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# Absence of Continuous Spectral Types for Certain Non-Stationary Random Schrödinger Operators

Anne Boutet de Monvel, Peter Stollmann and Günter Stolz

*In Memory of Robert M. Kauffman*

**Abstract.** We consider continuum random Schrödinger operators of the type $H_\omega = -\Delta + V_0 + V_\omega$ with a deterministic background potential $V_0$. We establish criteria for the absence of continuous and absolutely continuous spectrum, respectively, outside the spectrum of $-\Delta + V_0$. The models we treat include random surface potentials as well as sparse or slowly decaying random potentials. In particular, we establish absence of absolutely continuous surface spectrum for random potentials supported near a one-dimensional surface ("random tube") in arbitrary dimension.

## 1 Introduction

In this article we are concerned with spectral properties of certain non-stationary random operators. More specifically, we consider Schrödinger operators of the form $H_\omega = -\Delta + V_0 + V_\omega$ in $L^2(\mathbb{R}^d)$. Here $V_0$ is a deterministic background potential and $V_\omega$ an Anderson-type random potential which is either sparse near infinity, or concentrated near a lower dimensional surface, or both. This type of models has attracted considerable interest as it allows to study a transition from pure point to continuous spectrum. Here, we are mainly concerned with the former phenomenon.

We obtain our results by essentially "deterministic" techniques from [27, 22, 28], establishing conditions on $V_\omega$ such that $H_\omega$ has no absolutely continuous spectrum or no continuous spectrum outside the spectrum of $-\Delta + V_0$. This gives us considerable flexibility in the choice of our model. In particular, we are able to avoid some of the typical technical restrictions that come with the usual multiscale analysis or fractional moments proofs of localization. E.g., we can allow for perturbations of changing sign and single site distributions without any continuity. On the other hand, we need decaying randomness in the sense that near infinity the random perturbation is not too effective. That excludes identically distributed random parameters in most cases. An important exception is our result on 1-D "surfaces" (rather tubes) in arbitrary dimensions, see Theorem 4.1 below.

The paper is organized in the following way: In Section 2 we present the deterministic techniques we use, recalling the relevant notions and results from [27, 22, 28]; in fact we will need results that are a little stronger than what is explicitly stated in the above cited articles. The common flavor of these methods is that they provide comparison criteria for the absence of continuous and absolutely continuous spectra, respectively. These criteria are formulated in the following way: We consider Schrödinger operators with two potentials that differ only on a set that is "small near infinity in a certain geometrical sense". Then the spectrum of the first operator has no absolutely continuous component on the resolvent set of the second one. To exclude continuous spectrum one needs a bit more complicated assumptions involving randomization.

In Sections 3 and 4 we state and prove our main new results, Theorems 3.1, 4.1 and 4.3.

In Section 3 we are dealing with sparse random potentials. The framework we introduce is fairly general and includes as special cases the sparse random models considered in [7], e.g., random scatterers are distributed quite arbitrarily in space and the single site perturbations are assumed to be picked with probabilities that tend to zero near infinity. Then, throughout the resolvent set of the unperturbed operator there is no absolutely continuous spectrum (3.1(a)). Since we can treat quite general unperturbed operators, this includes cases with gaps in the spectrum of the unperturbed operator, a case that is completely new. In the proof we combine elementary combinatorial arguments, Lemma 3.2, with the methods discussed above.

In the same fashion, under a bit more incisive conditions concerning the background and at least one random scatterer but with the same condition concerning the decay of probabilities near infinity, we can even deduce absence of continuous spectrum outside the spectrum of the unperturbed operator (3.1(b)). That is, all the new spectrum generated by the random perturbation is pure point. This is quite different from what one can obtain with the usual localization proofs, which require a large disorder condition, or apply to energies near the spectral boundaries of the perturbed operator only (with the exception of the one-dimensional case).

In Section 4 we study surface-like structures. This means we consider potentials that are concentrated near a subset of lower dimension. Our strongest result, Theorem 4.1, concerns what we call quasi-1D surfaces. There is quite some literature on surface potentials. Most are dealing with the discrete case [4, 5, 8, 9, 11, 10, 13, 14] while in [3, 7] and the present paper continuum models are treated. Here again, our goal was to be able to exclude absolutely continuous spectrum on all of the unperturbed resolvent set and not just near band edges. Theorem 4.3 deals with absence of absolutely continuous and continuous spectrum, respectively, for $m$-dimensional surface potentials in $\mathbb{R}^d$ under an additional sparseness assumption.

In the last section we conclude with a discussion of some possible extensions of our results and a comparison with other works, in particular the results in [10] and [7].

## 2 Comparison criteria for absence of (absolutely) continuous spectrum

In this section we present our methods of proof, essentially taken from [27, 22, 28]. These methods rely on comparison of the spectral properties of Schrödinger operators

$$H_1 = -\Delta + V_1 \text{ and } H_2 = -\Delta + V_2$$

whose "difference" is "small" in the sense that the set

$$\{V_1 \neq V_2\} := \{x \in \mathbb{R}^d \mid V_1(x) \neq V_2(x)\}$$

is sufficiently sparse. To this end we introduce the following concept, following [27]:

**Definition.** A sequence $(S_n)_{n\in\mathbb{N}}$ of compact subsets of $\mathbb{R}^d$ with Lebesgue measure $|S_n| = 0$ $(n \in \mathbb{N})$ is called a *total decomposition* if there exists a family $(U_i)_{i\in I}$ of disjoint, open, bounded sets such that

$$\mathbb{R}^d \setminus \bigcup_{n\in\mathbb{N}} S_n = \bigcup_{i\in I} U_i.$$

A typical example would be $S_n = \partial B(0, n)$, where $B(x, r)$ denotes the closed ball of radius $r$, centered at $x$. (Let us stress that the $S_n$'s need not be pairwise disjoint.)

The sparseness of $\{V_1 \neq V_2\}$ will be expressed by the existence of a total decomposition $(S_n)_{n\in\mathbb{N}}$ with sufficient distance of $S_n$ to $\{V_1 \neq V_2\}$ compared with the *size* of $S_n$. An appropriate notion of size is given by the *generalized surface area* of a set, a notion introduced in [22] in the following way; here $S \subset \mathbb{R}^d$ is compact:

$$\sigma(S) := \sup_{r\geq 0} \frac{|\{x \in \mathbb{R}^d \mid r \leq \operatorname{dist}(x, S) \leq r + 1\}|}{r^d + 1}.$$

It is easily seen that

$$\sigma(S) \leq C\,((\operatorname{diam} S)^d + 1), \tag{2.1}$$

i.e., $\sigma(S)$ is at worst a volume, while for sufficiently regular surfaces it is a surface area measure, for example

$$\sigma(\partial B(x, r)) \leq C(r^{d-1} + 1).$$

We cite the following result, essentially taken from [27]:

**Theorem 2.1** *Assume that for each $\gamma > 0$ there exists a total decomposition $(S_n)_{n\in\mathbb{N}} = (S_n^{(\gamma)})_{n\in\mathbb{N}}$ such that*

$$\delta_n = \delta_n^{(\gamma)} := \operatorname{dist}(\{V_1 \neq V_2\}, S_n) \to \infty \text{ as } n \to \infty \tag{2.2}$$

*and*

$$\sum_n \sigma(S_n)\mathrm{e}^{-\gamma\delta_n} < \infty. \tag{2.3}$$

*Then*

$$\sigma_{\mathrm{ac}}(H_1) \cap \varrho(H_2) = \varnothing.$$

The following figure is to help visualizing the geometry one is confronted with in the Theorem.

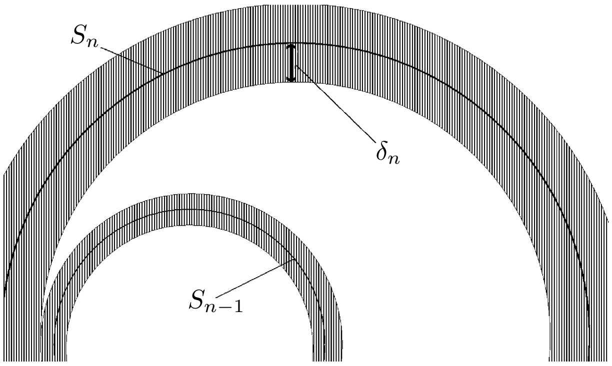


Figure 1. $\{V_1 \neq V_2\}$ must not intersect the shaded region.

Here, and in what follows, all potentials $V$ are assumed to be locally uniformly in $L^p$, where $p \geq 2$ if $d \leq 3$ and $p > d/2$ if $d > 3$, i.e.,

$$\|V\|_{p,\mathrm{unif}}^p := \sup_x \int_{B(x,1)} |V(y)|^p \,\mathrm{d}y < \infty. \tag{2.4}$$

Theorem 2.1 is essentially Theorem 4.2 from [27]. We will need the slightly stronger version provided above in which the decomposition $S_n$ may vary with $\gamma$. The proof provided in [27] goes through under this weaker assumption. This is roughly seen as follows: It suffices to show that

$$\sigma_{\mathrm{ac}}(H_1) \cap J = \varnothing \tag{2.5}$$

for all compact subsets $J$ of $\varrho(H_2)$. For fixed $J$ the argument in [27] provides a $\gamma > 0$ (roughly the exponential decay rate in a Combes-Thomas type bound on the resolvent of $H_2$ for energies in $J$) such that the validity of (2.2) and (2.3) for a suitable decomposition will imply (2.5).

Also, in [27] all potentials are assumed to have locally integrable positive parts and negative parts in the Kato class. Our $L^p$-type assumptions are a special case.

The second result we use is taken from [28] and excludes continuous spectrum. It is clear that a statement of the form of Theorem 2.1 above has to be false, since dense pure point spectrum is extremely unstable and can be destroyed by "tiny" perturbations [26]. The geometry is somewhat similar to what we had above but more restrictive. Namely, consider an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of bounded open sets with $\bigcup_n A_n = \mathbb{R}^d$. Then $S_n := \partial A_n$ is a total decomposition. For the arguments in [28] it is not necessary that $|\partial S_n| = 0$, but this will be the case in all our applications.

We assume that

$$\delta'_n := \min\{\mathrm{dist}(S_n, \{V_1 \neq V_2\}), \tfrac{1}{2}\,\mathrm{dist}(S_n, S_{n-1} \cup S_{n+1})\} > 0.$$

**Theorem 2.2** *Assume that* $V_1 \in L^{\frac{d+1}{2}}_{\mathrm{loc}}(\mathbb{R}^d)$, $W \in L^\infty$ *with compact support, of fixed sign and such that* $|W| \geq c\chi_{B(0,s)}$ *for suitable* $c > 0$ *and* $s > 0$. *Moreover, assume that for every* $\gamma > 0$ *there exist* $A_n = A_n(\gamma)$ *as above such that* $\delta'_n = \delta'_n(\gamma) \to \infty$ *and*

$$\sum_n |A_{n+1} \setminus A_{n-1}|\, \mathrm{e}^{-\gamma\delta'_n} < \infty. \tag{2.6}$$

*Then for the family* $H_\lambda := H_1 + \lambda W$, $\lambda \in \mathbb{R}$ *there exists a measurable subset* $M_0 \subset \mathbb{R}$ *such that* $|\mathbb{R} \setminus M_0| = 0$ *and*

$$\sigma_{\mathrm{c}}(H_\lambda) \cap \varrho(H_2) = \varnothing \text{ for all } \lambda \in M_0.$$

See [28] for the proof which extends to the case of $W$ as specified above.

Again, as with Theorem 2.1 above, the possible $\gamma$-dependence of the sets $A_n$ is not explicitly stated in [28], but allowed for by the proof provided there.

The requirement that the summability conditions (2.3), (2.6) have to hold for *all* $\gamma > 0$ (and suitable decompositions) comes from the fact that we want to exclude (absolutely) continuous spectrum up to the edges of $\sigma(H_2)$. It is possible to quantify and refine the results in a way which says that validity of (2.3), (2.6) for a fixed $\gamma$ implies absence of (absolutely) continuous spectrum in regions above a certain ($\gamma$-dependent) distance from $\sigma(H_2)$.

## 3 Sparse random models

In this section we will show how to use the methods from the preceding section to prove absence of continuous or absolutely continuous spectrum for sparse random potentials. As mentioned in the introduction, these models have been set up to study situations in which a transition from singular to absolutely continuous spectrum occurs.

This has attracted some interest in the last decade as can be seen in the articles [15, 16, 19, 20, 21, 23, 24] dealing with discrete Schrödinger operators and [7] for the continuum case.

We will be concerned mainly with absence of a continuous spectral component away from the spectrum of the unperturbed operator. For this reason we state our results in a generality that does include cases in which no absolutely continuous spectrum survives. As model examples, let us mention two families of models that have been treated in [7].

Specific models of sparse random potentials, as considered in [7], are

**Model I**

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} \xi_i(\omega) f(x-i), \quad \omega \in \Omega$$

where $f$ is a compactly supported single site potential and the $\xi_i$ are independent Bernoulli variables. Set $p_i := \mathbb{P}(\xi_i = 1)$. If $p_i \to 0$ as $|i| \to \infty$ the random potential will no longer be stationary. In fact, it will be sparse in the sense that almost surely large islands near $\infty$ will occur where $V_\omega$ vanishes.

For the second model $f$ and the $\xi_i$, $p_i$ will have the same meaning and, additionally, the $q_i$ are i.i.d. nonnegative random variables.

**Model II**

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i(\omega)\xi_i(\omega) f(x-i).$$

Again, $V_\omega$ is sparse in the above sense. Of course, for $p_i \equiv 1$ we would get the usual Anderson model. Hundertmark and Kirsch study in [7] the metal insulator transition for $H(\omega) = -\Delta + V_\omega$ in $L^2(\mathbb{R}^d)$ for the case that $p_i \to 0$ as $|i| \to \infty$ but not too fast in order to make sure that $\sigma_{\mathrm{ess}}(H(\omega)) \cap (-\infty, 0) \neq \varnothing$.

**Our Model**

In the following we consider:

(A$_1$) $V_0 \colon \mathbb{R}^d \to \mathbb{R}$ which is locally uniformly $L^p$ with $p \geq 2$ if $d \leq 3$ and $p > d/2$ if $d > 3$.

(A$_2$) $\Sigma \subset \mathbb{R}^d$ a set of sites that is uniformly discrete in the sense that

$$\inf\{|j-i| \mid j, i \in \Sigma,\ j \neq i\} =: r_\Sigma > 0.$$

(A$_3$) For each $i \in \Sigma$ a single site potential $f_i \in L^p$ such that, for finite constants $\rho$ and $M$,

$$\operatorname{supp} f_i \subset B(0,\rho) \text{ and } \|f_i\|_p \leq M.$$

(A$_4$)

$$V_\omega(x) = \sum_{i\in\Sigma} \omega_i f_i(x-i)$$

where $\omega = (\omega_i)_{i\in\Sigma} \in (\Omega, \mathbb{P}) = (\mathbb{R}^\Sigma, \bigotimes_{i\in\Sigma} \mu_i)$, i.e., the $\omega_i$ are independent random variables with distribution $\mu_i$, and $\operatorname{supp} \mu_i \subset [0,1]$ for all $i \in \Sigma$.

For our results on absence of continuous spectrum, in order to apply Theorem 2.2, we will also require

($A_5$) Let $V_0$, $f_i \in L^{(d+1)/2}_{\mathrm{loc}}(\mathbb{R}^d)$ for all $i \in \Sigma$. There exists one $k \in \Sigma$ with $f_k$ of definite sign, bounded, and such that $|f_k| \geq c\chi_{B(0,s)}$ for some $c > 0$ and $s > 0$.

For further reference denote

$$p_i(\varepsilon) := \mu_i([\varepsilon, 1]) = \mathbb{P}\{\omega_i \geq \varepsilon\}. \tag{3.1}$$

Also, denote by

$$m_k := (\mu_k)_{\mathrm{ac}}([0, 1]) \tag{3.2}$$

the total mass of the absolutely continuous component $(\mu_k)_{\mathrm{ac}}$ of $\mu_k$. We will only use this for the fixed $k \in \Sigma$ given in ($A_5$).

We consider the self-adjoint random Schrödinger operator

$$H(\omega) = H_0 + V_\omega \text{ in } L^2(\mathbb{R}^d) \tag{3.3}$$

where $H_0 = -\Delta + V_0$. Our assumptions guarantee that the local $L^p$-bounds (2.4) for $V_0 + V_\omega$ are uniform not only in $x$, but also in $\omega$.

Of course, our model contains Models I and II above as special cases and $p_i(\varepsilon) \leq p_i$ for any $\varepsilon > 0$ in these cases. We have the following result:

**Theorem 3.1** *Let $H(\omega)$ be as above, satisfying ($A_1$) to ($A_4$), and assume that for all $\varepsilon > 0$,*

$$p_i(\varepsilon) = \mathrm{o}(|i|^{-(d-1)}) \text{ as } |i| \to \infty. \tag{3.4}$$

*Then*

(a) $\sigma_{\mathrm{ac}}(H(\omega)) \cap \varrho(H_0) = \varnothing$ *almost surely.*

(b) *Assume, moreover, that ($A_5$) holds. Then, with $k$ as in ($A_5$),*

$$\mathbb{P}\{\sigma_{\mathrm{c}}(H(\omega)) \cap \varrho(H_0) = \varnothing\} \geq m_k. \tag{3.5}$$

In particular, $\sigma_{\mathrm{c}}(H(\omega)) \cap \varrho(H_0) = \varnothing$ holds almost surely if $\mu_k$ is purely absolutely continuous, without any assumption on the distribution at the other sites.

In order to apply the results from Section 2 we need to find sufficiently many and sufficiently large regions in which the random potential $V_\omega$ is small and thus $H_\omega$ close to $H_0$. We start by showing that these regions appear with probability one.

**Definition.** Call a set $U$ *$\varepsilon$-free* for $\omega$ if $\omega_i \leq \varepsilon$ for all $i \in \Sigma \cap U$.

Denote by

$$A_{r,R} = B(0, R) \setminus B(0, r) \tag{3.6}$$

the annulus with inner radius $r$ and outer radius $R$.

**Lemma 3.2** *Fix $\varepsilon > 0$ and $a > 1$. For $n \in \mathbb{N}$ let*

$$a_n := \mathbb{P}\left(A_{r,r+n} \text{ is not } \varepsilon\text{-free for all } r \in [a^n, a^{n+1} - n]\right). \tag{3.7}$$

*Then $\sum_n a_n < \infty$.*

*Proof.* Choose $\eta > 0$ such that $a(1-\eta) > 1$. Using uniform discreteness of $\Sigma$ we get that for all $n \in \mathbb{N}$ and $r \geq 1$,

$$\#(A_{r,r+n} \cap \Sigma) \leq Cnr^{d-1}, \tag{3.8}$$

where $C$ depends on $d$ and $r_\Sigma$. Here $\# A$ is the cardinality of a set $A$. With $C$ from (3.8) choose $\delta \in (0, \eta/(Ca^{d-1}))$.

By (3.4), $p_i(\varepsilon) \leq \delta |i|^{-(d-1)}$ for $i$ sufficiently large. Thus, for sufficiently large $n$ and each $r \in [a^n, a^{n+1} - n]$,

$$\begin{aligned} \mathbb{P}(A_{r,r+n} \text{ is } \varepsilon\text{-free}) &= \prod_{i \in A_{r,r+n} \cap \Sigma} (1 - p_i(\varepsilon)) \\ &\geq (1 - \delta |i|^{-(d-1)})^{\#(A_{r,r+n} \cap \Sigma)} \\ &\geq (1 - \delta a^{-n(d-1)})^{Cna^{(n+1)(d-1)}} \\ &\geq (1 - C\delta a^{d-1})^n \;\geq\; (1-\eta)^n. \end{aligned} \tag{3.9}$$

$A_{a^n, a^{n+1}}$ contains at least $\frac{1}{n}(a^{n+1} - a^n) - 1$ disjoint annuli $A_j := A_{r_j, r_j+n}$ of width $n$. Thus, using independence and (3.9),

$$\begin{aligned} a_n &\leq \mathbb{P}(\text{no } A_j \text{ is } \varepsilon\text{-free}) \\ &= \prod_j \mathbb{P}(A_j \text{ is not } \varepsilon\text{-free}) \\ &\leq (1 - (1-\eta)^n)^{n^{-1}(a^{n+1} - a^n) - 1} \\ &\leq \mathrm{e}^{-(1-\eta)^n (n^{-1} a^n (a-1) - 1)}. \end{aligned} \tag{3.10}$$

As $(1-\eta)a > 1$, the $a_n$ are summable. □

By the Borel-Cantelli lemma we conclude $\mathbb{P}(\Omega_{\varepsilon,a}) = 1$, where

$$\begin{aligned} \Omega_{\varepsilon,a} := \big\{\omega \in \Sigma :\ & \text{For each sufficiently large } n \text{ the annulus } A_{a^n, a^{n+1}} \\ & \text{contains a sub-annulus } A_{r_n, r_n+n} \text{ which is } \varepsilon\text{-free for } \omega\big\}. \end{aligned} \tag{3.11}$$

Therefore

$$\Omega_\varepsilon = \bigcap_{\ell \in \mathbb{N}} \Omega_{\varepsilon, 1+1/\ell} \tag{3.12}$$

also has full measure.

Based on this we can now complete the

*Proof of Theorem* 3.1. Fix a compact $K \subset \varrho(H_0)$. Since $\varrho(H_0)$ can be exhausted by an increasing sequence of compact subsets, it suffices to prove that

$$\sigma_{\mathrm{ac}}(H(\omega)) \cap K = \varnothing \text{ almost surely.} \tag{3.13}$$

It can be shown, using the general theory of uniformly local $L^p$ potentials, e.g., [25], that there is an $\varepsilon' > 0$ such that

$$\sigma(H_0 + V) \cap K = \varnothing, \tag{3.14}$$

for each $V$ with $\|V\|_{p,\mathrm{unif}} \leq \varepsilon'$. Thus, by the properties of $\Sigma$ and $f_i$, there is an $\varepsilon > 0$ such that

$$\sigma\Big(H_0 + \sum_{i\in\Sigma} \delta_i f_i(x-i)\Big) \cap K = \varnothing \tag{3.15}$$

if $|\delta_i| \leq \varepsilon$ for all $i \in \Sigma$.

Fix this $\varepsilon > 0$ and let $\Omega_\varepsilon$ be the full measure set found above. For given $\omega \in \Omega_\varepsilon$ let $\tilde{\omega}_i := \min\{\omega_i, \varepsilon\}$, $i \in \Sigma$, and

$$V_2(x) := \sum_{i\in\Sigma} \tilde{\omega}_i f_i(x-i).$$

By (3.15) we have $\sigma(H_0 + V_2) \cap K = \varnothing$. Thus, in order to apply Theorem 2.1 and conclude (3.13), it suffices to find for every $\gamma > 0$ a total decomposition $(S_n^{(\gamma)})$ of $\{V_\omega \neq V_2\}$ which satisfies (2.2) and (2.3).

For given $\gamma > 0$ choose an integer $\ell > 2(d-1)/\gamma$. This implies $(d-1)\log a < \gamma/2$, where $a := 1 + 1/\ell$. As $\omega \in \Omega_{\varepsilon,a}$, for each sufficiently large $n$ the annulus $A_{a^n,a^{n+1}}$ contains an $\varepsilon$-free annulus $A_{r_n,r_n+n}$.

Choose $S_n^{(\gamma)} := \partial B(0, r_n + \frac{n}{2})$. Then

$$\delta_n^{(\gamma)} := \mathrm{dist}(\{V_\omega \neq V_2\}, S_n^{(\gamma)}) \geq \frac{n}{2} - \rho$$

since $A_{r_n,r_n+n}$ is $\varepsilon$-free (recall that supp $f_k \subset B(0,\rho)$). Thus $\delta_n^{(\gamma)} \to \infty$. Also using that $\sigma(S_n^{(\gamma)}) \leq C a^{n(d-1)}$, we conclude

$$\sum_n \sigma(S_n^{(\gamma)}) \mathrm{e}^{-\gamma\delta_n^{(\gamma)}} \leq C\mathrm{e}^{\gamma\rho} \sum_n \mathrm{e}^{n((d-1)\log a - \gamma/2)} < \infty.$$

This proves part (a) of Theorem 3.1.

In order to apply Theorem 2.2 to *prove part (b)* we slightly modify the above construction, essentially replacing $\Sigma$ by $\Sigma \setminus \{k\}$.

Let $\Omega' := \mathbb{R}^{\Sigma\setminus\{k\}}$ with measure $\mathbb{P}' = \otimes_{i\in\Sigma\setminus\{k\}}\mu_i$. As the property defining $\Omega_{\varepsilon,a}$ in (3.11) does not depend on the value of $\omega_k$, we get that also $\mathbb{P}'(\Omega'_{\varepsilon,a}) = \mathbb{P}'(\Omega'_\varepsilon) = 1$, where $\Omega'_{\varepsilon,a}$ and $\Omega'_\varepsilon$ are defined as in (3.11) and (3.12), but as subsets of $\Omega'$.

For compact $K \subset \varrho(H_0)$ choose $\varepsilon > 0$ as in the proof of part (a). For $\omega' \in \Omega'_\varepsilon$ let $\tilde{\omega}'_i := \min\{\omega'_i, \varepsilon\}$ ($i \in \Sigma \setminus \{k\}$). Also let $V_{\omega'}(x) = \sum_{i \in \Sigma \setminus \{k\}} \omega'_i f_i(x - i)$ and $V'_2(x) = \sum_{i \in \Sigma \setminus \{k\}} \tilde{\omega}'_i f_i(x - i)$. As before, $\sigma(H_0 + V'_2) \cap K = \varnothing$.

For $\gamma > 0$ choose $\ell > 2d/\gamma$, $a = 1 + 1/\ell$. With $r_n$ from (3.11), let $A_n = B(0, r_n + \frac{n}{2})$ and $S_n = \partial A_n$. This yields

$$|A_{n+1} \setminus A_{n-1}| \leq c_d a^{(n+2)d}$$

and

$$\delta'_n = \min\left\{ \operatorname{dist}(S_n, \{V_{\omega'} \neq V_2\}), \frac{1}{2} \operatorname{dist}(S_n, S_{n-1} \cup S_{n+1}) \right\} \geq \frac{n}{2} - \rho.$$

The choice of $a$ guarantees that $\sum_n |A_{n+1} \setminus A_{n-1}| \, \mathrm{e}^{-\gamma \delta'_n} < \infty$. By Theorem 2.2 this proves the existence of a measurable subset $M_{0,\omega'} \subset \mathbb{R}$ with $|\mathbb{R} \setminus M_{0,\omega'}| = 0$ and such that

$$\sigma_{\mathrm{c}}(H(\lambda, \omega')) \cap K \subset \sigma_{\mathrm{c}}(H(\lambda, \omega')) \cap \varrho(H_0 + V'_2) = \varnothing$$

for all $\lambda \in M_{0,\omega'}$, where $H(\lambda, \omega') = H_0 + \lambda f_k(x - k) + V_{\omega'}(x)$.

As $\mu_k(M_{0,\omega'}) \geq (\mu_k)_{\mathrm{ac}}(M_{0,\omega'}) = (\mu_k)_{\mathrm{ac}}(\mathbb{R}) = m_k$ it follows by Fubini that $\mathbb{P}\{\omega \in \Omega : \sigma_{\mathrm{c}}(H(\omega)) \cap K = \varnothing\} \geq m_k$. Since this bound is independent of $K$ and we can exhaust $\varrho(H_0)$ by an increasing sequence $K_n$ we arrive at the assertion. This completes the proof of Theorem 3.1. □

**Remarks.** (1) While the "volume" term $|A_{n+1} \setminus A_{n-1}|$ in (2.6) has to be considered larger than the "surface" term $\sigma(S_n)$ in (2.3), this did not make a significant difference in the above proof. The same total decomposition $S_n$ can be used to prove absence of absolutely continuous spectrum and absence of continuous spectrum. The difference will become more significant for the quasi-1D surfaces considered in the next section.

(2) Crucial for our method to apply is the almost sure appearance of a sequence of $\varepsilon$-free annular regions which must (i) grow in thickness and (ii) not be too far apart, as found in Lemma 3.2. In Theorem 3.1 this was enforced through the assumptions on the distribution of the coupling constants. In Section 4 it will follow from sparseness of the single site set $\Sigma$.

(3) Note that our methods are sufficiently "soft" to allow for considerable flexibility of the model. The single site potentials $f_i$ may depend on the site, do not need to be sign definite, and may include $L^p$-type singularities. We can deal with quite arbitrary single site distributions. Only for the proof of absence of continuous spectrum we need one of the distributions to be absolutely continuous. These assumptions are weaker than what usually enters into the proof of localization properties through the multiscale analysis or fractional moment methods.

(4) The assumption $\operatorname{supp} \mu_j \subset [0, 1]$ is just a normalization. For our methods to apply, the random potentials have to obey uniform bounds, e.g., in the sense of $\| \cdot \|_{p,\mathrm{unif}}$ from (2.4).

Let us finally state the following result for our model which easily follows from the "Almost surely free lunch Theorem" in [7]. For the case $V_0 = 0$ it can be combined with Theorem 3.1 to provide examples with purely singular or pure point (while not discrete) spectrum below zero and an absolutely continuous spectral component above zero.

**Theorem 3.3** *Let $\mu_k$, $f_k$, $V_0$ be as above, $V_0 = 0$ and assume that, additionally, the $\|f_k\|_\infty$ are uniformly bounded and that the second moments of the $\eta_k$ obey*

$$\mathbb{E}(\eta_k^2) = \int_0^1 x^2 \mathrm{d}\mu_k \leq C|k|^{-\beta}$$

*for some $\beta > 2$. Then*

$$\sigma_{\mathrm{ac}}(H(\omega)) \supset [0, \infty) \ \ \mathbb{P}\text{-a.s.}$$

*Proof.* The assumptions clearly make sure that

$$W(x) := \mathbb{E}(V_\omega(x)^2)^{\frac{1}{2}} \leq C(1 + |x|)^{-(1+\varepsilon)}$$

so that we can apply Theorem 2.4 from [7] to see that Cook's criterion is applicable for $\mathbb{P}$-a.e. $\omega \in \Omega$. □

For general $V_0$ the corresponding result, namely that $\sigma_{\mathrm{ac}}(-\Delta + V_0) \subset \sigma_{\mathrm{ac}}(H_\omega)$ almost surely, is probably false. It should be true for certain periodic potentials, see [2, 6, 29].

## 4 Quasi-1D surfaces

In Section 3 sparseness of the potential $V_\omega$ in ($\mathrm{A}_4$) resulted from an assumption on decaying randomness, e.g., (3.4). In the present section we will modify our methods and results for the case where sparseness of $V_\omega$ arises directly through sparseness of the deterministic set $\Sigma$. By this we mean situations where $\Sigma$ does not have positive $d$-dimensional density in $\mathbb{R}^d$, i.e., $\#(\Sigma \cap B(0, R)) = \mathrm{o}(R^d)$ as $R \to \infty$. A special case would be an $m$-dimensional sublattice, e.g., $\Sigma = \mathbb{Z}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{d-m}$, $0 < m < d$, in which case $V_\omega$ would model a random surface potential. Our most interesting result holds for $m = 1$, where our methods cover the following more general situation:

**Definition.** A uniformly discrete subset $\Sigma$ of $\mathbb{R}^d$ is called *quasi-one-dimensional* (quasi-1D) if there exists $C < \infty$ such that

$$\#(\Sigma \cap A_{R,R+1}) \leq C \tag{4.1}$$

for all $R \geq 0$.

**Theorem 4.1** *Let $H(\omega) = H_0 + V_\omega$ satisfy ($A_1$) to ($A_4$). In addition, assume that $\Sigma$ is quasi-1D and that*

$$\sup_{i \in \Sigma} p_i(\varepsilon) < 1 \tag{4.2}$$

*for every $\varepsilon > 0$. Then $\sigma_{\mathrm{ac}}(H(\omega)) \cap \varrho(H_0) = \varnothing$ almost surely.*

If $\Sigma$ is quasi-1D, then by Theorem 4.1, no spatial decay in the randomness of the $\eta_i$ is required to conclude absence of absolutely continuous spectrum in gaps of $\sigma(H_0)$. For example, (4.2) is satisfied for independent, identically distributed random variables $\eta_i$ such that $0 \in \operatorname{supp}\mu$ for their common distribution $\mu$. In particular, as every uniformly discrete $\Sigma \subset \mathbb{R}$ is quasi-1D, this strengthens Theorem 3.1(a) in the case $d = 1$, which would require $p_i(\varepsilon) = \mathrm{o}(1)$ as $k \to \infty$. Of course, in the case $d = 1$ our result is hardly new as (essentially) much stronger results are known for one-dimensional random potentials.

More interesting is the case $d > 1$, where special cases of quasi-1D sets include discrete tubes of the form $\Sigma = \mathbb{Z} \times S$, with $S$ a bounded subset of $\mathbb{Z}^{d-1}$. Theorem 4.1 shows the absence of absolute continuity in the "surface spectrum" generated by the random (1D) surface potential $V(\omega)$. Also, within certain limitations, we can allow for curvature in the tubes $\Sigma$, thus covering rather general "random sausages".

*Proof.* We start with a modification of Lemma 3.2.

**Lemma 4.2** *Fix $\varepsilon > 0$. Let $\delta = \sup_i p_i(\varepsilon) < 1$, $C$ as in (4.1) and $a > \frac{1}{(1-\delta)^C}$. Then the $a_n$, as defined in* (3.7)*, are summable.*

*Proof.* This follows with the same argument as in the proof of Lemma 3.2, using that now $\mathbb{P}(A_{r,r+n}$ is $\varepsilon$-free$) \geq (1-\delta)^{Cn}$. Thus the set $\Omega_{\varepsilon,a}$, defined as in (3.11), has full $\mathbb{P}$-measure. $\square$

Fix $K \subset \varrho(H_0)$ compact and argue as in the proof of Theorem 3.1 to find $\varepsilon > 0$ such that $\sigma(H_0 + V_2) \cap K = \varnothing$, where $V_2(x) = \sum_{i\in\Sigma} \tilde{\omega}_i f_i(x-i)$, $\tilde{\omega}_i = \min\{\omega_i, \varepsilon\}$. Choose $a > 1$ as in Lemma 4.2 and $\omega \in \Omega_{\varepsilon,a}$, i.e., $A_{a^n,a^{n+1}}$ contains $\varepsilon$-free $A_{r_n,r_n+n}$ for all sufficiently large $n$.

As before, the spheres $S_n = \partial B(0, r_n + \frac{n}{2})$ give a total decomposition with $\operatorname{dist}(\{V_\omega \neq V_2\}, S_n) \geq \frac{n}{2} - \rho$. But, as Lemma 4.2 prevents us from choosing $a$ arbitrarily close to 1, this will not yield convergence of (2.3) for *all* $\gamma > 0$. We will therefore refine our construction by splitting the $S_n$ in two parts. One part is a union of spherical caps for which, due to points of $\Sigma$ close to $A_{r_n,r_n+n}$, the distance $\frac{n}{2} - \rho$ from $\{V_\omega \neq V_2\}$ can't be improved. The second part (the remaining "Swiss cheese") has much bigger distance to $\{V_\omega \neq V_2\}$ and, due to the sparseness of $\Sigma$, contains most of $S_n$. The details of this construction are as follows:

Fix $\alpha > 1$. Let

$$P_n := (A_{r_n - n^\alpha, r_n} \cup A_{r_n+n, r_n+n+n^\alpha}) \cap \Sigma \tag{4.3}$$

be the points of $\Sigma$ in the $n^\alpha$-neighborhood of $A_{r_n,r_n+n}$ (but outside $A_{r_n,r_n+n}$). For each $j \in P_n$ define the spherical cap

$$S_{n,j} := S_n \cap B\Big(\big(r_n + \frac{n}{2}\big)\frac{j}{|j|}, n^\alpha\Big). \tag{4.4}$$

Also let

$$S_n' := \overline{S_n \setminus \bigcup_j S_{n,j}}.$$

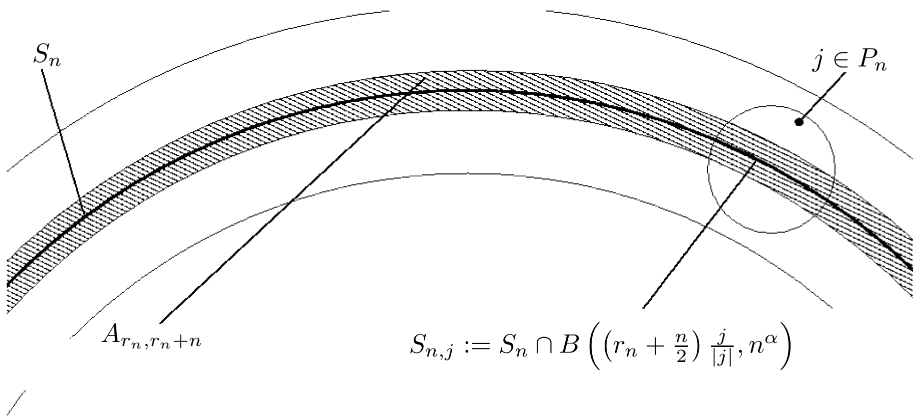


Figure 2. The geometry in the proof of Lemma 4.2: the bold face line shows a part of $S_n$, the shaded region is $A_{r_n,r_n+n}$, the point in the small circle a $j \in P_n$ and the small circle the boundary of $B((r_n + \frac{n}{2})\frac{j}{|j|}, n^\alpha)$.

Since $S'_n \cup \bigcup_j S_{n,j} = S_n$, we have that

$$\{S_{n,j} : n \in \mathbb{N}, j \in P_n\} \cup \{S'_n : n \in \mathbb{N}\} \tag{4.5}$$

is a total decomposition of $\mathbb{R}^d$. As above, since $A_{r_n,r_n+n}$ is $\varepsilon$-free,

$$\delta_{n,j} := \operatorname{dist}(\{V_\omega \neq V_2\}, S_{n,j}) \geq \frac{n}{2} - \rho. \tag{4.6}$$

If $x \in S'_n$ and $j \in \Sigma \cap (A_{r_n,r_n+n})^{\mathrm{c}}$, then, by elementary geometric considerations, $\operatorname{dist}(x,j) \geq n^\alpha$ for sufficiently large $n$. Using this and again that $A_{r_n,r_n+n}$ is $\varepsilon$-free, we find

$$\delta'_n := \operatorname{dist}(\{V_\omega \neq V_2\}, S'_n) \geq n^\alpha - \rho. \tag{4.7}$$

From the simple volume bound (2.1) on the generalized surface area one gets

$$\sigma(S_{n,j}) \leq C n^{d\alpha}, \tag{4.8}$$

$$\sigma(S'_n) \leq C a^{dn}. \tag{4.9}$$

Checking (2.3) for the partition (4.5) amounts to proving that

$$\sum_n \sigma(S'_n)\,\mathrm{e}^{-\gamma\delta'_n} < \infty \tag{4.10}$$

and

$$\sum_n \sum_{j \in P_n} \sigma(S_{n,j})\,\mathrm{e}^{-\gamma\delta_{n,j}} < \infty \tag{4.11}$$

for each $\gamma > 0$. (4.10) follows from (4.7) and (4.9) since $\alpha > 1$. (4.11) follows from (4.6) and (4.8), noting that $\#P_n \leq 2n^\alpha + 2$ since $\Sigma$ is quasi-1D. From Theorem 2.1 we conclude $\sigma_{\mathrm{ac}}(H(\omega)) \cap K \subset \sigma_{\mathrm{ac}}(H(\omega)) \cap \varrho(H_0 + V_2) = \varnothing$. □

**Remark.** It is possible to prove Theorem 4.1 under a slightly weaker assumption on the set $\Sigma$, namely that there exists $C < \infty$ such that

$$\#(\Sigma \cap B(0,R)) \leq CR \tag{4.12}$$

for all $R \geq 1$. (4.12) is weaker than (4.1) in that it allows the number of points in $\Sigma \cap A_{R,R+1}$ to be unbounded with respect to $R$. (4.12) is also somewhat more natural as it doesn't depend on the norm used to define $B(0,R)$ nor on the choice of the center of the ball.

A simple counting argument shows that, under the assumption (4.12), for each annulus of the form $A_{a^n,a^{n+1}}$ most sub-annuli $A_{R,R+n}$ satisfy a bound $\#(\Sigma \cap A_{R,R+n}) \leq Cn$. Here "most" means at least a non-vanishing fraction. One finds sufficiently many disjoint such annuli to construct $\varepsilon$-free regions as before. Moreover, by an additional counting argument, one argues that most of these annuli do not have more than $C'n^\alpha$ points of $\Sigma$ in their $n^\alpha$-neighborhoods. Based on this one can construct a partition $\{S_n', S_{n,j}\}$ as above and carry through the proof. We skip the somewhat tedious details of this generalization.

We are not able to prove a result like Theorem 3.1(b), i.e., absence of continuous spectrum in $\varrho(H_0)$ with positive probability, under the assumptions of Theorem 4.1 (plus ($A_5$)). For the partition $S_n = \partial A_n$, $A_n = B(0, r_n + \frac{n}{2})$ the volumes $|A_{n+1} \setminus A_{n-1}|$ grow too fast to get validity of (2.6) for all $\gamma > 0$. A trick like the introduction of $\{S_n', S_{n,j}\}$ as above is not applicable here since in Theorem 2.2 the $S_n$ need to arise as boundaries of a growing sequence $A_n$.

However, if one replaces (4.2) by $p_i(\varepsilon) = \mathrm{o}(1)$ as $|i| \to \infty$ for all $\varepsilon > 0$, then Lemma 4.2 will hold for any $a > 1$, which allows for an application of Theorem 2.2 with a $\gamma$-dependent choice of the $S_n$, as in the proof of Theorem 3.1(b). Sparseness of the random potential is achieved here through a combination of sparseness of $\Sigma$ and decaying randomness $p_i(\varepsilon) = \mathrm{o}(1)$, as opposed to Theorem 3.1, where sparseness follows exclusively from stronger decay $p_i(\varepsilon) = \mathrm{o}(|i|^{-(d-1)})$.

In fact, the correlation between the degree of sparseness of $\Sigma$ and the rate of decay of $p_i(\varepsilon)$ can be made more specific. For this, call a uniformly discrete set $\Sigma \subset \mathbb{R}^d$ *quasi-m-dimensional* ($1 \leq m \leq d$, not necessarily integer) if for some $C < \infty$ and all $R \geq 0$,

$$\#(\Sigma \cap A_{R,R+1}) \leq CR^{m-1}. \tag{4.13}$$

Then the following result is found with the same methods as above:

**Theorem 4.3** *Let $H(\omega)$ satisfy ($A_1$) to ($A_4$), $\Sigma$ be quasi-m-dimensional and, for all $\varepsilon > 0$,*

$$p_i(\varepsilon) = \mathrm{o}(|i|^{-(m-1)}) \ \text{as}\ |i| \to \infty, \tag{4.14}$$

*then $\sigma_{\mathrm{ac}}(H(\omega)) \cap \varrho(H_0) = \varnothing$ almost surely.*

*If, moreover, ($A_5$) holds, then $\mathbb{P}\{\sigma_{\mathrm{c}}(H(\omega)) \cap \varrho(H_0) = \varnothing\} \geq m_k$.*

## 5 Concluding remarks

Among the known results for discrete surface models, the one most closely related to Theorem 4.1 above is the result of Jakšić and Molchanov [10]. They consider the discrete Laplacian on $\mathbb{Z} \times \mathbb{Z}_+$ with random boundary condition $\psi(n, -1) = V_\omega(n)\psi(n, 0)$, where the $V_\omega(n)$ are i.i.d. random variables. They show that the spectrum outside $[-4, 4]$, i.e., outside the spectrum of the two-dimensional discrete Laplacian, is almost surely pure point. This is stronger than our continuum analogue in the sense that we can only prove absence of absolute continuity outside the spectrum of the deterministic background operator $H_0$.

The proof in [10] requires a technical tour de force. The two-dimensional problem can be reduced to a one-dimensional problem with long range interactions. Anderson localization for the latter has been proven in [12] with methods based on an approach developed in [17] (which is also behind Theorem 2.2 above). The one-dimensional problem depends nonlinearly on the spectral parameter, a difficulty which is resolved by adapting some ideas from the Aizenman-Molchanov fractional moment method [1].

Our methods are comparatively soft. In particular, they work directly in the multi-dimensional PDE setting and do not require a reduction to $d = 1$. One-dimensionality of the random surface only enters through its probabilistic consequences (Lemma 4.2) for the frequency of the appearance of $\varepsilon$-free regions, which constitute the "potential barriers" required in Theorem 2.1.

This makes our methods very flexible. In addition to the extension to continuum models, they allow for rather general quasi-1D surfaces (e.g., curved tubes, unions of tubes), work in arbitrary dimension $d$ and allow for the presence of an additional deterministic background potential $V_0$. It is possible to adapt our methods to lattice operators and prove absence of absolutely continuous spectrum outside the spectrum of the discrete Laplacian for much more general geometries than the half-plane considered in [10].

Also, our methods can easily be adjusted to work for operators of the type (3.3) on $L^2(\Omega)$, $\Omega \neq \mathbb{R}^d$. For example, for $H(\omega) = -\Delta + V_\omega$ in $L^2((0, a) \times \mathbb{R}^{d-1})$ with Dirichlet boundary conditions and $V_\omega$ given through ($\mathrm{A}_2$) to ($\mathrm{A}_4$) with i.i.d. coupling constants $\omega_i$, we would get that $\sigma_{\mathrm{ac}}(H(\omega)) \cap (-\infty, 0) = \varnothing$ almost surely. Of course, for this physically one-dimensional operator (with no bulk space), one would expect the much stronger result that $\sigma_{\mathrm{c}}(H(\omega)) = \varnothing$. But the corresponding result for discrete strips, e.g., [18], does not seem to extend easily to the continuum.

Finally, we mention that Hundertmark and Kirsch [7] announce some results on pure point spectrum for continuum models similar to the ones studied here. They will use suitable adaptations of multiscale analysis to show that the negative spectrum of $-\Delta + V_\omega$ is almost surely pure point. Here $V_\omega$ is either of the type of Model II above or a random potential at the surface of a half space Schrödinger operator. In situations where the multiscale analysis can be carried out, their results should be stronger than ours.

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Anne Boutet de Monvel
Institut de Mathématiques de Jussieu
Université Paris 7
2, place Jussieu, case 7012
F-75251 Paris
France
email: aboutet@math.jussieu.fr

Peter Stollmann
Fakultät für Mathematik
Technische Universität
D-09107 Chemnitz
Germany
email: P.Stollmann@mathematik.tu-chemnitz.de

Günter Stolz
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294
USA
email: stolz@math.uab.edu

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