d$ where $*_H$ is the Hodge star. Completeness (plus boundedness from below of the Ricci curvature) is needed to have conservation of probability [2, 9]:

$$\int_M \mu_g(p)\ K_t(p,p') = 1,$$

where $K_t := K_{e^{-t\Delta}}$ is the heat kernel of the manifold. Recall that $K_t(p,p')$ for $t>0$ is a smooth strictly positive symmetric function on $M \times M$. The restriction to manifolds without boundary is required to have a simple (with vanishing of the odd terms [17]) on-diagonal expansion of the heat kernel

$$K_t(p,p) \simeq (4\pi t)^{-n/2} \sum_{l\in\mathbb{N}} t^l\ a_{2l}(p), \quad t \to 0, \tag{2.6}$$

where $a_l(p)$ are the so called Seeley-De Witt coefficients.

It is proved in [16] that for non-compact non-periodic deformations (the statement being immediate in the periodic case) $L_f \equiv L_f^\Theta$ (resp. $R_f \equiv R_f^\Theta$), the operator of left (resp. right) twisted multiplication by f, defined by $L_f\psi = f\star_\Theta\psi$ (resp.

$R_f\psi = \psi\star_\Theta f$), for $\psi \in \mathcal{H} := L^2(M, \mu_g)$, is bounded for any $f \in C_c^\infty(M)$. This will be also true for smooth functions decreasing fast enough at infinity.

Denote by V_z the induced action of $\mathbb{R}^l$ on $L^2(M, \mu_g)$ by unitary operators

$$V_z\psi(p) := \psi(-z.p);$$

then one can alternatively define L_f and R_f by an operator valued integral

$$L_f = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, V_{\frac{1}{2}\Theta y}\, M_f\, V_{-z}, \tag{2.7}$$

$$R_f = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, V_{-z}\, M_f\, V_{\frac{1}{2}\Theta y}, \tag{2.8}$$

where M_f denotes the operator of pointwise multiplication by f.
Such integrals do not define Böchner integrals in the vector space $\mathcal{L}(\mathcal{H})$. Indeed, the operatorial norm of the integrands in (2.7) and (2.8) are not integrable functions on $\mathbb{R}^{2l}$, since they depend on y and z only through unitary operators. Actually, the latter must be understood as $\mathcal{L}(\mathcal{H})$-valued oscillatory integrals [28].

Formulas (2.7) and (2.8) can be easily derived from (2.1) using

$$V_z M_f V_{-z} = M_{\alpha_z(f)}$$

and the translation $z \to z - \frac{1}{2}\Theta y$ which leaves invariant the phase due to the skewsymmetry of the deformation matrix. Note that they can be used to define (left and right) 'Moyal multiplications' of any bounded operator on $\mathcal{H}$, taking the place of M_f in the formulas. Within this presentation, it is straightforward to check that L and R are two commuting representations (in fact R is an anti-representation):

$$[L_f, R_h] = 0, \ \ \forall f, h \in C_c^\infty(M).$$

Thus formulas (2.7) and (2.8) provide an other way to check the associativity of the twisted product, which is equivalent to the commutativity of the left and right regular representations.

Using the trace property (2.2), one can also prove that the adjoint of the left (resp. right) twisted multiplication by f equals the left (resp. right) twisted multiplication by the complex conjugate of f:

$$(L_f)^* = L_{f^*}, \ \ (R_f)^* = R_{f^*}.$$

Again, this fact can be directly checked using formulas (2.7) and (2.8). For L_f it reads

$$\begin{aligned}(L_f)^* &= (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{i<y,z>}\, V_z\, M_{f^*}\, V_{-\frac{1}{2}\Theta y}\\ &= (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, V_{\frac{1}{2}\Theta z}\, M_{f^*}\, V_{-y},\end{aligned}$$

where the changes of variable $z \to \frac{1}{2}\Theta z, y \to 2\Theta^{-1}y$ and the relation $< \Theta^{-1}y, \Theta z >$ $= - < y, z >$ have been used.

The primary example of such a space is the n-dimensional Moyal plane $\mathbb{R}^n_\Theta$. In this case, the manifold is the flat Euclidean space $\mathbb{R}^n, l = n$, and $\mathbb{R}^n$ acts on itself by translation. Another interesting non-compact space which carries a smooth action of $\mathbb{R}^{n-1}$ by isometry is the n-dimensional hyperbolic space $\mathbb{H}^n$, that we can make into noncommutative $\mathbb{H}^n_\Theta$ by the previous prescription.

For periodic actions, there is a lattice $L = \beta\mathbb{Z}^l, \beta \in M_l(\mathbb{Z})$ in the kernel of α which factors through a torus $\mathbb{T}^l_\beta := \mathbb{R}^l/\beta\mathbb{Z}^l$. This quotient is a compact space if and only if the rank of β equals l. In this case, we have a spectral subspace (Peter-Weyl) decomposition (see [6,28,33] for details): for any bounded operator A which is α-norm smooth (the map $z \in \mathbb{T}^l_\beta \mapsto V_z A V_{-z}$ is smooth for the norm topology of $\mathcal{L}(\mathcal{H})$), one can define a l-grading by declaring A of l-degree $r = (r_1, \dots, r_l) \in \beta\mathbb{Z}^l$ when

$$V_z A V_{-z} = e^{-i(r_1 z_1 + \cdots + r_l z_l)} A, \ \forall z \in \mathbb{T}^l_\beta.$$

Then, any α-norm smooth operator can be uniquely written as a norm convergent sum

$$A = \sum_{r \in \beta\mathbb{Z}^l} A_r,$$

where each A_r is of l-degree $(r_1, \dots, r_l)$.
This is in particular the case for the operator of pointwise multiplication by any function $f \in C^\infty_c(M)$, since M_f lies inside the smooth domains of the derivations $\delta_j(.) := [X_j, .]$. This assertion is obtained iterating the relation

$$\|[X_j, M_f]\| = \|M_{X_j(f)}\| = \|X_j(f)\|_\infty,$$

which is finite since $f \in C^\infty_c(M)$ and because the X_j are ordinary smooth vector fields.
Writing the spectral subspace decomposition of such operator, we find the Peter-Weyl decomposition of any $f \in C^\infty_c(M)$, as $f = \sum_{r \in \beta\mathbb{Z}^l} f_r$, where f_r satisfies $\alpha_z(f_r) = e^{-i(r_1 z_1 + \cdots + r_l z_l)} f_r$. The twisted product of homogeneous components satisfies the noncommutative torus relation:

$$f_r \star_\Theta h_s = e^{-\frac{i}{2}<r, \Theta s>} f_r \, h_s. \tag{2.9}$$

Noncommutative tori $\mathbb{T}^n_\Theta$, odd and even Connes-Landi spheres $\mathbb{S}^{2n+1}_\theta, \mathbb{S}^{2n}_\theta$ [7] are examples of such compact noncommutative spaces; and the ambient space of $\mathbb{S}^{n-1}_\theta$ is a non-compact periodic deformation.

In summary, it is clear that the noncommutative structures of isospectral deformations are inherited from the NC tori or Moyal planes one's, depending whether the deformation is periodic or not.

When Θ is not invertible, the deformed product reduces to another twisted product associated with the restricted action $\sigma := \alpha|_{V^\perp}$, where V is the null space of Θ – see for example [28]. Hence, one can handle non-invertible deformation matrices without any trouble. But of course, the "effective" deformation is always of even rank.

Finally, in the non-periodic case only, properness of α implies that it is also free. To see that, recall that properness of any G-action is equivalent to $\{g \in G | g.X \cap Y \neq \emptyset\}$ is compact for any X, Y compact subset of M – see [25]. So, taking $X = Y = \{p_0\}$ for any $p_0 \in M$, its isotropy group $H_{p_0} = \{z \in \mathbb{R}^l | z.p_0 = p_0\} = \{z \in \mathbb{R}^l | z.\{p_0\} \cap \{p_0\} \neq \emptyset\}$ is compact as well. But the only compact subgroup of $\mathbb{R}^l$ is $\{0\}$, hence the action is automatically free. This implies that the quotient map $\pi : M \to M/\mathbb{R}^l$ defines a $\mathbb{R}^l$-principal bundle projection.

In the periodic case, the action is no longer automatically free, and the set M_{sing} of points with non-trivial isotropy groups can give rise to additional divergences in the effective action. This will be shown to constitute a new feature of the UV/IR mixing on isospectral deformation manifolds.

3 $\varphi^{\star_\Theta 4}$ theory on 4-d isospectral deformations

3.1 The effective action at one-loop

For the sake of simplicity, we now restrict to the four-dimensional case; $n = \dim(M) = 4$. It will be clear, nevertheless, that our techniques apply to higher dimensions without essential modifications. We consider the classical functional action for a real scalar field φ:

$$S[\varphi] := \int_M \mu_g \Big[\tfrac{1}{2}(\nabla^\mu\varphi)\star_\Theta(\nabla_\mu\varphi) + \tfrac{1}{2}m^2\varphi\star_\Theta\varphi + \frac{\lambda}{4!}\varphi^{\star_\Theta 4}\Big]. \tag{3.1}$$

We could add a coupling with gravitation of the type $\xi R(\varphi\star_\Theta\varphi)$ (or even $\xi R\star_\Theta\varphi\star_\Theta\varphi$), where R is the scalar curvature and ξ a coupling constant, without change in our conclusions. Indeed, this term is not modified by the deformation: due to the α-invariance of the scalar curvature, we have $R\star_\Theta f = R.f$ for any $f \in C_c^\infty(M)$, thus

$$\int_M \mu_g\, R.(\varphi\star_\Theta\varphi) = \int_M \mu_g\, R\star_\Theta\varphi\star_\Theta\varphi = \int_M \mu_g\, (R\star_\Theta\varphi).\varphi = \int_M \mu_g\, R\,.\varphi\,.\varphi.$$

Similarly, thanks to the trace property (2.2), $S[\varphi]$ can be rewritten as

$$S[\varphi] = \int_M \mu_g \Big[\tfrac{1}{2}\varphi\Delta\varphi \ + \tfrac{1}{2}m^2\varphi\,\varphi + \frac{\lambda}{4!}(\varphi\star_\Theta\varphi)\,(\varphi\star_\Theta\varphi)\Big], \tag{3.2}$$

so that, as in the falt cases, the kinetic part is not affected by the deformation. Recall that in our conventions the Laplacian is positive: $\Delta = -\nabla^\mu\nabla_\mu$.

We aim to compute the divergent part of the effective action $\Gamma_{1l}[\varphi]$ associated to $S[\varphi]$ at one loop. This is formally given by $\frac{1}{2}\ln(\det H)$, where H is the effective potential. In our case (as in the commutative one) it will be seen that $H = \Delta + m^2 +$

B, where B is positive and bounded; so that when the manifold is not compact H has a non empty essential spectrum (typically the whole interval $[m^2,+\infty[$). In order to deal with operators having pure-point spectrum (discret with finite multiplicity), we need first (independently of any regularization scheme) to redefine formally the one-loop effective action as:

$$\Gamma_{1l}[\varphi] := \tfrac{1}{2}\ln\det\big(HH_0^{-1}\big),$$

where $H_0^{-1} := (\Delta+m^2)^{-1}$ is the free propagator. We are "not so far" from having a well-defined determinant since:

$$HH_0^{-1} = (H_0+B)H_0^{-1} = 1 + BH_0^{-1},$$

and BH_0^{-1} is 'small': not trace-class in general, but compact; more precisely BH_0^{-1} lies inside the p-th Schatten-class for all $p>2$ (see below for the concrete expression of B and [16] for a proof of this claim). Physically, to replace H by HH_0^{-1} corresponds to remove the vacuum-to-vacuum amplitudes. We then define the logarithm of the determinant by the Schwinger "proper time" representation:

$$\Gamma_{1l}[\varphi] = \frac{1}{2}\ln\big(\det(HH_0^{-1})\big) := -\frac{1}{2}\int_0^\infty \frac{dt}{t}\ \mathrm{Tr}\left(e^{-tH}-e^{-tH_0}\right). \tag{3.3}$$

Before giving a precise meaning to the previous expression, that is to choose a regularization scheme, we go through the computation of the effective potential H. For that, the following definition will be useful.

Definition 3.1. Let $(X,d\mu)$ a measure space. A kernel operator on $\mathcal{E}$, a functions space on X, is a linear map $A:\mathcal{E}\to\mathcal{E}$ which can be written as

$$\big(Af\big)(p) = \int_X d\mu(q)\, K_A(p,q)\, f(q), \qquad f\in\mathcal{E},\ p,q\in X,$$

where K_A is the kernel of A. This definition leads to the following rules for the product of two kernel operators and for the kernel of the adjoint:

$$K_{AB}(p,q) = \int_X d\mu(u)\, K_A(p,u)\, K_B(u,q), \quad \text{and} \quad K_{A^*}(p,q) = K_A(q,p)^*. \tag{3.4}$$

In our case, $(X,d\mu)\equiv(M,\mu_g)$ as a measure space, $\mathcal{E}\equiv C_c^\infty(M)$ and we will only be interested on distributional kernels, that is those K_A lying on $C_c^\infty(M\times M)'$, the space of distributions on $M\times M$.

Recall that the effective potential (see for example [36]) is the operator whose distributional kernel is given by the second functional derivative of the classical action:

$$K_H(p,p') := \frac{\delta^2 S[\varphi]}{\delta\varphi(p)\delta\varphi(p')}, \qquad K_{H_0}(p,p') := \left.\frac{\delta^2 S[\varphi]}{\delta\varphi(p)\delta\varphi(p')}\right|_{\lambda=0},$$

with functional derivatives defined as usual in the weak sense

$$\left\langle \frac{\delta S[\varphi]}{\delta\varphi}, \psi \right\rangle := \left. \frac{dS[\varphi + t\psi]}{dt} \right|_{t=0},$$

where the coupling is given by the integral with Riemannian volume form $\langle f, h\rangle = \int_M \mu_g\, f\, h$.

Using the trace property (2.2) we find out:

$$\left. \frac{dS[\varphi + t\psi]}{dt} \right|_{t=0} = \left\langle \Delta\varphi + m^2\varphi + \frac{\lambda}{3!}\varphi^{\star_\Theta 3}, \psi \right\rangle.$$

Hence,

$$\tilde{S}_p[\varphi] := \frac{\delta S[\varphi]}{\delta\varphi(p)} = \Delta\varphi(p) + m^2\varphi(p) + \frac{\lambda}{3!}\varphi^{\star_\Theta 3}(p).$$

The second functional derivative reads

$$\begin{aligned} \left\langle \frac{\delta^2 S[\varphi]}{\delta\varphi(p)\delta\varphi}, \psi \right\rangle &:= \left. \frac{d\tilde{S}_p[\varphi + t\psi]}{dt} \right|_{t=0} \\ &= \left\langle \big(\Delta + m^2 + \frac{\lambda}{3!}(L_{\varphi\star_\Theta\varphi} + R_{\varphi\star_\Theta\varphi} + R_\varphi L_\varphi)\big)\delta^g_p, \psi \right\rangle, \end{aligned}$$

where δ^g_p is the distribution defined by $\langle \delta^g_q, \phi\rangle = \int_M \mu_g(p)\delta^g_q(p)\phi(p) = \phi(q)$, for any test function $\phi \in C^\infty_c(M)$.
In conclusion, the explicit form of the operator H is:

$$H = \Delta + m^2 + \frac{\lambda}{3!}(L_{\varphi\star_\Theta\varphi} + R_{\varphi\star_\Theta\varphi} + R_\varphi L_\varphi).$$

Because φ is real, the operators L_φ and R_φ are self-adjoint, and we can check directly the strict positivity of H:

$$L_{\varphi\star_\Theta\varphi} + R_{\varphi\star_\Theta\varphi} + L_\varphi R_\varphi = \tfrac{1}{2}(L_\varphi + R_\varphi)^*(L_\varphi + R_\varphi) + \tfrac{1}{2}L^*_\varphi L_\varphi + \tfrac{1}{2}R^*_\varphi R_\varphi.$$

We are come to an important point: the existence of UV/IR mixing for field theory on isospectral deformations comes from the simultaneous presence of left and right twisted multiplications in the effective potential. Precisely, we wish to illustrate the smearing nature of the product of left and right twisted multiplication operator $L_f R_h$. The crucial consequence, employed in subsection 3.3, is that the trace of $L_f\, R_h\, e^{-t(\Delta+m^2)}$ is regular when t goes to zero, contrary to $\mathrm{Tr}(L_f\, e^{-t(\Delta+m^2)}), \mathrm{Tr}(R_f\, e^{-t(\Delta+m^2)}), \mathrm{Tr}(M_f\, e^{-t(\Delta+m^2)})$, which in n dimensions behave as $t^{-n/2}$ when $t \to 0$ (In fact the three latter traces are identical).

Remark 3.2. For a $\frac{\lambda}{3!}\varphi^{\star_\Theta 3}$ theory on a six dimensional manifold, the effective potential reads:

$$H = \Delta + m^2 + \frac{\lambda}{2!}(L_\varphi + R_\varphi).$$

Even in the lack of the 'mixed' term $R_\varphi L_\varphi$, those theories have a non-planar sector, but which will be present only at the level of the two-point function; the tadpole is not affected by the mixing.

Consider the non-degenerate ($n = 2N, \Theta$ invertible) Moyal plane case. The operator $L_f R_h$ turns out to be trace-class whenever $f, h \in \mathcal{S}(\mathbb{R}^{2N})$, say. This fact is known to the experts, but rarely mentioned – to the knowledge of the author, its first mention in writing is in [1]. We do a little disgression to see how it comes about. Recall [12] that there is an orthonormal basis for $L^2(\mathbb{R}^{2N}, d^{2N}x)$, the harmonic oscillator eigentransitions $(2\pi\theta)^{-N/2}\{f_{mn}\}_{m,n\in\mathbb{N}^N}, \theta := (\det\Theta)^{1/2N}$ which are matrix units for the Moyal product:

$$f_{mn}\star_\Theta f_{kl} = \delta_{nk} f_{ml}.$$

Expanding $f, h \in \mathcal{S}(\mathbb{R}^{2N})$ in this basis: $f = \sum_{m,n} c_{mn} f_{mn}, h = \sum_{m,n} d_{mn} f_{mn}$, we obtain:

$$\begin{aligned}
\operatorname{Tr}\left(L_f R_h\right) &= (2\pi\theta)^{-N} \sum_{m,n,k,l,s,t} c_{kl}\, d_{st}\, \langle f_{mn}, f_{kl}\star_\Theta f_{mn}\star_\Theta f_{st}\rangle \\
&= (2\pi\theta)^{-N} \sum_{m,n,k,t} c_{km}\, d_{nt}\, \langle f_{mn}, f_{kt}\rangle \\
&= \sum_{m,n} c_{mm}\, d_{nn} \\
&= (2\pi\theta)^{-N} \int d^{2N}x\, f(x) \int d^{2N}y\, h(y) < \infty.
\end{aligned}$$

Then in this case one can factorize HH_0^{-1} and extract a finite part in the effective action. We have

$$\begin{aligned}
H_0 H^{-1} = &\Big(1 - \frac{\lambda}{3!}(L_{\varphi\star_\theta\varphi} + R_{\varphi\star_\theta\varphi}) \frac{1}{\Delta + m^2 + \frac{\lambda}{3!}(L_{\varphi\star_\theta\varphi} + R_{\varphi\star_\theta\varphi})}\Big) \\
&\times \Big(1 - \frac{\lambda}{3!} L_\varphi R_\varphi \frac{1}{\Delta + m^2 + \frac{\lambda}{3!}(L_{\varphi\star_\theta\varphi} + R_{\varphi\star_\theta\varphi} + L_\varphi R_\varphi)}\Big). \qquad (3.5)
\end{aligned}$$

Now,

$$1 - \frac{\lambda}{3!} L_\varphi R_\varphi \frac{1}{\Delta + M^2 + \frac{\lambda}{3!}(L_{\varphi\star_\theta\varphi} + R_{\varphi\star_\theta\varphi} + L_\varphi R_\varphi)} \in 1 + \mathcal{L}^1(\mathcal{H}),$$

so that its determinant is well defined. Thus only the determinant of the first piece of (3.5) needs to be regularized. The determinant of the second piece of (3.5) contains the whole non-planar contribution to the two-point function, while for the four-point function the finite non-planar part lies in both pieces.

The structure of the effective potential, i.e., the presence of mixed products of left and right twisted multiplication operators, and thus the existence of two distinct sectors in the theory is fairly general: for noncommutative scalar field theories whose classical field counterparts are regarded as elements of a noncommutative algebra, and a classical action built from a trace on the algebra, the effective potential will contain in general sums and mixed products of left and right regular representation operators.

Let us go back to the computation of $\Gamma_{1l}[\varphi]$. The t-integral in (3.3) is divergent because of the small-t behavior of the heat kernel on the diagonal. We thus define a one-loop regularized effective action by:

$$\Gamma^{\epsilon}_{1l}[\varphi] := -\frac{1}{2}\int_{\epsilon}^{\infty} \frac{dt}{t} \ \mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right). \tag{3.6}$$

One can invoke less rough regularization schemes, for example a ζ-function regularization

$$\Gamma^{\sigma;\mu}_{1l}[\varphi] := -\frac{1}{2}\int_{0}^{\infty} \frac{dt}{t} \left(t\mu^2\right)^{\sigma} \ \mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right), \tag{3.7}$$

akin to dimensional regularization. However, for the purposes of this article (3.6) will do. One can think of ϵ as of the inverse square of Λ, with Λ a momentum space cutoff.

To show that the expressions (3.6) and (3.7) are now well defined, we have to prove that $e^{-tH} - e^{-tH_0}$ is trace-class for all $t > 0$. Note that for $t \to \infty$ convergence is ensured by the global e^{-tm^2} factor, and that when the spectrum of the Laplacian has a strictly positive lower bound one can construct massless, IR divergence-free NCQFT. That is the case for the twisted hyperbolic planes $\mathbb{H}^n_\Theta$ since the L^2-spectrum of Δ on $\mathbb{H}^n$ is the whole half line $[n^2/4, \infty[$.

Lemma 3.3. *The semigroup difference $e^{-tH} - e^{-tH_0}$ is trace-class for all $t > 0$.*

Proof. Using positivity of H and H_0, the semigroup property and the holomorphic functional calculus with a path γ surrounding both the spectrum $\mathrm{sp}(H) \subset \mathbb{R}^+$ and $\mathrm{sp}(H_0) \subset \mathbb{R}^+$, we have

$$\begin{aligned} e^{-tH} - e^{-tH_0} = {} & \frac{1}{(2i\pi)^2} \\ & \int_{\gamma\times\gamma} dz_1\, dz_2\, e^{-t(z_1+z_2)/2} \ \left(R_H(z_1)R_H(z_2) - R_{H_0}(z_1)R_{H_0}(z_2)\right), \end{aligned}$$

where $R_A(z) = (z - A)^{-1}$ denotes the resolvent of A. But $H = H_0 + B$ where B is bounded. Using next $R_H(z) = R_{H_0}(z)(1 + BR_H(z))$, we find

$$\begin{aligned} R_H(z_1)R_H(z_2) - R_{H_0}(z_1)R_{H_0}(z_2) = {} & R_{H_0}(z_1)R_{H_0}(z_2)BR_H(z_2) \\ & + R_{H_0}(z_1)BR_H(z_1)R_{H_0}(z_2) \\ & + R_{H_0}(z_1)BR_H(z_1)R_{H_0}(z_2)BR_H(z_2). \end{aligned}$$

The first resolvent equation and the fact that $L_f(z-\Delta)^{-k}, R_f(z-\Delta)^{-k} \in \mathcal{L}^p(\mathcal{H})$, for $p > 2/k, f \in C^\infty_c(M)$ [16], together with the Hölder inequality for Schatten classes, yield

$$\int_{\gamma\times\gamma} dz_1\, dz_2\, e^{-t(z_1+z_2)/2} R_{H_0}(z_1) R_{H_0}(z_2) B R_H(z_2)$$

is absolutely convergent for the trace norm. Similarly for the other terms. So $e^{-tH} - e^{-tH_0}$ is trace-class as required. □

3.2 Field expansion

We now tackle the ϵ-behavior of $\Gamma^{\epsilon}_{1l}[\varphi]$ to describe the divergences. We will then show that, as for the Moyal planes and noncommutative tori, there exist for general isospectral deformations two kind of contributions to the Green functions, the planar one giving rise to ordinary singularities and the non-planar one exhibiting the UV/IR mixing phenomenon. Note that, since we are in a curved background, we can no longer work with Feynman diagrams in momentum space. However, by abuse of language we continue to speak about planar and non-planar contributions, because there is a splitting at the operator level which coincides with the splitting of planar and non-planar Feynman graphs in the known flat cases. This point will become clearer in subsequent subsections.

As we are only interested in the ϵ-behavior of $\Gamma^{\epsilon}_{1l}[\varphi]$ (we only consider the potentially divergent part of the regularized effective action), we need a small t-expansion for $\mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right)$. This expansion will be managed in the same vein as the ones obtained in [13,35]. The Baker-Campbell-Hausdorff formula is written:

$$e^{-tH} = e^{-tB + \frac{t^2}{2}[\Delta,B] - \frac{t^3}{6}[\Delta,[\Delta,B]] - \frac{t^3}{12}[B,[\Delta,B]] + \cdots}\, e^{-tH_0}. \tag{3.8}$$

We now expand the first exponential up to factors which, after taking the trace, give terms of order less or equal to zero in t. Only a few terms will be important: We have first to take into account that (in n dimensions)

$$\begin{aligned} &\mathrm{Tr}(L_f \Delta^k e^{-t\Delta}) \simeq t^{-n/2-k},\ t \to 0,\\ &\mathrm{Tr}(R_f \Delta^k e^{-t\Delta}) \simeq t^{-n/2-k},\ t \to 0. \end{aligned} \tag{3.9}$$

Indeed, for the "left" case (the right one being similar) since, as proved in [16], one has $L_f(1+\Delta)^{-k} \in \mathcal{L}^p(\mathcal{H})$ for any $p > n/2k$ and any $f \in C_c^\infty(M)$, we conclude for all $\epsilon > 0$:

$$\begin{aligned} \|L_f \Delta^k e^{-t\Delta}\|_1 &\le \|L_f(1+\Delta)^{-n/2-\epsilon}\|_1 \, \|\frac{\Delta^k}{(1+\Delta)^k}\| \|(1+\Delta)^{n/2+k+\epsilon} e^{-t\Delta}\| \\ &\le C(\epsilon) t^{-(n/2+k+\epsilon)}. \end{aligned}$$

The last estimate follows from functional calculus. Therefore, in the field expansion we need to correct the power in t by the order of the differential operator appearing when we expand the first exponential in the equation (3.8).

Secondly, we have to notice that the commutators $[\Delta, L_f]$, $[\Delta, R_f]$ (and also $[\Delta, R_f L_h]$) reduce by one the order of the differential operator (cf. equation (3.10) below). To see this, we compute the commutators $[\Delta, L_f]$, $[\Delta, R_f]$ and $[\Delta, R_f L_h]$. The simplest way is to use the formulas (2.7) and (2.8). By $[V_z, \Delta] = 0$ for all $z \in \mathbb{R}^l$ (from the isometry property of α) and choosing a local coordinate system $\{x^\mu\}$, one obtains

$$\begin{aligned}
[\Delta, L_f] &= (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, V_{\frac{1}{2}\Theta y}\, [\Delta, M_f]\, V_{-\frac{1}{2}\Theta y - z} \\
&= (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, V_{\frac{1}{2}\Theta y}\, \left(M_{\Delta f} - 2M_{\nabla^\mu f}\nabla_\mu\right)\, V_{-\frac{1}{2}\Theta y - z} \\
&= L_{\Delta f} - 2L_{\nabla^\mu f}\nabla_\mu, \qquad (3.10)
\end{aligned}$$

and similarly,

$$\begin{aligned}
[\Delta, R_f] &= R_{\Delta f} - 2R_{\nabla^\mu f}\nabla_\mu, \qquad (3.11)\\
[\Delta, R_f L_h] &= R_f[\Delta, L_h] + [\Delta, R_f]L_h \\
&= R_f L_{\Delta(h)} + R_{\Delta(f)}L_h - 2R_{\nabla^\mu f}L_{\nabla_\mu h} - 2\big(R_f L_{\nabla^\mu h} + R_{\nabla^\mu f}L_h\big)\nabla_\mu. \qquad (3.12)
\end{aligned}$$

The local coordinate system used must be compatible with the deformation, that is, defined on some α-invariant open neighborhood $U \subset M$. To obtain one such, choose any open covering $\{U_I\}_{i\in I}$ of M and define $\{\widetilde{U}_I\}_{i\in I}$ by letting $\mathbb{R}^l$ act on it: $\widetilde{U}_i := \mathbb{R}^l.U_i$.

This implies that in n dimensions, one only needs to use the BCH formula up to order $n-2$ to capture the divergent structure of the effective action.

Moreover, that the commutators decrease the degree of the differential operator is a necessary condition to make the BCH expansion meaningfull: In [15], we consider a field theory on a noncommutative 4-plane with an (associative) position-dependant Moyal product (coming from a rank-2 Poisson structure on $\mathbb{R}^4$). It turns out that the commutators $[\Delta, L_f]$ and $[\Delta, R_f]$ contain now a term with an order two differential operator. This makes the BCH development useless since the k-times iterated commutator $[t\Delta, [\cdots, [t\Delta, tL_f]\cdots]]$ contains a term which gives after the exponential expansion a contribution of order $t^{-n/2+1}$, independently of k, the number of commutators involved. Thus, in this case the whole BCH serie will be needed to capture the divergences.

Putting all together, we finally obtain:

$$e^{-tH} = \left(1 - tB + \frac{t^2}{2}[\Delta, B] - \frac{t^3}{6}[\Delta, [\Delta, B]] + \frac{t^2}{2}B^2\right) e^{-tH_0} \; + O(t);$$

we mean by this estimate that we have a small-t expansion:

$$\mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right) = \\ \mathrm{Tr}\left(\left(-tB + \frac{t^2}{2}[\Delta, B] - \frac{t^3}{6}[\Delta,[\Delta,B]] + \frac{t^2}{2}B^2\right)e^{-tH_0}\right) + O(t). \quad (3.13)$$

We now show that in fact, the commutators in the expression (3.13) give no contribution to the effective action. Indeed, if each terms $C\Delta e^{-t\Delta}$ and $\Delta C e^{-t\Delta}$ are trace-class, with $C = B$ or $C = [\Delta, B]$, then by the cyclicity of the trace and the fact that the Laplacian commutes with the heat semigroup, one gets

$$\mathrm{Tr}\left(\Delta\, C\, e^{-t\Delta} - C\,\Delta\, e^{-t\Delta}\right) = \mathrm{Tr}\left(C\,\Delta\, e^{-t\Delta} - C\,\Delta\, e^{-t\Delta}\right) = 0 \quad (3.14)$$

That $C\Delta e^{-t\Delta}$ is trace-class is obvious from functional calculus and using the same arguments than those used to obtain the estimate (3.9). For $\Delta C e^{-t\Delta}$, it is a little bit less immediate since the latter appears as a product of a trace-class operator ($Ce^{-t\Delta}$) times an unbounded one (Δ). Actually, using the tautological relation

$$\Delta\, C\, e^{-t\Delta} = C\,\Delta\, e^{-t\Delta} + [\Delta, C]\, e^{-t\Delta},$$

and the equations (3.10) and (3.11) (iterated once more when $C = [\Delta, B]$), one sees that this term appears also as a sum of trace-class operators. Hence (3.14) is proved and we are left with

$$\begin{aligned}\mathrm{Tr}\left(e^{-tH} - e^{-tH_0}\right) = & -t\frac{\lambda}{3!}\,\mathrm{Tr}\left(\left(L_{\varphi\star_\Theta\varphi} + R_{\varphi\star_\Theta\varphi} + R_\varphi L_\varphi\right)e^{-t(\Delta+m^2)}\right) \\ & + \frac{t^2}{2}\frac{\lambda^2}{(3!)^2}\,\mathrm{Tr}\left(\left(L_{\varphi^{\star_\Theta 4}} + R_{\varphi^{\star_\Theta 4}} + 3R_{\varphi\star_\Theta\varphi}L_{\varphi\star_\Theta\varphi}\right.\right. \\ & \left.\left. + 2R_\varphi L_{\varphi^{\star_\Theta 3}} + 2R_{\varphi^{\star_\Theta 3}}L_\varphi\right)e^{-t(\Delta+m^2)}\right) + O(t).\end{aligned}$$

3.3 Planar and non-planar contributions

We split the previous expansion in two parts. In the first one, we only keep terms like $L_f e^{-t\Delta}$ and $R_f e^{-t\Delta}$. Those belong to the "planar part", since they give commutative-like contributions as easily seen from equation (3.15) below. The second contribution, corresponding to the "non-planar part", consists of crossed terms like $L_f R_h e^{-t\Delta}$.

The planar contribution to the effective action is

$$\begin{aligned}\Gamma^\epsilon_{1l,P}[\varphi] := \frac{1}{2}\int_\epsilon^\infty dt\, e^{-tm^2}\Big\{ & \frac{\lambda}{3!}\,\mathrm{Tr}\left(\left(L_{\varphi\star_\Theta\varphi} + R_{\varphi\star_\Theta\varphi}\right)e^{-t\Delta}\right) \\ & - \frac{t}{2}\frac{\lambda^2}{(3!)^2}\,\mathrm{Tr}\left(\left(L_{\varphi^{\star_\Theta 4}} + R_{\varphi^{\star_\Theta 4}}\right)e^{-t\Delta}\right)\Big\} + O(\epsilon^0).\end{aligned}$$

To compute those traces, let us show that first the trace is a dequantizer for the deformed product

$$\mathrm{Tr}\left(L_f\, e^{-t\Delta}\right) = \mathrm{Tr}\left(R_f\, e^{-t\Delta}\right) = \mathrm{Tr}\left(M_f\, e^{-t\Delta}\right), \tag{3.15}$$

whenever $M_f\, e^{-t\Delta}$ is trace-class. Here M_f still denotes the operator of pointwise multiplication by f. We only treat the L_f case, since for the R_f case the arguments are similar. From the definition 2.1 and the product rule (3.4) for kernel operators, a little calculation gives the following expression for the Schwartz kernel of $L_f\, e^{-t\Delta}$:

$$K_{L_f\, e^{-t\Delta}}(p,p') = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, f(-\tfrac{1}{2}\Theta y.p)\, K_t(z.p,p').$$

Then

$$\begin{aligned}\mathrm{Tr}\left(L_f\, e^{-t\Delta}\right) &= \int_M \mu_g(p)\, K_{L_f\, e^{-t\Delta}}(p,p)\\ &= (2\pi)^{-l} \int_M \mu_g(p) \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, f(-\tfrac{1}{2}\Theta y.p)\, K_t(z.p,p).\end{aligned}$$

Using next the invariance of the volume form under the isometry $p \to \frac{1}{2}\Theta y.p$ and the fact that $[e^{-t\Delta}, V_z] = 0$, translated in terms of invariance of its kernel

$$K_t(z.p, z.p') = K_t(p,p'), \tag{3.16}$$

the claim follows after a plane waves integration:

$$\begin{aligned}\mathrm{Tr}\left(L_f\, e^{-t\Delta}\right) &= (2\pi)^{-l} \int_M \mu_g(p) \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, f(p)\, K_t(z.p,p)\\ &= \int_M \mu_g(p)\, f(p) \int_{\mathbb{R}^l} d^l z\, \delta(z)\, K_t(z.p,p)\\ &= \int_M \mu_g(p)\, f(p)\, K_t(p,p) = \mathrm{Tr}\left(M_f\, e^{-t\Delta}\right).\end{aligned}$$

Hence, the planar part of the one loop effective action reads:

$$\Gamma^{\epsilon}_{1l,P}[\varphi] = \int_{\epsilon}^{\infty} dt\, e^{-tm^2} \Big\{ \frac{\lambda}{3!}\, \mathrm{Tr}\left(M_{\varphi\star_\Theta\varphi} e^{-t\Delta}\right) - \frac{t}{2}\frac{\lambda^2}{(3!)^2}\, \mathrm{Tr}\left(M_{\varphi^{\star_\Theta 4}} e^{-t\Delta}\right)\Big\} + O(\epsilon^0).$$

Using the on-diagonal heat kernel expansion up to order one

$$K_t(x,x) = (4\pi t)^{-2}\big(1 - \frac{t}{6}R(x)\big) + O(t^0),$$

where R is the scalar curvature, together with the relation

$$K_{M_f e^{-t\Delta}}(x,x) = f(x) K_t(x,x),$$

one obtains at ϵ^0 order:

$$\Gamma^\epsilon_{1l,P}[\varphi] = \int_\epsilon^\infty \frac{dt\, e^{-tm^2}}{(4\pi t)^2} \int_M \mu_g \Big(\frac{\lambda}{3!} \varphi \star_\Theta \varphi - t\Big(\frac{1}{6}\frac{\lambda}{3!} (\varphi \star_\Theta \varphi) R + \frac{1}{2} \frac{\lambda^2}{(3!)^2} \varphi^{\star_\Theta 4} \Big)\Big). \tag{3.17}$$

The planar part thus yields ordinary $\frac{1}{\epsilon}$ and $|\ln \epsilon|$ divergences. They can be substracted adding local counter-terms to the original action.

The contribution for the non-planar part is

$$\Gamma^\epsilon_{1l,NP}[\varphi] := \frac{1}{2} \int_\epsilon^\infty dt\, e^{-tm^2} \Big\{ \frac{\lambda}{3!} \operatorname{Tr}\Big(R_\varphi L_\varphi\, e^{-t\Delta}\Big)$$
$$- \frac{t}{2} \frac{\lambda^2}{(3!)^2} \operatorname{Tr}\Big(\big(3R_{\varphi\star_\Theta\varphi} L_{\varphi\star_\Theta\varphi} + 2R_\varphi L_{\varphi^{\star_\Theta 3}} + 2R_{\varphi^{\star_\Theta 3}} L_\varphi \big) e^{-t\Delta} \Big) \Big\} + O(\epsilon^0).$$

We now simplify this expression. By the definition of the twisted product (2.1) and using the identity $\psi(z.p) = \int_M \mu_g(p')\, \delta^g_{z.p}(p')\, \psi(p')$, one can easily derive the Schwartz kernel of the left and right twisted multiplication operators:

$$K_{L_f}(p,p') = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>} f(-\tfrac{1}{2}\Theta y.p)\, \delta^g_{z.p}(p'),$$

and

$$K_{R_f}(p,p') = (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>} f(z.p)\, \delta^g_{-\frac{1}{2}\Theta y.p}(p').$$

By the kernel composition rule (3.4), we obtain after few changes of variables and a plane waves integration, the kernel of $L_f R_h e^{-t\Delta}$ in term of the heat kernel K_t:

$$K_{L_f R_h e^{-t\Delta}}(p,p') =$$
$$(2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, f((-\tfrac{1}{2}\Theta y - z).p)\, h(z.p)\, K_t(-\tfrac{1}{2}\Theta y.p, p').$$

Hence, the trace of $L_f R_h e^{-t\Delta}$ reads (with a few changes of variable):

$$\operatorname{Tr}\big(L_f R_h e^{-t\Delta}\big) = (2\pi)^{-l} \int_M \mu_g(p) \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, f(p)\, h(z.p)\, K_t(-\Theta y.p, p)$$
$$= \operatorname{Tr}\big(R_f L_h e^{-t\Delta}\big). \tag{3.18}$$

To obtain the last equality, we used the fact that K_t is symmetric, its invariance under α and the isometry $p \mapsto -z.p$. Invoking formula (3.18), we obtain for $\Gamma^\epsilon_{1l,NP}[\varphi]$:

$$\Gamma^\epsilon_{1l,NP}[\varphi] = (2\pi)^{-l} \frac{1}{2} \int_\epsilon^\infty dt\, e^{-tm^2} \int_M \mu_g(p) \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>} \Big\{ \frac{\lambda}{3!} \varphi(p)\varphi(z.p)$$
$$- \frac{t}{2} \frac{\lambda^2}{(3!)^2} \Big(3\varphi\star_\Theta\varphi(p)\varphi\star_\Theta\varphi(z.p) + 4\varphi(p)\varphi^{\star_\Theta 3}(z.p) \Big) \Big\} K_t(-\Theta y.p, p) + O(\epsilon^0)\ .$$

We shall see that the better ϵ-behavior of the non-planar part and the UR/IV entanglement phenomenon come from the presence of the off-diagonal heat kernel in the previous expression. Depending on the precise geometric setup, the non-planar contributions could still be divergent. In the unfavorable cases, the divergences are non-local as shown is the next subsections. This makes the renormalization problematic.

4 Non-periodic deformations

4.1 NCQFT on the Moyal plane in configuration space

When $M = \mathbb{R}^4$ with the flat metric, $l = 4$ and $\mathbb{R}^4$ acting on itself by translation, isospectral deformation gives $\mathbb{R}^4_\Theta$. In this case, the heat kernel is exactly given by

$$K_t(x,y) = (4\pi t)^{-2} e^{-\frac{|x-y|^2}{4t}},$$

so we can explicitly compute $\Gamma^\epsilon_{1l,P}(\varphi)$ and $\Gamma^\epsilon_{1l,NP}(\varphi)$. For the planar part, we obtain from (3.17)

$$\Gamma^\epsilon_{1l,P}[\varphi] = \int_\epsilon^\infty dt \frac{e^{-tm^2}}{(4\pi t)^2} \int_{\mathbb{R}^4} d^4x \Big(\frac{\lambda}{3!}\varphi^2(x) - \frac{t}{2}\frac{\lambda^2}{(3!)^2}(\varphi\star_\Theta\varphi)^2(x)\Big) + O(\epsilon^0),$$

that will give the ordinary ϵ^{-1} and $|\ln\epsilon|$ divergences for the respectively planar two- and four-point functions.

The non-planar part is given by:

$$\Gamma^\epsilon_{1l,NP}[\varphi] = (2\pi)^{-4} \int_\epsilon^\infty dt\, \frac{e^{-tm^2}}{(4\pi t)^2} \int_{\mathbb{R}^{12}} d^4x\, d^4y\, d^4z\, e^{-i<y,z>}\, e^{-\frac{|\Theta y|^2}{4t}}$$
$$\times\Big(\frac{1}{2}\frac{\lambda}{3!}\varphi(x)\varphi(x+z) - \frac{\lambda^2}{(3!)^2}\frac{t}{4}\big(3\varphi\star_\Theta\varphi(x)\varphi\star_\Theta\varphi(x+z) + 4\varphi(x)\varphi^{\star_\Theta 3}(x+z)\big)\Big) + O(\epsilon^0).$$

The Gaussian y-integration can be performed to obtain:

$$\Gamma^\epsilon_{1l,NP}[\varphi] = (2\pi\theta)^{-4} \int_\epsilon^\infty dt\, e^{-tm^2} \int_{\mathbb{R}^8} d^4x\, d^4z\, e^{-t|\Theta^{-1}(z-x)|^2}$$
$$\times\Big(\frac{1}{2}\frac{\lambda}{3!}\varphi(x)\varphi(z) - \frac{\lambda^2}{(3!)^2}\frac{t}{4}\big(3\varphi\star_\Theta\varphi(x)\varphi\star_\Theta\varphi(z) + 4\varphi(x)\varphi^{\star_\Theta 3}(z)\big)\Big) + O(\epsilon^0),$$

where $\theta := (\det\Theta)^{1/4}$. Finally, the t-integration gives

$$\Gamma^\epsilon_{1l,NP}[\varphi] = (2\pi\theta)^{-4} \int_{\mathbb{R}^8} d^4x\, d^4z\, \frac{e^{-\epsilon(m^2+|\Theta^{-1}(z-x)|^2)}}{m^2+|\Theta^{-1}(z-x)|^2}$$
$$\times\Big(\frac{\lambda}{2.3!}\varphi(x)\varphi(z) - \frac{\lambda^2}{(3!)^2}\frac{3\varphi\star_\Theta\varphi(x)\varphi\star_\Theta\varphi(z) + 4\varphi(x)\varphi^{\star_\Theta 3}(z)}{4(m^2+|\Theta^{-1}(z-x)|^2)}\Big) + O(\epsilon^0).$$

This expression is regular when ϵ goes to zero – we are now in the full noncommutative picture.

From the previous formula one reads off the associated (non-planar) two- and four-point functions in configuration space in the limit $\epsilon \to 0$:

$$G^2_{1l,NP}(x,y) = (\pi\theta)^{-4}\frac{\lambda}{96}\frac{1}{m^2+|\Theta^{-1}(x-y)|^2},$$

$$G^4_{1l,NP}(x,y,z,u) = -(\pi\theta)^{-8}\frac{\lambda^2}{24}\Big(\frac{3}{2}\delta(x-y+z-u)\int d^4v\frac{e^{2i<v,\Theta^{-1}(u-z)>}}{(m^2+|\Theta^{-1}(z-v-x)|^2)^2} + \frac{e^{2i<x-y,\Theta^{-1}(z-y)>}}{(m^2+|\Theta^{-1}(x-y+z-u)|^2)^2}\Big).$$

We see that the UV/IR mixing in configuration space manifests itself in the long-range behavior of the correlation functions. The slow decreasing at infinity of the two- and four-point functions is equivalent to a IR singularity in momentum space, as shown by a Fourier transform:

$$\widehat{G^2}_{1l,NP}(\xi,\eta) \propto \frac{m}{\theta|\xi|}K_1(m|\Theta\xi|)\delta(\xi+\eta).$$

Here $K_n(z)$ denotes the n-th modified Bessel function. We retrieve the known UV/IR mixing (se for example [29]):

$$\frac{m}{\theta|\xi|}K_1(m|\Theta\xi|) \sim (\theta|\xi|)^{-2},\ |\xi| \to 0.$$

This last result at one loop in the Moyal (translation-invariant) context is usually obtained by means of Feynman diagrams in momentum space – see for example [29]. We just checked that the Fourier transform for the two-point function coincides with the standard calculation's result.

However, this is not the end of the story. The behavior of the amplitudes as $\theta \downarrow 0$ presents interesting differences in configuration and momentum spaces. Assume that Θ has been put in the canonical form

$$\Theta = \begin{pmatrix} & \theta & & \\ -\theta & & & \\ & & & \theta' \\ & & -\theta' & \end{pmatrix},$$

and choose $\theta' = \theta$ for simplicity. In effect, developing the two-point expression in terms of θ, we find

$$\frac{1}{\theta^4 m^2+\theta^2|x|^2} = \frac{1}{\theta^2|x|^2}\Big(1-\frac{\theta^2m^2}{|x|^2}+\frac{\theta^4m^4}{|x|^4}-\cdots\Big).$$

First of all, we remark that the logarithmic dependence on θ of the UV/IR mixing in momentum space (in addition to its quadratic divergence) found in [29] is apparently absent here. Now, with the sole exception of the first term, the previous series is made of functions that are not tempered distributions, and so they have no Fourier transform. In other words, the passage to the "commutative limit" does not commute with taking Fourier transforms.

The question is subtler, though. We can ask ourselves to which kind of divergences the terms of the last development are associated to. The answer is that first term is infrared divergent in configuration space; the second one is both ultraviolet and infrared divergent, and the following are all ultraviolet divergent. It is perhaps surprising that there is a way to recover the exact result from that nearly nonsensical infinite series; this involves precisely the correction to the indicated UV divergences. Indeed we can "renormalize" (in the sense of Epstein and Glaser) the $1/|x|^{2k+4}$ functions, with the result that the redefined distributions $[1/|x|^{2k+4}]_R$ are tempered. Those $[.]_R$ distributions depend on a mass scale parameter. Their Fourier transforms $[1/\widehat{|x|^{2k+4}}]_R$ (making a long history short) have been calculated as well [18, 30], with the result

$$[1/\widehat{|x|^{2k+4}}]_R(\xi) = \frac{(-)^{k+1}|\xi|^{2k}}{4^{k+1}k!(k+1)!}\left[2\ln\frac{|\xi|}{2\mu} - \Psi(k+1) - \Psi(k+2)\right],$$

Now, a natural mass scale parameter in our context is $1/\theta m$. This is where $\ln\theta$ can sneak back in. Upon substituting this for μ in the previous formula, and summing the series of Fourier transforms, we recover on the nose the exact result:

$$\frac{1}{\theta^2|\xi|^2} + \frac{m^2}{2}\sum_{n=0}^{\infty}\frac{\theta^{2n}m^{2n}|\xi|^{2n}}{4^n\,n!(n+1)!}\left(\ln\frac{\theta m|\xi|}{2} - \Psi(n+1) - \Psi(n+2)\right) = \frac{m}{\theta|\xi|}K_1(\theta m|\xi|).$$

For the four-point function, again in the $\theta \downarrow 0$ limit no dependence on $\ln(\theta)$ is apparent in configuration space. The resulting expression is however (UV- and) IR-divergent, and its redefinition *à la* Epstein and Glaser allows one to reintroduce the $\ln\theta$.

The effect of the rank of Θ becomes clearer in position space. Indeed, for a generic n-dimensional Moyal plane with a deformation matrix of rank $l \leq n$, the two-point function in momentum space is always finite and behaves as $|\Theta\xi|^{-n+2}$, when $\xi \to 0$. However, since $\Theta\xi \in Im(\Theta) = \mathbb{R}^l$, the IR singularity is not locally integrable if $l \leq n-2$. It follows that the two-point Green function does not have a Fourier transform since it is not a temperate distribution. Thus in the four-dimensional case, the non-planar contribution to the tadpole in position space remains infinite if $l = 2$! The four-point function has a Fourier transform, its IR singularity in momentum space being of the ln type, and the Green function in position space is finite whenever $l \neq 0$. For example, had we treated $\mathbb{R}^2_\theta \times \mathbb{R}^2$ instead of $\mathbb{R}^4_\Theta$, we would have found that the four-point part of $\Gamma^\epsilon_{1l,NP}[\varphi]$ is convergent, while the two-point part diverges as $\ln\epsilon$.
This point is discussed in details in [14], where we use the ζ-regularization scheme and the Duhamel asymptotic expansion (instead of the BCH one), in order to compare our results with those present in the literature.

These features of the UV/IR mixing phenomenon on position space reappear in the general non-periodic case, where the effective action will still be divergent for $l = 2$. This is shown in the next subsection.

4.2 The divergences of the general non-periodic case

Assume $\varphi \in C_c^\infty(M)$. We have also to make some more precise assumptions on the behavior of the geometry at infinity in order to control the heat kernel. In [2,9], it is proved that if M is non-compact, complete, with Ricci curvature bounded from below (plus either uniform boundness of the inverse of the volume or of the inverse of the isoperimetric constant of the Riemannian ball for some fixed radius), then the heat kernel satisfies

$$\begin{aligned}(4\pi t)^{-2} e^{-d_g^2(p,p')/4t} &\leq K_t(p,p') \\ &\leq C(4\pi t)^{-2} e^{-d_g^2(p,p')/4(1+c)t},\end{aligned} \tag{4.1}$$

where d_g is the Riemannian distance and C, c are strictly positive constants.

In the general periodic case, we have shown that $\Gamma^\epsilon_{1l,NP}[\varphi]$ is given by:

$$\begin{aligned}\Gamma^\epsilon_{1l,NP}[\varphi] = \frac{1}{2(2\pi)^l}\int_\epsilon^\infty dt\, e^{-tm^2}\int_M \mu_g(p)\int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, K_t(-\Theta y.p,p) \\ \times\Big\{\frac{\lambda}{3!}\varphi(p)\varphi(z.p) - \frac{t}{2}\frac{\lambda^2}{(3!)^2}\Big(3\varphi\star_\Theta\varphi(p)\varphi\star_\Theta\varphi(z.p) + 4\varphi(p)\varphi^{\star_\Theta 3}(z.p)\Big)\Big\} + O(\epsilon^0).\end{aligned}$$

We now show that this expression cannot produce more important divergences than the planar contribution. Again, the regularity of those integrals depends only on l (that we may call the effective noncommutative dimension), and on the metric through the Riemannian distance function.

Before estimating the two-point part of $\Gamma^\epsilon_{1l,NP}[\varphi]$, which is our main purpose in this section, we make the following remark: in our present setting, the two-point non-planar Green function reads

$$G^\epsilon_{1l,NP,2P}(p,p') = \frac{\lambda}{6(2\pi)^l}\int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\int_\epsilon^\infty dt\, e^{-tm^2}\, K_t(-\Theta y.p,p)\,\delta^g_{z.p}(p').$$

Now, one can qualitatively see in this distributional expression the UV/IR entanglement phenomenon: thanks to the estimate (4.1), we have

$$\begin{aligned}\int_0^\infty dt\, e^{-tm^2}\, K_t(\Theta y.p,p) &\leq C\int_0^\infty dt\,\frac{e^{-tm^2}}{(4\pi t)^2} e^{-d_g^2(\Theta y.p,p)/4(1+c)t} \\ &= \frac{C}{16\pi^2}\frac{4m\sqrt{1+c}}{d_g(\Theta y.p,p)}\, K_1\big(\frac{m\, d_g(\Theta y.p,p)}{\sqrt{1+c}}\big) \\ &\sim C'\, d_g^{-2}(\Theta y.p,p), \quad y \to 0,\end{aligned}$$

and the reverse inequality also holds

$$\int_0^\infty dt\, e^{-tm^2}\, K_t(\Theta y.p, p) \geq C''\, d_g^{-2}(\Theta y.p, p)$$

which points precisely to the UV/IR mixing, since $y \in \widehat{\mathbb{R}^l}$ has to be interpreted as a momentum.

For the two-point part of $\Gamma^\epsilon_{1l,NP}[\varphi]$ we have

$$\begin{aligned}
\left|\Gamma^\epsilon_{1l,NP,2P}[\varphi]\right| \leq{}& \frac{C\,\lambda}{12(2\pi)^l} \sup_{p\in M}\left\{\int_{\mathbb{R}^l} d^l z\, |\varphi(z.p)|\right\} \\
&\times \int_\epsilon^\infty dt\, \frac{e^{-tm^2}}{(4\pi t)^2} \int_M \mu_g(p)\ |\varphi(p)| \int_{\mathbb{R}^l} d^l y\, e^{-d_g^2(-\Theta y.p,p)/4(1+c)t} \\
\leq{}& \frac{C\,\lambda}{12(2\pi)^l} \sup_{p\in M}\left\{\int_{\mathbb{R}^l} d^l z\, |\varphi(z.p)|\right\} \|\varphi\|_1 \\
&\times \sup_{p\in \mathrm{supp}(\varphi)} \left\{\int_\epsilon^\infty dt\, \frac{e^{-tm^2}}{(4\pi t)^2} \int_{\mathbb{R}^l} d^l y\, e^{-d_g^2(-\Theta y.p,p)/4(1+c)t}\right\}.
\end{aligned}$$

By the properness of α, $\int_{\mathbb{R}^l} d^l z\, |\varphi(z.p)|$ is finite for all $p \in M$ since $\{z \in \mathbb{R}^l : z.p \in \mathrm{supp}(\varphi)\}$ is compact for each $p \in M$ because φ has compact support. Thus, $\tilde\varphi(p) := \int_{\mathbb{R}^l} d^l z\, |\varphi(z.p)|$ is constant and finite on each orbit of α, and if we denote $\pi : M \to M/\mathbb{R}^l$ the projection on the orbit space, then $\tilde\varphi$ factors through π to give a map $\bar\varphi$ defined by $\bar\varphi(\pi(p)) := \tilde\varphi(p)$. Finally, $\bar\varphi \in C_c^\infty(M/\mathbb{R}^l)$ because if $p \notin \mathbb{R}^l.\mathrm{supp}(\varphi)$, so that $\pi(p)$ is not in the compact set $\pi(\mathrm{supp}(\varphi))$, then $\bar\varphi(\pi(p)) = 0$. This proves that $\sup_{p\in M}\left\{\int_{\mathbb{R}^l} d^l z\, |\varphi(z.p)|\right\} < \infty$. Furthermore, since α acts isometrically the induced metric $\tilde g$ on the orbits (which are closed submanifolds since the action is proper [25]) is constant, so

$$d_g^2(y.p, p) = \sum_{i,j=1}^{l} \tilde g_{ij}(p) y^i y^j.$$

Here, $\tilde g_{ij}(p)$ (which depend only on the the orbit of p) are strictly positive continuous functions since in the non-periodic case the action is free, and then $\{(0,p) \in \mathbb{R}^l \times M\}$ is the only set for which $F(y,p) := d_g(y.p,p)$ vanish. Note that we can use a global coordinate system (on one orbit) given by a suitable basis of $\mathbb{R}^l$ in such a way that $\tilde g_{ij}(p)$ is diagonal. Thus, with $\theta := (\det\Theta)^{1/l}$, we have:

$$\int_{\mathbb{R}^l} d^l y\, e^{-d_g^2(-\Theta y.p,p)/4(1+c)t} = \left(\frac{4\pi(1+c)t}{\theta^2}\right)^{l/2} (\det \tilde g(p))^{-1/2}.$$

Hence, one obtains

$$\left|\Gamma^\epsilon_{1l,NP,2P}[\varphi]\right| \leq \frac{\lambda}{6} C(l, \tilde g, \varphi, \varphi)\, \theta^{-l} \int_\epsilon^\infty dt\, t^{l/2-2} e^{-tm^2},$$

where

$$C(l,\tilde g,\varphi_1,\varphi_2) :=$$

$$\frac{C(4\pi)^{l/2-2}(1+c)^{l/2}}{2(2\pi)^l}\|\varphi_1\|_1 \sup_{p\in M}\left\{\int_{\mathbb{R}^l} d^l z\,|\varphi_2(z.p)|\right\}\sup_{p\in\mathrm{supp}(\varphi_1)}\left\{(\det\tilde g(p))^{-1/2}\right\}.$$

Four the four-point part, similar estimates read:

$$\left|\Gamma^{\epsilon}_{1l,NP,4P}[\varphi]\right| \le \frac{\lambda^2}{72}\,\theta^{-l}\left(3C(l,\tilde g,\varphi\star_\Theta\varphi,\varphi\star_\Theta\varphi)+4C(l,\tilde g,\varphi,\varphi\star_\Theta\varphi\star_\Theta\varphi)\right)\int_\epsilon^\infty dt\,t^{l/2-1}e^{-tm^2}.$$

We then have proved the following:

Theorem 4.1. *When M is non-compact, satisfying all assumptions on the behavior of the geometry at infinity displayed above and endowed with a smooth proper isometric action of $\mathbb{R}^l$, then for $\varphi\in C_c^\infty(M)$ we have:*

i) $$\left|\Gamma^{\epsilon}_{1l,NP,2P}[\varphi]\right| \le \begin{cases} C_1(\varphi,\Theta) & \textit{for } l=4,\\ C_2(\varphi,\Theta)|\ln\epsilon| & \textit{for } l=2,\end{cases}$$

ii) $$\left|\Gamma^{\epsilon}_{1l,NP,4P}[\varphi]\right| \le C_3(\varphi,\Theta)\quad \textit{for } l=4 \textit{ or } l=2.$$

The possible remaining divergence for $l=2$ refers to the fact that the IR singularity might be not integrable, as illustrated previously. In this case, the two-point non-planar Green function does not define a distribution and the theory is not renormalizable by addition of local counter-terms, already in its one-loop approximation order.

5 Periodic deformations

Periodic deformations (when the kernel of α is an integer lattice) behave rather differently from non-periodic ones. In the following, we consider $\ker\alpha=\beta\mathbb{Z}^l$ with β a $l\times l$ integer matrix of rank l, so that $\mathbb{R}^l/\beta\mathbb{Z}^l =: \mathbb{T}^l_\beta$ is compact. For the sake of simplicity, we will often suppress the subscript β.
Momentum space (the dual group of $\mathbb{T}^l_\beta$) being discrete, IR problems only occur for some values of the momentum. In favorable cases one can extract the divergent field configurations in the non-planar part (which are often finite in number when $(2\pi)^{-1}\Theta$ has irrational entries) and renormalize them like the planar contributions; then there is no really UV/IR mixing. When $(2\pi)^{-1}\Theta$ has rational entries, the theory is equivalent to the undeformed one, in the sense that there are infinitely many divergent field configurations.

Although in all periodic cases we have a Peter-Weyl decomposition for fields, only in the compact manifold case shall we be able to describe the individual behavior of non-planar "Feynman graphs", defined through that isotypic decomposition. Both in the compact and in the non-compact case, by means of the off-diagonal heat kernel estimate (4.1), we show in the second subsection how, for periodic deformations, the arithmetical nature of the entries of Θ, more precisely, the existence or nonexistence of a Diophantine condition on Θ, plays a role in determining the analytical nature of $\Gamma^{\epsilon}_{1l,NP}[\varphi]$.

5.1 Periodic compact case and the individual behavior of non-planar graphs

Because everything is explicit, we look first at the flat compact case. Let $M = \mathbb{T}^4$ with the flat metric, let $\mathbb{R}^4$ act on it by rotation (so and $l = 4$ and we are in the 'fully noncommutative picture'). With the orthonormal basis $\left\{\frac{e^{i<k,x>}}{(2\pi)^2}\right\}_{k\in\mathbb{Z}^4}$ of $L^2(\mathbb{T}^4, d^4x)$ the heat kernel is written

$$K_t(x,y) = (2\pi)^{-4} \sum_{k\in\mathbb{Z}^4} e^{-t|k|^2} e^{i<k,x-y>},$$

and we have

$$e^{i<k,x>} \star_\Theta e^{i<q,x>} = e^{-\frac{i}{2}\Theta(k,q)}\, e^{i<k+q,x>},$$

with $\Theta(k,q) := \langle k, \Theta q\rangle$. Expanding the background field φ in Fourier modes $\varphi = \sum_{k\in\mathbb{Z}^4} c_k\, e^{i<k,x>}$, with $\{c_k\}_{k\in\mathbb{Z}^4} \in \mathcal{S}(\mathbb{Z}^4)$ whenever $\varphi \in C^\infty(\mathbb{T}^4)$, we obtain:

$$\Gamma^{\epsilon}_{NP}[\varphi] = \frac{1}{2}\sum_k \frac{e^{-\epsilon(m^2+|k|^2)}}{m^2+|k|^2}\Big\{\frac{\lambda}{3!}\sum_r c_r\, c_{-r}\, e^{i\Theta(k,r)} - \frac{\lambda^2}{2(3!)^2}\frac{1}{m^2+|k|^2}$$
$$\times \sum_{r,s,u} c_r\, c_s\, c_{u-s}\, c_{-r-u} e^{-\frac{i}{2}\Theta(r+s,u)}\Big(3\, e^{i\Theta(k,r+s)} + 4\, e^{i\Theta(k,r+u)}\Big)\Big\} + O(\epsilon^0).$$

We can now analyze the individual behavior of non-planar Feynman diagrams. One sees that, thanks to the phase factors, the sum over k is finite when ϵ goes to zero, whenever $(2\pi)^{-1}\Theta$ has irrational entries and $r \neq 0$ for the two-point part, or $r+s \neq 0$ and $r+u \neq 0$ for the four-point part. In effect, returning to the Schwinger parametrization (which exchanges large momentum divergences with small-t ones) and applying the Poisson summation formula with respect to the sum over k we get:

$$\sum_{k\in\mathbb{Z}^4} \frac{e^{i\Theta(k,r)}}{m^2+|k|^2} = \sum_{k\in\mathbb{Z}^4}\int_0^\infty dt\, \frac{e^{-tm^2}}{(4\pi t)^2}\, e^{-|2\pi k-\Theta r|^2/4t}.$$

Hence, the t-integral is finite whenever $r \neq 0$ and $\frac{\Theta r}{2\pi} \notin \mathbb{Q}^l$. Essentially the same conclusion holds for the four-point part.

We now go to the general periodic compact case. In order to be able to calculate, we make explicit use of the invariance of the heat kernel under α. Let us decompose $\mathcal{H} = L^2(M, \mu_g)$ in spectral subspaces with respect to the group action:

$$\mathcal{H} = \bigoplus_{k\in\mathbb{Z}^l} \mathcal{H}_k.$$

Each $\mathcal{H}_k$ is stable under V_z (recall that V_z denotes the induced action on $\mathcal{H}$) for all $z \in \mathbb{R}^l$; and furthermore all $\psi \in \mathcal{H}_k$ satisfy $V_z\psi = e^{-i<z,k>}\psi$. Note that if $\psi \in \mathcal{H}_k$ then $|\psi| \in \mathcal{H}_0$. Let P_k be the orthogonal projection on $\mathcal{H}_k$. Because the Laplacian commutes with V_z, the heat operator also commutes with P_k; hence $e^{-t\Delta}$ is block diagonalizable with respect to the decomposition $\mathcal{H} = \bigoplus_{k\in\mathbb{Z}^l} \mathcal{H}_k$:

$$e^{-t\Delta} = \sum_{k\in\mathbb{Z}^l} P_k\, e^{-t\Delta}\, P_k.$$

In each $\mathcal{H}_k$ the operator $0 \leq P_k\, e^{-t\Delta}\, P_k$ is trace-class, so it can be written as

$$P_k\, e^{-t\Delta}\, P_k = \sum_{n\in\mathbb{N}} e^{-t\lambda_{k,n}} |\psi_{k,n}\rangle\langle\psi_{k,n}|,$$

where $\{\psi_{k,n}\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_k$ consisting of eigenvectors of $P_k\Delta P_k$ with eigenvalue $\lambda_{k,n}$. The heat semigroup being Hilbert-Schmidt, its kernel can be written as a ($L^2(M \times M, \mu_g \times \mu_g)$-convergent) sum:

$$K_t(p,p') = \sum_{k\in\mathbb{Z}^l}\sum_{n\in\mathbb{N}} e^{-t\lambda_{k,n}} \psi_{k,n}(p)\overline{\psi_{k,n}}(p'). \tag{5.1}$$

Because each $\psi_{k,n}(p)$ lies in $\mathcal{H}_k$, the invariance property (3.16) $K_t(z.p, z.p') = K_t(p,p')$ is explicit.

Any $\varphi \in C^\infty(M)$ has a Fourier decomposition $\varphi = \sum_{r\in\mathbb{Z}^l} \varphi_r$, such that $\{\|\varphi_r\|_\infty\} \in \mathcal{S}(\mathbb{Z}^l)$ and $\alpha_z(\varphi_r) = e^{-i<z,r>}\varphi_r$. Furthermore, this decomposition provides a notion of Feynman diagrams, that is of amplitude associated with a fixed field configuration. The non-planar one-loop regularized effective action reads:

$$\begin{aligned}
\Gamma^\epsilon_{1l,NP}[\varphi] &= \frac{1}{2}\int_M \mu_g(p) \sum_{k\in\mathbb{Z}^l}\sum_{n\in\mathbb{N}} \frac{e^{-\epsilon(m^2+\lambda_{k,n})}}{m^2+\lambda_{k,n}}\, |\psi_{k,n}|^2(p) \Big\{ \frac{\lambda}{3!} \\
&\sum_{r,s\in\mathbb{Z}^l} \varphi_r(p)\,\varphi_s(p)\, e^{-i\Theta(k,s)} - \frac{\lambda^2}{2(3!)^2}\frac{1}{m^2+\lambda_{k,n}} \sum_{r,s,u,v\in\mathbb{Z}^l} \varphi_r(p)\,\varphi_s(p)\,\varphi_u(p)\,\varphi_v(p) \\
&\quad \times \Big(3\, e^{-\frac{i}{2}(\Theta(r,s)+\Theta(u,v))} e^{-i\Theta(k,u+v)} + 4\, e^{-\frac{i}{2}\Theta(r+s,u+s)} e^{-i\Theta(k,v)} \Big) \Big\} + O(\epsilon^0).
\end{aligned}$$

Although we do not know the explicit form of the $\psi_{k,n}$, we can by momentum conservation reduce the sums exactly as in the NC torus case, as shown in the following lemma.

Lemma 5.1 (Momentum conservation). *Let $\psi_i \in \mathcal{H}_{k_i} \cap L^q(M,\mu_g)$ for $i=1,\dots,q$. Then:*

$$\int_M \mu_g\, \psi_1 \dots \psi_q = C(\psi_1,\dots,\psi_q)\, \delta_{k_1+\cdots+k_q,0}.$$

Proof. By the α-invariance of μ_g and with the relation $\alpha_z(\psi_i) = e^{-i<z,k_i>}\psi_i$ we have

$$\int_M \mu_g\, \psi_1 \dots \psi_q = e^{i<z,k_1+\cdots+k_q>} \int_M \mu_g\, \psi_1 \dots \psi_q,$$

for all $z \in \mathbb{R}^l$; the result follows. □

Because $|\psi_{k,n}|^2(p)$ is constant on the orbits of α and $\varphi_r \in C^\infty(M) \subset L^q(M,\mu_g)$ for all $q \geq 1$, Lemma 5.1 gives

$$\Gamma^\epsilon_{1l,NP}[\varphi] = \frac{1}{2}\int_M \mu_g(p) \sum_{k\in\mathbb{Z}^l}\sum_{n\in\mathbb{N}} \frac{e^{-\epsilon(m^2+\lambda_{k,n})}}{m^2+\lambda_{k,n}}\, |\psi_{k,n}|^2(p) \Big\{ \frac{\lambda}{3!}\sum_{r\in\mathbb{Z}^l}$$
$$\varphi_r(p)\,\varphi_{-r}(p)\, e^{i<k,\Theta r>} - \frac{\lambda^2}{2(3!)^2}\frac{1}{m^2+\lambda_{k,n}} \sum_{r,s,u\in\mathbb{Z}^l} \varphi_r(p)\,\varphi_s(p)\,\varphi_{u-s}(p)\,\varphi_{-r-u}(p)$$
$$\times\, e^{-\frac{i}{2}\Theta(r+s,u)}\Big(3\, e^{i\Theta(k,r+s)} + 4\, e^{i\Theta(k,r+u)}\Big)\Big\} + O(\epsilon^0). \quad (5.2)$$

To analyze the divergences when $\epsilon \to 0$ for a fixed field configuration, note that if we re-index $\lambda_{k,n}$ in a standard way ($\lambda_0 \leq \cdots \leq \lambda_n \leq \cdots$), Weyl's estimate asserts that $\lambda_n \sim n^{1/2}$, hence

$$\sum_{k\in\mathbb{Z}^l}\sum_{n\in\mathbb{N}} \frac{|\psi_{k,n}|^2(p)}{(m^2+\lambda_{k,n})^N} = \sum_{n\in\mathbb{N}} \frac{|\psi_n(p)|^2}{(m^2+\lambda_n)^N} = K_{(m^2+\Delta)^{-N}}(p,p),$$

is finite if and only if $N > 2$. We see that the sum over n and k in (5.2) diverges in the limit $\epsilon \to 0$ for certain values of the momenta ($r=0$ for the two-point part, $r+s=0$ and $r+u=0$ for the four-point part) if $(2\pi)^{-1}\Theta$ has irrational entries. When the entries $(2\pi)^{-1}\Theta$ are rational, there are infinitely many divergent field configurations since $e^{-i<k,\Theta r>} = 1$ for infinitely many k whenever $\frac{\Theta r}{2\pi} \in \mathbb{Q}^l$. For other configurations, convergence is guaranteed by the estimate (4.1), as shown in the next subsection.

In summary, we have shown that the behavior of an individual field configuration in the non-planar sector for any periodic compact deformation reproduces the main features of the noncommutative torus.

In the next paragraph, the arithmetic nature of the entries of Θ gets into the act; also we show there that the possible existence of fixed points for the action may give rise to additional divergences.

5.2 General periodic case and the Diophantine condition

Assume now that α periodic, but M can be compact or not (within the hypothesis of section 4.2 when M is not compact). In this general setup, the Peter-Weyl decomposition still exists, but the heat operator, not being a priori compact, cannot be written as (5.1). Thus we return to the off-diagonal heat kernel estimate. In this case, using Lemma 5.1 and the α-invariance of K_t, we obtain:

$$\begin{aligned}\Gamma^{\epsilon}_{1l,NP}[\varphi] = \frac{1}{2}\int_\epsilon^\infty dt\, e^{-tm^2}\int_M \mu_g(p)\Big\{\frac{\lambda}{3!}\sum_{r\in\mathbb{Z}^l} K_t(\Theta r.p,p)\varphi_r(p)\,\varphi_{-r}(p)\\ -\frac{t\,\lambda^2}{2(3!)^2}\sum_{r,s,u\in\mathbb{Z}^l}\varphi_r(p)\,\varphi_s(p)\,\varphi_{u-s}(p)\,\varphi_{-r-u}(p)\; e^{-\frac{i}{2}\Theta(r+s,u)}\\ \times\Big(3\,K_t(\Theta(r+s).p,p)+4\,K_t(\Theta(r+u).p,p)\Big)\Big\}+O(\epsilon^0).\end{aligned}$$

We consider only the case $(2\pi)^{-1}\Theta$ has irrational entries, from now on. Then divergences appear when $r=0$ for the two-point function and $r+s=0, r+u=0$ for the four-point functions. This leads us to introduce a *reduced* non-planar one-loop effective action $\Gamma^{\epsilon,red}_{1l,NP}[\varphi]$ by subtracting the divergent field configurations; for renormalization purposes, they have to be treated together with the planar sector.

$$\begin{aligned}\Gamma^{\epsilon,red}_{1l,NP}[\varphi] := \frac{1}{2}\int_\epsilon^\infty dt\, e^{-tm^2}\int_M \mu_g(p)\Big\{\frac{\lambda}{3!}{\sum}' K_t(\Theta r.p,p)\varphi_r(p)\,\varphi_{-r}(p)\\ -\frac{t\,\lambda^2}{2(3!)^2}{\sum}'\,\varphi_r(p)\,\varphi_s(p)\,\varphi_{u-s}(p)\,\varphi_{-r-u}(p)e^{-\frac{i}{2}\Theta(r+s,u)}\\ \times\Big(3\,K_t(\Theta(r+s).p,p)+4\,K_t(\Theta(r+u).p,p)\Big)\Big\}.\end{aligned}$$

Here ${\sum}'$ is the notation for $\sum_{r\in\mathbb{Z}^l,\, r\neq 0}$ in the two-point part, $\sum_{r,s,u\in\mathbb{Z}^l,\, r+s\neq 0}$ and $\sum_{r,s,u\in\mathbb{Z}^l,\, r+u\neq 0}$ in respectively the first and second piece of the four-point part. Using now the estimate (4.1) and performing the t-integration, we obtain:

$$\begin{aligned}\lim_{\epsilon\to 0}\Big|\Gamma^{\epsilon,red}_{1l,NP}[\varphi]\Big| \le \frac{C}{32\pi^2}\int_M \mu_g(p)\Big\{\frac{\lambda}{3!}{\sum}'\,|\varphi_r(p)|\,|\varphi_{-r}(p)|\frac{4m\sqrt{1+c}}{d_g(\Theta r.p,p)}\\ K_1\big(\frac{m\,d_g(\Theta r.p,p)}{\sqrt{1+c}}\big)+\frac{\lambda^2}{2(3!)^2}{\sum}'\,|\varphi_r(p)|\,|\varphi_s(p)|\,|\varphi_{u-s}(p)|\,|\varphi_{-r-u}(p)|\\ \times\Big(3K_0\big(\frac{m\,d_g(\Theta(r+s).p,p)}{\sqrt{1+c}}\big)+4K_0\big(\frac{m\,d_g(\Theta(r+u).p,p)}{\sqrt{1+c}}\big)\Big)\Big\}.\end{aligned}\tag{5.3}$$

Definition 5.2. $\theta\in\mathbb{R}^l\setminus\mathbb{Q}^l$ satisfies a Diophantine condition if there exists $C>0$, $\beta\ge 0$ such that for all $n\in\mathbb{Z}^l_{\setminus\{0\}}$:

$$\|n\theta\|_{\mathbb{T}^l} := \inf_{k\in\mathbb{Z}^l}|n\theta+k| \ge C|n|^{-(l+\beta)}.$$

Diophantine conditions constitute a way to characterize and classify irrational numbers which are "far from the rationals" in the sense of being badly approximated by rationals. The set of numbers satisfying a Diophantine condition is 'big' (of full Lebesgue measure) in the sense of measure theory, but 'small' (of first category) in the sense of category theory [27]. Again because the metric is constant on the orbits we have:

$$d_g^2(y.p,p) = \inf_{k\in\mathbb{Z}^l}\left(\sum_{i,j=1}^{l}\tilde{g}_{ij}(p)(y^i+k^i)(y^j+k^j)\right).$$

Recall also that the modified Bessel functions have the following behavior near the origin

$$K_1(x) = \frac{1}{x} + O(x^0), \qquad K_0(x) = -\gamma + \ln(2) - \ln(x) + O(x),$$

where γ is the Euler constant. Thus, in view of $\{\|\varphi_r\|_\infty\} \in \mathcal{S}(\mathbb{Z}^l)$, and provided the integral over the manifold with the measure μ_g can be carried out, in (5.3) we have convergence if and only if $d_g^{-2}(\Theta r.p,p) \in \mathcal{S}'(\mathbb{Z}^l)$, that is, if and only if the entries of Θ satisfy a Diophantine condition. This result seems to be new, although the pertinence of Diophantine conditions in NCQFT had been conjectured by Connes long ago. Recently, these conditions have been found to play a role in Melvin models with irrational twist parameter in conformal field theory [22].

We said above: "provided the integral over the manifold with the measure μ_g can be carried out". This because $d_g^{-2}(\alpha_y(.),.)$ for a non-zero $y \in \mathbb{T}^l$ might not be locally integrable with respect to the measure given by the Riemannian volume form. Problems may appear on a neighborhood of the set of points with non-trivial isotropy groups. In fact, by simple dimensional analysis, we expect serious trouble when the isotropy group is one dimensional. For $p \in M$ let H_p its isotropy group and let $M_{sing} := \{p \in M : H_p \neq \{0\}\}$. Recall that M_{sing} is closed and of zero-measure in M since the action is proper (see [25]), and note that for a non-zero $y \in \mathbb{T}^l$, $d_g(y.p,p) = 0$ if and only if $p \in M_{sing}$ and $y \in H_p$. On $M_{reg} := M \setminus M_{sing}$ (the set of principal orbit type), since the action is free, one can define normal coordinates on a tubular neighborhood of an orbit $\mathbb{T}^l.p$. Let $(\hat{x}^\mu, \tilde{x}^i)$, $\mu = 1,\dots,n-l$, $i = 1,\dots,l$ be respectively the transverse and the torus coordinates of a point $p \in M_{reg}$. Because the action is isometric, in this coordinate system the metric takes the form

$$g(\hat{x},\tilde{x}) = \begin{pmatrix} h(\hat{x}) & l(\hat{x}) \\ l(\hat{x}) & \tilde{g}(\hat{x}) \end{pmatrix},$$

where $\tilde{g}$ is the induced (constant) metric on the orbit. Such coordinate system is singular with singularities located at each point of M_{sing}, and when $x \equiv (\hat{x}^\mu, \tilde{x}^i)$ approach $p_0 \in M_{sing}$, $\tilde{g}(\hat{x})$ collapses to a $l - dim(H_{p_0})$ rank matrix. Since in this coordinate system $\mu_g(p)\, d_g^{-2}(y.p,p)$ equals

$$\frac{\sqrt{\det g(\hat{x})}}{\sum_{i,j=1}^{l} \tilde{g}_{ij}(\hat{x}) y^i y^j} d^l \tilde{x}\, d^{n-l}\hat{x},$$

when $dim(H_{p_0}) = 1$ the singularity of $d_g^{-2}(y.p,p)$ for $p \to p_0$ cannot be cancelled by $\sqrt{\det g}$. This is a new feature of the UV/IR mixing for generic periodic isospectral deformations which needs to be investigated in detail in each model; it occurs, for instance, for the Connes-Landi spheres and their ambient spaces. Let us summarize:

Theorem 5.3. *For M compact or not (within the assumptions displayed in section 4.2 in the non-compact case), endowed with a smooth isometric action of the compact group $\mathbb{T}^l$, $l = 2$ or $l = 4$ and with a deformation matrix whose entries satisfy a Diophantine condition, then for any external field $\varphi \in C_c^\infty(M,)$ vanishing in a neighborhood M_{sing} the one-loop non-planar reduced effective action is finite.*

In other words, if the Diophantine condition is not satisfied or if $d_g^{-2}(\alpha_y(.),.) \notin L^1_{loc}(M,\mu_g)$ then the reduced non-planar two-point function does not define a distribution and the theory is not renormalizable, already at one-loop, by addition of local counter-terms.

6 Summary and perspectives

We have shown the existence of the UV/IR mixing for isospectral deformations of curved spaces.

For periodic deformations the entanglement only concerns (at the level of the two-point function) the 0-th component of the field in the spectral subspace decomposition induced by the torus action. In this case, the UV/IR mixing does not generate much trouble since one can treat it for renormalization purposes together with the planar sector.

In the non-periodic situation, we obtain non-planar Green functions which present the mixing in a similar form to the Moyal plane paradigms.

Our approach gives an algebraic way to understand the presence of the non-planar sector for those theories: it comes from the product of left and right regular representation operators. As a byproduct of our trace computations, we obtain that the better behavior of the non-planar sector is due to the presence of the off-diagonal heat kernel in the integrals.

However, its regularizing character depends highly on the geometric data. For non-periodic deformations, the conclusion is that when the noncommutative rank is equal to two, the non-planar 1PI two-point Green function does not define a distribution and the associated effective action remains divergent [14]. Only the group action of rank four gives rise to a UV divergent-free non-planar sector in the 4-dimensional manifold case. When the action is periodic, we have shown that it is necessary that the entries of $(2\pi)^{-1}\Theta$ satisfy a Diophantine condition to ensure finiteness of the reduced non-planar effective action, i.e., in order that the reduced non-planar 1PI two-point Green function define a distribution. Additional divergences may exist due to the possible fixed points structure of the action α.

Our treatment of the generic UV/IR behavior, can be generalized to higher dimensional isospectral deformations and/or to gauge theories. Also, we have restricted ourselves to the 4-dimensional case, for the sake of simplicity and physical interest, but it is clear that the heat kernel techniques employed here apply to higher dimensional scalar theories.

For gauge theory on (any dimensional) isospectral deformations manifolds, there is an intrinsic way to define noncommutative actions of the Yang-Mills type. For any $\omega \in \Omega^p(M), \eta \in \Omega^q(M)$ (say compactly supported and smooth with respect to α) one can set

$$\omega \wedge_\Theta \eta := (2\pi)^{-l} \int_{\mathbb{R}^{2l}} d^l y\, d^l z\, e^{-i<y,z>}\, (\alpha^*_{-\frac{1}{2}\Theta y}\omega) \wedge (\alpha^*_z \eta),$$

where α^*_z is the pull-back of α_z on forms. Given now an associated vector bundle $\pi : E \to M$ with compact structure group $G \subset U(N)$, and a connection $A \in \Omega^1(M, Lie(G))$ we define the NC analogue of the YM action

$$S_{YM}(A) := \int_M \mathrm{tr}(F_\Theta \wedge_\Theta *_H F_\Theta),$$

where $F_\Theta := dA + A \wedge_\Theta A$. In this context, one can prove a trace property, namely:

$$\int_M \omega \wedge_\Theta *_H \eta = \int_M \omega \wedge *_H \eta,\ \forall \omega, \eta \in \Omega^p(M).$$

Hence $S_{YM}(A)$ equals $\int_M \mathrm{tr}(F_\Theta \wedge *_H F_\Theta)$. To manage the quantization, one can once again use the background field method in the background gauge, and if we ignore the Gribov ambiguity, the one-loop effective action reduce to the computation of determinants of operators (quadratic part in A of $S_{YM}+S_{gf}$ and Faddeev-Popov determinant) which can be locally expressed as

$$(\nabla_\mu + L_{A_\mu} - R_{A_\mu})(\nabla^\mu + L_{A^\mu} - R_{A^\mu}) + B,$$

where B is bounded and contains left, right and a product of left and right twisted multiplication operators. It is then clear that UV/IR mixing will appears in the same form as in the flat situations (see [21, 23, 24]).

A further interesting task is be to look at what happens for a Grosse-Wulkenhaar like model for the non-compact case. In [20] it is proved that if we add a confining potential (harmonic oscillator in their work) in the usual $\lambda\varphi^{\star_\Theta 4}$ theory on the four dimensional Moyal plane, i.e., the Grosse-Wulkenhaar action

$$S_{GW}[\varphi] := \int d^4x \Big[\tfrac{1}{2}(\partial_\mu\varphi \star_\theta \partial^\mu\varphi)(x) + 2\frac{\Omega^2}{\theta^2}(x_\mu\varphi)\star_\theta(x^\mu\varphi) + \frac{m^2}{2}\varphi\star_\theta\varphi(x) + \frac{\lambda}{4!}\varphi\star_\theta\varphi\star_\theta\varphi\star_\theta\varphi(x)\Big],$$

then the theory is perturbatively renormalizable to all orders in λ. The deep meaning of this result is not yet fully understood, but some explanations can be mentioned. First, to add a confining potential is in some sense equivalent to a compactification of the Moyal plane and in the second hand, the particular choice of the potential corresponds to a Moyal-deformation of both the configuration and the momentum space. This can be seen by the invariance (up to a rescaling) of this action under $p_\mu \leftrightarrow 2(\theta^{-1})_{\mu\nu}x^\nu$, $\widehat{\varphi}(p) \leftrightarrow (\pi\theta)^2\varphi(x)$. This point needs to be clarified. It would be good to know whether their renormalizability conclusion (UV/IR decoupling) holds in the general context when one adds a coupling with a confining potential in the scalar theory.

Last, but not least, it remains to see whether the UV/IR entanglement concerns only θ-deformations or not. Connes-Dubois-Violette [7] 3-spheres and 4-planes, whose defining algebras are related to Sklyanin algebras, are good candidates to test this point.

Acknowledgments

I am very grateful to J. M. Gracia-Bondía, J. C. Várilly and my advisor B. Iochum for their help. I also would like to thank M. Grasseau, T. Krajewski, F. Ruiz Ruiz and R. Zentner for fruitful discussions and/or suggestions. Special thanks are also due to the Departamento de Física Teórica I of the Universidad Complutense de Madrid for its hospitality during the final stages of this work. I finally would like to thank the Referee for his enlightened remarks.

References

[1] G. Braunss, On the regular Hilbert space representation of a Moyal quantization, *J. Math. Phys.* **35**, 2045–2056 (1994).

[2] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, London and San Diego, 1984.

[3] I. Chepelev and R. Roiban, Renormalization of Quantum Fields Theories on Noncommutative $\mathbb{R}^d$. I. Scalar, *J. High Energy Phys.* **5**, 137–168 (2000).

[4] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.

[5] A. Connes, Gravity coupled with matter and the foundation of noncommutative geometry, *Commun. Math. Phys.* **182**, 155–176 (1996).

[6] A. Connes and G. Landi, Noncommutative manifolds, the instanton algebra and isospectral deformations, *Commun. Math. Phys.* **221**, 141–159 (2001).

[7] A. Connes and M. Dubois-Violette, Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples, *Commun. Math. Phys.* **230**), 539–579 (2002.

[8] T. Coulhon, E. Russ and V. Tardivel-Nachef, Sobolev algebras on Lie groups and Riemannian manifolds, *Amer. J. Math.* **123**, 283–342 (2001).

[9] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[10] M.R. Douglas and N.A. Nekrasov, Noncommutative Fields Theory, *Rev. Modern Phys.* **73**, 977–1024 (2001).

[11] T. Filk, Divergences in a Field Theory on Quantum Space, *Phys. Lett. B* **376**, 53–58 (1996).

[12] V. Gayral, J.M. Gracia-Bondía, B. Iochum, T. Schücker and J.C. Várilly, Moyal planes are spectral triples, *Commun. Math. Phys.* **246**, 569–623 (2004).

[13] V. Gayral and B. Iochum, The spectral action for Moyal planes, *J. Math. Phys.* **46**, 043503 (2005).

[14] V. Gayral, J.M. Gracia-Bondía and F. Ruiz Ruiz, Trouble with space and nonconstant noncommutativity field theory, *Phys. Lett. B* **610**, 141–146 (2005).

[15] V. Gayral, J.M. Gracia-Bondía and F. Ruiz Ruiz, Position-dependent noncommutative products: classical construction and field theory, hep-th/0504022.

[16] V. Gayral, B. Iochum and J.C. Várilly, Dixmier trace on non-compact isospectral deformations, in preparation.

[17] P.B. Gilkey, *Invariance Theory, the Heat Equation and the Atiyah-Singer Index Theorem*, 2nd edition, CRC Press, Boca Raton, FL, 1995.

[18] J.M. Gracia-Bondía, Improved Epstein-Glaser renormalization in coordinate space I. Euclidean framework, *Math. Phys. Analysis Geom.* **6**, 59–88 (2003).

[19] J.M. Gracia-Bondía, J.C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston, 2001.

[20] H. Grosse and R. Wulkenhaar, Renormalisation of ϕ^4-Theory on Noncommutative $\mathbb{R}^4$ to all order, hep-th 0403232.

[21] T. Krajewski and R. Wulkenhaar, Perturbative Quamtum Gauge Fields on the Noncommutative Torus, *Int. J. Mod. Phys. A* **15**, 1011–1030 (2000).

[22] D. Kurasov, J. Marklof and G.W. Moore, Melvin models and Diophantine approximation, hep-th 0407150.

[23] C.P. Martin and D. Sanchez-Ruiz, The One Loop UV Divergent stucture of $U(1)$ Yang-Mills Theory on Noncommutative $\mathbb{R}^4$, *Phys. Rev. Lett.* **83**, 476–479 (1999).

[24] S. Minwalla, M.V. Raamsdonk and N. Seiberg, Noncommutative Perturbative Dynamics, *J. High Energy Phys.* **2**, 20–31 (2000).

[25] P.W. Michor, *Isometric actions of Lie groups and invariants*, Notes of a lecture course at the University of Vienna, July 1997.

[26] S.B. Myers and N. Steenrod, On the group of isometries of a Riemannian manifold, *Ann. Math.* **40**, 406–416 (1939).

[27] J.C. Oxtoby, *Measure and Category*, Springer, Berlin, 1972.

[28] M.A. Rieffel, *Deformation Quantization for Actions of* $\mathbb{R}^d$, Memoirs Amer. Math. Soc. **506**, Providence, RI, 1993.

[29] F. Ruiz Ruiz, UV/IR mixing and the Goldstone theorem in noncommutative field theory, *Nucl. Phys.* **B637**, 143–167 (2002).

[30] O. Schnetz, Natural renormalization, *J. Math. Phys.* **38**, 738–758 (1997).

[31] A. Sitarz, Rieffel's deformation quantization and isospectral deformations, *Int. J. Theor. Phys.* **40**, 1693–1696 (2001).

[32] R.J. Szabo, Quantum Fields Theory on Noncommutative space, *Phys. Rep.* **37**, 207–299 (2003).

[33] J.C. Várilly, Quantum symmetry groups of noncommutative spheres, *Commun. Math. Phys.* **221**, 511–523 (2001).

[34] J.C. Várilly and J.M. Gracia-Bondía, On the ultraviolet behavior of quantum fields over noncommutative manifolds, *Int. J. Mod. Phys.* **A14**, 1305–1323 (1999).

[35] D.V. Vassilevich, Non-commutative heat kernel, *Lett. Math. Phys.* **67**, 185–194 (2004).

[36] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, fourth edition, Clarendon Press, Oxford, 2002.

Victor Gayral
CPT-CNRS
UMR 6207
Luminy Case 907
F-13288 Marseille Cedex 9
France
email: gayral@cpt.univ-mrs.fr

Communicated by Vincent Rivasseau
submitted 22/12/04, accepted 22/03/05