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# Spin, Statistics, and Reflections I. Rotation Invariance

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**Abstract.** The universal covering of $SO(3)$ is modelled as a reflection group $\mathbf{G}_R$ in a representation independent fashion. For relativistic quantum fields, the Unruh effect of vacuum states is known to imply an intrinsic form of reflection symmetry, which is referred to as *modular* $P_1CT$*-symmetry* [1, 2, 11]. This symmetry is used to construct a representation of $\mathbf{G}_R$ by pairs of modular $\mathrm{P_1CT}$-operators. The representation thus obtained satisfies Pauli's spin-statistics relation.

## 1 Introduction

A vacuum state of a quantum field theory usually exhibits the Unruh effect, i.e., a uniformly accelerated observer experiences it as a thermal state whose temperature is proportional to his acceleration [27]. This has been shown by Bisognano and Wichmann [1, 2] for finite-component quantum fields (in the Wightman setting). For general quantum fields, it has recently been derived from the mere condition that each vacuum state exhibits passivity to each inertial or uniformly accelerated observer [18], i.e., that in the observer's rest frame, no engine can extract energy from the state by cyclic processes.[1]

By the theorem of Bisognano and Wichmann mentioned above, all familiar quantum fields also exhibit an intrinsic form of PCT-symmetry.[2] Namely, one can assign to each Rindler wedge $\mathcal{W}$, i.e., the set $\mathcal{W}_1 := \{x_1 \geq |x_0|\}$ or its image under some Poincaré transformation, an antiunitary involution $J_\mathcal{W}$. This assignment is an intrinsic construction using the vacuum vector and the field operators only. It is also basic to the so-called modular theory due to Tomita and Takesaki, where an operator like $J_\mathcal{W}$ is called a *modular conjugation*. $J_\mathcal{W}$ then implements a $P_1CT$*-symmetry*, i.e., a linear reflection in charge and at the edge of $\mathcal{W}$. This property is called *modular* $P_1CT$*-symmetry*. Note as an aside that this symmetry is a typical property of $1+2$-dimensional quantum fields as well, whereas these fields do not exhibit PCT-symmetry as a whole [23].

Modular $\mathrm{P_1CT}$-symmetry is a consequence of the Unruh effect [11], but the converse implication does not hold: There are examples of $\mathrm{P_1CT}$-symmetric quantum fields that do not exhibit the Unruh property [4].

[1] Two related uniqueness results can be found in Refs. 16 and 17.
[2] cf. also Refs. 11, 16, and 17.

Guido and Longo have derived Pauli's spin-statistics relation from the Unruh effect for general quantum fields in $1+3$ dimensions [11].[3] Independently from this, the present author derived the spin-statistics relation making use of modular $\mathrm{P_1CT}$-symmetry only [15].

This symmetry was assumed for the field's observables only, but since use of a theorem due to Doplicher and Roberts [8] was made later on, the result of Ref. 15 is confined to the massive-particle excitations of the vacuum.

In Ref. 11 the Unruh effect was assumed for the whole field on the one hand. On the other hand, no use of the Doplicher-Roberts theorem was made, so a much larger class of fields and states was included; even fields that are covariant with respect to *more than one* representation of the universal covering group of $L_+^\uparrow$, among which there may be both representations satisfying and violating Pauli's relation [24]. What one did obtain was a unique representation satisfying the Unruh effect. This representation exhibits Pauli's spin-statistics connection. All spin-statistics theorems obtained before did not admit this extent of generality.

This paper is the first of two that generalize the result of Ref. 15 in this spirit as well. Assuming $\mathrm{P_1CT}$-symmetry with respect to all Rindler wedges whose edges are two-dimensional planes in a given tim-zero plane, a covariant unitary representation $\tilde{W}$ of the rotation group's universal covering is constructed. This representation satisfies Pauli's spin-statistics relation. The argument does not make use of the Doplicher-Roberts theorem and applies to general relativistic quantum fields.

Like its predecessor in Ref. 15, the argument is crucially based on the fact that each rotation in $\mathbb{R}^3$ can be implemented by combining two reflections at planes. This is, as such, well known for both $SO(3)$ and $L_+^\uparrow$. A corresponding result for the universal coverings of these groups is, however, less elementary to obtain.

In Section 2, a model $\mathbf{G}_R \cong SU(2)$ of the universal covering group $\widetilde{SO(3)}$ of $SO(3)$ will be constructed from nothing except pairs of "reflections along normal vectors", i.e., from the family $(j_a)_{a\in S^2}$, where $j_a$ is the reflection at the plane $a^\perp$. This representation-independent construction is set up according to the needs of the spin-statistics theorem to be proved later on. A model $\mathbf{G}_L \cong SL(2,\mathbb{C})$ of $\widetilde{L_+^\uparrow}$ will be constructed in a forthcoming paper. It is to be expected that the universal coverings of other Lie groups could be constructed the same way.

Recently it has been shown by Buchholz, Dreyer, Florig, and Summers that this structure has a representation theoretic consequence: unitary representations of $L_+^\uparrow$ can be constructed from a system of reflections satisfying a minimum of covariance conditions, as they are satisfied by the modular conjugations of a quantum field with modular $\mathrm{P_1CT}$-symmetry [4, 9, 5]. This raises the question how to generalize these results to $\mathbf{G}_R$ and $\mathbf{G}_L$, the goal being a considerable generalization of the spin-statistics analysis in Ref. 15.

In Section 3, it is shown that this can, indeed, be accomplished for $\mathbf{G}_R$; the group $\mathbf{G}_L$ will be treated in the forthcoming paper. If a quantum field exhibits

[3] cf. also Refs. 10, 12, and 13.

modular $\mathrm{P}_1$CT-symmetry, then it is elementary to build a distinguished representation $\tilde{W}$ of $\mathbf{G}_R$ from the modular conjugations that implement $\mathrm{P}_1$CT-symmetry. This representation can, eventually, easily be shown to conform with Pauli's spin-statistics principle.

It is well known that not all $\mathbf{G}_R$-covariant quantum fields exhibit the spin-statistics relation, and it should be remarked that even for Lorentz covariant fields there are counterexamples [24]. This means that some condition specifying the representation or field under consideration is needed for whatever spin-statistics theorem. In the early spin-statistics theorems, this condition was that the number of internal degrees of freedom is finite, in this paper the condition is that the representation is constructed from modular $\mathrm{P}_1$CT-operators. At the moment, such sufficient conditions are all one has in the relativistic setting; only in the setting of nonrelativistic quantum mechanics, a both sufficient and necessary condition has been established [19, 20].

## 2 $\widetilde{SO(3)}$ as a reflection group

There are many ways to model the universal covering group of the rotation group $SO(3) =: R$. Among topologists, "the" universal covering group is the group $\widetilde{SO(3)}$ of homotopy classes of curves starting at some base point, physicists are more familiar with $SU(2)$, but these are, of course, not the only examples of simply connected covering groups. As a new model, a group $\mathbf{G}_R$ will be constructed in this section from pairs of "reflections along normal vectors", i.e., from the family $(j_a)_{a\in S^2}$, where $j_a$ is the reflection at the plane $a^\perp$.

Let $\mathbf{M}_R$ be the pair groupoid of $S^2$, i.e., the set $S^2 \times S^2$ endowed with the concatenation $(a,b)\circ(b,c) := (a,c)$. Then the map $\rho : \mathbf{M}_R \to R$ defined by $\rho(a,b) := j_a j_b$ is well known to be surjective. Namely, $\rho(a,a) = 1$ for all $a \in S^2$. For $\sigma \neq 1$, choose $\tau \in R$ such that $\tau^2 = \sigma$; if $a \in S^2$ is perpendicular to the axis of $\sigma$, then $\rho(\tau a, a) = \sigma$.

Call $(a,b)$ and $(c,d)$ equivalent if $\rho(a,b) = \rho(c,d)$ and if there exists a $\sigma \in R$ commuting with $\rho(a,b)$ and satisfying $(a,b) = (\sigma c, \sigma d)$. Let $\mathbf{G}_R$ be the quotient space $\mathbf{M}_R/\sim$ associated with this equivalence relation, and let $\pi : \mathbf{M}_R \to \mathbf{G}_R$ denote the corresponding canonical projection. Define $\tilde{\rho} : \mathbf{G}_R \to R$ by $\tilde{\rho}(\pi(\underline{m})) := \rho(\underline{m})$ for all $\underline{m} \in \mathbf{M}_R$. Then the diagram

$$\begin{array}{ccc} \mathbf{M}_R & \xrightarrow{\pi} & \mathbf{G}_R \\ \rho\downarrow & \swarrow \tilde{\rho} & \\ R & & \end{array} \tag{1}$$

commutes by construction. All maps in this diagram are continuous: $\pi$ is continuous by definition, and continuity of $\rho$ is elementary to show. The proof for $\tilde{\rho}$ is elementary as well: given any open set $M \subset R$, the pre-image $\tilde{\rho}^{-1}(M)$ is open if and only if $\pi^{-1}(\tilde{\rho}^{-1}(M))$ is open. This set coincides with $\rho^{-1}(M)$, which is open by continuity of $\rho$.

Defining $\pm 1 := \pi(a, \pm a)$ for arbitrary $a \in S^2$, and $-\pi(a,b) := \pi(a,-b)$ for $(a,b) \in \mathbf{M}_R$, one verifies that $\tilde{\rho}^{-1}(\sigma)$ consists of two equivalence classes for each $\sigma \in R$.

**Lemma 1**

(i) $\mathbf{G}_R$ *is a Hausdorff space.*

(ii) $\tilde{\rho}$ *is a two-sheeted covering map.*

Before proving this lemma, we introduce some notation.

*Notation.* Denote the set $R\backslash\{1\}$ by $\dot{R}$. For each $\sigma \in \dot{R}$, let $A(\sigma)$ be the rotation axis of $\sigma$. If $\mathbf{a} \in A(\sigma)$ is one of the two unit vectors in $A(\sigma)$, then there is a unique $\alpha \in (0, 2\pi)$ such that $\sigma$ is the right-handed rotation around $\mathbf{a}$ by the angle $\alpha$. The vector $\mathbf{a}$ and the angle $\alpha$ determine $\sigma$, and occasionally we use the notation $[\mathbf{a}, \alpha]$ for $\sigma$. Note that $[\mathbf{a}, \alpha] = [-\mathbf{a}, 2\pi - \alpha]$.

Denote the set $\rho^{-1}(\dot{R})$ by $\dot{\mathbf{M}}_R$. To each $(a,b) \in \dot{\mathbf{M}}_R$, assign the axial unit vector $\mathbf{a}(a,b) := \frac{a \times b}{|a \times b|}$, and denote by $\sphericalangle(a,b) \in (0,\pi)$ the angle between $a$ and $b$. Note that $\rho(\underline{m}) = [\mathbf{a}(\underline{m}), 2\sphericalangle(\underline{m})]$ for all $\underline{m} \in \dot{\mathbf{M}}_R$.

Denote the set $\tilde{\rho}^{-1}(\dot{R})$ by $\dot{\mathbf{G}}_R$. Since $\underline{m} \sim \underline{n}$ implies

$$\mathbf{a}(\underline{m}) = \mathbf{a}(\underline{n}) \quad \text{and} \quad \sphericalangle(\underline{m}) = \sphericalangle(\underline{n})$$

one can define

$$\tilde{\mathbf{a}}(\pi(\underline{m})) := \mathbf{a}(\underline{m}) \quad \text{and} \quad \tilde{\sphericalangle}(\pi(\underline{m})) := \sphericalangle(\underline{m}).$$

Note that $\tilde{\rho}(g) = [\tilde{\mathbf{a}}(g), 2\tilde{\sphericalangle}(g)]$ for all $g \in \dot{\mathbf{G}}_R$.

*Proof of Lemma* 1.(i). Define $\dot{\mathbb{B}}_\pi := \{x \in \mathbb{R}^3 : |x| \in (0,\pi)\}$, and assign to each $x \in \dot{\mathbb{B}}_\pi$ the rotation $\tau(x) := [x/|x|, |x|]$. Choose any $x \in \dot{\mathbb{B}}_\pi$ and an $a \in S^2 \cap x^\perp$, and put $\xi_a(x) := \pi(\tau(x)a, a) \in \dot{\mathbf{G}}_R$. One then obtains $\xi_a(x) = \xi_b(x)$ for all $b \in S^2 \cap x^\perp$, so a map $\xi : \dot{\mathbb{B}}_\pi \to \dot{\mathbf{G}}_R$ is well defined by $\xi(x) := \xi_a(x)$, where $a \in S^2 \cap x^\perp$ is arbitrary.

$\xi$ is inverse to the map $\eta : \dot{\mathbf{G}}_R \to \dot{\mathbb{B}}_\pi$ defined by $\eta(g) := -\tilde{\sphericalangle}(g)\tilde{\mathbf{a}}(g)$. Namely, since $b \perp a \times b$ for all $(a,b) \in \dot{\mathbf{M}}_R$, one has

$$\begin{aligned}
\xi(\eta(\pi(a,b))) &= \xi\left(\sphericalangle(a,b) \cdot \mathbf{a}(a,b)\right) \\
&= \pi\left(\,\tau\left(-\sphericalangle(a,b)\mathbf{a}(a,b)\right)\, b, b\right) \\
&= \pi\left([-\mathbf{a}(a,b), \sphericalangle(a,b)]\, b, b\right) \\
&= \pi(a,b).
\end{aligned}$$

So $\eta$ is continuous, surjective, and has a continuous inverse, so $\eta$ is a homeomorphism, and $\dot{\mathbf{G}}_R$ is a Hausdorff space.

It remains to construct disjoint neighborhoods of two distinct points $g, h \in \mathbf{G}_R$ for the case that $g = \pm 1$ and $h \in \mathbf{G}_R$ is arbitrary.

If $g = 1$, then $\tilde{\sphericalangle}(h) \neq 0$, so there exist disjoint open neighborhoods $X$ and $Y$ of 0 and $\tilde{\sphericalangle}(h)$ in the topological space $[0, \pi]$, respectively. Since the map $\tilde{\sphericalangle}$ is continuous, the sets $U := \tilde{\sphericalangle}^{-1}(X)$ and $V := \tilde{\sphericalangle}^{-1}(Y)$ are disjoint neighborhoods of 1 and $h$. If $g = -1$, there exist disjoint neighborhoods $U'$ and $V'$ of $-g$ and $-h$, so $-U'$ and $-V'$ are disjoint neighborhoods of $g$ and $h$, respectively.
*Proof of (ii).* Define $\hat{\rho} : \dot{\mathbb{B}}_\pi \to \dot{R}$ by $\hat{\rho}(x) := [x/|x|, 2|x|]$. Then the diagram

$$\begin{tikzcd} & \dot{\mathbf{G}}_R \arrow[dl, "\tilde{\rho}|_{\dot{\mathbf{G}}_R}"'] \arrow[d, "\eta"] \\ \dot{R} & \dot{\mathbb{B}}_\pi \arrow[l, "\hat{\rho}"'] \end{tikzcd} \tag{2}$$

commutes. $\hat{\rho}$ is a two-sheeted covering map, and $\eta$ is a homeomorphism, so $\tilde{\rho}|_{\dot{\mathbf{G}}_R} = \hat{\rho} \circ \eta$ is a two-sheeted covering map.

In order to prove that $\tilde{\rho}$ as a whole is a covering map, it remains to be shown that $\tilde{\rho}$ is open not only on $\dot{\mathbf{G}}_R$, but also in $\pm 1$. Since $\mathbf{G}_R$ is Hausdorff, since $\dot{\mathbf{G}}_R$ is a two-sheeted covering space of $\dot{R}$, and since $\tilde{\rho}^{-1}(1) = \{\pm 1\}$ contains, like all other fibers of $\tilde{\rho}$, precisely two elements, it then follows that $\tilde{\rho}$ has continuous local inverses everywhere.

So let $(\sigma_n)_n$ be any sequence in $\dot{R}$ converging to 1, then some sequence $(g_n)_n$ in $\dot{\mathbf{G}}_R$ needs to be found with $\tilde{\rho}(g_n) = \sigma_n$ for all $n$ and $g_n \to 1$; note that $(-g_n)_n$ then satisfies $\tilde{\rho}(-g_n) = \sigma_n$ as well and converges to $-1$.

For each $g \in \dot{\mathbf{G}}_R$, one has $\tilde{\sphericalangle}(g) \leq \pi/2$ or $\tilde{\sphericalangle}(-g) \leq \pi/2$. It follows that for each $n$ some $g_n \in \tilde{\rho}^{-1}(\sigma_n)$ can be chosen such that $\tilde{\sphericalangle}(g_n) \leq \pi/2$. Since $[0, \pi/2]$ is compact, the sequence $(\tilde{\sphericalangle}(g_n))_n$ has at least one accumulation point, and since $\sigma_n$ tends to 1, the only possible accumulation point in the interval $[0, \pi/2]$ is zero. It follows that $\tilde{\sphericalangle}(g_n)$ tends to zero and, hence, that $g_n$ tends to 1, proving that $\tilde{\rho}$ is open. □

The reason why this proof is nontrivial is that $\rho$ and $\pi$ are not open. If this were the case, $\mathbf{G}_R$ would directly inherit the Hausdorff property from $\mathbf{M}_R$, and the proof that $\tilde{\rho}$ is a covering map would be elementary. But neither $\rho$ nor $\pi$ is open.

In order to see this, let $(\sigma_n)_n$ be any sequence of rotations around some fixed $a \in S^2$, and suppose this sequence to converge to 1. If $\rho$ were open, one would have to find, for each $\underline{m} \in \pi^{-1}(1)$ a sequence $(\underline{m}_n)_n$ converging to $\underline{m}$ and satisfying $\rho(\underline{m}_n) = \sigma_n$ for all $n$. Now choose $\underline{m} = (a, a)$. Since $a \in A(\sigma_n)$ for all $n$, one knows for all $(b_n, c_n) \in \rho^{-1}(\sigma_n)$ that both $b_n$ and $c_n$ are perpendicular to $a$. As a consequence, no sequence $(\underline{m}_n)_n$ with $\rho(\underline{m}_n) = \sigma_n$ for all $n$ can coverge to $\underline{m} = (a, a)$.

$\pi$ cannot be open either, since this would, by diagram 1 and the preceding Lemma, imply that $\rho$ is open. Only the restrictions of $\rho$ and $\pi$ to $\rho^{-1}(\dot{R})$ are open.

**Theorem 2**

(i) $\mathbf{G}_R$ *is simply connected.*

(ii) *There is a unique group product* $\odot$ *on* $\mathbf{G}_R$ *such that the diagram*

$$\begin{array}{ccc} \mathbf{M}_R \times \mathbf{M}_R & \xrightarrow{\circ} & \mathbf{M}_R \\ \downarrow \pi \times \pi & & \downarrow \pi \\ \mathbf{G}_R \times \mathbf{G}_R & \xrightarrow{\odot} & \mathbf{G}_R \\ \downarrow \tilde{\rho} \times \tilde{\rho} & & \downarrow \tilde{\rho} \\ R \times R & \xrightarrow{\cdot} & R \end{array} \tag{3}$$

*commutes.*

*Proof of* (i). $\mathbf{G}_R = \pi(\mathbf{M}_R)$ is pathwise connected because $\mathbf{M}_R = S^2 \times S^2$ and because $\pi$ is continuous. Together with Lemma 1, this implies the statement, since the fundamental group of $R$ is $\mathbb{Z}_2$.

*Proof of* (ii). The outer arrows of the diagram commute, so it suffices to prove the existence and uniqueness of a group product conforming with the lower part. But it is well known that each simply connected covering space $\tilde{G}$ of an arbitrary topological group $G$ can be endowed with a unique group product $\odot$ such that $G$ is a covering group.[4] □

## 3 Spin and statistics

The preceding section has provided the basis of a general spin-statistics theorem, which is the subject of this section. From an intrinsic form of symmetry under a charge conjugation combined with a time inversion and the reflection in *one* spatial direction, which is referred to as *modular* $P_1CT$*-symmetry*, a strongly continuous unitary representation $\tilde{W}$ of $\mathbf{G}_R$ will be constructed using the above and related reasoning. It is, then, elementary to show that $\tilde{W}$ exhibits Pauli's spin-statistics relation.

In order to make the notion of rotation meaningful, fix a distinguished time direction by choosing a future-directed timelike unit vector $e_0$. The 2-sphere of unit vectors in the time-zero plane $e_0^\perp$ will be called $S^2$.

Let $F$ be an arbitrary quantum field on $\mathbb{R}^{1+3}$ in a Hilbert space $\mathcal{H}$. The following standard properties of relativistic quantum fields will be used here.

(A) *Algebra of field operators.* Let $\mathfrak{C}$ be a linear space of arbitrary dimension,[5] and denote by $\mathfrak{D}$ the space $C_0^\infty(\mathbb{R}^{1+3})$ of test functions on $\mathbb{R}^{1+3}$. The field

[4] See, e.g., Props. 5 and 6 in Sect. I.VIII. in Ref. 7.

[5] $\mathfrak{C}$ is the "component space", and its dimension equals the number of components, which may be infinite in what follows.

$F$ is a linear function that assigns to each $\Phi \in \mathfrak{C} \otimes \mathfrak{D}$ a linear operator $F(\Phi)$ in a separable Hilbert space $\mathcal{H}$.

(A.1) $F$ is free from redundancies in $\mathfrak{C}$, i.e., if $\mathfrak{c}, \mathfrak{d} \in \mathfrak{C}$ and if $F(\mathfrak{c} \otimes \varphi)) = F(\mathfrak{d} \otimes \varphi)$ for all $\varphi \in \mathfrak{D}$, then $\mathfrak{c} = \mathfrak{d}$.

(A.2) Each field operator $F(\Phi)$ and its adjoint $F(\Phi)^\dagger$ are densely defined. There exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ contained in the domains of $F(\Phi)$ and $F(\Phi)^\dagger$ and satisfying $F(\Phi)\mathcal{D} \subset \mathcal{D}$ and $F(\Phi)^\dagger\mathcal{D} \subset \mathcal{D}$ for all $\Phi \in \mathfrak{C} \otimes \mathfrak{D}$.

Denote by $\mathbf{F}$ the algebra generated by all $F(\Phi)|_{\mathcal{D}}$ and all $F(\Phi)^\dagger|_{\mathcal{D}}$. Defining an involution $*$ on $\mathbf{F}$ by $A^* := A^\dagger|_{\mathcal{D}}$, the algebra $\mathbf{F}$ is endowed with the structure of a $*$-algebra.

For each $a \in S^2$, denote by $\mathcal{W}_a := \{x \in \mathbb{R}^{1+3} : xa > |xe_0|\}$ the *Rindler wedge* associated with $a$,[6] and let $\mathbf{F}(a)$ be the algebra generated by all $F(\mathfrak{c} \otimes \varphi)|_{\mathcal{D}}$ and all $F(\mathfrak{c} \otimes \varphi)^\dagger|_{\mathcal{D}}$ with $\mathrm{supp}(\varphi) \subset \mathcal{W}_a$. The algebra $\mathbf{F}(a)$ inherits the structure of a $*$-algebra from $\mathbf{F}$ by restriction of $*$.

(A.3) $\mathbf{F}(a)$ is nonabelian for each $a$, and $a \neq b$ implies $\mathbf{F}(a) \neq \mathbf{F}(b)$.

(B) *Cyclic vacuum vector.* There exists a vector $\Omega \in \mathcal{H}$ that is cyclic with respect to each $\mathbf{F}(a)$.

(C) *Normal commutation relations.* There exists a unitary and self-adjoint operator $k$ on $\mathcal{H}$ with $k\Omega = \Omega$ and with $k\mathbf{F}(a)k = \mathbf{F}(a)$ for all $a \in S^2$. Define $F_\pm := \frac{1}{2}(F \pm kFk)$. If $\mathfrak{c}$ and $\mathfrak{d}$ are arbitrary elements of $\mathfrak{C}$ and if $\varphi, \psi \in \mathfrak{D}$ have spacelike separated supports, then

$$\begin{aligned} &F_+(\mathfrak{c} \otimes \varphi)F_+(\mathfrak{d} \otimes \psi) = F_+(\mathfrak{d} \otimes \psi)F_+(\mathfrak{c} \otimes \varphi), \\ &F_+(\mathfrak{c} \otimes \varphi)F_-(\mathfrak{d} \otimes \psi) = F_-(\mathfrak{d} \otimes \psi)F_+(\mathfrak{c} \otimes \varphi), \quad \text{and} \\ &F_-(\mathfrak{c} \otimes \varphi)F_-(\mathfrak{d} \otimes \psi) = -F_-(\mathfrak{d} \otimes \psi)F_-(\mathfrak{c} \otimes \varphi). \end{aligned}$$

The involution $k$ is the *statistics operator*, and $F_\pm$ are the bosonic and fermionic components of $F$, respectively. Defining $\kappa := (1 + ik)/(1 + i)$ and $F^t(\mathfrak{d} \otimes \psi) := \kappa F(\mathfrak{d} \otimes \psi)\kappa^\dagger$, the normal commutation relations read

$$[F(\mathfrak{c} \otimes \varphi), F^t(\mathfrak{d} \otimes \psi)] = 0.$$

This property is referred to as *twisted locality.* Denote $\mathbf{F}(a)^t := \kappa\mathbf{F}(a)\kappa^\dagger$.

These properties imply that $\Omega$ is *separating* with respect to each algebra $\mathbf{F}(a)$, i.e., for each $A \in \mathbf{F}(a)$, the condition $A\Omega = 0$ implies $A = 0$.[7]

[6] An observer who is uniformly accelerated in the direction $a$ can interact with precisely the events in $\mathcal{W}_a$.

[7] If $A\Omega = 0$ and $B, C \in \mathbf{F}(-a)^t$, then $0 = \langle BC\Omega, A\Omega\rangle = \langle C\Omega, AB^*\Omega\rangle$, so $A = 0$ by cyclicity of $\Omega$.

As a consequence, an antilinear operator $R_a : \mathbf{F}(a)\Omega \to \mathbf{F}(a)\Omega$ is defined by $R_a A\Omega := A^*\Omega$. This operator is closable.[8] Its closed extension $S_a$ has a unique polar decomposition $S_a = J_a \Delta_a^{1/2}$ into an antiunitary operator $J_a$, which is called the *modular conjugation*, and a positive operator $\Delta_a^{1/2}$, which is called the *modular operator*. $J_a$ is an involution.[9] $S_a$, $J_a$, and $\Delta_a^{1/2}$ are the objects of the so-called *modular theory* developed by Tomita and Takesaki.[10]

For each $a \in S^2$, let $j_a$ be the orthogonal reflection at the plane $a^\perp \cap e_0^\perp$,[11] and for each $\varphi \in \mathfrak{D}$, define the test function $j_a\varphi \in \mathfrak{D}$ by $j_a\varphi(x) := \varphi(j_a x)$.

(D) *Modular $P_1CT$-symmetry.* For each $a \in S^2$, there exists an antilinear involution $C_a$ in $\mathfrak{C}$ such that for all $\mathfrak{c} \in \mathfrak{C}$ and $\varphi \in \mathfrak{D}$, one has

$$J_a F(\mathfrak{c} \otimes \varphi) J_a = F^t(C_a\mathfrak{c} \otimes \overline{j_a\varphi}).$$

The map $a \mapsto J_a$ is strongly continuous.[12]

It will now be shown that pairs of modular $\mathrm{P_1CT}$-reflections give rise to a strongly continuous representation of $\mathbf{G}_R$ which exhibits Pauli's spin-statistics connection.

**Lemma 3** *Let $K$ be a unitary or antiunitary operator in $\mathcal{H}$ such that $K\mathcal{D} = \mathcal{D}$ and $K\Omega = \Omega$, and suppose there are $a, b \in S^2$ such that $K\mathbf{F}(a)K^\dagger = \mathbf{F}(b)$. Then $KJ_aK^\dagger = J_b$, and $K\Delta_aK^\dagger = \Delta_b$.*

*Proof.* If $A \in \mathbf{F}(b)$, then $KS_aK^\dagger A\Omega = KS_a \underbrace{K^\dagger AK}_{\in\mathbf{F}(a)}\Omega = A^*\Omega = S_bA\Omega$. The statement now follows by the uniqueness of the polar decomposition. □

In particular, this lemma implies

$$kJ_ak = J_a, \quad \text{whence} \quad J_a\kappa = \kappa^\dagger J_a \tag{4}$$

by definition of $k$. Using twisted locality, the lemma also implies

$$\kappa J_a \kappa^\dagger = \kappa^\dagger J_a \kappa = J_{-a} \tag{5}$$

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[8] By twisted locality, the operator $\kappa R_{-a}\kappa^\dagger$ is formally adjoint to $R_a$. Namely, if $A \in \kappa\mathbf{F}(-a)\kappa^\dagger$ and $B \in \mathbf{F}(a)$, then $\langle A\Omega, R_aB\Omega\rangle = \langle A\Omega, B^*\Omega\rangle = \langle B\Omega, A^*\Omega\rangle = \langle B\Omega, \kappa R_{-a}\kappa^\dagger A\Omega\rangle$. Since $\kappa R_{-a}\kappa^\dagger$ is densely defined, it follows that $R_a$ is closable.

[9] $R_a^2 = 1$ implies $S_a^2 = 1$, so $J_a\Delta_a^{1/2} = S_a = S_a^{-1} = \Delta_a^{-1/2}J_a^*$, i.e., $J_a^2\Delta_a^{1/2} = J_a\Delta_a^{-1/2}J_a^*$. Since $J_a\Delta_a^{-1/2}J_a^*$ is positive, one obtains $J_a^2 = 1$ and $J_a\Delta^{-1/2}J_a = \Delta^{1/2}$ from the uniqueness of the polar decomposition [3].

[10] The original work [26] directly applies to von-Neumann algebras, which are normed. But also for the present setting this structure has been applied earlier, e.g., in the classical papers of Bisognano and Wichmann [1, 2]. See, also, Ref. 14 for a monograph on the Tomita-Takesaki theory of unbounded-operator algebras.

[11] i.e., the linear reflection with $j_a a = -a$, $\quad j_a e_0 = -e_0$, and $j_a x = x$ for all $x \in a^\perp \cap e_0^\perp$.

[12] If one assumes covariance with respect to some strongly continuous representation of $G_R$ (which may also violate the spin-statistics connection), this is straightforward to derive; cf. Lemma 3. But covariance, as such, is not needed.

which, in turn, implies

$$J_a J_b J_a = J_{-j_a b} = J_{j_a j_b b} = J_{\rho(a,b)b} \tag{6}$$

by modular $\mathrm{P}_1$CT-symmetry.

Define a map $W$ from $\mathbf{M}_R$ into the unitary group of $\mathcal{H}$ by $W(a,b) := J_a J_b$.

**Lemma 4**
(i) *$\underline{m} \sim \underline{n}$ implies $W(\underline{m}) = W(\underline{n})$.*
(ii) *$W(\underline{m}) = W(\underline{n})$ implies $\rho(\underline{m}) = \rho(\underline{n})$.*

*Proof of* (i). The proof of Lemma 2.4 in Ref. 5 can be taken without any relevant changes. Despite the fact that the Buchholz-Summers paper is confined to bosonic fields, which, in particular, implies $J_a = J_{-a}$, it is straightforward to translate their proof to the present setting. This will not be spelled out here. The proof makes use of the continuous dependence of $J_a$ from $a$ assumed in Assumption (D).

*Proof of* (ii). $\rho(\underline{m}) \neq \rho(\underline{n})$ would imply that there is some $b \in S^2$ such that $\rho(\underline{m})b \neq \rho(\underline{n})b$, so $\mathbf{F}(\rho(\underline{m})b) \neq \mathbf{F}(\rho(\underline{n})b)$ by Assumption (A), whence $W(\underline{m})\mathbf{F}(b)W(\underline{m})^* \neq W(\underline{n})\mathbf{F}(b)W(\underline{n})^*$ by Assumption (D), i.e., $W(\underline{m}) \neq W(\underline{n})$. □

By this lemma, a map $\tilde{W} : \mathbf{G}_R \to W(\mathbf{M}_R)$ is defined by $\tilde{W}(\pi(\underline{m})) := W(\underline{m})$, and another map $\rho_W : W(\mathbf{M}_R) \to R$ is defined by $\rho_W(W(\underline{m})) = \rho(\underline{m})$. The diagrams

$$(A) \quad \begin{array}{ccc} \mathbf{M}_R & \xrightarrow{\pi} & \mathbf{G}_R \\ \rho\downarrow & \overset{W}{\searrow} & \downarrow \tilde{W} \\ R & \xleftarrow{\rho_W} & W(\mathbf{M}_R) \end{array} \quad \text{and} \quad (B) \quad \begin{array}{ccc} \mathbf{M}_R & \xrightarrow{\pi} & \mathbf{G}_R \\ \rho\downarrow & \overset{\tilde{\rho}}{\swarrow} & \downarrow \tilde{W} \\ R & \xleftarrow{\rho_W} & W(\mathbf{M}_R) \end{array} \tag{7}$$

commute.

**Theorem 5**
(i) *There is a unique group product $\odot_W$ on $W(\mathbf{M}_R)$ with the property that the diagram*

$$\begin{array}{ccc} \mathbf{M}_R \times \mathbf{M}_R & \xrightarrow{\circ} & \mathbf{M}_R \\ \downarrow \pi\times\pi & & \downarrow \pi \\ \mathbf{G}_R \times \mathbf{G}_R & \xrightarrow{\odot} & \mathbf{G}_R \\ \downarrow \tilde{W}\times\tilde{W} & & \downarrow \tilde{W} \\ W(\mathbf{M}_R)\times W(\mathbf{M}_R) & \xrightarrow{\odot_W} & W(\mathbf{M}_R) \\ \downarrow \rho_W\times\rho_W & & \downarrow \rho_W \\ R\times R & \xrightarrow{\cdot} & R \end{array} \tag{8}$$

*commutes, i.e., $\tilde{W}$ is a homomorphism.*

(ii) $\odot_W$ *is the operator product in the algebra* $\mathcal{B}(\mathcal{H})$ *of bounded operators on* $\mathcal{H}$*, i.e.,* $\tilde{W}$ *is a representation.*

(iii) *There is a representation* $\tilde{D}$ *of* $G_R$ *in* $\mathfrak{C}$ *such that*

$$\tilde{W}(g)F(\mathfrak{c}\otimes\varphi)\tilde{W}(g)^* = F(\tilde{D}(g)\mathfrak{c}\otimes\tilde{\rho}(g)\varphi) \quad \textit{for all } g,\mathfrak{c},\varphi, \tag{9}$$

*where* $\tilde{\rho}(g)\varphi := \varphi(\tilde{\rho}(g)^{-1}\,\cdot)$*.*

*Proof of* (i). The diagram already commutes if the arrow representing $\odot_W$ is omitted.

For each $g \in \mathbf{G}_R$ and each $(a,b) \in \pi^{-1}(g)$ one has

$$\tilde{W}(\pm 1)\tilde{W}(\pi(a,b)) = W((\pm a,a)\circ(a,b)) = W(\pm a,b) = \tilde{W}(\pm(\pi(a,b))),$$

so $\tilde{W}(\pm 1) \cong \mathbb{Z}_2$ if and only if $\tilde{W}(\pm g) \cong \mathbb{Z}_2$.

If $\tilde{W}(\pm 1) \cong \mathbb{Z}_2$, then $\tilde{W}$ is a bijection, so $\odot_W$ is defined by

$$U \odot_W V := \tilde{W}(\tilde{W}^{-1}(U)\odot\tilde{W}^{-1}(V)).$$

If $\tilde{W}(\pm 1) \cong \{1\}$, then $\rho_W$ is a bijection, so $\odot_W$ is defined by

$$U \odot_W V := \rho_W^{-1}(\rho_W(U)\cdot\rho_W(V)).$$

*Proof of* (ii). The statement is nontrivial only on $\dot{\mathbf{G}}_R$. Given $g,h \in \dot{\mathbf{G}}_R$, the planes $\tilde{\mathbf{a}}(g)^\perp$ and $\tilde{\mathbf{a}}(h)^\perp$ intersect in an at least one-dimensional subspace, so one can choose $(a,b)\in\pi^{-1}(g)$ and $(c,d)\in\pi^{-1}(h)$ such that $b=c$ is in this intersection. Then

$$\begin{aligned}\tilde{W}(\pi(a,b)\odot\pi(c,d)) &= \tilde{W}(\pi((a,b)\circ(b,d)))\\ &= \tilde{W}(\pi(a,d)) = W(a,d)\\ &= J_aJ_d = J_aJ_bJ_bJ_d\\ &= W(a,b)W(b,d) = \tilde{W}(\pi(a,b))\tilde{W}(\pi(b,d))\\ &= \tilde{W}(\pi(a,b))\tilde{W}(\pi(c,d)).\end{aligned}$$

*Proof of* (iii). Define a map $D$ from $\mathbf{M}_R$ into the automorphism group $\mathrm{Aut}(\mathfrak{C})$ of $\mathfrak{C}$ by $D(a,b) := C_aC_b$. If $(a,b)\sim(c,d)$, then modular $\mathrm{P}_1\mathrm{CT}$-symmetry implies

$$\begin{aligned}F(C_aC_b\mathfrak{c}\otimes j_aj_b\varphi) &= W(a,b)F(\mathfrak{c}\otimes\varphi)W(a,b)^*\\ &= W(c,d)F(\mathfrak{c}\otimes\varphi)W(c,d)^*\\ &= F(C_cC_d\mathfrak{c}\otimes j_cj_d\varphi)\\ &= F(C_cC_d\mathfrak{c}\otimes j_aj_b\varphi)\end{aligned}$$

for all $\mathfrak{c}$ and all $\varphi$. Using Assumption (A.1), one obtains $C_aC_b\mathfrak{c} = C_cC_d\mathfrak{c}$ for all $\mathfrak{c}$, so $D(a,b) = D(c,d)$, and a map $\tilde{D}:\ \mathbf{G}_R \to \mathrm{Aut}(\mathfrak{C})$ is defined by $\tilde{D}(\pi(\underline{m})) := D(\underline{m})$. This map $\tilde{D}$ now inherits the representation property from $\tilde{W}$. □

**Theorem 6 (Spin-statistics connection)**

$$F_{\pm}(\mathfrak{c} \otimes \varphi) = \frac{1}{2}(1 \pm F(\tilde{D}(-1)\mathfrak{c} \otimes \varphi))$$

*for all* $\mathfrak{c}$ *and all* $\varphi$.

*Proof.* For each $a \in S^2$ one has

$$\tilde{W}(-1) = J_a J_{-a} = J_a \kappa J_a \kappa^\dagger = J_a^2 (\kappa^\dagger)^2 = k,$$

so

$$\begin{aligned} F(\mathfrak{c} \otimes \varphi) &= kF(\mathfrak{c} \otimes \varphi)k \\ &= \tilde{W}(-1)F(\mathfrak{c} \otimes \varphi)\tilde{W}(-1) \\ &= \tilde{W}(-1)F(\mathfrak{c} \otimes \varphi)\tilde{W}(-1)^\dagger \\ &= F(\tilde{D}(-1)\mathfrak{c} \otimes \varphi). \end{aligned}$$ □

If, in particular, $\tilde{D}$ is irreducible with spin $s$, then $\tilde{D}(-1) = e^{2\pi i s}$, so $F_- = 0$ for integer $s$ and $F_+ = 0$ for half-integer $s$.

## 4 PCT-symmetry

In order to justify the term "modular $\mathrm{P}_1$CT-symmetry", one should show that this condition yields, at least in 1+3 dimensions, a full PCT-operator in a base-independent fashion.

**Theorem 7 (PCT-symmetry)** *There exists an antiunitary involution* $\Theta$ *with the properties*

(i) $J_a J_b J_c = \Theta$ *for each right-handed orthogonal basis* $(a, b, c)$ *of* $e_0^\perp$.

(ii) *There exists an antilinear involution* $C$ *such that*

$$\Theta F(\mathfrak{c} \otimes \varphi)\Theta = F(C\mathfrak{c} \otimes \overline{\varphi}(-\,\cdot)).$$

*Proof.* Let $(a', b', c')$ be a second right-handed orthonormal base, and define $\Theta' := J_{a'} J_{b'} J_{c'}$. Then it follows from modular symmetry that

$$\begin{aligned} \Theta'\Theta F(\mathfrak{c} \otimes \varphi)\Omega &= \Theta'\Theta F(\mathfrak{c} \otimes \varphi)\Theta\Theta'\Omega \\ &= F(C_{a'} C_{b'} C_{c'} C_a C_b C_c \mathfrak{c} \otimes \varphi)\Omega \\ &= F(\tilde{D}(1)\mathfrak{c} \otimes \varphi)\Omega \\ &= F(\mathfrak{c} \otimes \varphi)\Omega. \end{aligned}$$

Since $\Omega$ is cyclic, this implies the statement. □

If $(a, b, c)$ is right-handed and $(a', b', c')$ is left-handed, then $\tilde{D}(1)$ has to be replaced by $\tilde{D}(-1)$ in the above computation. Since $J_{-a} J_{-b} J_{-c} = \kappa J_a J_b J_c \kappa^\dagger$, this is no surprise.

## Conclusion

Both the classical geometry and the fundamental quantum field theoretic representations of the rotation group $SO(3)$ and its universal covering group are based on reflection symmetries. At the classical level, the universal covering group $\mathbf{G}_R$ can be constructed from $\mathrm{P}_1$T-reflections. For a quantum field $F$ with $\widetilde{SO(3)}$-symmetry, a class of antiunitary $\mathrm{P}_1$CT-operators exists that are fixed by the intrinsic structure of the respective field. Along precisely the same lines of argument used for the construction of $\mathbf{G}_R$, a covariant unitary representation $\tilde{W}$ of $\mathbf{G}_R$ is constructed. $\tilde{W}$ exhibits Pauli's spin-statistics connection.

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## Appendix. $SU(2)$ versus $\mathbf{G}_R$

The isomorphism between the models $SU(2)$ and $\mathbf{G}_R$ of $\widetilde{SO(3)}$ can be described as follows.

First recall the standard representation of $SU(2)$ on $\mathbb{R}^3$. Denote by $\sigma_1, \ldots, \sigma_3$ the Pauli matrices, and define $\hat{x} := \sum_\nu x_\nu \sigma_\nu$, $x \in \mathbb{R}^3$. For each $\nu$, the map $\hat{x} \mapsto \mathrm{Ad}(\pm i\sigma_\nu)\hat{x}$ is well known to implement the rotation $[e_\nu, \pi]$. Since the parity transformation $P$ is implemented by the map $\hat{x} \mapsto -\hat{x}$, one finds that for each $\nu$, the map $\hat{x} \mapsto -\mathrm{Ad}(\pm\sigma_\nu)\hat{x}$ implements the reflection $j_\nu$. The determinants of the Pauli matrices equal $-1$, and all of them are involutions.

Now one can define an isomorphism $\mathfrak{J}$ from $S^2$ onto the unitary matrices with determinant $-1$ by $\mathfrak{J}(a) := a\vec{\sigma}$. The products of pairs of unitary matrices with determinant $-1$ yield all of $SU(2)$.

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