

# Geometric Modular Action and Spontaneous Symmetry Breaking

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Dedicated to the memory of Siegfried Schlieder

**Abstract.** We study spontaneous symmetry breaking for field algebras on Minkowski space in the presence of a condition of geometric modular action (CGMA) proposed earlier as a selection criterion for vacuum states on general space-times. We show that any internal symmetry group must commute with the representation of the Poincaré group (whose existence is assured by the CGMA) and each translation-invariant vector is also Poincaré invariant. The subspace of these vectors can be centrally decomposed into pure invariant states and the CGMA holds in the resulting sectors. As positivity of the energy is not assumed, similar results may be expected to hold for other space-times.

## 1 Introduction

There are a number of physically relevant mechanisms which entail a degeneracy of the vacuum state in quantum field theory. Primary among these is the spontaneous symmetry breaking of an internal symmetry group. Initiated by Borchers and by Reeh and Schlieder, systematic study [1, 5, 6, 19, 29, 30, 35] in quantum field theories satisfying the Wightman axioms [36] or the standard axioms of algebraic quantum field theory [22, 23] has shown that the presence of multiple vacua determines much of the global structure of the theory. Common to these approaches is the assumption of the positivity of the energy, with its concomitant analyticity properties. In Minkowski space the spectrum condition is a natural and physically meaningful assumption, but in other space-times it is neither.

It is therefore of interest to revisit both spontaneous symmetry breaking and the structural consequences of degenerate vacua with the standard axioms for Minkowski space theories replaced by a recently proposed condition of geometric modular action (CGMA) [10, 13]. This condition is designed to characterize those elements in the state space of a quantum system which admit an interpretation as a “vacuum”. It is expressed in terms of the modular conjugations associated to the state and given family of algebras indexed by suitable subregions (wedges) of the underlying space-time and, in principle, can be applied to theories on any space-time manifold. For a motivation of this condition and applications to theories in Minkowski, de Sitter, anti-de Sitter and a class of Robertson–Walker space-times, we refer the interested reader to [11–15, 17]. In this paper we shall restrict our

attention to four-dimensional Minkowski space, but the arguments are applicable to other space-times, yielding similar results.

We shall consider an arbitrary group  $G$  as the internal symmetry group of a quantum field theory formulated in the algebraic context [23]. Hence, we shall assume there exists a net  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  of von Neumann algebras indexed by the set  $\mathcal{W}$  of all wedges (Poincaré transforms of the set  $\{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_1 > |x_0|\}$ ) in Minkowski space and acting upon a separable Hilbert space  $\mathcal{H}$ , and that there exists a unitary representation  $V$  of the group  $G$  such that

$$V(g)\mathcal{R}(W)V(g)^{-1} = \mathcal{R}(W) \quad , \quad g \in G, W \in \mathcal{W}.$$

We shall assume that there is a vacuum vector  $\Omega_0 \in \mathcal{H}$  invariant under  $V(G)$  but make no assumption about the invariance properties of the other vacua. Indeed, one of the situations we are interested in including in our analysis is the case where the various vacua are permuted among themselves by the action of  $V(G)$ .

After specifying the working assumptions of this paper in Section 2, we shall show that in the presence of the CGMA, the internal symmetries must commute with the representation of the Poincaré group, whose existence is assured by the CGMA and which is constructed using the modular conjugations. In Section 3 we shall investigate the global structure of the observable algebras and prove that any translation-invariant vectors must also be Poincaré-invariant, in contrast to what is known about vectors invariant under representations of the translation group which satisfy the spectrum condition but do not arise from modular objects [1, 19]. We then prove that under the central decomposition of the global observable algebra all relevant structures are preserved. Finally, in Section 4 we show that the CGMA and the modular stability condition introduced in [13] manifest some remarkable rigidity properties.

## 2 Modular action and internal symmetries

Although the arguments presented here apply more generally, for convenience we assume that the net  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  is locally generated in the sense defined in [16] with a generating family  $\mathcal{C}$  of convex compact space-time regions  $\mathcal{O}$ . Roughly speaking, this means that every algebra  $\mathcal{R}(W)$  is generated by the family of all algebras  $\mathcal{R}(\mathcal{O})$  with  $\mathcal{O} \in \mathcal{C}$  and  $\mathcal{O} \subset W$ . This subsumes such familiar examples as nets generated by algebras associated with the set of double cones. Note that nets affiliated with quantum field theories satisfying the Wightman axioms are locally generated in this sense [37]. For notational simplicity, we shall only consider bosonic theories here.

We shall be assuming that the  $V(G)$ -invariant unit vector  $\Omega_0 \in \mathcal{H}$  is cyclic and separating for  $\mathcal{R}(W)$ , for every  $W \in \mathcal{W}$ .<sup>1</sup> Thus, the Tomita–Takesaki modular theory will be applicable, cf. [8, 25]. In the following  $J_W$ , resp.  $\Delta_W$ , will

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<sup>1</sup>The fundamental insight that under physically motivated conditions the vacuum vector is cyclic and separating for the quantum fields localized in wedge regions is due to Reeh and Schlieder [34].

denote the modular conjugation, resp. the modular operator, associated to the pair  $(\mathcal{R}(W), \Omega_0)$  by the modular theory. Also, we shall use  $\mathcal{J}$  to represent the group generated by the set  $\{J_W \mid W \in \mathcal{W}\}$ .

The following are included in the standing assumptions of this paper.

- (a)  $W \mapsto \mathcal{R}(W)$  is an order-preserving bijection.
- (b)  $\Omega_0$  is cyclic and separating for  $\mathcal{R}(W)$ , given any  $W \in \mathcal{W}$ .
- (c) For all  $W_0, W \in \mathcal{W}$ ,  $J_{W_0}\mathcal{R}(W)J_{W_0} = \mathcal{R}(\lambda_{W_0}W)$ , where  $\lambda_{W_0} \in \mathcal{P}_+$  is the reflection through the edge of the wedge  $W_0$ .

In [13,16] the Condition of Geometric Modular Action (CGMA), formulated solely in terms of the vector  $\Omega_0$  and the net  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  without any *a priori* assumptions about the specific form of the adjoint action of the modular conjugations on  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  or even the existence of an isometry group, was shown to entail conditions (a)–(c). It has also been shown in [3] that (c) must hold for any nets  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  locally associated with finite-component quantum fields satisfying the Wightman axioms. Note that condition (c) implies that the adjoint action of any modular conjugation  $J_W$  leaves the set  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  invariant. As the surjectivity of the map in (a) is automatic and the order-preserving property is just the operationally motivated condition of isotony, only the significance of the injectivity assumption is not immediately clear. It is shown in the Appendix that if the injectivity condition is dropped, the remaining assumptions imply that the algebras  $\mathcal{R}(W)$  are all Abelian and independent of localization region  $W$ . Such a situation is of no interest in quantum field theory. Hence, there is no loss of physical generality to include in our standing assumptions the requirement that in no subrepresentation of the net  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  are the wedge algebras abelian.

Condition (c) and modular theory imply  $\mathcal{R}(W)' = J_W\mathcal{R}(W)J_W = \mathcal{R}(W')$  for any wedge  $W \in \mathcal{W}$ , where  $W' \in \mathcal{W}$  denotes the causal complement of  $W$ . Thus, the net fulfills wedge duality and hence *a fortiori* locality. An immediate consequence of this fact is the following result about the type of the global algebra generated by the wedge algebras. It is in perfect concord with the idea that the CGMA characterizes elementary states.

**Proposition 2.1** *Let  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega_0$  be a net and vector satisfying the standing assumptions, and let  $\mathcal{R} = \bigvee_{W \in \mathcal{W}} \mathcal{R}(W)$ . Then  $\mathcal{R}' \subset \mathcal{R}$  and  $\mathcal{R}$  is of type I.*

*Proof.* Because of wedge duality,  $\mathcal{R}' = \bigwedge_{W \in \mathcal{W}} \mathcal{R}(W)' = \bigwedge_{W \in \mathcal{W}} \mathcal{R}(W') \subset \mathcal{R}$ . Hence  $\mathcal{R}'$  coincides with the center of  $\mathcal{R}$ , proving that  $\mathcal{R}$  is of type I.       $\square$

As shown in [13,16], the standing assumptions also imply that there exists a strongly continuous (anti)unitary representation  $U$  of the proper Poincaré group  $\mathcal{P}_+$  on four-dimensional Minkowski space, which is constructed in a canonical manner from products of the modular conjugations  $J_W \in \mathcal{J}$  so that  $U(\lambda_W) = J_W$ ,

for all  $W \in \mathcal{W}$ . Indeed, one has  $\mathcal{J} = U(\mathcal{P}_+)$ , so that  $\mathcal{J}$  is closed in the strong- $*$ -topology. One therefore has  $U(\lambda)\Omega_0 = \Omega_0$ , for all  $\lambda \in \mathcal{P}_+$ . The representation  $U$  acts covariantly upon the net:

$$U(\lambda)\mathcal{R}(W)U(\lambda)^{-1} = \mathcal{R}(\lambda W),$$

for all  $\lambda \in \mathcal{P}_+$ ,  $W \in \mathcal{W}$ .

Since the representation of the Poincaré group is constructed out of modular involutions, a number of results which are difficult or not possible to obtain in other settings follow easily in the presence of the CGMA. Indeed, since  $V(g)\Omega_0 = \Omega_0$  and  $V(g)\mathcal{R}(W)V(g)^{-1} = \mathcal{R}(W)$ , for all  $g \in G$  and  $W \in \mathcal{W}$ , a basic result of modular theory entails that  $V(g)$  commutes with all modular involutions  $J_W$ ,  $W \in \mathcal{W}$ , cf. [8, Corollary 2.5.32]. The commutation of  $V(G)$  with  $U(\mathcal{P}_+)$  is therefore immediate.

**Theorem 2.2** *If the standing assumptions are fulfilled,  $V(g)U(\lambda) = U(\lambda)V(g)$ , for all  $g \in G$  and  $\lambda \in \mathcal{P}_+$ .*

We note that the CGMA, and hence also the standing assumptions, can be satisfied by examples in which the spectrum condition is violated [13]. Landau and Wichmann showed that in the context of a local net in an irreducible vacuum representation (with spectrum condition) the internal symmetry group must commute with the representation of the translation group [28]. With the further assumptions that there is a mass gap in the theory and that for each particle in the theory there exists a field with non-zero matrix elements between the vacuum and the one-particle states, Landau proved that the internal symmetry group must commute with the representation of the Poincaré group [27]. From another more technical set of assumptions, Bisognano and Wichmann [4] were able to derive the same conclusion. Common to all these earlier approaches is the assumption of the spectrum condition.

### 3 Invariance and decomposition

Let  $\mathcal{Z}$  denote the center of the algebra  $\mathcal{R} = \bigvee_{W \in \mathcal{W}} \mathcal{R}(W)$  and  $\mathcal{Z}(W)$  denote the center of  $\mathcal{R}(W)$ . Furthermore, let  $\mathcal{Z}_s$  represent the set of all self-adjoint elements of  $\mathcal{Z}$ . We recall from Proposition 2.1 that  $\mathcal{Z} = \mathcal{R}'$  and continue with some useful properties of these algebras.

**Proposition 3.1** *Under the standing assumptions,  $\mathcal{Z} \subset U(\mathcal{P}_+^\uparrow)'$ ,  $\mathcal{Z}_s \subset U(\mathcal{P}_+)'$  and  $\mathcal{Z} \subset \mathcal{Z}(W)$ , for all  $W \in \mathcal{W}$ .*

**Remark.** Since  $\mathcal{Z}' = \mathcal{R}$ , it follows from this result that the unitaries  $U(\mathcal{P}_+^\uparrow)$  are elements of the global algebra  $\mathcal{R}$ , i.e., the Poincaré transformations are weakly inner. Again, this is in accord with the idea that the CGMA characterizes elementary states.

*Proof.* As  $\mathcal{Z} = \mathcal{R}' \subset \mathcal{R}(W)' = \mathcal{R}(W')$  for any  $W \in \mathcal{W}$ , one obtains  $\mathcal{Z} \subset \mathcal{Z}(W)$  for all  $W \in \mathcal{W}$ . But one knows from [2, Lemma 3] that  $J_W A J_W = A^*$ , for all  $A \in \mathcal{Z}(W)$ . Since every element of  $U(\mathcal{P}_+^\uparrow)$  is a product of an even number of modular conjugations and every element of  $U(\mathcal{P}_+)$  is the product of  $J_W$  and an element of  $U(\mathcal{P}_+^\uparrow)$  [13], the remaining claims follow at once.  $\square$

Let  $E_0$  be the orthogonal projection onto the subspace of  $\mathcal{H}$  consisting of  $U(\mathbb{R}^4)$ -invariant vectors. Hence, we have  $\Omega_0 \in E_0\mathcal{H}$ . It therefore follows from the preceding result that  $\overline{\mathcal{Z}\Omega_0} \subset E_0\mathcal{H}$ . We shall see that the converse also holds. But, first, we adapt classic arguments [1, 5, 19, 35] to prepare some intermediate results. Let  $a \in \mathbb{R}^4$  be a spacelike translation, and set  $a_n = na$ , for each  $n \in \mathbb{N}$ . Let  $\mathcal{O} \in \mathcal{C}$ ,  $A \in \mathcal{R}(\mathcal{O})$  and  $A(a_n) = U(a_n)AU(a_n)^{-1}$ . Since the sequence  $\{A(a_n)\}$  is uniformly bounded in norm and  $\mathcal{H}$  is separable, there exists a subsequence  $\{A(a_{n_k})\}$  which is weakly convergent. By the standing assumption on  $\mathcal{C}$ , for any  $\tilde{\mathcal{O}} \in \mathcal{C}$  there exists an  $N \in \mathbb{N}$  and a wedge  $W_N \in \mathcal{W}$  such that  $\tilde{\mathcal{O}} \subset W_N$  and, for every  $n \geq N$ ,  $\mathcal{O} + na \subset W_N'$ , i.e.,  $A(a_n) \in \mathcal{R}(W_N') = \mathcal{R}(W_N)' \subset \mathcal{R}(\tilde{\mathcal{O}})'$ . Since  $\mathcal{R}$  is generated by the algebras  $\mathcal{R}(\tilde{\mathcal{O}})$ ,  $\tilde{\mathcal{O}} \in \mathcal{C}$ , the weak limit of the corresponding subsequence  $\{A(a_{n_k})\}$ , call it  $A_\infty$ , is an element of  $\mathcal{R}' = \mathcal{Z}$ . Moreover, [5, Lemma 4] implies

$$A_\infty\Omega_0 = w - \lim_{k \rightarrow \infty} A(a_{n_k})\Omega_0 = w - \lim_{k \rightarrow \infty} U(a_{n_k})A\Omega_0 = E_0A\Omega_0. \tag{3.1}$$

Let  $\mathcal{Y} = \{A_\infty \mid A \in \mathcal{R}(\mathcal{O}), \mathcal{O} \in \mathcal{C}\} \subset \mathcal{Z}$  denote the set of all such weak limit points. Since  $\Omega_0$  is cyclic for  $\mathcal{R}$  it follows from relation (3.1) that  $\mathcal{Y}\Omega_0$  is a dense subset of  $E_0\mathcal{H}$ . Thus, since  $\mathcal{Y} \subset \mathcal{Z}$  and  $\mathcal{Z}\Omega_0 \subset E_0\mathcal{H}$  we arrive at the following statement.

**Proposition 3.2** *Under the standing assumptions, one has  $E_0\mathcal{H} = \overline{\mathcal{Z}\Omega_0} = \overline{\mathcal{Y}\Omega_0}$ .*

The following result is an easy consequence of the preceding proposition and the inclusion  $\mathcal{Y} \subset \mathcal{Z}$ , established before.

**Corollary 3.3** *Given the standing assumptions, one has the equality  $\mathcal{Y} = \mathcal{Z}$ .*

*Proof.* It was shown in Proposition 3.2 that  $\overline{\mathcal{Y}\Omega_0} = E_0\mathcal{H}$ . Thus, the restriction of the abelian algebra  $\mathcal{Y}$  to the subspace  $E_0\mathcal{H}$  has  $\Omega_0$  as a cyclic vector. It follows that  $\mathcal{Y}$  is maximally abelian on  $E_0\mathcal{H}$ . Since  $\mathcal{Y}$  is contained in the abelian algebra  $\mathcal{Z}$ , the restrictions of  $\mathcal{Y}$  and  $\mathcal{Z}$  to  $E_0\mathcal{H}$  must coincide. The desired assertion then follows, because  $\Omega_0$  is separating for  $\mathcal{Z}$ .  $\square$

In a vacuum representation fulfilling the standard assumptions, including the spectrum condition, it is known [19] that  $\mathcal{Z} = \mathcal{Z}(W)$ , for all  $W \in \mathcal{W}$ , but this need not be the case in the setting considered here.

After these preparations, we proceed to the central decomposition of  $\mathcal{R}$ . Since the center  $\mathcal{Z}$  of  $\mathcal{R}$  coincides with the commutant  $\mathcal{R}'$ , this amounts to a decomposition of the underlying Hilbert space into irreducible subsectors. Moreover, as the

Poincaré transformations are weakly inner, the representation  $U(\mathcal{P}_+)$  decomposes into a continuous unitary representation of  $\mathcal{P}_+$  in each sector. In particular, the Lorentz group is not spontaneously broken by this decomposition, which is to be contrasted with the existence of examples of nets in vacuum representations (satisfying the spectrum condition but not the CGMA) in which the Lorentz group is spontaneously broken in the central decomposition of  $\mathcal{R}$  [1, 19]. In [19] it was shown that modular covariance (see below for a definition) prevents spontaneous breaking of the Lorentz group; here it is the CGMA which assures the stability of each vacuum sector under the action of the Lorentz group. Note that the CGMA is known to hold more generally than modular covariance does [13].

The proof of our decomposition theorem rests upon the theory of direct integral decomposition of a von Neumann algebra presented in [18]. The algebra  $\mathcal{R}$  is decomposed with respect to the abelian algebra  $\mathcal{Z}$  to yield a standard Borel measure space  $(S, \nu)$  and measurable families  $\zeta \mapsto \mathcal{H}(\zeta)$  of Hilbert spaces and  $\zeta \mapsto \mathcal{R}(\zeta)$  of von Neumann algebras such that

$$\mathcal{H} = \int_S^\oplus \mathcal{H}(\zeta) \, d\nu(\zeta), \quad \mathcal{R} = \int_S^\oplus \mathcal{R}(\zeta) \, d\nu(\zeta).$$

For  $\nu$ -almost all  $\zeta$ ,  $\mathcal{R}(\zeta)$  is a factor [18, Thm. II.3.3].

But here we are concerned with the decomposition of a great deal more structure. Though it is clear from Proposition 3.1 and 3.2 that the algebras  $\mathcal{R}(W)$ ,  $W \in \mathcal{W}$ , and the group  $U(\mathcal{P}_+) = \mathcal{J}$  also decompose, it is necessary to find a set  $N \subset S$  with  $\nu(N) = 0$  such that for every  $\zeta \in S \setminus N$  all of the decomposed structures still have the original properties. However, this involves *prima facie* uncountably many conditions, which could lead to a zero-set catastrophe.

The standard technique to handle this technical problem is to impose only countably many of these conditions, each of which would hold for all  $\zeta$  except in a set of measure zero. Since  $\nu$  is countably additive, all countably many conditions would hold except in a possibly larger set  $N$  of measure zero. One then employs a suitable limit argument to assure that the remaining conditions also hold for all  $\zeta \in S \setminus N$ . Of course, a countable union of countable sets is countable, and it is only a matter of taste or convenience whether one imposes in the argument the countable union of conditions at once or each countable subset after the other. Since the decomposition of many of the structures we are concerned with here has already been carefully treated in the literature, we shall only indicate details which seem to involve new arguments.

We recall some facts from [13, 16]. Making use of the fact that  $\mathcal{P}_+^\uparrow$  acts transitively on  $\mathcal{W}$ , we identify  $\mathcal{W}$ , as a topological space, with the quotient space  $\mathcal{P}_+^\uparrow/\mathcal{P}_0$ , where  $\mathcal{P}_0 \subset \mathcal{P}_+^\uparrow$  is the invariance subgroup of any given wedge  $W_0 \in \mathcal{W}$ ; note that the topology does not depend on the choice of  $W_0$ . As  $\mathcal{P}_+^\uparrow/\mathcal{P}_0$  is separable, so is  $\mathcal{W}$ . In order to successfully decompose all the structures of interest in such a manner that the zero set catastrophe is avoided, we need to be able to choose a countable,

dense subgroup  $\widehat{\mathcal{P}} \subset \mathcal{P}_+^\uparrow$  and a countable, dense subset  $\widehat{\mathcal{W}} \subset \mathcal{W}$  satisfying the following conditions:

- (i) The elements of  $\widehat{\mathcal{P}}$  leave  $\widehat{\mathcal{W}}$  stable and  $\widehat{\mathcal{P}}$  acts transitively upon  $\widehat{\mathcal{W}}$ .
- (ii) For any  $W_1, W_2 \in \mathcal{W}$  such that  $W_1 \subset W_2$ , there exist two sequences  $\{W_{1,n}\}, \{W_{2,n}\} \subset \widehat{\mathcal{W}}$  such that  $\{W_{i,n}\}$  converges to  $W_i$ ,  $i = 1, 2$ , and  $W_{1,n} \subset W_{2,n}$ , for all  $n \in \mathbb{N}$ .

The reader may verify that these conditions are fulfilled if  $\widehat{\mathcal{P}}$  is chosen to be the semi-direct product of rational translations with the image under the canonical projection homomorphism of the subgroup of the covering group  $SL(2, \mathbb{C})$  whose elements have entries with only rational real and imaginary parts, and  $\widehat{\mathcal{W}}$  is chosen to be  $\widehat{\mathcal{P}}W_0$  for some fixed wedge  $W_0$ .

**Theorem 3.4** *Under the standing assumptions, the central decomposition of  $\mathcal{R}$  leads to a unique<sup>2</sup> integral decomposition of the given structures into irreducible, Poincaré-covariant nets. Precisely, there exists a measure  $\nu$  on the spectrum  $S$  of  $\mathcal{Z}$  and measurable families of Hilbert spaces  $\zeta \rightarrow \mathcal{H}(\zeta)$ , von Neumann algebras  $\zeta \rightarrow \mathcal{R}(\zeta) \subset \mathcal{B}(\mathcal{H}(\zeta))$ , and strongly continuous (anti)unitary representations of the proper Poincaré group  $\zeta \rightarrow U(\mathcal{P}_+)(\zeta)$  such that*

$$\mathcal{H} = \int_S^\oplus \mathcal{H}(\zeta) \, d\nu(\zeta), \quad \mathcal{R} = \int_S^\oplus \mathcal{R}(\zeta) \, d\nu(\zeta), \quad U(\lambda) = \int_S^\oplus U(\lambda)(\zeta) \, d\nu(\zeta),$$

for all  $\lambda \in \mathcal{P}_+$ . Moreover, for each  $W \in \mathcal{W}$ , there exists a measurable family of von Neumann algebras  $\zeta \rightarrow \mathcal{R}(W)(\zeta) \subset \mathcal{B}(\mathcal{H}(\zeta))$  such that

$$\mathcal{R}(W) = \int_S^\oplus \mathcal{R}(W)(\zeta) \, d\nu(\zeta), \tag{3.2}$$

and such that isotony is satisfied by  $\{\mathcal{R}(W)(\zeta)\}_{W \in \mathcal{W}}$   $\nu$ -almost everywhere. For  $\nu$ -almost all  $\zeta$ ,  $\mathcal{R}(\zeta) = \mathcal{B}(\mathcal{H}(\zeta))$ ,  $E_0(\zeta)\mathcal{H}(\zeta) = (E_0\mathcal{H})(\zeta)$  is one-dimensional, and

$$U(\lambda)(\zeta) \mathcal{R}(W)(\zeta) U(\lambda)(\zeta)^{-1} = \mathcal{R}(\lambda W)(\zeta), \tag{3.3}$$

for all  $\lambda \in \mathcal{P}_+$  and  $W \in \mathcal{W}$ .

*Proof.* The decomposition of the Hilbert space and algebra  $\mathcal{R}$  is explained in [18]. As already mentioned, the factorial components  $\mathcal{R}(\zeta)$  in the central decomposition of  $\mathcal{R}$  indeed act irreducibly on the respective subspaces  $\mathcal{H}(\zeta)$  since  $\mathcal{Z} = \mathcal{R}'$  by Proposition 2.1. The representation  $U(\mathcal{P}_+^\uparrow)$  of the identity component of the Poincaré group and the subspace  $E_0\mathcal{H}$  are decomposed in [19], and the attendant

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<sup>2</sup>The measure space  $(S, \nu)$  is unique up to isomorphism, and given  $(S, \nu)$  the measurable fields are unique up to unitary equivalence. See Section II.6.3 in [18] for details.

assertions made above are proven there, using results in [24]. Although the net  $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$  was also decomposed in [19], there the argument was framed for locally generated nets for which  $\mathcal{C}$  is the set of double cones; a concrete choice of a countable “dense” subcollection of double cone algebras was given there. To obtain the assertion in the generality made here, one must provide another argument.

Instead, here one decomposes the elements of the countable set  $\{\mathcal{R}(W)\}_{W \in \widehat{\mathcal{W}}}$  to obtain for each  $W \in \widehat{\mathcal{W}}$  a measurable family  $\zeta \mapsto \mathcal{R}(W)(\zeta)$  such that (3.2) holds. By enlarging the zero set  $N$ , if necessary, the covariance (3.3) in  $\nu$ -almost all sectors holds for all  $W \in \widehat{\mathcal{W}}$  and  $\lambda \in \widehat{\mathcal{P}}$ .

Theorem II.3.1 in [18] guarantees that for a fixed pair of wedges such that  $W_1 \subset W_2$ , the containment  $\mathcal{R}(W_1)(\zeta) \subset \mathcal{R}(W_2)(\zeta)$  holds  $\nu$ -almost everywhere. After a possible change of the set  $N$ , the same is true for all  $W_1, W_2 \in \widehat{\mathcal{W}}$  with  $W_1 \subset W_2$ .

For an arbitrary  $W \in \mathcal{W}$ , there exists an element  $\lambda_0 \in \mathcal{P}_+^\uparrow$  such that  $W = \lambda_0 W_0$ . By construction, there exists a sequence  $\{\lambda_n\} \subset \widehat{\mathcal{P}}$  which converges to  $\lambda_0$ . Define

$$\mathcal{R}(W)(\zeta) = \{w - \lim_{n \rightarrow \infty} U(\lambda_n)(\zeta)A(\zeta)U(\lambda_n)(\zeta)^{-1} \mid A(\zeta) \in \mathcal{R}(W_0)(\zeta)\}.$$

The strong continuity of  $U(\mathcal{P}_+^\uparrow)(\zeta)$  in these sectors entails that  $\mathcal{R}(W)(\zeta)$  is independent of the choice of such a sequence. Moreover, the same continuity implies that (3.3) is valid for all  $W \in \mathcal{W}$ ,  $\lambda \in \mathcal{P}_+^\uparrow$ , and the definition of  $\mathcal{R}(W)(\zeta)$  is compatible with all elements of the construction. In particular  $\mathcal{R}(W)(\zeta) = U(\lambda_0)(\zeta)\mathcal{R}(W_0)(\zeta)U(\lambda_0)(\zeta)^{-1}$ . By the measurability of  $\zeta \mapsto U(\mathcal{P}_+^\uparrow)(\zeta)$  and the covariance of the original net, it follows that the family  $\zeta \mapsto \mathcal{R}(W)(\zeta)$  is measurable and that (3.2) holds for all  $W \in \mathcal{W}$ . The isotony in  $\nu$ -almost all sectors for wedge algebras indexed by the elements of  $\mathcal{W}$  now follows easily from property (ii) above and the already-established isotony for wedge algebras indexed by  $\widehat{\mathcal{W}}$ .

Finally, as  $U(\mathcal{P}_+^\downarrow) = U(\lambda_W)U(\mathcal{P}_+^\uparrow)$ , for fixed  $W \in \mathcal{W}$ , the assertion concerning  $U(\mathcal{P}_+)$  follows, since the complex antilinearity of  $U(\lambda_W) = J_W$ , *i.e.*, the fact that  $U(\lambda_W)$  commutes with  $\mathcal{Z}_s$  but not with  $\mathcal{Z}$ , poses no problems [24, Thm. III.2]. □

It is noteworthy that, in our general setting, the above central decomposition always results in irreducible sectors even though the spectrum condition need not hold. This is in contrast to the situation in the Wightman formalism where the extremal states resulting from a corresponding decomposition need not be pure states [6, 7] – cf. [20] for a discussion of this matter.

We close this section with a comment about unbroken symmetries in the internal symmetry group  $G$ . The group  $G$  will be unitarily implemented in a given sector if and only if  $V(G)$  commutes with the corresponding projection in  $\mathcal{Z}$ . On the other hand, if  $G$  is a separable topological group, the representation  $g \mapsto V(g)$  is strongly continuous, and there exists a subgroup  $H \subset G$  such that  $\mathcal{Z} \subset V(H)'$ ,



then the above arguments entail that there exists a measurable family of strongly continuous unitary representations  $\zeta \rightarrow V(H)(\zeta)$  such that

$$V(h) = \int_S^{\oplus} V(h)(\zeta) \, d\nu(\zeta),$$

and

$$V(h)(\zeta) \mathcal{R}(W)(\zeta) V(h)(\zeta)^{-1} = \mathcal{R}(W)(\zeta),$$

for all  $h \in H$ ,  $W \in \mathcal{W}$  and  $\nu$ -almost all  $\zeta$ .

In the next section, we prove that the modular structure associated with pairs  $(\mathcal{R}(W), \Omega_0)$ ,  $W \in \mathcal{W}$ , also decomposes in such a manner that conditions (a)–(c) are satisfied in  $\nu$ -almost all sectors.

### 4 The rigidity of geometric modular action

We maintain the standing assumptions in this section and turn our attention to the modular structures, their properties and their behavior under the central decomposition carried out above. Let

$$\mathcal{P}_W^{\natural} = \overline{\{\Delta_W^{1/4} A \Omega_0 \mid A \in \mathcal{R}(W)_+\}} \tag{4.1}$$

denote the natural positive cone corresponding to the pair  $(\mathcal{R}(W), \Omega_0)$ , where  $\mathcal{R}(W)_+$  is the set of all positive elements in  $\mathcal{R}(W)$ , and let

$$\mathcal{P}_0 = \bigcap_{W \in \mathcal{W}} \mathcal{P}_W^{\natural}.$$

Of course, we have  $\Omega_0 \in \mathcal{P}_0$ . As shown in [2], every vector  $\Phi \in \mathcal{P}_W^{\natural}$ , which is either cyclic or separating for  $\mathcal{R}(W)$ , is both cyclic and separating for  $\mathcal{R}(W)$ . Moreover, the modular conjugation  $J_W^{\Phi}$  corresponding to the pair  $(\mathcal{R}(W), \Phi)$  coincides with  $J_W$  [2, Thm. 4]. Hence, if  $\Omega \in \mathcal{P}_0$  is cyclic or separating for all  $\mathcal{R}(W)$ ,  $W \in \mathcal{W}$ , then  $J_W^{\Omega} = J_W$ , for every  $W \in \mathcal{W}$ . Thus, the pair  $(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega)$  must also fulfill conditions (a)–(c), if  $(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega_0)$  does. The CGMA therefore selects state vectors which lie in  $\mathcal{P}_0$ , and so we wish to investigate the structure of  $\mathcal{P}_0$  and the properties of the states determined by the elements of  $\mathcal{P}_0$ .

We begin with the following lemma.

**Lemma 4.1** *Under the standing assumptions,  $\mathcal{P}_0$  is a pointed, weakly closed convex cone such that*

$$\langle \Omega, A J_W A \Omega_0 \rangle \geq 0, \tag{4.2}$$

for all  $\Omega \in \mathcal{P}_0$ ,  $W \in \mathcal{W}$  and  $A \in \mathcal{R}(W)$ .

*Proof.* It is shown in [2, Thm. 4] that  $\mathcal{P}_W^{\natural}$  is a pointed, weakly closed, selfdual convex cone. Since  $\mathcal{P}_0$  is an intersection of these cones, it is clearly a weakly closed

convex cone. Moreover, if  $\Omega$  and  $-\Omega$  are contained in  $\mathcal{P}_0$ , they are also in  $\mathcal{P}_W^\natural$ ; hence,  $\Omega = 0$ . In the same theorem it is shown that  $\langle \Omega, AJ_W A \Omega_0 \rangle \geq 0$ , for all  $\Omega \in \mathcal{P}_W^\natural$  and  $A \in \mathcal{R}(W)$ . Since  $\mathcal{P}_0 \subset \mathcal{P}_W^\natural$ , for all  $W \in \mathcal{W}$ , the final assertion follows. □

This lemma enables us to prove the following result.

**Proposition 4.2** *Under the standing assumptions, every element of  $\mathcal{P}_0$  is invariant under  $U(\mathcal{P}_+)$ ; in particular,  $\mathcal{P}_0 \subset E_0\mathcal{H}$ . In fact,  $E_0\mathcal{H}$  is the linear span of  $\mathcal{P}_0$  and  $\mathcal{P}_0 = \overline{\mathcal{Z}_+\Omega_0}$ .*

*Proof.* Theorem 4 (2) in [2] entails that if  $\Omega \in \mathcal{P}_0$ , then  $J_W \Omega = \Omega$ , for all  $W \in \mathcal{W}$ . Since  $U(\mathcal{P}_+) = \mathcal{J}$ , one has  $U(\lambda)\Omega = \Omega$ , for every  $\lambda \in \mathcal{P}_+$ . Thus, in particular,  $\mathcal{P}_0 \subset E_0\mathcal{H}$ . A basic result of modular theory (cf. [8, Lemma 3.2.16]) entails that every element of the center  $\mathcal{Z}(W)$  is left invariant by the adjoint action of the modular unitaries  $\Delta_W^{it}$ ,  $t \in \mathbb{R}$ . Hence, Proposition 3.1 implies that for every  $Z \in \mathcal{Z}_+$  one has  $\Delta_W^{1/4} Z \Omega_0 = Z \Omega_0$ , and thus  $Z \Omega_0 \in \mathcal{P}_W^\natural$ , for every  $W \in \mathcal{W}$ , by (4.1). This entails the inclusion  $\mathcal{Z}_+ \Omega_0 \subset \mathcal{P}_0$ . Proposition 3.2 then implies that  $E_0\mathcal{H}$  is the linear span of  $\mathcal{P}_0$ .

Since  $\mathcal{P}_0 \subset E_0\mathcal{H} = \overline{\mathcal{Z}\Omega_0}$ , there exists a normal operator  $Z$  affiliated with  $\mathcal{Z}$  such that  $\Omega = Z \Omega_0$ . But for any  $A \in \mathcal{Z}_+$  one has  $A = A^{1/2} J_W A^{1/2} J_W$ , so that (4.2) yields  $\langle \Omega, A \Omega_0 \rangle \geq 0$ , for all  $A \in \mathcal{Z}_+$ . Setting  $A = B^* B$ ,  $B \in \mathcal{Z}$ , this implies

$$0 \leq \langle Z \Omega_0, B^* B \Omega_0 \rangle = \langle Z B \Omega_0, B \Omega_0 \rangle.$$

The restriction of  $Z$  to  $\mathcal{Z}\Omega_0$  is therefore positive. But  $Z$  can be decomposed into four positive operators  $Z_+, Z_-, \tilde{Z}_+, \tilde{Z}_-$  affiliated with  $\mathcal{Z}$  such that  $Z = Z_+ - Z_- + i(\tilde{Z}_+ - \tilde{Z}_-)$ , and since  $\Omega_0$  is separating for  $\mathcal{Z}$ , it follows that  $Z = Z_+$ . □

Although the modular conjugations associated with a given von Neumann algebra and different cyclic and separating vectors from  $\mathcal{P}^\natural$  coincide, typically the corresponding modular unitaries differ from vector to vector. However, the rigidity of the structure investigated here carries through also to the modular operators.

**Corollary 4.3** *Under the standing assumptions, if  $\Omega \in \mathcal{P}_0$  is cyclic or separating for  $\mathcal{R}(W)$  and  $\Delta_W^\Omega$  is the associated modular operator, then  $\Delta_W^\Omega = \Delta_W$ .*

*Proof.* By Proposition 4.2, there exists a positive operator  $Z$  affiliated with  $\mathcal{Z}$  such that  $\Omega = Z \Omega_0$ . This operator, just as every positive element of  $\mathcal{Z} \subset \mathcal{Z}(W)$ , commutes with the antiunitary  $J_W$ , the algebra  $\mathcal{R}(W) \vee \mathcal{R}(W)'$  and with any modular group associated with  $\mathcal{R}(W)$ . Hence, for any  $A \in \mathcal{R}(W)$  one has

$$\begin{aligned} (\Delta_W^\Omega)^{1/2} A \Omega &= J_W A^* \Omega = J_W A^* Z \Omega_0 = J_W A^* J_W Z \Omega_0 \\ &= Z J_W A^* \Omega_0 = Z \Delta_W^{1/2} A \Omega_0 = \Delta_W^{1/2} Z A \Omega_0 \\ &= \Delta_W^{1/2} A Z \Omega_0 = \Delta_W^{1/2} A \Omega, \end{aligned}$$

where  $J_W \mathcal{R}(W) J_W = \mathcal{R}(W)'$  has also been used. Thus, one concludes  $\Delta_W^\Omega \subset \Delta_W$ . A similar argument interchanging the roles of  $\Omega$  and  $\Omega_0$  completes the proof.  $\square$

In light of the fact that any normal state on  $\mathcal{B}(\mathcal{H})$ , when restricted to  $\mathcal{R}(W)$ , can be implemented on  $\mathcal{R}(W)$  by a suitable vector in  $\mathcal{P}_W^h$  [2, Theorem 6], it is noteworthy that these different implementers sit in the various natural positive cones  $\mathcal{P}_W^h$  in such a way that only very well-behaved states are determined by the vectors left in the intersection  $\mathcal{P}_0$ .

Proposition 4.2 also entails that under the central decomposition of  $\mathcal{R}$ , in  $\nu$ -almost all  $\mathcal{H}(\zeta)$  the corresponding set  $\mathcal{P}_0(\zeta)$  contains only vectors proportional to  $\Omega_0(\zeta)$ . Hence, in each irreducible vacuum sector at most one state can satisfy the CGMA in the form of conditions (a)–(c).

The next theorem establishes the properties under central decomposition of the various modular structures of concern to us.

**Theorem 4.4** *Under the standing assumptions, in reference to the structures discussed in Theorem 3.4, let, for each  $W \in \mathcal{W}$ ,  $J_W(\zeta), \Delta_W(\zeta), \mathcal{P}_W^h(\zeta)$  represent the modular objects associated with the pair  $(\mathcal{R}(W)(\zeta), \Omega_0(\zeta))$ , where  $\Omega_0 = \int_S^\oplus \Omega_0(\zeta) d\nu(\zeta)$ . Then for each  $W \in \mathcal{W}, t \in \mathbb{R}$ , the fields  $\zeta \mapsto J_W(\zeta), \zeta \mapsto \Delta_W^{it}(\zeta)$  and  $\zeta \mapsto \mathcal{P}_W^h(\zeta)$  are measurable and*

$$J_W = \int_S^\oplus J_W(\zeta) d\nu(\zeta), \Delta_W^{it} = \int_S^\oplus \Delta_W^{it}(\zeta) d\nu(\zeta), \mathcal{P}_W^h = \int_S^\oplus \mathcal{P}_W^h(\zeta) d\nu(\zeta).$$

Conditions (a)–(c) hold in  $\nu$ -almost all sectors. If, moreover,  $\mathcal{P}_0(\zeta) = \bigcap_{W \in \mathcal{W}} \mathcal{P}_W^h(\zeta)$ , then also  $\zeta \mapsto \mathcal{P}_0(\zeta)$  is measurable and

$$\mathcal{P}_0 = \int_S^\oplus \mathcal{P}_0(\zeta) d\nu(\zeta).$$

For  $\nu$ -almost all  $\zeta$ ,  $\mathcal{P}_0(\zeta) = \{c\Omega_0(\zeta) \mid c \in [0, \infty)\}$ .

*Proof.* For every  $W \in \mathcal{W}$ , the measurability of the fields  $\zeta \mapsto J_W(\zeta), \zeta \mapsto \Delta_W^{it}(\zeta)$  and the equalities

$$J_W = \int_S^\oplus J_W(\zeta) d\nu(\zeta), \Delta_W^{it} = \int_S^\oplus \Delta_W^{it}(\zeta) d\nu(\zeta)$$

are assured by [24, Thm. III.2]. From Theorem 3.4 it follows that

$$J_W = U(\lambda_W) = \int_S^\oplus U(\lambda_W)(\zeta) d\nu(\zeta),$$

for every  $W \in \mathcal{W}$ . Corollary II.2.2 in [18] then yields the equality  $J_W(\zeta) = U(\lambda_W)(\zeta)$  for  $\nu$ -almost all  $\zeta$ . With a possible change in the zero set  $N$ , this equality may be assured for all  $W \in \widehat{\mathcal{W}}$ . In Section 3 of [16] it was shown that for a

locally generated net satisfying Haag duality the map  $W \mapsto J_W$  from the space of wedges to the topological group  $\mathcal{J}$  is continuous, as is the map  $\lambda_W \mapsto J_W$  from  $\mathcal{P}_+$  to  $\mathcal{J}$ . This continuity and the continuity of the representation  $U(\mathcal{P}_+)$  entail then that condition (c) holds in  $\nu$ -almost all sectors.

The isotony in  $\nu$ -almost every sector was established in Theorem 3.4. From [24, Prop. II.2] it follows that for a fixed  $W_0 \in \mathcal{W}$ ,  $\Omega_0(\zeta)$  is cyclic and separating for  $\mathcal{R}(W_0)(\zeta)$  for  $\nu$ -almost all  $\zeta$ . In view of the covariant action of the unitaries  $U(\lambda)(\zeta)$  on the wedge algebras  $\mathcal{R}(W)(\zeta)$  proven in Theorem 3.4 and the transitivity of  $\mathcal{P}_+^\uparrow$  on  $\mathcal{W}$ , this is therefore true for all  $W \in \mathcal{W}$  and the same set of  $\zeta$ . Hence, conditions (a)–(c) with the possible exception of the injectivity in (a) hold in  $\nu$ -almost all sectors.

By Proposition A.1, if the map  $W \mapsto \mathcal{R}(W)(\zeta)$  is not injective, then  $U(\mathcal{P}_+^\uparrow)(\zeta)$  is trivial and  $\mathcal{R}(W)(\zeta)$  is abelian and independent of  $W \in \mathcal{W}$ . But then  $\mathcal{R}(\zeta)$  is an abelian factor with cyclic vector. If this were true for all  $\zeta$  in a measurable set  $M \subset S$  with positive  $\nu$ -measure, then  $\int_M^\oplus \mathcal{H}(\zeta) \, d\nu(\zeta)$  would be a subspace of  $\mathcal{H}$  on which the corresponding subrepresentation of  $U(\mathcal{P}_+^\uparrow)$  was trivial and of  $\mathcal{R}$  was abelian. This degenerate situation has been excluded by the standing assumptions.

Since  $\Delta_W$  commutes with  $\mathcal{Z}(W)$ , Proposition 3.1 implies that it commutes with  $\mathcal{Z}$  and hence is also decomposable. Appealing to [24, Thm. I.8, Thm. III.2], it follows that

$$\Delta_W^{1/4} A \Omega_0 = \int_S^\oplus \Delta_W^{1/4}(\zeta) A(\zeta) \Omega_0(\zeta) \, d\nu(\zeta),$$

for all  $A \in \mathcal{R}(W)$ , and therefore that

$$\mathcal{P}_W^\natural = \int_S^\oplus \mathcal{P}_W^\natural(\zeta) \, d\nu(\zeta),$$

for all  $W \in \mathcal{W}$ .

As the standing assumptions hold in  $\nu$ -almost all sectors, one may apply Proposition 4.2 in each sector to conclude  $\mathcal{P}_0(\zeta) = \mathcal{Z}(\zeta)_+ \Omega_0 = \mathcal{Z}_+(\zeta) \Omega_0$ , and thereby also  $\mathcal{P}_0 = \int_S^\oplus \mathcal{P}_0(\zeta) \, d\nu(\zeta)$ . Since for  $\nu$ -almost all  $\zeta$  the elements of  $\mathcal{Z}_+(\zeta)$  are positive multiples of the identity operator on  $\mathcal{H}(\zeta)$ , the final assertion is immediate.  $\square$

We remark that also all of the modular unitaries  $\{\Delta_W^{it}\}_{t \in \mathbb{R}}$  can be reunited in  $\nu$ -almost all sectors as above by first decomposing the operators  $\Delta_W^{it}$ , for  $t$  rational and  $W \in \widehat{\mathcal{W}}$ , and then using the strong continuity to reconstruct  $\Delta_W^{it}(\zeta)$  for all  $t \in \mathbb{R}$ . From Section 3 of [16] and [13, Prop. 4.6] one knows that also the map  $W \mapsto \Delta_W^{it}$  is strongly continuous, given our standing assumptions. This is then employed to reconstruct  $\Delta_W^{it}(\zeta)$  for all  $W \in \mathcal{W}$ .

A conceptually simple and quite general criterion for stable states on general space-times is the Modular Stability Condition, proposed in [13]. We recall this condition here for the convenience of the reader.

- (d) For any  $W \in \mathcal{W}$ , the elements  $\Delta_W^{it}$ ,  $t \in \mathbb{R}$ , of the modular group corresponding to  $(\mathcal{R}(W), \Omega_0)$  are contained in the group  $\mathcal{J}$  generated by all finite products of the modular involutions  $\{J_W\}_{W \in \mathcal{W}}$ .

We refer the interested reader to [11, 13] for a discussion of the background of this condition and a brief account of other interesting approaches towards an algebraic characterization of ground states on general space-times. As shown in [13], if the standing assumptions of this paper and the Modular Stability Condition hold, then modular covariance obtains:  $\Delta_W^{it} = U(\lambda_W(2\pi t))$ , for all  $t \in \mathbb{R}$  and  $W \in \mathcal{W}$ , where  $\{\lambda_W(2\pi t) \mid t \in \mathbb{R}\}$  is the one-parameter subgroup of boosts leaving  $W$  invariant. In addition, the spectrum condition holds.

We close this section with a theorem which summarizes the consequences of the Modular Stability Condition for the topics under consideration here.

**Theorem 4.5** *If the standing assumptions and the Modular Stability Condition hold for  $\Omega_0$ , then the conditions (a)–(d) also obtain for any  $\Omega \in \mathcal{P}_0$  which is cyclic or separating for all wedge algebras  $\mathcal{R}(W)$ . In addition,  $\mathcal{R}' = \mathcal{Z} = \mathcal{Z}(W)$ , for every  $W \in \mathcal{W}$ . Hence, the central decomposition in Theorem 3.4 results in irreducible vacuum sectors in which the Modular Stability Condition is satisfied in  $\nu$ -almost every sector, as is modular covariance and the spectrum condition. In  $\nu$ -almost all sectors,  $\mathcal{R}(W)(\zeta)$  is a type III<sub>1</sub> factor, for all  $W \in \mathcal{W}$ .*

*Proof.* It has already been shown that conditions (a)–(c) hold for every  $\Omega$  as described. Corollary 4.3 entails that also condition (d) is satisfied by the modular unitaries associated to each wedge algebra by such vectors  $\Omega$  (and, of course, they manifest modular covariance). Together, Proposition 5.1 and the proof of Theorem 5.1 in [13] entail that  $U(\mathbb{R}^4)$  fulfills the spectrum condition. It then follows from [19, Prop. 3.1] that  $\mathcal{Z}(W) = \mathcal{Z} = \mathcal{R}'$ , for every  $W \in \mathcal{W}$ . Therefore, in the central decomposition in Theorem 3.4 one has the spectrum condition for  $U(\mathbb{R}^4)(\zeta)$ , for  $\nu$ -almost all  $\zeta$  [19, Thm. 4.1]. Furthermore, from the proof of Lemma 3.2 in [19] one may conclude that  $\mathcal{R}(W)(\zeta)$  is a type III<sub>1</sub> factor, for all  $W \in \mathcal{W}$ .

From the proof of Theorem 4.4 and [18, Cor. II.2.2], it follows that for  $\nu$ -almost all  $\zeta$  one has  $\Delta_W^{it}(\zeta) = U(\lambda_W(2\pi t))(\zeta)$ , for all  $t \in \mathbb{R}$  and  $W \in \mathcal{W}$ , *i.e.*, modular covariance holds in  $\nu$ -almost all sectors. From the proof of Theorem 4.4 it also follows that  $U(\mathcal{P}_+)(\zeta) = \mathcal{J}(\zeta)$ , for  $\nu$ -almost all  $\zeta$ . Therefore, the Modular Stability Condition also holds in  $\nu$ -almost all sectors.  $\square$

We mention that if the hypothesis of Theorem 4.5 holds, then one can show using [13, Thm. 5.1] and [31, Thm. 1.2] that any  $\Omega \in \mathcal{P}_0$  and the corresponding  $\Omega(\zeta)$ , for  $\nu$ -almost all  $\zeta$ , determine passive states on their respective nets with respect to all uniformly accelerated observers. Hence, the CGMA and the Modular Stability Condition select particularly stable states.

## 5 Final comments

A number of different criteria [9, 21, 26, 32, 33] have been proposed to select physically relevant states for quantum field theories on curved space-times, where translation covariance and the spectrum condition are simply not applicable. However, these criteria, when they obtain, are valid for an entire folium of states and therefore beg the question of which state (or states) of the respective folium is to be regarded as fundamental, *i.e.*, as a reference or ground state [13].

We emphasized in [13] that the CGMA is a selection criterion for states and not an entire folium. But the CGMA explicitly places constraints only on the algebras  $\mathcal{R}(W)$ ,  $W \in \mathcal{W}$ , and the modular conjugations  $J_W$ ,  $W \in \mathcal{W}$  – the algebras are state-independent and each modular conjugation  $J_W$  is common to every state vector in the natural cone  $\mathcal{P}_W^h$ , which is itself so large that it spans the Hilbert space  $\mathcal{H}$ . However, the CGMA is a condition on the entire set  $\{J_W \mid W \in \mathcal{W}\}$ , and therefore the vectors selected by the CGMA are those in  $\mathcal{P}_0$ .

We have shown in this paper that the vectors remaining in the intersection  $\mathcal{P}_0$  share the properties one would desire of reference states, without any appeal to the spectrum condition, and that the structures associated with the CGMA and the Modular Stability Condition are gratifyingly rigid. Moreover, we have shown that these conclusions do not rely upon the more technical assumptions of the CGMA in Minkowski space [13], which were designed to assure not only the existence of the representation of the Poincaré group discussed above, but also to derive the Poincaré group and its action upon Minkowski space from the initial data  $(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega_0)$ . Already the conditions (a)–(c), themselves consequences of the CGMA in Minkowski space, are sufficient to assure the above-mentioned conclusions.

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## A Nets of wedge algebras

We show that in the presence of the other standing assumptions, the injectivity of the map  $W \mapsto \mathcal{R}(W)$  can fail only in the most extreme manner.

**Proposition A.1** *Let all of the standing assumptions hold, except condition (a). If the map  $W \mapsto \mathcal{R}(W)$  is order-preserving but not injective, then the representation  $U(\mathcal{P}_+^\uparrow)$  is trivial, and  $\mathcal{R}(W)$  is abelian and independent of  $W \in \mathcal{W}$ .*

*Proof.* Let  $W_1, W_2 \in \mathcal{W}$  be distinct wedges such that  $\mathcal{R}(W_1) = \mathcal{R}(W_2)$ . Then the corresponding modular conjugations must coincide, *i.e.*,  $J_{W_1} = J_{W_2}$ . Since  $W_1 \neq W_2$ , condition (c) and  $U(\mathcal{P}_+) = \mathcal{J}$  entail the existence of a nontrivial element  $\lambda_0 = \lambda_{W_1} \lambda_{W_2} \in \mathcal{P}_+^\dagger$  such that  $U(\lambda_0) = J_{W_1} J_{W_2} = 1$ . With  $\lambda_0 = (\Lambda_0, x_0)$ ,  $\Lambda_0 \in \mathcal{L}_+^\dagger$ ,  $x_0 \in \mathbb{R}^4$ , one would then have

$$U(x_0)^{-1} = U(x_0)^{-1} U(\lambda_0) = U(\Lambda_0).$$

Hence,  $U(\Lambda_0) \in U(\mathbb{R}^4)$ , and since  $U(\mathbb{R}^4)$  is a normal subgroup of  $U(\mathcal{P}_+^\dagger)$  it then follows that

$$U(\Lambda \Lambda_0 \Lambda^{-1}) = U(\Lambda) U(\Lambda_0) U(\Lambda)^{-1} \in U(\mathbb{R}^4),$$

for every  $\Lambda \in \mathcal{L}_+^\dagger$ . The elements  $\{\Lambda \Lambda_0 \Lambda^{-1} \mid \Lambda \in \mathcal{L}_+^\dagger\}$  generate a (nontrivial) normal subgroup of  $\mathcal{L}_+^\dagger$ . But  $\mathcal{L}_+^\dagger$  is a simple group, so one deduces that  $U(\mathcal{L}_+^\dagger) \subset U(\mathbb{R}^4)$ . The representation  $U(\mathcal{L}_+^\dagger)$  is therefore abelian and hence trivial. But then for every  $x \in \mathbb{R}^4$  and  $\Lambda \in \mathcal{L}_+^\dagger$  one has  $U(x) = U(\Lambda) U(x) U(\Lambda)^{-1} = U(\Lambda x)$ , so that it follows that  $U(x)$  is independent of  $x \in \mathbb{R}^4$ . Thus,  $U(\mathcal{P}_+^\dagger)$  is trivial. But the covariance of the net under the adjoint action of  $U(\mathcal{P}_+^\dagger)$  then entails that  $\mathcal{R}(W) = \mathcal{R}(\lambda W)$ , for all  $\lambda \in \mathcal{P}_+^\dagger$ . Thus, one must conclude, in particular, that  $\mathcal{R}(W) = \mathcal{R}(\lambda_W W) = \mathcal{R}(W') = \mathcal{R}(W)'$ , for all  $W \in \mathcal{W}$ . As  $\mathcal{P}_+^\dagger$  acts transitively upon  $\mathcal{W}$ , the proof is completed.  $\square$

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