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Integrable Renormalization II: The General Case

Kurusch Ebrahimi-Fard, Li Guo and Dirk Kreimer

Abstract. We extend the results we obtained in an earlier work [1]. The cocommutative case of ladders is generalized to a full Hopf algebra of (decorated) rooted trees. For Hopf algebra characters with target space of Rota-Baxter type, the Birkhoff decomposition of renormalization theory is derived by using the double Rota-Baxter construction, respectively Atkinson's theorem. We also outline the extension to the Hopf algebra of Feynman graphs via decorated rooted trees.

1 Introduction

The perturbative approach to quantum field theory (QFT) has been spectacularly successful in the past. It is based on a priori formal series expansions of Green functions in orders of a coupling constant, measuring the strength of the corresponding interaction. Terms in these series expansions are indexed by Feynman diagrams, a graphical shorthand for the corresponding Feynman integrals. Physically relevant quantum field theories when treated perturbatively develop short-distance singularities present in all superficially divergent contributions to the perturbative expansion. Renormalization theory [2] allows nevertheless for a consistent way to treat these divergent Feynman integrals in perturbative QFT. The intricate combinatorial, algebraic and analytic structure of renormalization theory within QFT is by now known for almost 70 years. Within the physics community the subject reached its final and satisfying form through the work of Bogoliubov, Parasiuk, Hepp, and Zimmermann. It was very recently that one of us in [3] discovered a unifying scheme in terms of Hopf algebras and its duals underlying the combinatorial and algebraic structure. This Hopf algebraic approach to renormalization theory, as well as the related Lie algebra structures, were exploited in subsequent work [4, 5, 6, 7, 8, 9, 10, 11].

The focus of an earlier work of us [1][1] and this article is on the algebraic Birkhoff decomposition discovered first in [4, 8, 9], and the related Lie algebra of rooted trees, respectively Feynman graphs [10, 11]. The Rota-Baxter algebra structure on the target space of (regularized) Hopf algebra characters showed to be of crucial importance with respect to the Birkhoff decomposition (in [4] this relation appeared under the name multiplicativity constraint).
Using a classical r-matrix ansatz, coming simply from the Rota-Baxter map, we were able to derive in (I) the formulae for the factors $\phi_\pm$ [8] for the decomposition of a Hopf algebra character ϕ in the case of the Hopf subalgebra of rooted ladder

[1]For the rest of this work we will cite paper [1] by (I).

trees. Bogoliubov's $\bar{R}$-map finds its natural formulation in terms of a character with values in the double Rota-Baxter algebra of the above target space Rota-Baxter algebra. The counterterm $S_R^\phi = \phi_-$ and the renormalized character ϕ_+ simply lie in the images of the group homomorphisms $\mathcal{R}$, $-\tilde{\mathcal{R}} = \mathcal{R} - id$, respectively, of the Bogoliubov character.

In this work we would like to extend these results to the general case, i.e., the full Hopf algebra of arbitrary rooted trees. The main difference lies in the fact that in the rooted ladder tree case we worked with a cocommutative Hopf algebra, or dually, with the universal enveloping algebra of an Abelian Lie algebra. In the general case the Lie algebra of infinitesimal characters is non-Abelian, correspondingly the Hopf algebra is non-cocommutative, necessitating a more elaborate treatment due to contributions from the Baker-Campbell-Hausdorff (BCH) formula. The modifications coming from these BCH contributions have to be subtracted order by order in the grading of the Hopf algebra. Hence we will define in a recursive manner an infinitesimal character in the Lie algebra which allows for the Birkhoff decomposition of the Feynman rules regarded as an element of the character group of the Hopf algebra. The factors of the derived decomposition give the formulae for the renormalized character ϕ_+ and the counterterm $S_R^\phi = \phi_-$ introduced in [3, 4, 8]. Bogoliubov's $\bar{R}$-map becomes a character with values in the double of the target space Rota-Baxter algebra. It should be underlined that the above ansatz in terms of an r-matrix solely depends on the algebraic structure of the Lie algebra of infinitesimal characters, i.e., the dual of the Hopf algebra of (decorated) rooted trees or Feynman graphs, and on the Rota-Baxter structure underlying the target space of characters. When specializing the Rota-Baxter algebra to be the algebra of Laurent series with pole part of finite order, this approach naturally reduces to the minimal subtraction scheme in dimensional regularization, or to the momentum scheme, which are both widely used in perturbative QFT and thoroughly explored in [3, 4, 5, 6], which extend to non-perturbative aspects still using the Hopf algebra [7].

The paper is organized as follows. In the following section, we introduce the notion of Rota-Baxter algebras and recall some related basic algebraic facts, like the relation to the notion of classical Yang-Baxter type identities, the double Rota-Baxter construction and Atkinson's theorem.

After that, we review the notion of a renormalization Hopf algebra by introducing the universal object [8] for such Hopf algebras, the Hopf algebra of rooted trees. Having this at our disposal, generalizations to the Hopf algebra of decorated rooted trees or Feynman graphs are a straightforward generalization which we will outline later on. Its dual, containing the Lie group of Hopf algebra characters, and the related Lie algebra of its generators, i.e., the infinitesimal characters, is introduced without repeating the details which are by now standard [8].

In section four, which contains the main part of this paper, the notion of a regularized character is introduced as a character with values in a Rota-Baxter algebra. Note that Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization falls into this class, even though it makes no use of a regulator, but of a Taylor operator on the integrand instead, which provides a Rota-Baxter map similarly. This is immediate upon recognizing that disjoint one-particle irreducible graphs allow for independent Taylor expansions in masses and momenta. This allows us to lift the Lie algebra of infinitesimal characters to a Rota-Baxter Lie algebra, giving the notion of a classical r-matrix on this Lie algebra. We then review briefly the results of (I), i.e., the Birkhoff factorization in the cocommutative case.

Motivated by this result for the simple case of rooted ladder trees, we solve here the factorization problem for the non-cocommutative Hopf algebra of rooted trees by defining a BCH-modified infinitesimal character.

Section four closes with some calculations using the notion of normal coordinates intended to make the construction of the modified character in terms of the BCH-corrections more explicit, and a remark on decorated, non-planar rooted trees and Feynman graphs.

2 Rota-Baxter algebras: from Baxter to Baxter

The Rota-Baxter (RB) relation first appeared in 1960 in the work of the American mathematician Glen Baxter [12]. Later it was explored especially by the mathematicians F.A. Atkinson, G.-C. Rota and P. Cartier [13, 14, 15]. In particular, Rota underlined its importance in various fields of mathematics, especially within combinatorics [16]. But it was very recently that after a period of dormancy it showed to be of considerable interest in several so far somewhat disconnected areas like Loday type algebras [17, 18, 19, 20, 21], q-shuffle and q-analogs of special functions through the Jackson integral [22], differential algebras [23, 24], number theory [25], and the Hopf algebraic approach to renormalization theory in perturbative QFT [1, 4, 9]. In particular, in collaboration with Connes, the connection to Birkhoff decompositions based on Rota-Baxter maps was introduced in [9, 10]. It is the latter aspect on which we will focus in this work.

In its Lie algebraic version the RB relation found one of its most important applications within the theory of integrable systems, where it was rediscovered in the 1980s under the name of (operator form of the) classical and modified classical Yang-Baxter[2] equation [26, 27, 28]. There some of its main features, already mentioned in [13], and Atkinson's theorem itself, were analyzed in greater detail. Especially the double Rota-Baxter construction introduced in the work of Semenov-Tian-Shansky [26, 27], and the related factorization theorems [29] will be of interest to us.

In the following we will collect a few basic results on Rota-Baxter algebras, some of which we will need later, and some of which we state just to indicate interesting relations of these algebras to other areas of mathematics. Of course, the list is by no means complete, and a more exhaustive treatment needs to be done.

[2]Referring to C.N. Yang and the Australian physicist Rodney Baxter.

Let us start with the definition of a Rota-Baxter algebra [16, 23, 24]. Suppose $\mathbb{K}$ is a field of characteristic 0. A $\mathbb{K}$-algebra neither needs to be associative, nor commutative, nor unital unless stated otherwise.

Definition 2.1 *Let $\mathcal{A}$ be a $\mathbb{K}$-algebra with a $\mathbb{K}$-linear map $R : \mathcal{A} \to \mathcal{A}$. We call $\mathcal{A}$ a Rota-Baxter (RB) $\mathbb{K}$-algebra and R a Rota-Baxter map (of weight $\theta \in \mathbb{K}$) if the operator R holds the following Rota-Baxter relation of weight $\theta \in \mathbb{K}$* [3]*:*

$$R(x)R(y) + \theta R(xy) = R\big(R(x)y + xR(y)\big), \ \forall x, y \in \mathcal{A}. \tag{1}$$

Remark 2.2

1) For $\theta \neq 0$ a simple scale transformation $R \to \theta^{-1}R$ gives the so-called standard form:
$$R(x)R(y) + R(xy) = R\big(R(x)y + xR(y)\big). \tag{2}$$
For the rest of the paper we will always assume the Rota-Baxter map to be of weight $\theta = 1$, i.e., to be in standard form.
2) If R fulfills relation (2) then $\tilde{R} := id - R$ fulfills the same Rota-Baxter relation.
3) The images of R and $id - R$ give subalgebras in $\mathcal{A}$.
4) The free associative, commutative, unital RB algebra is given by the mixable shuffle algebra [23] which is an extension of Hoffman's quasi-shuffle algebra [30, 31].
5) The case $\theta = 0$, $R(x)R(y) = R\big(R(x)y + xR(y)\big)$, naturally translates into the ordinary shuffle relation, and finds its most prominent example in the integration by parts rule for the Riemann integral.
6) A relation of similar form is given by the associative Nijenhuis identity [32]:
$$N(x)N(y) + N^2(xy) = N\big(N(x)y + xN(y)\big). \tag{3}$$
Given a RB algebra with an idempotent RB map R, the operator $N_\gamma := R - \gamma\tilde{R}$, $\gamma \in \mathbb{K}$ fulfills relation (3). See [20, 33, 34] for recent results with respect to this relation.

Example 2.3 1) The q-analog of the Riemann integral, or Jackson-integral [16, 22], on a well-chosen function algebra $\mathcal{F}$ is given by:

$$\begin{aligned} J[f](x) \ &:= \ \int_0^x f(y)d_qy \\ &:= \ (1-q)\sum_{n\geq 0} f(xq^n)xq^n. \end{aligned} \tag{4}$$

It may be written in a more algebraic version, using the operator:

$$P_q[f] := \sum_{n>0} E_q^n[f], \tag{5}$$

[3] Some authors denote this relation in the form $R(x)R(y) = R\big(R(x)y + xR(y) + \lambda xy\big)$. So $\lambda = -\theta$.

where $E_q[f](x) := f(qx),\ f \in \mathcal{F}$. P_q and $id + P_q =: \hat{P}_q$ are RB operators of weight $-1,\ 1$, respectively. Now let us define a multiplication operator $M_f : \mathcal{F} \to \mathcal{F},\ f \in \mathcal{F}$, $M_f[g](x) := [f\,g](x) = f(x)g(x)$ which fulfills the associative Nijenhuis relation (3). The Jackson integral is given in terms of the above operators as:

$$J[f](x) = (1-q)\hat{P}_q M_{id}\,[f](x), \tag{6}$$

and fulfills the following mixed RB relation

$$J[f]\,J[g] + (1-q)JM_{id}[f\,g] = J\Big[J[f]\,g + f\,J[g]\Big]. \tag{7}$$

In a forthcoming work two of us (K.E.-F., L.G.) will report some interesting implications of this fact with respect to some recent results on q-analog of multiple-zeta-values [31].

2) A rich class of Rota-Baxter maps is given by certain projectors. Within renormalization theory, dimensional regularization together with the minimal subtraction scheme play an important rôle. Here the RB map R_{MS} is of weight $\theta = 1$ and defined on the algebra of Laurent series $\mathbb{C}\big[\big[\epsilon, \epsilon^{-1}\big]$ [4] with finite pole part. For $\sum_{k=-m}^{\infty} c_k\epsilon^k \in \mathbb{C}\big[\big[\epsilon, \epsilon^{-1}\big]$ it gives:

$$R_{MS}\big(\sum_{k=-m}^{\infty} c_k\epsilon^k\big) := \sum_{k=-m}^{-1} c_k\epsilon^k. \tag{8}$$

Of equal importance is the projector which keeps the finite part, closely related to the momentum scheme.

We now introduce the modified Rota-Baxter relation. Its Lie algebraic version already appeared in [26, 28].

Definition 2.4 *Let $\mathcal{A}$ be a Rota-Baxter algebra, R its Rota-Baxter map. Define the operator $B : \mathcal{A} \to \mathcal{A},\ \ B := id - 2R$ to be the modified Rota-Baxter map and call the corresponding relation fulfilled by B:*

$$B(x)B(y) = B\big(B(x)y + xB(y)\big) - xy,\ \forall x, y \in \mathcal{A} \tag{9}$$

the modified Rota-Baxter relation.

Remark 2.5 In the following proposition (2.6), we mention the notion of pre-Lie algebras. Let us state briefly its definition. A (left) pre-Lie $\mathbb{K}$-algebra $\mathcal{A}$ is a $\mathbb{K}$-vector space, together with a bilinear pre-Lie product $\centerdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, holding the (left) pre-Lie relation:

$$a \centerdot (b \centerdot c) - (a \centerdot b) \centerdot c = b \centerdot (a \centerdot c) - (b \centerdot a) \centerdot c, \quad \forall a, b, c \in \mathcal{A}.$$

The commutator $[a, b] := a \centerdot b - b \centerdot a,\ \ \forall a, b \in \mathcal{A}$ fulfills the Jacobi identity.

Proposition 2.6 *For the Rota-Baxter algebra $\mathcal{A}$ to be either an associative or pre-Lie $\mathbb{K}$-algebra, the (modified) Rota-Baxter relation naturally extends to the Lie algebra $\mathcal{L}_{\mathcal{A}}$ with commutator bracket $[x, y] := xy - yx$, $\forall x, y \in \mathcal{A}$:*

$$[R(x), R(y)] \ + \ R([x, y]) = R\big([R(x), y] + [x, R(y)]\big) \tag{10}$$
$$[B(x), B(y)] \ = \ B\big([B(x), y] + [x, B(y)]\big) - [x, y]. \tag{11}$$

The proof is a straightforward calculation. The relations (10) and (11) are well known as the (operator form of the) classical Yang-Baxter and modified Yang-Baxter equation, respectively.

Remark 2.7

1) The same is true for the associative Nijenhuis relation (3). In its Lie algebraic version, identity (3) was investigated in [35, 36].
2) Let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra. We regard $\mathcal{A}\otimes\mathcal{A}$ as an $\mathcal{A}$-bimodule, $x\otimes y \in \mathcal{A}\otimes\mathcal{A}$ and $a(x\otimes y)b = (ax\otimes y)b = ax\otimes yb$. A solution $r := \sum_i s_{(i)}\otimes t_{(i)} \in \mathcal{A}\otimes\mathcal{A}$ of the extended associative classical Yang-Baxter relation:
$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = \theta r_{13}, \quad \theta \in \mathbb{K}$$
gives a RB map $\beta : \mathcal{A} \to \mathcal{A}$ of weight θ, defined by $\beta(x) := \sum_i s_{(i)}\, x\, t_{(i)}$. The notation r_{ij} means, for instance, $r_{13} := \sum_i s_{(i)}\otimes 1\otimes t_{(i)}$. This example implies many more interesting results with respect to unital infinitesimal bialgebras, which will be presented elsewhere. The case $\theta = 0$ was already treated in [21], implying a RB map of weight 0.

Atkinson gave in [13] a very nice characterization of general RB $\mathbb{K}$-algebras in terms of a so-called subdirect Birkhoff decomposition:

Theorem 2.8 (Atkinson [13]): *For a $\mathbb{K}$-algebra $\mathcal{A}$ with a linear map $R : \mathcal{A} \to \mathcal{A}$ to be a Rota-Baxter $\mathbb{K}$-algebra, it is necessary and sufficient that $\mathcal{A}$ has a subdirect Birkhoff decomposition.*

The proof of this theorem may be found in [13] and will not be given here. Essentially, the subdirect Birkhoff decomposition in this case means that the Cartesian product $\mathcal{D} := (R(\mathcal{A}), -\tilde{R}(\mathcal{A})) \subset \mathcal{A} \times \mathcal{A}$ is a subalgebra in $\mathcal{A} \times \mathcal{A}$ and that every element $x \in \mathcal{A}$ has a unique decomposition $x = R(x) + \tilde{R}(x)$. This should be compared to the results in the Lie algebra case (10) to be found in [26, 27, 29, 37].

We come now to one of the main facts about RB algebras. In the following we assume every RB algebra $\mathcal{A}$ to be either an associative algebra or a pre-Lie or Lie algebra. The RB relation then implies furthermore a possibly infinite hierarchy of the same RB structure in each of the former cases. We call this the double Rota-Baxter construction of the RB-hierarchy on the RB algebra $\mathcal{A}$, given as follows.

Proposition 2.9 *Let $\mathcal{A}$ be a Rota-Baxter algebra with (modified) Rota-Baxter map R, set $B = id - 2R$. Equipped with the new product:*

$$\begin{aligned} a *_R b &:= R(a)b + aR(b) - ab && (12)\\ &= -\frac{1}{2}\big(B(a)b + aB(b)\big), && (13)\end{aligned}$$

$\mathcal{A}$ is again a Rota-Baxter algebra of the same type, denoted by $\mathcal{A}_R$.

The proof of this proposition is immediate by the definition of $*_R$. Following the terminology in [26, 27], we call this new Rota-Baxter algebra $\mathcal{A}_R$ the double RB algebra of $\mathcal{A}$. It is also in [26] where already the notion of the double RB structure for associative algebras equipped with a modified Rota-Baxter operator was suggested.

Remark 2.10

1) Let $\mathcal{A}$ be an associative RB algebra. The composition $a \diamond b := R(a)b - bR(a) + ab$ defines a pre-Lie structure on $\mathcal{A}$. This aspect becomes more apparent in the context of Loday's dendriform structures, for which associative RB algebras give a rich class of interesting examples, see [17, 18, 19, 20, 21] and references therein.
2) It is obvious that Proposition 2.9 implies a whole, possibly infinite, hierarchy of double RB algebras $\mathcal{A}_R^{(i)}$ (here, $* = *_R^{(0)}$ and $*_R = *_R^{(1)}$):
$$\mathcal{A}_R^{(0)} := \mathcal{A},\ \mathcal{A}_R^{(1)} := (\mathcal{A}, *_R),\ \ldots,\ \mathcal{A}_R^{(i)} := (\mathcal{A}, *_R^{(i)}), \ldots$$
$$a *_R^{(i)} b := \frac{d^i}{dt^i}_{|t=0} e^{-\frac{1}{2}tB}(a)\, e^{-\frac{1}{2}tB}(b),\ a, b \in \mathcal{A}.$$
Let us call $\mathcal{A}_R^{(i)}$ the ith double RB algebra of $\mathcal{A}$, or equivalently the double of $\mathcal{A}_R^{(i-1)}$. The following diagram serves to visualize the so-called RB-hierarchy:
$$\mathcal{A} \xrightarrow{*_R^{(1)}} \mathcal{A}_R^{(1)} \xrightarrow{*_R^{(2)}} \mathcal{A}_R^{(2)} \xrightarrow{*_R^{(3)}} \mathcal{A}_R^{(3)} \to \cdots$$
3) The RB-hierarchy becomes cyclic of period 2 at level $i = 3$, for R being an idempotent RB map, $R^2 = R$, i.e., the kth double product $*_R^{(k)} = *_R^{(k+2)}$.
4) The Rota-Baxter map R becomes an $\mathbb{K}$-algebra homomorphism between $\mathcal{A}_R^{(i)}$ and $\mathcal{A}_R^{(i-1)}$, $i \in \mathbb{N}$:
$$R(a *_R^{(i)} b) = R(a) *_R^{(i-1)} R(b). \qquad (14)$$
5) For the Rota-Baxter map $\tilde{R} := id - R$, we have
$$\tilde{R}(a *_R^{(i)} b) = -\tilde{R}(a) *_R^{(i-1)} \tilde{R}(b). \qquad (15)$$
We therefore have the following diagram of $\mathbb{K}$-algebra homomorphisms:
$$\mathcal{A} \xleftarrow{R,\tilde{R}} \mathcal{A}_R^{(1)} \xleftarrow{R,\tilde{R}} \mathcal{A}_R^{(2)} \xleftarrow{R,\tilde{R}} \mathcal{A}_R^{(3)} \leftarrow \cdots$$

We introduce the following composition, using the shuffle product notion formally. For an associative $\mathbb{K}$-algebra $\mathcal{A}$ and $a, b \in \mathbb{K}$, we define $a \sqcup\!\sqcup_{\mathcal{A}} b := ab + ba$. For fixed $a_i, b_j \in \mathcal{A}$, $1 \leq i \leq m$, $1 \leq j \leq n$, define recursively

$$\begin{aligned}(a_1 a_2 \cdots a_m) \sqcup\!\sqcup_{\mathcal{A}} (b_1 b_2 \cdots b_n) &= a_1\big((a_2 \cdots a_m) \sqcup\!\sqcup_{\mathcal{A}} (b_1 \cdots b_n)\big) \\ &\quad + b_1\big((a_1 \cdots a_m) \sqcup\!\sqcup_{\mathcal{A}} (b_2 \cdots b_n)\big),\end{aligned}$$

Proposition 2.11 *Let $\mathcal{A}$ be an associative Rota-Baxter algebra. For $n \in \mathbb{N}$, $x \in \mathcal{A}$ we have*

1) *integer powers of $R(x)$ and $\tilde{R}(x)$ can be written explicitly as:*

$$\begin{aligned}(-R(x))^n &= (-1)^n R\big(x^{*_R n}\big) \\ &= -R\big(x^n + \sum_{k=1}^{n-1} (-R(x))^{n-k} \sqcup\!\sqcup_{\mathcal{A}} x^k\big) \qquad (16) \\ \tilde{R}(x)^n &= \tilde{R}\big((-1)^{(n-1)} x^{*_R n}\big) \\ &= \tilde{R}\big(x^n + \sum_{k=1}^{n-1} (-R(x))^{n-k} \sqcup\!\sqcup_{\mathcal{A}} x^k\big). \qquad (17)\end{aligned}$$

2) *for $\mathcal{A}$ also being commutative the above formulae simplify to:*

$$(-R(x))^n = -R\big(x^n + \sum_{k=1}^{n-1} \binom{n}{k} (-R(x))^{(n-k)}\, x^k\big), \qquad (18)$$

$$\tilde{R}(x)^n = \tilde{R}\big(x^n + \sum_{k=1}^{n-1} \binom{n}{k} (-R(x))^{(n-k)}\, x^k\big). \qquad (19)$$

The proof of this proposition follows by induction on n.

3 The Hopf algebra of rooted trees

Rooted trees naturally give a convenient way to denote the hierarchical structure of subdivergences appearing in a Feynman diagram [3], and the structure maps of their Hopf algebras describe the combinatorics of renormalization of local interactions, encapsulating Zimmermann's forest formula. For a renormalizable theory, the hierarchy of subdivergences can always be resolved into decorated rooted trees (the parenthesized words of [3]) upon resolving overlapping divergences using maximal forests [38] corresponding to Hepp sectors. This amounts to a determination of the closed Hochschild one-cocycles of the Hopf algebra of renormalization for a given quantum field theory. This is always possible as the rooted trees Hopf algebra with its one-cocycle B_+ is the universal object [8] of graded commutative Hopf algebras. Hence it suffices to study this universal object, while the details of a specific Hopf algebra of renormalization of a chosen quantum field theory only provide additional notational excesses, albeit cumbersome, see [5, 6, 7] for applications.

The main ingredient of this universal commutative Hopf algebra of rooted trees is given by a well-suited non-cocommutative coproduct, defined in terms of admissible cuts on these rooted trees. The aforementioned forest formula is then given essentially by the recursively defined antipode of this Hopf algebra, coming for free from mathematical structure.

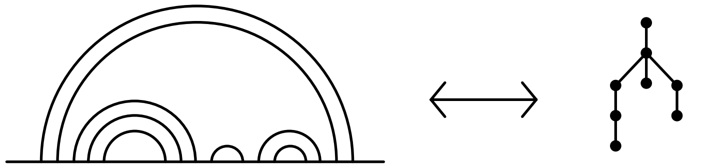
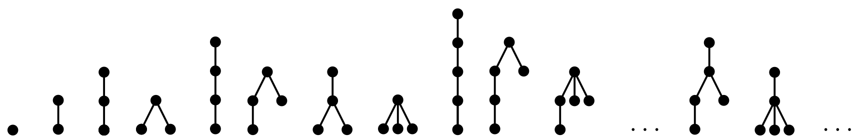


Figure 1. A rainbow diagram and corresponding rooted tree of weight 8.

Having the Hopf algebra of rooted trees, organizing the algebraic and combinatorial aspects of renormalization, the description of the analytical structure in terms of the group of so-called (regularized) Hopf algebra characters takes place within the dual of this Hopf algebra, being an associative algebra with respect to convolution.

Let us introduce the Hopf algebra of rooted trees [8, 39, 40], which we will denote as $\mathcal{H}_{rt}$. The base field $\mathbb{K}$ is assumed once and for all to be of characteristic zero. By definition a rooted tree T is made out of vertices and nonintersecting oriented edges, such that all but one vertex have exactly one incoming line. We denote the set of vertices and edges of a rooted tree by $V(T)$, $E(T)$ respectively. The root is the only vertex with no incoming line. Each rooted tree is effectively a representative of an isomorphism class, and the set of all isomorphism classes will be denoted by $\mathcal{T}_{rt}$.



Definition 3.1 *The commutative, unital, associative $\mathbb{K}$-algebra of rooted trees $\mathcal{A}_{rt}$ is the polynomial algebra, generated by the symbols T, each representing an isomorphism class in $\mathcal{T}_{rt}$. The unit is the empty tree, denoted by* 1, *and the product of rooted trees is denoted by concatenation, i.e.,* $m_{\mathcal{A}_{rt}}(T, T') =: TT'$.

We define a grading on the rooted tree algebra $\mathcal{A}_{rt}$ in terms of the number of vertices of a rooted tree, $\#(T) := |V(T)|$. This is extended to monomials, i.e., so-called forests of rooted trees, by $\#(T_1 \cdots T_n) := \sum_{i=1}^{n} \#(T_i)$, so that $\mathcal{A}_{rt} = \bigoplus_{n \geq 0} \mathcal{A}_{rt}^{(n)}$ becomes a graded, connected, unital, commutative, associative $\mathbb{K}$-algebra.

Let us introduce now the notion of admissible cuts on a rooted tree. A cut c_T of a rooted tree is a subset of the set of edges of T, $c_T \subset E(T)$. It becomes an admissible cut, if and only if along a path from the root to any of the leaves of the tree T, one meets at most one element of c_T. By removing the set c_T, $E(T) - c_T$, each admissible cut c_T produces a monomial of pruned trees, denoted by P_{c_T}. The rest, which is a rooted tree containing the original root, is denoted by R_{c_T}. We exclude the cases, where $c_T = \emptyset$, such that $R_{c_T} = T$, $P_{c_T} = \emptyset$ and the full cut, such that $R_{c_T} = \emptyset$, $P_{c_T} = T$. We extend the rooted tree algebra $\mathcal{A}_{rt}$ to a bialgebra $\mathcal{H}_{rt}$ by defining the co-unit $\epsilon : \mathcal{H}_{rt} \to \mathbb{K}$:

$$\epsilon(T_1 \cdots T_n) := \begin{cases} 0 & T_1 \cdots T_n \neq 1 \\ 1 & else. \end{cases} \tag{20}$$

The coproduct $\Delta : \mathcal{H}_{rt} \to \mathcal{H}_{rt} \otimes \mathcal{H}_{rt}$ is defined in terms of the set of all admissible cuts C_T of a rooted tree T:

$$\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_{c_T \in C_T} P_{c_T} \otimes R_{c_T}. \tag{21}$$

It is obvious, that this coproduct is non-cocommutative. We extend this by definition to an algebra morphism.

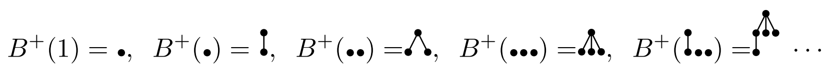
Definition 3.2 *The graded connected Hopf algebra $\mathcal{H}_{rt} := (\mathcal{A}_{rt}, \Delta, \epsilon)$ is defined as the algebra $\mathcal{A}_{rt}$ equipped with the above defined compatible coproduct $\Delta : \mathcal{H}_{rt} \to \mathcal{H}_{rt} \otimes \mathcal{H}_{rt}$ (21), and co-unit $\epsilon : \mathcal{H}_{rt} \to \mathbb{K}$ (20).*

Remark 3.3

1) The coproduct can be written in a recursive way, using the B^+ operator, which is a Hochschild 1-cocycle [4, 8, 39]:

$$\Delta(B^+(T_{i_1} \cdots T_{i_n})) = T \otimes 1 + \{id \otimes B^+\}\Delta(T_{i_1} \cdots T_{i_n}). \tag{22}$$

$B^+ : \mathcal{H}_{rt} \to \mathcal{H}_{rt}$ is a linear operator, mapping a (forest, i.e., monomial of) rooted tree(s) to a rooted tree, by connecting the root(s) to a new adjoined root:



It therefore raises the degree by 1. Every rooted tree lies in the image of the B^+ operator. Its conceptual importance with respect to fundamental notions of physics was illuminated recently in [41].

2) The Hopf algebra $\mathcal{H}_{rt}$ contains a commutative, cocommutative Hopf subalgebra $\mathcal{H}_{rt}^{l}$, generated by the so-called rooted ladder trees, denoted by the

symbol $t_n,\ n \in \mathbb{N}$ and recursively defined in terms of the B^+ operator, $t_0 := 1$, $t_m = {B^+}^m(1)$. The coproduct (21) therefore can be written as:

$$\Delta(t_n) = t_n \otimes 1 + 1 \otimes t_n + \sum_{i=1}^{n-1} t_i \otimes t_{n-i}. \tag{23}$$

The bialgebra $\mathcal{H}_{rt}$ actually is a graded connected Hopf algebra, since due to its grading and connectedness, it comes naturally equipped with an antipode $S : \mathcal{H}_{rt} \to \mathcal{H}_{rt}$, recursively defined by:

$$S(T) := -T - \sum_{c_T \in C_T} S(P_{c_T}) R_{c_T}. \tag{24}$$

We come now to the dual $\mathcal{H}^*_{rt}$ of the Hopf algebra of rooted trees, i.e., linear maps from $\mathcal{H}_{rt}$ into $\mathbb{K}$. It is convenient to denote $f(T) =: \langle f, T\rangle \in \mathbb{K}$, $f \in \mathcal{H}^*_{rt},\ T \in \mathcal{H}_{rt}$. Equipped with the convolution product:

$$\begin{aligned} f \star g(T) \quad &:= \quad m_{\mathbb{K}}(f \otimes g)\Delta(T), \quad T \in \mathcal{H}_{rt}. \\ &\overset{(21)}{=} \quad f(T) + g(T) + \sum_{c_T \in C_T} f(P_{c_T}) g(R_{c_T}) \end{aligned} \tag{25}$$

$$\mathcal{H}_{rt} \xrightarrow{\Delta} \mathcal{H}_{rt} \otimes \mathcal{H}_{rt} \xrightarrow{f \otimes g} \mathbb{K} \otimes \mathbb{K} \xrightarrow{m_{\mathbb{K}}} \mathbb{K}$$

it becomes an associative $\mathbb{K}$-algebra. Its unit is given by the co-unit ϵ.

Remark 3.4 Higher powers of the convolution product are defined as follows:

$$\begin{aligned} f_1 \star f_2 \star \cdots \star f_n &:= m_{\mathbb{K}}(f_1 \otimes f_2 \otimes \cdots \otimes f_n)\Delta^{(n-1)} \\ \Delta^{(0)} := id, \quad &\Delta^{(k)} := (id \otimes \Delta^{(k-1)}) \circ \Delta. \end{aligned} \tag{26}$$

$\mathcal{H}^*_{rt}$ contains the set $\operatorname{char}_{\mathbb{K}} \mathcal{H}_{rt}$ of Hopf algebra characters, i.e., multiplicative linear maps with values in the field $\mathbb{K}$.

Definition 3.5 *A linear map $\phi : \mathcal{H}_{rt} \to \mathbb{K}$ is called a character if $\phi(T_1T_2) = \phi(T_1)\phi(T_2)$, $T_i \in \mathcal{H}_{rt},\ i = 1, 2$, i.e., $\phi(1) = 1_{\mathbb{K}}$. We denote the set of characters by* $\operatorname{char}_{\mathbb{K}} \mathcal{H}_{rt}$.

Proposition 3.6 *The set of characters* $\operatorname{char}_{\mathbb{K}} \mathcal{H}_{rt}$ *forms a group with respect to the convolution product* (25). *The inverse of $\phi \in \operatorname{char}_{\mathbb{K}} \mathcal{H}_{rt}$ is given in terms of the antipode* (24)*, $\phi^{-1} := \phi \circ S$.*

Definition 3.7 *A linear map $Z : \mathcal{H}_{rt} \to \mathbb{K}$ is called derivation, or infinitesimal character if $Z(T_1T_2) = Z(T_1)\epsilon(T_2) + \epsilon(T_1)Z(T_2)$, $T_i \in \mathcal{H}_{rt},\ i = 1, 2$, i.e., $Z(1) = 0$. The set of infinitesimal characters is denoted by $\partial \operatorname{char}_{\mathbb{K}} \mathcal{H}_{rt}$.*

Lemma 3.8 *For any $Z \in \partial\,\mathrm{char}_{\mathbb{K}}\,\mathcal{H}_{rt}$ and $T \in \mathcal{H}_{rt}$ of degree $\#(T) = n < \infty$, we have for $m > n$, $Z^{\star m}(T) = 0$.*

Remark 3.9 The last result implies that the exponential $\exp^*(Z)(T) := \sum_{k\geq 0} \frac{Z^{\star k}}{k!}(T)$, $Z \in \partial\,\mathrm{char}_{\mathbb{K}}\,\mathcal{H}_{rt}$, is a finite sum, ending at $k = \#(T)$.

The above facts culminate into the following important results [8, 40]. Given the explicit base of rooted trees generating $\mathcal{H}_{rt}$, the set of derivations $\partial\,\mathrm{char}_{\mathbb{K}}\,\mathcal{H}_{rt}$ is generated by the dually defined infinitesimal characters, indexed by rooted trees:

$$Z_T(T') = \langle Z_T, T'\rangle := \delta_{T,T'}. \tag{27}$$

Proposition 3.10 *The set $\partial\,\mathrm{char}_{\mathbb{K}}\,\mathcal{H}_{rt}$ defines a Lie algebra, denoted by $\mathcal{L}_{\mathcal{H}_{rt}}$, and equipped with the commutator:*

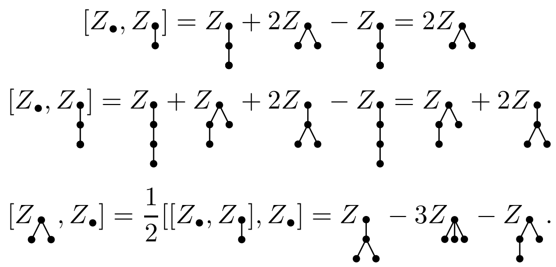
$$\begin{aligned} [Z_{T'}, Z_{T''}] &:= Z_{T'} \star Z_{T''} - Z_{T''} \star Z_{T'} \\ &= \sum_{T \in \mathcal{T}_{rt}} \big(n(T', T''; T) - n(T'', T'; T)\big) Z_T, \end{aligned} \tag{28}$$

where the $n(T', T''; T) \in \mathbb{N}$ denote so-called section coefficients, which count the number of single simple cuts, $|c_T| = 1$, such that $P_{c_T} = T'$ and $R_{c_T} = T''$.
The exponential map $\exp^* : \mathcal{L}_{\mathcal{H}_{rt}} \to \mathrm{char}_{\mathbb{K}}\,\mathcal{H}_{rt}$ defined in remark (3.9) is a bijection.

Generated by the infinitesimal characters Z_T (27), the Lie algebra $\mathcal{L}_{\mathcal{H}_{rt}}$ carries naturally a grading in terms of the grading of the rooted trees in $\mathcal{H}_{rt}$, $\deg(Z_T) := \#(T)$, and $\mathcal{L}_{\mathcal{H}_{rt}} = \bigoplus_{n>0} \mathcal{L}^{(n)}_{\mathcal{H}_{rt}}$. The commutator (28) implies then:

$$\big[\mathcal{L}^{(n)}_{\mathcal{H}_{rt}}, \mathcal{L}^{(m)}_{\mathcal{H}_{rt}}\big] \subset \mathcal{L}^{(m+n)}_{\mathcal{H}_{rt}}. \tag{29}$$

Let us calculate a few commutators, to get a better feeling for the structure of $\mathcal{L}_{\mathcal{H}_{rt}}$:



(30)

This Lie algebra received more attention recently [11, 42, 43], but needs further structural analysis, since it captures in an essential way the whole of renormalization and the structure of the equations of motion [7] in perturbative QFT. This remark is underlined by the results presented in the next section.

4 Classical r-Matrix and Birkhoff decomposition

For a renormalizable theory, the process of renormalization removes the short-distance singularities order by order in the coupling constant. For this to work one has to choose a renormalization scheme which determines the remaining finite part. This choice is of analytic nature but also contains an important algebraic combinatorial aspect, which lies at the heart of the Birkhoff factorization, found in [4, 9]. It is the goal of this section to clarify how this algebraic step implies the Birkhoff decomposition in a completely algebraic manner. We derive the corresponding theorem for graded connected Hopf algebras quite generically. The main ingredient is a generalized notion of regularization in terms of a Rota-Baxter structure, which is supposed to underlie the target space of the characters of $\mathcal{H}_{rt}$.

Following the Hopf algebraic approach to renormalization in perturbative QFT, we henceforth introduce the notion of regularized (infinitesimal) characters, maps from $\mathcal{H}_{rt}$ into a commutative, associative, unital Rota-Baxter algebra $\mathcal{A}$. The choice of the Rota-Baxter map is determined by the choice of the renormalization scheme, which can be a BPHZ scheme (Taylor subtractions of the integrand), the before-mentioned minimal subtraction and momentum schemes, and others, which all provide Rota-Baxter maps. Here is not the space to give a complete census of renormalization schemes in use in physics, but we simply assume Feynman rules and a Rota-Baxter map being given.

Let us mention that sometimes we write R-matrix, instead of the standard notation r-matrix, to underline its operator form, and origin in the Rota-Baxter relation.

We therefore generalize $\mathcal{H}_{rt}^*$ to $L(\mathcal{H}_{rt}, \mathcal{A})$, consisting of $\mathbb{K}$-linear maps from $\mathcal{H}_{rt}$ into the Rota-Baxter algebra $\mathcal{A}$, i.e., $\langle \phi, T \rangle \in \mathcal{A}$, $\phi \in L(\mathcal{H}_{rt}, \mathcal{A})$, $T \in \mathcal{H}_{rt}$. Due to the double RB structure on the Rota-Baxter algebra (12) we naturally get $L(\mathcal{H}_{rt}, \mathcal{A}_R)$. We then lift the Rota-Baxter map $R : \mathcal{A} \to \mathcal{A}$ to $L(\mathcal{H}_{rt}, \mathcal{A})$, which is possible since it is linear.

Proposition 4.1 *Define the linear map* $\mathcal{R} : L(\mathcal{H}_{rt}, \mathcal{A}) \to L(\mathcal{H}_{rt}, \mathcal{A})$ *by* $f \mapsto \mathcal{R}(f) := R \circ f : \mathcal{H}_{rt} \to R(\mathcal{A})$. *Then* $L(\mathcal{H}_{rt}, \mathcal{A})$ *becomes an associative, unital Rota-Baxter algebra. The Lie algebra of infinitesimal characters* $\mathcal{L}_{\mathcal{H}_{rt}} \subset L(\mathcal{H}_{rt}, \mathcal{A})$ *with bracket* (28) *becomes a Lie Rota-Baxter algebra, i.e., for* $Z', Z'' \in \partial\,\mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$*, we have the notion of a classical R-matrix respectively classical Yang-Baxter relation:*

$$[\mathcal{R}(Z'), \mathcal{R}(Z'')] = \mathcal{R}\big([Z', \mathcal{R}(Z'')]\big) + \mathcal{R}\big([\mathcal{R}(Z'), Z'']\big) - \mathcal{R}\big([Z', Z'']\big). \tag{31}$$

Notice that we replaced $\mathbb{K}$ by $\mathcal{A}$ for the target space of the regularized infinitesimal characters. The proof of this proposition was given in (I).

Using the double RB construction and Atkinson's theorem of Section 2 we have the following

Lemma 4.2 *The Rota-Baxter algebra* $L(\mathcal{H}_{rt}, \mathcal{A})$ *equipped with the convolution product:*

$$f \star_{\mathcal{R}} g = f \star \mathcal{R}(g) + \mathcal{R}(f) \star g - f \star g \tag{32}$$

gives a Rota-Baxter algebra structure on the set of linear functionals with values in the double RB algebra $\mathcal{A}_R$ of $\mathcal{A}$, denoted by $L(\mathcal{H}_{rt}, \mathcal{A}_R)$. An analog for $\mathcal{L}_{\mathcal{H}_{rt}}$ exists, denoted by $\mathcal{L}_{\mathcal{H}_{rt}\mathcal{R}}$, equipped with the $\mathcal{R}$-bracket:

$$\begin{aligned}[Z', Z'']_{\mathcal{R}} &= [Z', \mathcal{R}(Z'')] + [\mathcal{R}(Z'), Z''] - [Z', Z''] \\ &= \frac{-1}{2}\big([Z', \mathcal{B}(Z'')] + [\mathcal{B}(Z'), Z'']\big).\end{aligned}$$

The $\mathcal{R}$ map becomes a (Lie) algebra morphism ($\mathcal{L}_{\mathcal{H}_{rt}\mathcal{R}} \to \mathcal{L}_{\mathcal{H}_{rt}}$) $L(\mathcal{H}_{rt}, \mathcal{A}_R) \to L(\mathcal{H}_{rt}, \mathcal{A})$.

Remark 4.3

1) The above is also true for $\tilde{R} := id - R$, respectively $\tilde{\mathcal{R}} := id - \tilde{\mathcal{R}}$ (see Remark 2.10).
2) We will denote the Lie subalgebras $\mathcal{R}(\mathcal{L}_{\mathcal{H}_{rt}})$ by $\mathcal{L}^-_{\mathcal{H}_{rt}}$ and $\tilde{\mathcal{R}}(\mathcal{L}_{\mathcal{H}_{rt}})$ by $\mathcal{L}^+_{\mathcal{H}_{rt}}$.

We now apply Atkinson's theorem to the Lie algebra $\mathcal{L}_{\mathcal{H}_{rt}}$ of infinitesimal characters, the generators of the group of Hopf algebra characters $\mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$.

Lemma 4.4 *Every infinitesimal character $Z \in \mathcal{L}_{\mathcal{H}_{rt}}$ has a unique subdirect Birkhoff decomposition $Z = \mathcal{R}(Z) + \tilde{\mathcal{R}}(Z)$.*

Remark 4.5

1) In the case of an idempotent Rota-Baxter map R we have a direct decomposition $\mathcal{A} = \mathcal{A}_- + \mathcal{A}_+$ respectively $\mathcal{L}_{\mathcal{H}_{rt}} = \mathcal{L}^-_{\mathcal{H}_{rt}} + \mathcal{L}^+_{\mathcal{H}_{rt}}$.
2) Let $Z \in \mathcal{L}_{\mathcal{H}_{rt}}$ be the infinitesimal character generating the character $\phi = \exp^\star(Z) \in \mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$. Using the result in Proposition 2.11, we then see that for elements in $\ker(\epsilon)$, the augmentation ideal, we have

$$\exp^\star(-\mathcal{R}(Z)) = \mathcal{R}\big(\exp^{\star_{\mathcal{R}}}(-Z)\big), \quad \exp^\star(\tilde{\mathcal{R}}(Z)) = -\tilde{\mathcal{R}}\big(\exp^{\star_{\mathcal{R}}}(-Z)\big).$$

4.1 Review of the Ladder case

For the Hopf subalgebra of rooted ladder trees, introduced in the last section, we found in (I) the following simple factorization for a regularized character $\mathrm{char}_{\mathcal{A}}\,\mathcal{H}^l_{rt} \ni \phi = \exp^*(Z),\ Z \in \partial\,\mathrm{char}_{\mathcal{A}}\,\mathcal{H}^l_{rt}$ due to the abelianess of $\mathcal{L}_{\mathcal{H}^l_{rt}}$, and induced by Atkinson's theorem, i.e., the lifted Rota-Baxter map $\mathcal{R}$:

$$\begin{aligned}\phi &= \exp^*(Z) && (33)\\ &= \exp^*\big(\mathcal{R}(Z) + \tilde{\mathcal{R}}(Z)\big) && (34)\\ &= \phi_-^{-1} \star \phi_+ && (35)\end{aligned}$$

where:

$$\phi_- = \exp^\star(-\mathcal{R}(Z)) \quad \phi_+ = \exp^\star(\tilde{\mathcal{R}}(Z)), \tag{36}$$

and such that we arrive at the following formulae [8] for $\phi_\pm$, using Proposition 2.11:

Proposition 4.6 *In the rooted ladder tree case, $\mathcal{H}^l_{rt}$, we find for the factors* (36) *in the Birkhoff decomposition* (35) *the following explicit formulae:*

$$\begin{aligned} \phi_-(t_n) &= \mathcal{R}\big(\exp^{\star_{\mathcal{R}}}(-Z)\big)(t_n) && (37)\\ &= -R\big\{\phi(t_n)+\sum_{k=1}^{n-1}\phi_-(t_k)\phi(t_{n-k})\big\} && (38)\end{aligned}$$

$$\begin{aligned} \phi_+(t_n) &= -\tilde{\mathcal{R}}\big(\exp^{\star_{\mathcal{R}}}(-Z)\big)(t_n) && (39)\\ &= \tilde{R}\big\{\phi(t_n)+\sum_{k=1}^{n-1}\phi_-(t_k)\phi(t_{n-k})\big\}. && (40)\end{aligned}$$

We emphasize that the map:

$$\begin{aligned} b[\phi](t_n) &:= \exp^{\star_{\mathcal{R}}}(-Z)(t_n) && (41)\\ &= -\phi(t_n)-\sum_{k=1}^{n-1}\phi_-(t_k)\phi(t_{n-k}) && (42)\end{aligned}$$

which we will call Bogoliubov character, is a Hopf algebra character $\mathcal{H}^l_{rt}\to\mathcal{A}_R$, i.e., into the double RB algebra $\mathcal{A}_R$ of $\mathcal{A}$. This gives a natural algebraic expression for Bogoliubov's $\bar{R}$-map. In the next section, where we treat the general case, we will formally introduce $b[\phi]$. The Rota-Baxter maps $\mathcal{R}$ and $-\tilde{\mathcal{R}}$, i.e., the Lie algebra homomorphisms, become group homomorphisms $\mathrm{char}_{\mathcal{A}_R}\mathcal{H}_{rt}\xrightarrow{\mathcal{R},-\tilde{\mathcal{R}}}\mathrm{char}_{\mathcal{A}}\mathcal{H}_{rt}$. We will generalize this to arbitrary rooted trees in the following section, using equation (17).

4.2 The general case

As stated above, in (I) we introduced a classical R-matrix coming from the Rota-Baxter structure underlying the target space of regularized characters. We saw that in the case of the Hopf subalgebra of rooted ladder trees, the abelianess of the related Lie algebra implies a somehow simple Birkhoff factorization (35, 36) respectively the formulae for the factors $\phi_\pm$ (38, 40). The general, i.e., non-cocommutative case can be solved due to the graded, connectedness of the Hopf algebra of rooted trees.

Suppose we start with an infinitesimal character $Z\in\partial\,\mathrm{char}_{\mathcal{A}}\mathcal{H}_{rt}$ generating the regularized Hopf algebra character $\phi=\exp^*(Z)\in\mathrm{char}_{\mathcal{A}}\mathcal{H}_{rt}$. The above mentioned properties of the Hopf algebra allow for a recursive definition of an infinitesimal character $\chi=\chi(Z)\in\partial\,\mathrm{char}_{\mathcal{A}}\mathcal{H}_{rt}$, defined in terms of Z, using the lower central series of Lie algebra commutators.

Setting

$$\chi(Z)=Z+\sum_{k=1}^{\infty}\chi_Z^{(k)} \tag{43}$$

we proceed in the following manner. We first introduce the related series

$$\chi(u;Z) = Z + \sum_{k=1}^{\infty} u^k \chi_Z^{(k)}, \tag{44}$$

where we assume that $0 < u < 1$ is a real parameter. We also set $\chi_Z^{(0)} = Z$. We next introduce the Baker-Campbell-Hausdorff (BCH) series

$$BCH(A,B) := \frac{1}{2}[A,B] + \frac{1}{12}\Big(\big[A,[A,B]\big] - \big[B,[A,B]\big]\Big) + \cdots . \tag{45}$$

See [44] for more details on the BCH formula. Let us write equation (45) in the form

$$BCH(A,B) = \sum_{k=1}^{\infty} c_k K^{(k)}(A,B), \tag{46}$$

such that the $K^{(k)}$ are the appropriate nested- or multicommutators of depth $k \in \mathbb{N}$, i.e., $K^{(1)} = [A,B]$, $K^{(2)} = [A,[A,B]] - [B,[A,B]]$ and so on. Then, the $\chi_Z^{(k)}$ are defined as the solution of the fix point equation

$$\chi(u;Z) = Z - \sum_{k=1}^{\infty} c_k u^k K^{(k)}\left(\mathcal{R}(\chi(Z)), \tilde{\mathcal{R}}(\chi(Z))\right). \tag{47}$$

Note that $\chi(u;Z)(T)$ is a polynomial in u of degree $m-2$ for any finite tree T of degree $\#(T) = m$ say, i.e., with m vertices and therefore is well defined at $u = 1$. We hence set $\chi(Z) \equiv \chi(1;Z)$. Furthermore, $\chi(Z)$ is an infinitesimal character as it is a finite linear combination of infinitesimal characters, and thus the above definition on trees implies its action on forests as a derivation in the sense of definition (3.7).

It is immediate that $\chi_Z^{(k)}$ vanishes for all $k \geq 1$ when applied to cocommutative Hopf algebra elements.

Let us work out the cases $k = 1, 2$ as examples:

$$\chi^{(1)} = -\frac{1}{2}[\mathcal{R}(Z), \tilde{\mathcal{R}}(Z)] = -\frac{1}{2}[\mathcal{R}(Z), Z] \tag{48}$$

and for $k = 2$ we have:

$$\begin{aligned}
\chi^{(2)} &= -\frac{1}{2}[\mathcal{R}(\chi^{(1)}), \tilde{\mathcal{R}}(Z)] - \frac{1}{2}[\mathcal{R}(Z), \tilde{\mathcal{R}}(\chi^{(1)})] \\
&\qquad -\frac{1}{12}\big(\big[\mathcal{R}(Z),[\mathcal{R}(Z),Z]\big] - \big[\tilde{\mathcal{R}}(Z),[\mathcal{R}(Z),Z]\big]\big) \\
&= +\frac{1}{4}[\mathcal{R}([\mathcal{R}(Z),Z]), \tilde{\mathcal{R}}(Z)] + \frac{1}{4}[\mathcal{R}(Z), \tilde{\mathcal{R}}([\mathcal{R}(Z),Z])] \\
&\qquad -\frac{1}{12}\big(\big[\mathcal{R}(Z),[\mathcal{R}(Z),Z]\big] - \big[\tilde{\mathcal{R}}(Z),[\mathcal{R}(Z),Z]\big]\big)
\end{aligned}$$

$$= \frac{1}{4}\big[\mathcal{R}([\mathcal{R}(Z),Z]),Z\big] \\ +\frac{1}{12}\big(\big[\mathcal{R}(Z),[\mathcal{R}(Z),Z]\big]-\big[[\mathcal{R}(Z),Z],Z\big]\big). \tag{49}$$

where $\tilde{\mathcal{R}}$ has completely vanished. This nontrivial fact comes partly from $\tilde{\mathcal{R}} = \mathrm{id} - \mathcal{R}$, and in a moment we show that the $\chi_Z^{(k)}$ solve the simpler recursion:

$$\chi(u;Z) = Z + \sum_{k=1}^{\infty} c_k u^k K^{(k)}\left(-\mathcal{R}(\chi(Z)),Z\right). \tag{50}$$

Indeed, from the relation (47) and the recursive definition of the $\chi_Z^{(k)}$ we have the following factorization for group like elements in $\mathcal{H}_{rt}^*$:

Proposition 4.7 *Using the infinitesimal character $\chi \in \partial\,\mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$ defined in* (47)*, we have the following decomposition of a character $\phi \in \mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$ given in terms of its generating infinitesimal character $Z \in \partial\,\mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$:*

$$\exp^*(Z) = \exp^*\left(\mathcal{R}(\chi(Z))\right) \star \exp^*\left(\tilde{\mathcal{R}}(\chi(Z))\right). \tag{51}$$

This then implies the simpler recursion (50), in which the vanishing of $\tilde{\mathcal{R}}$ is apparent.

Remark 4.8 The above formal derivation of the factorization of $\mathcal{H}_{rt}$ characters using the BCH formula in (47) and (50) to define the infinitesimal character $\chi(Z)$ (43) may be summarized in a more suggestive manner by the following two recursive formulae:

$$\begin{aligned}\chi(Z) &= Z - BCH\big(\mathcal{R}(\chi(Z)),\tilde{\mathcal{R}}(\chi(Z))\big)\\ &= Z + BCH\big(-\mathcal{R}(\chi(Z)),Z\big).\end{aligned}$$

Let us define the factors, i.e., characters $\phi_\pm \in \mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$:

$$\phi_-^{-1} := \exp^*\left(\mathcal{R}(\chi(Z))\right), \quad \phi_+ := \exp^*\left(\tilde{\mathcal{R}}(\chi(Z))\right) \tag{52}$$

and introduce the Bogoliubov character now in general via the following definition.

Definition 4.9 *Let $\exp^*(Z) = \phi \in \mathrm{char}_{\mathcal{A}}\,\mathcal{H}_{rt}$. We define the following character $b[\phi] \in \mathrm{char}_{\mathcal{A}_R}\,\mathcal{H}_{rt}$ with values in the double RB algebra of $\mathcal{A}$, and call it Bogoliubov character:*

$$b[\phi] := \exp^{*R}(-\chi(Z)). \tag{53}$$

Remembering the crucial property of exponentiated Rota-Baxter maps, coming from (12), respectively (14, 15):

$$\exp^*\left(-\mathcal{R}(Z)\right) = \mathcal{R}\big(\exp^{*R}(-Z)\big) \tag{54}$$

$$\exp^*\left(\tilde{\mathcal{R}}(Z)\right) = -\tilde{\mathcal{R}}\big(\exp^{*R}(-Z)\big), \tag{55}$$

for elements in the augmentation ideal $\ker(\epsilon)$, we have:

$$\phi_- := \mathcal{R}(b[\phi]), \quad \phi_+ := -\tilde{\mathcal{R}}(b[\phi]). \tag{56}$$

We use the factorization (51) in proposition (4.7) to derive an explicit formula for the characters $b[\phi]$ respectively $\phi_\pm$. Let $T \in \ker(\epsilon)$, using the coproduct (21), we get:

$$\begin{aligned} -\tilde{\mathcal{R}}(b[\phi])(T) &= \exp^*(-\mathcal{R}(\chi(Z))) \star \exp^*(Z)(T) \qquad (57)\\ &= \exp^*(Z)(T) + \mathcal{R}(b[\phi])(T) \\ &\quad + \sum_{n\geq 0} \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} \mathcal{R}(-\chi(Z))^{\star(n-j)} \star Z^{\star j}(T), \qquad (58) \end{aligned}$$

where (57) again implies the simpler recursion equation (50).

Remark 4.10

1) All expressions are well defined since they reduce to finite sums for an element $T \in \ker(\epsilon)$ of finite order $\#(T) = m < \infty$.
2) In the last expression in equation (58), the primitive part in $\Delta(T)$ is mapped to zero, since only strictly positive powers of infinitesimal characters appear.

Continuing the above calculation, we get the following:

$$\begin{aligned} -\tilde{\mathcal{R}}(b[\phi])(T) - \mathcal{R}(b[\phi])(T) &= -b[\phi](T) \\ &= \exp^*(Z)(T) \\ &\quad + \sum_{n\geq 0} \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} \mathcal{R}(-\chi(Z))^{\star(n-j)} \star Z^{\star j}(T), \end{aligned}$$

and therefore we find the well-known formula:

$$\begin{aligned} \mathcal{R}\big(\exp^{*R}(-\chi(Z))\big)(T) &= -R\big(\exp^*(Z)(T) \\ &\quad + \sum_{n\geq 0} \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} \mathcal{R}(-\chi(Z))^{\star(n-j)} \star Z^{\star j}(T)\big). \end{aligned}$$

Finally, we rederive the results of [3, 4, 8] which gave the counterterm and the renormalized contribution as the image of the Bogoliubov character under the group homomorphisms $\mathcal{R}$ and $-\tilde{\mathcal{R}}$, now derived from the double RB construction for any algebraic Birkhoff decomposition based on a suitable $\mathcal{R}$, i.e., Rota-Baxter type map:

Theorem 4.11 *For $T \in \mathcal{H}_{rt}$, $\#(T) = m$ we have the following formulae for the factors in (51):*

$$\begin{aligned} \mathcal{R}(b[\phi])(T) &= \phi_-(T) \\ &= -R\big(\phi(T) + \sum_{n\geq 0}^{m} \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} \mathcal{R}(-\chi(Z))^{\star(n-j)} \star Z^{\star j}(T)\big). \\ &= -R\big(\phi(T) + \sum_{c_T \in C_T} \phi_-(P_{c_T})\phi(R_{c_T})\big). \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathcal{R}}\big(b[\phi]\big)(T) &= \phi_+(T) \\ &= \tilde{R}\big(\phi(T) + \sum_{n\geq 0}^{m} \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} \mathcal{R}(-\chi(Z))^{\star(n-j)} \star Z^{\star j}(T)\big). \\ &= \tilde{R}\big(\phi(T) + \sum_{c_T \in C_T} \phi_-(P_{c_T})\phi(R_{c_T})\big). \end{aligned}$$

This should be compared to the general equation (16) including the shuffle:

$$\begin{aligned} -\exp^{*R}(-\chi(Z))(T) &= \exp^*(Z)(T) + \sum_{n\geq 0} \frac{1}{n!} \\ &\quad \sum_{j=1}^{n-1} \binom{n}{j} \mathcal{R}(-\chi(Z))^{\star(n-j)} \star Z^{\star j}(T). \\ &= \exp^*(\chi(Z))(T) + \\ &\quad \sum_{n\geq 0} \frac{1}{n!} \sum_{j=1}^{n-1} \mathcal{R}(-\chi(Z))^{\star(n-j)} \sqcup\!\sqcup_\star \chi(Z)^{\star j}(T). \end{aligned}$$

It allows us to define the infinitesimal character $\chi = \chi(Z)$ to order $k > 0$ in another way recursively by using the $\chi_Z^{(j)}$, $j < k$. We therefore get to order k:

$$\begin{aligned} \chi_Z^{(k)} &= -\sum_{j=1}^{k-1} \chi_Z^{(j)} - \sum_{l=1}^{k+2} \frac{1}{l!} \chi(Z)^{\star\, l} + \sum_{l=1}^{k+2} \frac{1}{l!} Z^{\star\, l} \\ &\quad - \sum_{n\geq 0}^{k+2} \frac{1}{n!} \sum_{j=1}^{n-1} \mathcal{R}(-\chi(Z))^{\star(n-j)} \sqcup\!\sqcup_\star \chi(Z)^{\star j} \\ &\quad + \sum_{n\geq 0}^{k+2} \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} \mathcal{R}(-\chi(Z))^{\star(n-j)} \star Z^{\star j}. \end{aligned}$$

After these formal arguments based on the general results for Rota-Baxter operators and the structure of the rooted tree Hopf algebra, we end this section and the paper with a remark on calculational aspects.

When treating the rooted ladder case, we mentioned at the end of (I) the use of normal coordinates introduced by Chryssomalakos et al. in [43]. Given a regularized character $\phi \in \mathrm{char}_{\mathcal{A}}\, \mathcal{H}_{rt}$, this provided a very easy way to define the coefficients for its generator $Z \in \partial\, \mathrm{char}_{\mathcal{A}}\, \mathcal{H}_{rt}$:

$$Z := \sum_{n>0} \alpha^{(n)} Z_{t_n}, \quad \alpha^{(n)} \in \mathcal{A} \tag{59}$$

such that $\exp^*(Z)(t_n) = \phi(t_n) \in \mathcal{A}$.

Remark 4.12

1) Here and also later, we omit for notational reasons the tensor sign between the $\alpha^{(n)}$ and Z_{t_n}, i.e., $\alpha^{(n)} Z_{t_n} \in A \otimes \partial\, \mathrm{char}_{\mathcal{A}}\, \mathcal{H}_{rt},\ n > 0$.
2) The $\alpha^{(n)}$ were given in terms of Schur polynomials, i.e., $\alpha^{(n)} := \phi(P(t_1, \cdots, t_n))$. And ϕ_- was just $\exp^*(-\mathcal{R}(Z))$.

In the general case, i.e., for arbitrary rooted trees, the simple Schur polynomials get replaced by the following set of polynomial equations. Details may be found in [43]. Introducing the symbols x^T, indexed by rooted trees, and defining the new coordinates, which are characterized by $\exp^*(Z)(x^T) = \phi(x^T),\ \phi \in \mathrm{char}_{\mathcal{A}}\, \mathcal{H}_{rt}$, where:

$$Z = \sum_{T \in \mathcal{T}_{rt}} \alpha^{x^T} Z_T \in \partial\, \mathrm{char}_{\mathcal{A}}\, \mathcal{H}_{rt}, \tag{60}$$

we arrive at the following infinite set of coupled equations, expressing the coordinates T in terms of the new x^T:

$$T = \sum_{n\geq 0} \frac{1}{(n+1)!} m_{\mathcal{H}_{rt}}(\mathcal{P}^{\otimes n+1}) \Delta'^{(n)}(x^T), \quad T \in \mathcal{T}_{rt}. \tag{61}$$

Here, the map $\mathcal{P}$ denotes the projector into the augmentation ideal:

$$\mathcal{P}(x^{T_1} \cdots x^{T_n}) := \begin{cases} 0, & x^{T_1} \cdots x^{T_n} = 1 \\ x^{T_1} \cdots x^{T_n}, & else. \end{cases} \tag{62}$$

Δ' denotes the coproduct reduced to single simple cuts $|c_T| = 1$, and $\Delta'^{(n)} := (id \otimes \Delta'^{(n-1)}) \circ \Delta'$, such that $\Delta'^{(0)} := id,\ \Delta'^{(1)} = \Delta'$. One should compare this operation with the formal linear map $\exp^*(\hat{Z})(T)$ on $\mathcal{H}_{rt}$, where $\hat{Z} := \sum_{T \in \mathcal{T}_{rt}} T\, Z_T$, $T\, Z_T(T') = T \delta_{T,T'}$, and $T_1\, Z_{T_1} \star T_2\, Z_{T_2} := T_1 T_2\, Z_{T_1} \star Z_{T_2}$.

The first four equations for the rooted trees $\bullet$, $\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}$, $\begin{smallmatrix}\bullet\\ \vert\\ \bullet\\ \vert\\ \bullet\end{smallmatrix}$, $\overset{\bullet}{\bullet\wedge\bullet}$ are:

$$\begin{aligned}
\bullet &= x^{\bullet} \\
\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix} &= x^{\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}} + \frac{1}{2} x^{\bullet} x^{\bullet} \\
\begin{smallmatrix}\bullet\\ \vert\\ \bullet\\ \vert\\ \bullet\end{smallmatrix} &= x^{\begin{smallmatrix}\bullet\\ \vert\\ \bullet\\ \vert\\ \bullet\end{smallmatrix}} + x^{\bullet} x^{\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}} + \frac{1}{6} x^{\bullet} x^{\bullet} x^{\bullet}, \quad \overset{\bullet}{\bullet\wedge\bullet} = x^{\overset{\bullet}{\bullet\wedge\bullet}} + x^{\bullet} x^{\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}} + \frac{1}{3} x^{\bullet} x^{\bullet} x^{\bullet}
\end{aligned}$$

The final step, done in [43], is to invert the above equations, giving the x^T's in terms of the original rooted trees. In the ladder case we just get the Schur polynomials. In general, we have for example:

$$x^{\substack{\bullet\\ \bullet\\ \bullet}} = \substack{\bullet\\ \bullet\\ \bullet} - \bullet\substack{\bullet\\ \bullet} + \frac{1}{3}\bullet\bullet\bullet, \qquad x^{\wedge} = \wedge - \bullet\substack{\bullet\\ \bullet} + \frac{1}{6}\bullet\bullet\bullet.$$

Therefore, the coefficients $\alpha^{x^T} := \phi(x^T) \in \mathcal{A}$ in (60).

Let us briefly dwell on the generalization to decorated non-planar rooted trees, and to Feynman graphs, following [38, 45]. Every Feynman graph provides a number r of maximal forests. The integer r counts the number of terms p_i, $i = 1, \ldots, r$ in the coproduct which are primitive on the rhs, and in the augmentation ideal on the lhs of the coproduct on graphs. If $r > 1$, we call the graph overlapping divergent. It is then mapped to a linear combination of r decorated rooted trees, where each of those trees has a root decorated by one of the p_i. Iterating this procedure, one obtains a map from Feynman graphs to decorated rooted trees where the decorations are provided by subdivergence free skeleton contributions. Having resolved the overlapping sectors into trees, one then proceeds as before.

We close this paper with a study of a simple example on decorated rooted trees using two decorations. The generalization to Feynman graphs including form factor decompositions for theories with spin is somewhat excessive on the notational side, but provides no difficulty for the practitioner of quantum field theory, making full use of the Hopf and Lie algebra of Feynman graphs with external structures. See [5, 38, 45] where examples can be found.

We consider the example of vertices with a decoration $\mathcal{D}$ by 2 elements $\{a, b\}$. Let us denote them by a vertex $\bullet^{a}$ and a vertex $\bullet^{b}$. For the Lie bracket (28) of these two vertices we get:

$$[Z_{\bullet^a}, Z_{\bullet^b}] = Z_{\substack{\bullet b\\ \bullet a}} - Z_{\substack{\bullet a\\ \bullet b}}. \tag{63}$$

Note that though we have here the analog of a simple nesting of one graph in another, this has already a non-vanishing commutator in the Lie algebra. This fact makes it necessary to include the BCH-corrections (50) already at this level. For the above example (63), we have to add the correction (48).

Let us do the calculation of the counterterm ϕ_- explicitly for the decorated rooted ladder tree $\substack{\bullet b\\ \bullet a}$, using the normal coordinates in (60). We have to use $\chi = Z + \chi^{(1)}$ (43), with $\chi^{(1)}$ given in (48). The infinitesimal character Z generating the character ϕ is given to order 2 in terms of the normal coordinates x^T as:

$$Z = \phi(x^{\bullet^a})Z_{\bullet^a} + \phi(x^{\bullet^b})Z_{\bullet^b} + \phi(x^{\substack{\bullet a\\ \bullet b}})Z_{\substack{\bullet a\\ \bullet b}} + \phi(x^{\substack{\bullet b\\ \bullet a}})Z_{\substack{\bullet b\\ \bullet a}}, \tag{64}$$

where:

$$
\begin{aligned}
\phi(x^{\bullet^{a}}) &= \phi(\bullet^{a}), \quad \phi(x^{\bullet^{b}}) = \phi(\bullet^{b}) \\
\phi\Big(x^{\overset{\bullet^{a}}{\underset{\bullet^{b}}{|}}}\Big) &= \phi\Big(\overset{\bullet^{a}}{\underset{\bullet^{b}}{|}}\Big) - \frac{1}{2}\phi(\bullet^{a})\phi(\bullet^{b}), \quad \phi\Big(x^{\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}}\Big) = \phi\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) - \frac{1}{2}\phi(\bullet^{a})\phi(\bullet^{b}).
\end{aligned} \tag{65}
$$

And therefore we have for the infinitesimal character $\chi(Z)$ to order $k = 1$, i.e., including the first correction (48):

$$
\begin{aligned}
\chi &= \phi(x^{\bullet^{a}})Z_{\bullet^{a}} + \phi(x^{\bullet^{b}})Z_{\bullet^{b}} + \phi\Big(x^{\overset{\bullet^{a}}{\underset{\bullet^{b}}{|}}}\Big)Z_{\overset{\bullet^{a}}{\underset{\bullet^{b}}{|}}} + \phi\Big(x^{\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}}\Big)Z_{\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}} \\
&\quad -\frac{1}{2}\big([\mathcal{R}(\phi(x^{\bullet^{a}})Z_{\bullet^{a}}), \phi(x^{\bullet^{b}})Z_{\bullet^{b}}] + [\mathcal{R}(\phi(x^{\bullet^{b}})Z_{\bullet^{b}}), \phi(x^{\bullet^{a}})Z_{\bullet^{a}}]\big).
\end{aligned} \tag{66}
$$

So that when the counterterm character $\phi_- = \exp^*(-\mathcal{R}(\chi))$ is applied to $\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}$ we get:

$$
\begin{aligned}
\phi_-\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) &= \exp^*(-\mathcal{R}(\chi))\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) \\
&= -\mathcal{R}(\chi)\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) + \frac{1}{2}\mathcal{R}(\chi) \star \mathcal{R}(\chi)\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) && (67) \\
&= -R\Bigg(\phi\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big)Z_{\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}}\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) - \frac{1}{2}\phi(\bullet^{a})\phi(\bullet^{b})Z_{\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}}\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) \\
&\quad -\frac{1}{2}\big\{R(\phi(\bullet^{a})Z_{\bullet^{a}}(\bullet^{a}))\ \phi(\bullet^{b})Z_{\bullet^{b}}(\bullet^{b}) \\
&\qquad -\phi(\bullet^{a})Z_{\bullet^{a}}(\bullet^{a})\ R(\phi(\bullet^{b})Z_{\bullet^{b}}(\bullet^{b}))\big\}\Bigg) \\
&\qquad +\frac{1}{2}R(\phi(\bullet^{a})Z_{\bullet^{a}}(\bullet^{a}))R(\phi(\bullet^{b})Z_{\bullet^{b}}(\bullet^{b})) && (68) \\
&= -R\Big(\phi\Big(\overset{\bullet^{b}}{\underset{\bullet^{a}}{|}}\Big) + R(\phi(\bullet^{a}))\ \phi(\bullet^{b})\Big), && (69)
\end{aligned}
$$

which is the correct result. In line (67) no higher order terms can appear. In the next line we used relations (65). From (68) to the last equality (69) we used the RB relation:

$$
\begin{aligned}
&\frac{1}{2}R\big(\phi(\bullet^{a})Z_{\bullet^{a}}(\bullet^{a})\big)R\big(\phi(\bullet^{b})Z_{\bullet^{b}}(\bullet^{b})\big) + \frac{1}{2}R\big(\phi(\bullet^{a})Z_{\bullet^{a}}(\bullet^{a})\ \phi(\bullet^{b})Z_{\bullet^{b}}(\bullet^{b})\big) \\
&\qquad = \frac{1}{2}R\big\{R\big(\phi(\bullet^{a})Z_{\bullet^{a}}(\bullet^{a})\big)\ \phi(\bullet^{b})Z_{\bullet^{b}}(\bullet^{b}) \\
&\qquad\qquad +\phi(\bullet^{a})Z_{\bullet^{a}}(\bullet^{a})\ R\big(\phi(\bullet^{b})Z_{\bullet^{b}}(\bullet^{b})\big)\big\}.
\end{aligned}
$$

Remark 4.13

1) Note that now $\chi(u; Z)(T)$, equation (44), will be a polynomial of degree at most $m-1$ when acting on decorated trees with m vertices, as the tree studied above with two vertices differently decorated is already non-cocommutative under the coproduct.
2) Using the standard example of the QFT Φ^3_{6dim}, the result (69) should be compared to the counterterm for the Feynman graph , which has an additional factor two reflecting the fact that it is overlapping divergent, $r = 2$, and it resolves into two identical rooted trees [38].

5 Conclusion and outlook

In this work we generalized the results of (I) to arbitrary rooted trees, i.e., we showed how to derive the Birkhoff factorization for characters of the Hopf algebra of rooted trees. Using the Rota-Baxter structure underlying the target space of the characters of a renormalization Hopf algebra, the notion of a classical r-matrix was introduced on the corresponding Lie algebra defined on rooted trees. A couple of simple results for Rota-Baxter algebras were collected which allowed for a straightforward derivation of the twisted antipode formula, defined in [3, 8] concerning the study of the Hopf algebraic approach to perturbative QFT. This gives a firm algebraic basis to any renormalization scheme using an algebraic Birkhoff decomposition together with a suitable double RB construction.

We regard this work as a further step towards a more interesting connection to the realm of integrable systems. Sakakibara's result [46] also points into this direction. This connection was already apparent in [10], in which effectively the grading operator Y served as a Hamiltonian providing the "scaling evolution" of the coupling constant, and hence the renormalization group flow initiated by scaling transformations, and can and should be worked out for the corresponding flow of many other physical parameters of interest.

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Kurusch Ebrahimi-Fard* and Dirk Kreimer
Institut des Hautes Études Scientifiques
35, Route de Chartres
F-91440 Bures-sur-Yvette
France

and

*Universität Bonn
Physikalisches Institut
Nussallee 12
D-53115 Bonn
Germany
email: kurusch@ihes.fr

Li Guo
Rutgers University
Department of Mathematics and Computer Science
Newark, NJ 07102
USA
email: liguo@newark.rutgers.edu

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