Ann. Henri Poincaré 5 (2004) 1159 – 1180

1424-0637/04/061159-22
DOI 10.1007/s00023-004-0195-3

Annales Henri Poincaré

Perturbative Test of Single Parameter Scaling for $1D$ Random Media

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Abstract. Products of random matrices associated to one-dimensional random media satisfy a central limit theorem assuring convergence to a gaussian centered at the Lyapunov exponent. The hypothesis of single parameter scaling states that its variance is equal to the Lyapunov exponent. We settle discussions about its validity for a wide class of models by proving that, away from anomalies, single parameter scaling holds to lowest order perturbation theory in the disorder strength. However, it is generically violated at higher order. This is explicitly exhibited for the Anderson model.

1 Introduction and main result

One-dimensional quantum systems with a single channel can very efficiently be described by 2×2 transfer matrices. In a disordered medium, the transfer matrices are chosen to be random. The one-dimensional Anderson model is the proto-type for this class of models. The most important physical phenomena in these random media is localization due to multiple coherent wave scattering. It goes along with positivity of the Lyapunov exponents associated with products of the random matrices and the Lyapunov exponent is then interpreted as the inverse localization length of the system. Moreover, the fluctuations around this asymptotic behavior are gaussian. More precisely, Anderson, Thouless, Abraham and Fisher [ATAF] stated that the Landauer conductance follows asymptotically (in the system size) a log-normal distribution centered at the Lyapunov exponent. As pointed out by Johnston and Kunz [JK], this was a rediscovery of a mathematical result by Tutubalin [Tut] (refined by Le Page [LeP]). The paradigm of single parameter scaling [AALR, ATAF] is then that there is only one parameter describing the asymptotic behavior of the random system. For a one-dimensional model this means that the Lyapunov exponent and the variance of the gaussian should be in some relation and in fact simply be equal [ATAF]. The validity of single parameter scaling in this sense has been analyzed in various particular situations [ATAF, SAJ, CRS, DLA, ST]. The main result of the present work can roughly be resumed as follows: in a wide class of one-dimensional random models single parameter scaling is valid only perturbatively in a weak disorder regime and never holds in a strict sense (exceptional parameter values excluded).

For this purpose, we study a general class of one-parameter families of random transfer matrices exhibiting a so-called critical energy. This parameter is the effective size of the randomness. In the Anderson model, it is the coupling constant of the disordered potential while in the random dimer model [DWP] it is the distance in energy from the critical energy. We then develop a rigorous perturbation theory in this parameter. In case of the Lyapunov exponent, we do not appeal to the random phase approximation [ATAF, SAJ], but rather show that phase correlations give a contribution to lowest order perturbation theory (which, however, vanishes for the Anderson model). This generalizes arguments of [Tho, PF, JSS]. On the other hand, the perturbation theory for the variance is new to our best knowledge. One has to sum up the phase correlation decay by using adequate counter terms. This rigorous analysis is made possible by a result of Le Page (see Section 6). Furthermore, we extend the techniques of [SS] in order to calculate the scaling exponent of the expectation value of the Landauer conductance (this is sometimes also called a generalized Lyapunov exponent). This generalizes results of [Mol].

Comparing the coefficients to lowest (namely second) order perturbation theory away from Kappus-Wegner type anomalies [KW], we obtain that the Lyapunov exponent and the variance are equal while the scaling exponent of the averaged Landauer conductance is twice this value, just as predicted by [ATAF]. Calculation of higher orders is possible, but cumbersome in general. For the Anderson model it becomes feasible and is carried out in Section 9. We obtain that the next (namely forth) order contributions are *not* the same for the Lyapunov exponent and the variance and as a consequence they are not equal (but close) in the regime of weak disorder. For the regime of strong disorder, even large discrepancies have been observed numerically [SAJ]. Deviations from single parameter scaling even to lowest order were exhibited at the band center of the Anderson model, which is the prime example of a Kappus-Wegner anomaly [ST]. The Lloyd model analyzed in [DLA] does not fit in our framework because there the random variables do not have finite moments. The example of the Anderson model also allows to show that there does not exist a universal analytic function expressing the variance in terms of the Lyapunov exponent.

After this brief introduction, let us describe our results more precisely. The transfer matrices are supposed to be elements of the following subgroup of the general linear group $\mathrm{Gl}(2,\mathbb{C})$:

$$\mathrm{U}(1,1) \;=\; \{T \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid T^*JT = J\}\;, \qquad J \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\;. \tag{1}$$

Using the conjugation

$$C^*JC \;=\; \imath\,\Gamma\;, \qquad C \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \imath & \imath \\ 1 & -1 \end{pmatrix}\;, \qquad \Gamma \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\;.$$

one sees that $\mathrm{U}(1,1)$ is also isomorphic to the subgroup of matrices $T \in \mathrm{Gl}(2,\mathbb{C})$ satisfying $T^*\Gamma T = \Gamma$. This representation appears in some applications, but for our purposes it is more convenient to work with (1) because it contains the standard rotation matrix and the real subgroup $\mathrm{SL}(2,\mathbb{R})$.

We will study families $(T_{\lambda,\sigma})_{\lambda\in\mathbb{R},\sigma\in\Sigma}$ of transfer matrices in $\mathrm{U}(1,1)$ depending on a random variable σ in some probability space $(\Sigma,\mathbf{p})$ as well as a real coupling parameter λ. The dependence on λ is supposed to be smooth.

Definition 1 *The value $\lambda = 0$ is a critical point of the family $(T_{\lambda,\sigma})_{\lambda\in\mathbb{R},\sigma\in\Sigma}$ if for all $\sigma,\sigma'\in\Sigma$:*

$$\text{(i)}\ \ [\,T_{0,\sigma},T_{0,\sigma'}]\;=\;0\;, \qquad \text{(ii)}\ \ |\mathrm{Tr}(T_{0,\sigma})|\;<\;2\;. \tag{2}$$

Critical points appear in many applications like in the Anderson model and the random dimer model [DWP, JSS, Sed], but also continuous random Schrödinger operators where the transfer matrix is calculated from a single-site S-matrix [KS]. Condition (i) assures that there is no non-commutativity at $\lambda = 0$ (even though the matrices may be random), while by condition (ii) the matrices $T_{0,\sigma}$ are conjugated to rotations so that there is no *a priori* hyperbolicity in the system. The example of the Anderson model is studied in more detail in Section 9.

Associated to a given semi-infinite code $(\sigma_n)_{n\geq 1}$ is a sequence of matrices $(T_{\lambda,\sigma_n})_{n\geq 1}$. Codes are random and chosen independently according to the product law $\mathbf{p}^{\otimes\mathbb{N}}$. Averaging w.r.t. $\mathbf{p}^{\otimes\mathbb{N}}$ will be denoted by $\mathbf{E}$. We will suppose that all up to the 5th moment of $\mathbf{p}$ exist. In order to shorten notations, we will also write $T_{\lambda,n}$ for T_{λ,σ_n}. Of interest is the asymptotic behavior of the random products

$$\mathcal{T}_\lambda(N)\;=\;\prod_{n=1}^{N} T_{\lambda,n}\;.$$

It is first of all characterized by the Lyapunov exponent

$$\gamma(\lambda)\;=\;\lim_{N\to\infty}\frac{1}{N}\,\mathbf{E}\,\log\left(\|\mathcal{T}_\lambda(N)\|\right)\;. \tag{3}$$

The central limit theorem for products of random matrices now states that

$$\frac{1}{\sqrt{N}}\,\left(\,\log(\|\mathcal{T}_\lambda(N)e\|)-N\gamma(\lambda)\,\right)\;\underset{N\to\infty}{\longrightarrow}\;G_{\sigma(\lambda)} \tag{4}$$

where $G_{\sigma(\lambda)}$ is the centered Gaussian law of variance $\sigma(\lambda)$ and the convergence is in distribution independently of the initial unit vector e. This was first proven by Tutubalin under the hypothesis that the measure $\mathbf{p}$ has a density [Tut]. Le Page then proved it for arbitrary measures $\mathbf{p}$ [LeP]. Both proofs can be found in [BL]. As already discussed above, the single parameter scaling assumption is the equality $\sigma(\lambda) = \gamma(\lambda)$ [ATAF, CRS]. Apart from the Lyapunov exponent $\gamma(\lambda)$ and the variance $\sigma(\lambda)$, we are going to analyze the growth exponent of the average of the Landauer conductance defined by

$$\hat{\gamma}(\lambda)\;=\;\lim_{N\to\infty}\frac{1}{2N}\,\log\left(\mathbf{E}\,\mathrm{Tr}(\mathcal{T}_\lambda(N)^*\mathcal{T}_\lambda(N))\right)\;. \tag{5}$$

It follows immediately from Jensen's inequality that $\hat{\gamma}(\lambda) \geq \gamma(\lambda)$. Our main results are resumed in the following:

Theorem 1 *Introduce the phase* $\eta_\sigma \in [0, 2\pi)$ *by* $\cos(\eta_\sigma) = \frac{1}{2}\operatorname{Tr}(T_{0,\sigma})$. *Suppose* $\mathbf{E}(e^{2\imath j \eta_\sigma}) \neq 1$ *for* $j = 1, 2, 3$. *Then there is a constant* $D \geq 0$, *given in equation* (22) *below, such that near the critical point,*

$$\gamma(\lambda) \;=\; D\,\lambda^2 \;+\; \mathcal{O}(\lambda^3)\;, \qquad \hat{\gamma}(\lambda) \;=\; 2\,D\,\lambda^2 \;+\; \mathcal{O}(\lambda^3)\;. \tag{6}$$

If moreover $D > 0$, *then*

$$\sigma(\lambda) \;=\; D\,\lambda^2 \;+\; \mathcal{O}(\lambda^3)\;. \tag{7}$$

In Section 4, we also give criteria insuring that $D > 0$. The remainder of the paper contains the proof of this theorem as well as an analysis of higher orders for the Anderson model.

2 Normal form of transfer matrices near critical point

If $T \in \mathrm{U}(1,1)$, then $T^* J T = J$ implies that $\det(T) = e^{2\imath\xi}$ for some $\xi \in [0, \pi)$. Hence $e^{-\imath\xi} T \in \mathrm{SU}(1,1) = \{T \in \mathrm{U}(1,1) | \det(T) = 1\}$. This means $\mathrm{U}(1,1) = \mathrm{U}(1) \times \mathrm{SU}(1,1)$. Note also that $\operatorname{Tr}(T) = \det(T)\overline{\operatorname{Tr}(T)}$ for $T \in \mathrm{U}(1,1)$ so that $\operatorname{Tr}(T) \in \mathbb{R}$ for $T \in \mathrm{SU}(1,1)$. One easily verifies that this implies $\mathrm{SU}(1,1) = \mathrm{SL}(2,\mathbb{R})$ is a real subgroup of $\mathrm{Gl}(2,\mathbb{C})$. Its Lie algebra is well known:

$$\mathrm{sl}(2,\mathbb{R}) \;=\; \left\{ \left(\begin{array}{cc} a & b \\ c & -a \end{array} \right) \;\middle|\; a, b, c \in \mathbb{R} \right\}\;.$$

The eigenvalues of $T \in \mathrm{SL}(2,\mathbb{R})$ always come in pairs $\kappa, 1/\kappa$ where $\kappa = \operatorname{Tr}(T)/2 + \imath\sqrt{1 - \operatorname{Tr}(T)^2/4}$ and are hence complex conjugate of each other if $|\operatorname{Tr}(T)| < 2$ (which is the case of all transfer matrices near a critical energy). In this situation, the matrix $\operatorname{diag}(\kappa, 1/\kappa)$ is not in $\mathrm{SL}(2,\mathbb{R})$, however, the associated rotation matrix $R_\eta = \left(\begin{array}{cc} \cos(\eta) & -\sin(\eta) \\ \sin(\eta) & \cos(\eta) \end{array} \right)$ with $\kappa = e^{\imath\eta}$ is in $\mathrm{SL}(2,\mathbb{R})$ and can be attained with an adequate conjugation. Hence for any $T \in \mathrm{U}(1,1)$ with $|\operatorname{Tr}(T)| < 2$, there exists $M \in \mathrm{SL}(2,\mathbb{R})$ and η such that $MTM^{-1} = e^{\imath\xi} R_\eta$.

Let us now consider the family $(T_{\lambda,\sigma})_{\lambda\in\mathbb{C},\sigma\in\Sigma}$ in $\mathrm{U}(1,1)$ satisfying (2). Because they commute and have a trace less than 2 for $\lambda = 0$, they can simultaneously be conjugated to a rotation at that point. Using a Taylor expansion in λ, we therefore obtain the following:

$$M T_{\lambda,\sigma} M^{-1} \;=\; e^{\imath\xi_\sigma(\lambda)}\, R_{\eta_\sigma}\, \exp\left(\lambda P_\sigma + \lambda^2 Q_\sigma + \mathcal{O}(\lambda^3)\right)\;. \tag{8}$$

Here $e^{2\imath\xi_\sigma(\lambda)} = \det(T_{\lambda,\sigma})$ so that $\lambda P_\sigma + \lambda^2 Q_\sigma \in \mathrm{sl}(2,\mathbb{R})$ for all $\lambda \in \mathbb{R}$. In particular, $\mathrm{Tr}(P_\sigma) = \mathrm{Tr}(Q_\sigma) = 0$. The constant C in (6) only depends on the rotation angles η_σ and on P_σ through the constant

$$\beta_\sigma \;=\; \langle \overline{v}|P_\sigma|v\rangle \;, \qquad v \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\imath \end{pmatrix} \;. \tag{9}$$

Note that $\beta_\sigma = \langle \overline{v}|P_\sigma^*|v\rangle$, $|\beta_\sigma|^2 = \frac{1}{4}\mathrm{Tr}(|P_\sigma|^2 + P_\sigma^2)$ and that $P_\sigma^2 = (P_\sigma^*)^2$ is a multiple of the identity. Moreover, v and $\overline{v}$ are the eigenvectors of all rotations R_σ.

Given σ_n, let us now denote the associated random phases, rotations and perturbations by $\xi_n(\lambda), \eta_n, R_n, P_n, Q_n$, thus suppressing the dependence on the random variable σ_n.

In order to use the normal form (8) for the calculation of the Lyapunov and Landauer exponents, let us insert $M^{-1}M$ in between each pair of transfer matrices. As it only gives boundary contributions, one may also insert an M to the left and an M^{-1} to the right. Hence,

$$\gamma(\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N}\,\mathbf{E}\,\log\left(\left\|\prod_{n=1}^{N} MT_{\lambda,n}M^{-1}\right\|\right) \;,$$

as well as a similar formula for $\hat{\gamma}(\lambda)$. Now it is clear that there is no use in carrying along the phases $e^{\imath\xi_\sigma(\lambda)}$ in (8) if one is interested in calculating $\gamma(\lambda)$ and $\hat{\gamma}(\lambda)$. Therefore we may set from now on $\xi_\sigma(\lambda) = 0$. This is equivalent to supposing that $T_{\lambda,\sigma} \in \mathrm{SL}(2,\mathbb{R})$. Moreover, one may factor out the subgroup $\{\mathbf{1}, -\mathbf{1}\}$, namely even work with the projection of $MT_{\lambda,\sigma}M^{-1}$ into $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\mathbf{1},-\mathbf{1}\}$.

3 Further preliminaries and notations

Unit vectors in $\mathbb{R}^2$ will be denoted by:

$$e_\theta \;=\; \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \;, \qquad \theta \in [0, 2\pi) \;. \tag{10}$$

Each transfer matrix $T_{\lambda,\sigma}$ induces an action on unit vectors via

$$e_{\mathcal{S}_{\lambda,\sigma}(\theta)} \;=\; \frac{MT_{\lambda,\sigma}M^{-1}e_\theta}{\|MT_{\lambda,n}M^{-1}e_\theta\|} \;. \tag{11}$$

Using the vector v of (9), this is equivalent to

$$e^{2\imath\mathcal{S}_{\lambda,\sigma}(\theta)} \;=\; 2\,\frac{\langle v|MT_{\lambda,\sigma}M^{-1}|e_\theta\rangle^2}{\|MT_{\lambda,\sigma}M^{-1}e_\theta\|^2} \;=\; \frac{\langle v|MT_{\lambda,\sigma}M^{-1}|e_\theta\rangle}{\langle \overline{v}|MT_{\lambda,\sigma}M^{-1}|e_\theta\rangle} \;. \tag{12}$$

Now given an initial condition θ_0, a random sequence of phases θ_n associated to a code $(\sigma_n)_{n\in\mathbb{N}}$ is iteratively defined by

$$\theta_n \;=\; \mathcal{S}_{\lambda,\sigma_n}(\theta_{n-1}) \;. \tag{13}$$

A probability measure ν on S^1 is called invariant for this random dynamical system if

$$\int d\nu(\theta)\, f(\theta) \;=\; \int d\nu(\theta)\, \mathbf{E}_\sigma\, f(\mathcal{S}_{\lambda,\sigma}(\theta)) \;, \qquad f \in C(S^1) \;,$$

where $\mathbf{E}_\sigma$ denotes the average w.r.t. $\mathbf{p}$. Due to a theorem of Furstenberg [BL] ν exists and is unique whenever the Lyapunov exponent is positive. Positivity of the Lyapunov exponent for $\lambda \neq 0$ is guaranteed by condition (iii) of Definition 1 (we leave it to the reader to verify that the subgroup generated by the transfer matrices $T_{\lambda,\sigma}$ is then non-compact so that Furstenberg's criterium is satisfied [BL]). In the sequel, $\mathbf{E}_\nu$ will mean averaging w.r.t. ν as well as the whole code $(\sigma_n)_{n\in\mathbb{N}}$.

Next let us turn to the Lyapunov exponent. According to [BL, A.III.3.4] it is given by

$$\gamma(\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N}\, \mathbf{E}\, \log\left(\left\| \prod_{n=1}^{N} M T_{\lambda,n} M^{-1} e_\theta \right\|\right) \;. \tag{14}$$

As one can also insert the invariant measure, it is also given by the so-called Furstenberg formula:

$$\gamma(\lambda) \;=\; \int d\nu(\theta)\, \mathbf{E}_\sigma\, \log\left(\left\| M T_{\lambda,\sigma} M^{-1} e_\theta \right\|\right) \;.$$

Finally, one can use the random phase dynamics (13) in order to rewrite (14) as

$$\gamma(\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{E}_\nu\, \log\left(\left\| M T_{\lambda,n} M^{-1} e_{\theta_{n-1}} \right\|\right) \;. \tag{15}$$

4 Asymptotics of the Lyapunov exponent

Let us introduce the random variable

$$\gamma_n \;=\; \log\left(\left\| M T_{\lambda,n} M^{-1} e_{\theta_{n-1}} \right\|\right) \;. \tag{16}$$

Then the results cited in the previous section imply

$$\gamma(\lambda) \;=\; \lim_{N\to\infty} \frac{1}{N}\, \mathbf{E}_\nu \sum_{n=1}^{N} \gamma_n \;=\; \lim_{n\to\infty} \mathbf{E}\,(\gamma_n) \;. \tag{17}$$

Our first aim is to derive a perturbative formula for γ_n. Replacing (8), using that R_n is orthogonal and expanding the logarithm shows

$$\gamma_n \;=\; \frac{\lambda}{2} \langle e_{\theta_{n-1}} | \tilde{P}_n | e_{\theta_{n-1}} \rangle + \frac{\lambda^2}{2} \langle e_{\theta_{n-1}} | (\tilde{Q}_n + |P_n|^2 + P_n^2) | e_{\theta_{n-1}} \rangle - \frac{\lambda^2}{4} \langle e_{\theta_{n-1}} | \tilde{P}_n | e_{\theta_{n-1}} \rangle^2 \;+\; \mathcal{O}(\lambda^3) \;. \tag{18}$$

where we have used that $(P_n^2)^* = P_n^2$ is a multiple of the identity and set $\tilde{P}_n = P_n + P_n^*$ and $\tilde{Q}_n = Q_n + Q_n^*$. Now for any $T \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$, one has

$$\langle e_\theta | T | e_\theta \rangle \;=\; \frac{1}{2}\,\mathrm{Tr}(T) \;+\; \Re e\left(\langle \overline{v} | T | v\rangle\, e^{2\imath\theta}\right) \;.$$

Hence using the definition (9) and the remark following it, we deduce

$$\gamma_n \;=\; \frac{\lambda^2}{2}\,|\beta_n|^2 + \Re e \left(\lambda \beta_n e^{2\imath\theta_{n-1}} - \frac{\lambda^2}{2}\,\beta_n^2\, e^{4\imath\theta_{n-1}} \;+\; \frac{\lambda^2}{2}\,\langle \overline{v} | (|P_n|^2 + \tilde{Q}_n) | v\rangle\, e^{2\imath\theta_{n-1}}\right) \;+\; \mathcal{O}(\lambda^3)\;. \tag{19}$$

The so-called random phase approximation consists in supposing that the angles θ_{n-1} are distributed according the Lebesgue measure (*i.e.*, ν is the Lebesgue measure). Then only the non-oscillatory term in (19) would contribute so that one would get $\gamma(\lambda) = \frac{1}{2}\lambda \mathbf{E}_\sigma(|\beta_\sigma|^2) + \mathcal{O}(\lambda^3)$. In general, however, this is erroneous. Replacing (19) into (16), one has to calculate the following oscillatory sums (as in [JSS]).

Lemma 1 *For $j = 1, 2$, set*

$$I_j(N) \;=\; \mathbf{E}\,\frac{1}{N}\sum_{n=0}^{N-1} e^{2j\imath\theta_n}\;.$$

Suppose $\mathbf{E}_\sigma\left(e^{2j\imath\eta_\sigma}\right) \neq 1$ for $j = 1, 2$. Then

$$I_1(N) \;=\; \frac{\lambda\,\mathbf{E}_\sigma\left(\overline{\beta_\sigma}\, e^{2\imath\eta_\sigma}\right)}{1 - \mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right)} + \mathcal{O}(\lambda^2, N^{-1})\;, \qquad I_2(N) \;=\; \mathcal{O}(\lambda, N^{-1})\;.$$

Proof. It follows from (12) and (8) that

$$e^{2\imath\theta_n} \;=\; 2\,e^{2\imath\eta_n}\,\frac{\langle v | (\mathbf{1} + \lambda P_n) | e_{\theta_{n-1}}\rangle^2}{\langle (\mathbf{1} + \lambda\, P_n) e_{\theta_{n-1}} | (\mathbf{1} + \lambda P_n) e_{\theta_{n-1}}\rangle} \;+\; \mathcal{O}(\lambda^2)\;. \tag{20}$$

In particular, this implies

$$e^{2\imath\theta_n} \;=\; e^{2\imath(\eta_n + \theta_{n-1})} \;+\; \mathcal{O}(\lambda)\;,$$

so that, replacing this in each term, one obtains

$$I_j(N) \;=\; \mathbf{E}_\sigma\left(e^{2j\imath\eta_\sigma}\right)\; I_j(N) \;+\; \mathcal{O}(\lambda, N^{-1})\;.$$

The induction hypothesis therefore implies that $I_j(N) = \mathcal{O}(\lambda, N^{-1})$. In order to calculate the contribution of $\mathcal{O}(\lambda)$ to $I_1(N)$, let us expand (20). Some algebra shows that

$$e^{2\imath\theta_n} \;=\; e^{2\imath(\eta_n + \theta_{n-1})} \;-\; \lambda\, e^{2\imath(\eta_n + \theta_{n-1})} \left(e^{2\imath\theta_{n-1}}\beta_n - 2\langle v | P_n | v\rangle - e^{-2\imath\theta_{n-1}}\overline{\beta_n}\right) \;+\; \mathcal{O}(\lambda^2)\;. \tag{21}$$

From the last three terms, those containing still an oscillatory factor $e^{2\imath\theta_n}$ or $e^{4\imath\theta_n}$ will not contribute to leading order λ due to the above. Thus we deduce

$$I_1(N) \;=\; \mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right)\; I_1(N) \;+\; \lambda\,\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\,\overline{\beta_\sigma}\right) \;+\; \mathcal{O}(\lambda^2, N^{-1})\;.$$

This implies the result. □

As $\lim_{N\to\infty} I_j(N) = \mathbf{E}_\nu(e^{2\imath j\theta})$, the lemma shows that $\mathbf{E}_\nu(e^{2\imath\theta}) = \mathcal{O}(\lambda)$ and calculates the lowest order contribution. If β_σ is centered (as for the Anderson model), one even has $\mathbf{E}_\nu(e^{2\imath\theta}) = \mathcal{O}(\lambda^2)$. This shows in particular how far the invariant measure ν is away from the Lebesgue measure and that phase correlations are indeed present.

Now we can proceed with the calculation of $\gamma(\lambda)$. Carrying out the algebra shows $\gamma(\lambda) = D\,\lambda^2 + \mathcal{O}(\lambda^3)$ where

$$D \;=\; \frac{1}{2}\,\mathbf{E}_\sigma(|\beta_\sigma|^2) \;+\; \Re e\left(\frac{\mathbf{E}_\sigma(\beta_\sigma)\,\mathbf{E}_\sigma(\overline{\beta_\sigma}\,e^{2\imath\eta_\sigma})}{1-\mathbf{E}_\sigma(e^{2\imath\eta_\sigma})}\right)\;. \tag{22}$$

The second summand is due to phase correlations. It is important, *e.g.*, in the random polymer model [JSS]. In the Anderson model treated in Section 9, phase correlations only contribute to the forth order in λ. Because $\gamma(\lambda) \geq 0$, the coefficient D defined by (22) has to be non-negative. More precisely, we prove:

Proposition 1 *Suppose* $\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right) \neq 1$. *Then D is always non-negative. D vanishes if and only if one of the following two mutually excluding cases occurs:*

(i) *Both* $e^{2\imath\eta_\sigma}$ *and* β_σ *are* $\mathbf{p}$*-a.s. constant.*

(ii) $\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right) = 0$ *and* β_σ *is a constant multiple of* $1 - e^{2\imath\eta_\sigma}$*.*

Proof. Assume first that $e^{2\imath\eta_\sigma}$ is a.s. constant. Then $\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right) = e^{2\imath\eta_\sigma} \neq 1$ and it follows from $\Re e(1-e^{\imath\varphi})^{-1} = 1/2$ that $2\,D = \mathbf{E}_\sigma(|\beta_\sigma|^2) - |\mathbf{E}_\sigma(\beta_\sigma)|^2$. By the Cauchy-Schwarz inequality, $D \geq 0$ and $D = 0$ if and only if β_σ is a.s. constant. Now let us assume that $e^{2\imath\eta_\sigma}$ is not a.s. constant so that $|\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right)| < 1$. The proposition then follows from the following lemma by setting $\mathcal{H} = L^2(\mathbf{p})$, $\psi_1 = 1$ and $\psi_2 = e^{2\imath\eta_\sigma}$. □

Lemma 2 *Let ψ_1 and ψ_2 be two linearly independent unit vectors in a Hilbert space $\mathcal{H}$, implying $|\langle\psi_1|\psi_2\rangle| < 1$. Then the quadratic form*

$$\mathcal{Q}(\psi) \;=\; \langle\psi|\psi\rangle + \frac{1}{1-\langle\psi_1|\psi_2\rangle}\,\langle\psi|\psi_2\rangle\,\langle\psi_1|\psi\rangle + \frac{1}{1-\langle\psi_2|\psi_1\rangle}\,\langle\psi|\psi_1\rangle\,\langle\psi_2|\psi\rangle$$

on $\mathcal{H}$ is positive semi-definite. It is positive definite if and only if $\langle\psi_1|\psi_2\rangle \neq 0$. If $\langle\psi_1|\psi_2\rangle = 0$, then $\mathcal{Q}(\psi) = 0$ if and only if ψ is a multiple of $\psi_1 - \psi_2$.

Proof. Let $\mathcal{K}$ be the two-dimensional subspace of $\mathcal{H}$ spanned by ψ_1 and ψ_2 and $\mathcal{K}^\perp$ its orthogonal complement. For an arbitrary vector ψ, we write $\psi = \psi' + \psi''$ with

$\psi' \in \mathcal{K}$ and $\psi'' \in \mathcal{K}^\perp$. Then $\mathcal{Q}(\psi) = \mathcal{Q}(\psi') + \|\psi''\|^2$. Hence it suffices to assume $\psi = \psi' \in \mathcal{K}$, *i.e.*, we may restrict $\mathcal{Q}$ to $\mathcal{K}$. If we introduce the orthonormal basis

$$e_1 \;=\; \psi_1 \;, \qquad e_2 \;=\; \frac{1}{\sqrt{1-|\langle\psi_1|\psi_2\rangle|^2}} \;(\psi_2 - \langle\psi_1|\psi_2\rangle\,\psi_1)$$

in $\mathcal{K}$, then with respect to this orthonormal basis the quadratic form $\mathcal{Q}$ on $\mathcal{K}$ is represented by the 2×2 matrix

$$\mathbf{Q} \;=\; \begin{pmatrix} \frac{1+|\langle\psi_1|\psi_2\rangle|^2}{(1-\langle\psi_1|\psi_2\rangle)(1-\langle\psi_2|\psi_1\rangle)} & \frac{\sqrt{1-|\langle\psi_1|\psi_2\rangle|^2}}{1-\langle\psi_2|\psi_1\rangle} \\ \frac{\sqrt{1-|\langle\psi_1|\psi_2\rangle|^2}}{1-\langle\psi_1|\psi_2\rangle} & 1 \end{pmatrix} .$$

Now its trace is obviously strictly positive, while its determinant satisfies

$$\det\mathbf{Q} \;=\; \frac{2\,|\langle\psi_1|\psi_2\rangle|^2}{(1-\langle\psi_1|\psi_2\rangle)(1-\langle\psi_2|\psi_1\rangle)} \;\geq\; 0\,.$$

Hence both eigenvalues of $\mathbf{Q}$ are strictly positive whenever ψ_1 and ψ_2 are not orthogonal, while one eigenvalue is strictly positive and the other one zero if they are orthogonal. For this zero eigenvalue the corresponding eigenvector of $\mathbf{Q}$ is proportional to the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This concludes the proof of the lemma. □

5 Perturbation theory for the variance

Using telescoping as in (17), the variance is given by

$$\begin{aligned}\sigma(\lambda) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n,k=1}^{N}\mathbf{E}_\nu\left[(\gamma_n-\gamma)\,(\gamma_k-\gamma)\right] \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbf{E}_\nu\left[\gamma_n^2-\gamma^2+2\sum_{m=1}^{N-n}\Big(\gamma_n\gamma_{n+m}-\gamma^2\Big)\right],\end{aligned}$$

where we wrote γ for $\gamma(\lambda)$ for notational simplicity. Let us show that the sum over m is convergent even if $N\to\infty$. To this aim, we denote by $\mathbf{E}_n$ the expectation over all σ_m with $m \geq n$ such that with the previous notation $\mathbf{E}_1 = \mathbf{E}$. From the remark following Proposition 2 in Section 6 it follows that $\mathbf{E}_n(\gamma_{n+m})$ converges exponentially fast to γ for $m\to\infty$. Moreover, the summands of the sum over n converge in expectation so that, if ν is the unique invariant measure (integration w.r.t. θ_0), the variance is given by

$$\sigma(\lambda) = \mathbf{E}_\nu\left[\gamma_1^2-\gamma^2+2\sum_{m=2}^{\infty}\Big(\gamma_1\gamma_m-\gamma^2\Big)\right] = \mathbf{E}_\nu\left[\gamma_1^2-\gamma^2+2\gamma_1\mathbf{E}_2\sum_{m=2}^{\infty}(\gamma_m-\gamma)\right]. \tag{23}$$

Hence we need to evaluate the sums appearing in the following lemma. Its proof is deferred to Section 6.

Lemma 3 *Suppose* $\mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right) \neq 1$ *for* $j = 1, 2, 3$ *and* $D > 0$. *Then*

$$\mathbf{E}_2 \sum_{m=2}^{\infty} (\gamma_m - \gamma) \;=\; \lambda\, \Re e\, \frac{\mathbf{E}_\sigma(\beta_\sigma)}{1-\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right)}\, e^{2\imath(\theta_0+\eta_1)} \;+\; \mathcal{O}(\lambda^2)\;.$$

As $\gamma = \mathcal{O}(\lambda^2)$, the first summand γ^2 gives no contribution. Therefore we obtain

$$\sigma(\lambda) \;=\; \mathbf{E}_\nu \left[\gamma_1^2 \,+\, \Re e\, \frac{2\,\lambda\,\mathbf{E}_\sigma(\beta_\sigma)}{1-\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right)}\,\gamma_1\, e^{2\imath(\theta_0+\eta_1)}\right] \;+\; \mathcal{O}(\lambda^3)\;.$$

Let us calculate the first contribution supposing that ν is the invariant measure. Then extracting the linear coefficient in λ from (19) and using Lemma 1

$$\begin{aligned}
\mathbf{E}_\nu\left(\gamma_1^2\right) &= \lim_{N\to\infty} \frac{1}{N}\,\mathbf{E}_\nu \sum_{n=1}^{N} \gamma_n^2 \\
&= \frac{\lambda^2}{4} \lim_{N\to\infty} \frac{1}{N}\,\mathbf{E}_\nu \sum_{n=1}^{N} \left(\beta_n\, e^{2\imath\theta_{n-1}} + \overline{\beta_n}\, e^{-2\imath\theta_{n-1}}\right)^2 \,+\, \mathcal{O}(\lambda^3) \\
&= \frac{\lambda^2}{2}\,\mathbf{E}_\sigma\left(|\beta_\sigma|^2\right) \,+\, \mathcal{O}(\lambda^3)\;.
\end{aligned}$$

Similarly

$$\mathbf{E}_\nu\left(\gamma_1\, e^{2\imath(\theta_0+\eta_1)}\right) \;=\; \frac{\lambda}{2}\,\mathbf{E}_\sigma\left(\overline{\beta_\sigma}\, e^{2\imath\eta_\sigma}\right) \,+\, \mathcal{O}(\lambda^2)\;.$$

This implies

$$\sigma(\lambda) \;=\; \gamma(\lambda) \,+\, \mathcal{O}(\lambda^3)\;.$$

6 Estimates on the correlation decay

The main purpose of this section is to prove Lemma 3. We choose a slightly more general formulation, however, allowing to treat also other quantities. The first aim is to show how the estimates of [BL, Theorem A.V.2.5] (due to Le Page [LeP]) can be made quantitative when combined with Lemma 1. Throughout this section, we suppose that $D > 0$ and, for sake of notational simplicity, $\lambda \geq 0$. Let us introduce a distance on S^1 by

$$\delta(\theta,\psi) \;=\; \sqrt{1-\langle e_\theta|e_\psi\rangle^2} \;=\; \|e_\theta \wedge e_\psi\|\;,$$

where the second norm is in $\Lambda^2\mathbb{R}^2$. For $\alpha \in (0,1]$, the space of Hölder continuous functions $C_\alpha(S^1)$ is given by the continuous functions $f \in C(S^1)$ with finite Hölder norm $\|f\|_\alpha = \max\{\|f\|_\infty, m_\alpha(f)\}$, where

$$m_\alpha(f) \;=\; \sup_{\theta,\psi\in S^1} \frac{|f(\theta)-f(\psi)|}{\delta(\theta,\psi)^\alpha} \;.$$

Then we define $\mathcal{S},\mathcal{P} : C_\alpha(S^1) \to C_\alpha(S^1)$ by

$$(\mathcal{S}f)(\theta) \;=\; \mathbf{E}(f(\mathcal{S}_{\lambda,\sigma}(\theta)))\;, \qquad (\mathcal{P}f)(\theta) \;=\; \int d\nu(\psi)\, f(\psi)\;.$$

Here $\mathcal{P}$ should be thought of as the projection on the constant function with value given by the average w.r.t. the unique invariant measure. Both $\mathcal{S}$ and $\mathcal{P}$ depend on λ.

Proposition 2 *There exist a constants $c,d>0$ such that for $\alpha = d\lambda$*

$$\|(\mathcal{S}^N - \mathcal{P})(f)\|_\alpha \;\le\; \|f\|_\alpha\, e^{-c\,\lambda^3\,N}\;.$$

Proof. Let us introduce $\zeta : \mathrm{SL}(2,\mathbb{R}) \times S^1 \times S^1 \to \mathbb{R}$ by

$$\zeta(T,(\theta,\psi)) \;=\; \log\left(\frac{\delta(\mathcal{S}_T(\theta),\mathcal{S}_T(\psi))}{\delta(\theta,\psi)}\right)\;,$$

where $\mathcal{S}_T : S^1 \to S^1$ is defined as in (11) by $e_{\mathcal{S}_T(\theta)} = Te_\theta/\|Te_\theta\|$. Then ζ is a cocycle, namely it satisfies

$$\zeta(T'T,(\theta,\psi)) \;=\; \zeta(T',(\mathcal{S}_T(\theta),\mathcal{S}_T(\psi))) \;+\; \zeta(T,(\theta,\psi))\;.$$

Moreover, one has

$$\zeta(T,(\theta,\psi)) \;=\; \log\left(\frac{\|\Lambda^2 T\, e_\theta \wedge e_\psi\|}{\|Te_\theta\|\,\|Te_\psi\|\,\|e_\theta\wedge e_\psi\|}\right) \;=\; -\log(\|Te_\theta\|) - \log(\|Te_\psi\|)\;,$$

because $\|\Lambda^2 T\, e_\theta \wedge e_{\theta+\frac{\pi}{2}}\| \;=\; 1$. Therefore $|\zeta(MT_{\lambda,n}M^{-1},(\theta,\psi))| \le c_0\lambda$ and the cocycle property implies

$$|\zeta(\mathcal{T}_\lambda(n),(\theta,\psi))| \;\le\; c_1\,\lambda\,n\;.$$

Furthermore, invoking Lemma 1 shows that

$$\mathbf{E}\big(\zeta(\mathcal{T}_\lambda(n),(\theta,\psi))\big) \;\le\; -2D\,\lambda^2\,n + \; c_2\,\left(\lambda^3\,n+\lambda\right)\;,$$

where $D>0$ is the coefficient given in (22). Defining the sequence of angles ψ_n as in (13), but with initial condition $\psi_0 = \psi$, one can now infer from $e^x \le 1+x+x^2e^{|x|}/2$

that

$$\begin{aligned}\mathbf{E}\left(\frac{\delta(\theta_n,\psi_n)^\alpha}{\delta(\theta,\psi)^\alpha}\right) &\leq 1 + \alpha\,\mathbf{E}\big(\zeta(\mathcal{T}_\lambda(n),(\theta,\psi))\big) \\ &\qquad + \frac{1}{2}\,\alpha^2\,\mathbf{E}\big(\zeta(\mathcal{T}_\lambda(n),(\theta,\psi))^2 e^{\alpha\,|\zeta(\mathcal{T}_\lambda(n),(\theta,\psi))|}\big) \\ &\leq 1 - 2D\,\alpha\,\lambda^2\,n + c_2\,\alpha\,(\lambda^3\,n+\lambda) + \frac{1}{2}\,c_1^2\,(\alpha\lambda n)^2\,e^{c_1\,\alpha\lambda n}\,.\end{aligned}$$

Now one has on the one hand,

$$\begin{aligned}m_\alpha((\mathcal{S}^n-\mathcal{P})(f)) &= \sup_{\theta,\psi\in S^1}\frac{\big|\mathbf{E}\big(f(\theta_n)-f(\psi_n)\big)\big|}{\delta(\theta,\psi)^\alpha} \\ &\qquad\qquad \leq\; m_\alpha(f)\sup_{\theta,\psi\in S^1}\mathbf{E}\left(\frac{\delta(\theta_n,\psi_n)^\alpha}{\delta(\theta,\psi)^\alpha}\right)\,,\end{aligned}$$

and, furthermore, using the invariance of ν and then $\delta(\theta,\psi)\leq 1$,

$$\begin{aligned}\|(\mathcal{S}^n-\mathcal{P})(f)\|_\infty &= \sup_{\theta\in S^1}\left|\int d\nu(\psi)\;\;\mathbf{E}\big(f(\theta_n)-f(\psi_n)\big)\right| \\ &\leq m_\alpha(f)\;\big(1-2\,\alpha\,D\,\lambda 2\,n + c_2\,\alpha\,(\lambda^3 n+\lambda) + c_1 2\,(\alpha\lambda n)2\,e^{c_1\,\alpha\lambda n}\big)\,.\end{aligned}$$

Therefore, we can deduce

$$\|(\mathcal{S}^n-\mathcal{P})(f)\|_\alpha \;\leq\; m_\alpha(f)\;\big(1-\alpha\,D\,\lambda^2 + c_2\,(\lambda^3+\lambda n^{-1}) + c_1^2\,(\alpha\lambda n)^2\,e^{c_1\,\alpha\lambda n}\big)\,.$$

Now we choose $n=[c_3/\lambda]+1$ (as usual $[x]$ denotes the integer part of $x\in\mathbb{R}$) and $\alpha=c_4\lambda$ with adequate $c_3,c_4>0$ and note that $m_{\alpha'}(f)\leq m_\alpha(f)$ for $\alpha'\leq\alpha$, so that for some $c_5>0$

$$\|(\mathcal{S}^n-\mathcal{P})(f)\|_\alpha \;\leq\; \|f\|_\alpha\,(1-c_5\lambda^2)\,,\qquad n=\left[\frac{c_3}{\lambda}\right]+1\,.$$

Finally, we note that $\mathcal{S}^N-\mathcal{P}=(\mathcal{S}^n-\mathcal{P})^{N/n}$ because $\mathcal{SP}=\mathcal{PS}=\mathcal{P}$ so that iterating the last inequality completes the proof. □

When applied to the function $f(\theta)=\mathbf{E}_\sigma(\log(\|MT_{\lambda,\sigma}M^{-1}e_\theta\|))$ (which is Hölder continuous for any α) this estimate directly implies that $\mathbf{E}(\gamma_n)$ converges exponentially fast to γ when $n\to\infty$. Furthermore, using $m_\alpha(f)\leq\|\partial_\theta f\|_\infty$, one now gets:

Corollary 1 *There is a constant $c>0$ such that*

$$\mathbf{E}\sum_{m=1}^{\infty}\big(f(\theta_m)-\mathbf{E}_\nu(f(\theta))\big)\;\leq\; c\,\max\{\|f\|_\infty,\|\partial_\theta f\|_\infty\}\,\lambda^{-3}\,.$$

We will now consider an algebra $\mathcal{A}$ of functions which are analytic on some neighborhood of $\{0\} \times S^1 \subset \mathbb{C}^2$ and have a power series of the form

$$F(\lambda, z) \;=\; \sum_{k\geq 0} \lambda^k \sum_{|l|\leq k} F_{k,l}\, z^{2l} \;=\; \sum_{k\geq 0} \lambda^k\, F_k(z)\;, \tag{24}$$

with complex coefficients $F_{k,l}$. The only elementary fact needed here is that $F\cdot G \in \mathcal{A}$ whenever $F, G \in \mathcal{A}$. We will mainly be interested in the values on $\mathbb{R} \times S^1$ and there also write $F(\lambda, \theta) = F(\lambda, e^{2\imath\theta})$. Let us give two examples for elements in $\mathcal{A}$. One is the dynamics defined in (11), namely $(\lambda, \theta) \mapsto e^{2\imath(\mathcal{S}_{\lambda,\sigma}(\theta)-\theta)}$ is a function in $\mathcal{A}$ because it is a quotient of analytic functions in (λ, θ) of the form (24). Taking its jth power, one obtains the representation

$$e^{2\imath j \mathcal{S}_{\lambda,\sigma}(\theta)} \;=\; e^{2\imath j(\theta+\eta_\sigma)} \left(1 + \sum_{k\geq 1} \lambda^k \sum_{|l|\leq k} c^{j}_{k,l}(\sigma)\, e^{2\imath\, l\, \theta}\right)\;, \tag{25}$$

for adequate complex coefficients $c^j_{k,l}(\sigma)$. Comparing with (21), we see that $c^1_{1,1}(\sigma) = -\beta_\sigma$, $c^1_{1,0}(\sigma) = 2\,\langle v|P_\sigma|v\rangle$ and $c^1_{1,-1}(\sigma) = -\overline{\beta_\sigma}$.

Our second example is the function $F(\lambda, \theta) = \mathbf{E}_\sigma(\log(\|MT_{\lambda,\sigma}M^{-1}e_\theta\|))$. With this function, one has $\mathbf{E}_m(\gamma_m) = F(\lambda, \theta_{m-1})$ and $\gamma(\lambda) = \mathbf{E}_\nu(F(\lambda, \theta))$. This function appears in Lemma 3.

Generalizing Lemma 3, we are led to evaluate (perturbatively in λ) the summed up correlation decay for an arbitrary function $F \in \mathcal{A}$:

$$\mathrm{Cor}(F)(\lambda) \;=\; \mathbf{E}_2 \sum_{m=1}^{\infty} \big(F(\lambda, \theta_m) - \mathbf{E}_\nu(F(\lambda, \theta))\big)\;.$$

From Corollary 1 follows the *a priori* estimate $|\mathrm{Cor}(F)(\lambda)| \leq C\,\lambda^{-3}$. For simplicity, let us now calculate $\mathrm{Cor}(F)(\lambda)$ to order λ, namely discard terms of order $\mathcal{O}(\lambda^2)$. This is the situation covered by Lemma 3. Consider $G(\lambda, \theta) = F(\lambda, \theta) - \sum_{k=0}^{4} \lambda^k F_k(\theta) = \mathcal{O}(\lambda^5)$ so that $\|\partial_\theta G(\lambda, \,.\,)\|_\infty = \mathcal{O}(\lambda^5)$. Combined with Corollary 1, it follows:

$$\begin{aligned}
&\mathrm{Cor}(F)(\lambda)\\
&= \mathbf{E}_2 \sum_{m=1}^{\infty} \left(G(\lambda, \theta_m) + \sum_{k=0}^{4} \lambda^k F_k(\theta_m) - \sum_{k=0}^{4} \lambda^k\, \mathbf{E}_\nu(F_k(\theta)) - \mathbf{E}_\nu(G(\lambda, \theta))\right)\\
&= \sum_{k=0}^{4} \lambda^k\, \mathbf{E}_2 \sum_{m=1}^{\infty} \big(F_k(\theta_m) - \mathbf{E}_\nu(F_k(\theta))\big) \;+\; \mathcal{O}(\lambda^2)\\
&= \sum_{k=1}^{4} \lambda^k \sum_{|l|\leq k} F_{k,l}\, \mathbf{E}_2 \sum_{m=1}^{\infty} \big(e^{2\imath l\theta_m} - \mathbf{E}_\nu(e^{2\imath l\theta})\big) \;+\; \mathcal{O}(\lambda^2)\;.
\end{aligned}$$

The appearing sum over m is finite (again due to Corollary 1) and, moreover, we can calculate its value perturbatively.

Lemma 4 *Let*

$$J_j \;=\; \mathbf{E}_2 \sum_{m=1}^{\infty} \left(e^{2\imath j\theta_m} - \mathbf{E}_\nu(e^{2\imath j\theta})\right) , \qquad j \in \mathbb{Z} .$$

Suppose $\mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right) \neq 1$ *for* $j = 1,\ldots,4$. *Then* $J_4 = \mathcal{O}(\lambda^{-2})$, $J_3 = \mathcal{O}(\lambda^{-1})$, $J_2 = \mathcal{O}(1)$ *and*

$$J_1 \;=\; \frac{1}{1-\mathbf{E}_\sigma\left(e^{2\imath\eta_\sigma}\right)}\, e^{2\imath(\theta_0+\eta_1)} \;+\; \mathcal{O}(\lambda) \; . \tag{26}$$

Proof. We use (25) in order to express $e^{2\imath j\theta_m}$ in terms of $e^{2\imath j\theta_{m-1}}$, but truncate the expansion at $\mathcal{O}(\lambda^K)$. Using again Corollary 1, we deduce

$$\begin{aligned}
J_j \;&=\; \mathbf{E}_2 \sum_{m=1}^{\infty}\left[e^{2\imath j(\theta_{m-1}+\eta_m)}\left(1+\sum_{k=1}^{K}\lambda^k\sum_{|l|\le k}\mathbf{E}_\sigma(c^j_{k,l}(\sigma))e^{2\imath l\theta_{m-1}}\right)\right.\\
&\quad\left. -\,\mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right)\mathbf{E}_\nu\left(e^{2\imath j\theta}\left(1+\sum_{k=1}^{K}\lambda^k\sum_{|l|\le k}\mathbf{E}_\sigma(c^j_{k,l}(\sigma))e^{2\imath l\theta}\right)\right)\right] \;+\; \mathcal{O}(\lambda^{K-2})\\
&=\; \mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right)J_j \;+\; e^{2\imath j(\theta_0+\eta_1)} \;+\; \mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right)\sum_{k=1}^{K}\lambda^k\sum_{|l|\le k}\mathbf{E}_\sigma(c^j_{k,l}(\sigma))\,J_{j+l}\\
&\qquad + \mathcal{O}(\lambda^{K-2},\lambda)\\
&=\; \frac{1}{1-\mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right)}\left[e^{2\imath j(\theta_0+\eta_1)}+\mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right)\sum_{k=1}^{K}\lambda^k\sum_{|l|\le k}\mathbf{E}_\sigma(c^j_{k,l}(\sigma))\,J_{j+l}\right]\\
&\qquad + \mathcal{O}(\lambda^{K-2},\lambda)\;,
\end{aligned}$$

where $\mathcal{O}(\lambda^{K-2},\lambda) = \mathcal{O}(\lambda^{K-2})+\mathcal{O}(\lambda)$ and in the second equality we used the fact $\mathbf{E}_\nu(e^{2\imath j\theta}) = \mathcal{O}(\lambda)$ (due to Lemma 1). As we know that $J_0 = 0$ and that the *a priori* estimate $J_j = \mathcal{O}(\lambda^{-3})$ holds, this calculation shows that $J_j = \mathcal{O}(\lambda^{-2})$ if $\mathbf{E}_\sigma\left(e^{2\imath j\eta_\sigma}\right) \neq 0$. The estimate $J_3 = \mathcal{O}(\lambda^{-1})$ now follows by choosing $K = 1$ and replacing $J_4 = \mathcal{O}(\lambda^{-2})$ and $J_2 = \mathcal{O}(\lambda^{-2})$. The same way one deduces $J_2 = \mathcal{O}(\lambda^{-1})$ and $J_1 = \mathcal{O}(\lambda^{-1})$. Choosing $K = 2$, a similarly argument shows in the next step $J_2 = \mathcal{O}(1)$ and $J_1 = \mathcal{O}(1)$. In order to establish (26), let us choose $K = 3$. The first term in the last square bracket gives the desired contribution, while the sum is $\mathcal{O}(\lambda)$ because $J_2 = \mathcal{O}(1)$, $J_3 = \mathcal{O}(\lambda^{-1})$ and $J_3 = \mathcal{O}(\lambda^{-2})$. □

In order to calculate $\mathrm{Cor}(F)(\lambda)$, let us recall that $J_0 = 0$ and $J_{-j} = \overline{J_j}$. Thus

$$\mathrm{Cor}(F)(\lambda) \;=\; \lambda\,F_{1,1}\,J_1 \;+\; \lambda\,F_{1,-1}\,\overline{J_1} \;+\; \mathcal{O}(\lambda^2)\;,$$

with J_1 given by Lemma 4. For the function $F(\lambda,\theta) = \mathbf{E}_\sigma(\log(\|MT_{\lambda,\sigma}M^{-1}e_\theta\|))$, equation (19) implies $F_{1,1} = \frac{1}{2}\mathbf{E}_\sigma(\beta_\sigma) = \overline{F_{1,-1}}$. This gives Lemma 3.

7 Some identities linked to the adjoint representation

The adjoint representation of $\mathrm{SL}(2,\mathbb{R})$ on its Lie algebra is defined by $\mathrm{Ad}_T(t) = TtT^{-1}$, $t \in \mathrm{sl}(2,\mathbb{R})$. It leaves invariant the quadratic form $q(t,s) = \frac{1}{2}\,\mathrm{Tr}(ts)$ of signature $(2,1)$ (note that $q(t,t) = -\det(t)$). A basis $\mathcal{B} = \{b_1, b_2, b_3\}$ of $\mathrm{sl}(2,\mathbb{R})$ is given by

$$b_1 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \qquad b_2 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad b_3 \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$$

It is orthonormal w.r.t. the scalar product $\langle s|t\rangle = \frac{1}{2}\mathrm{Tr}(s^*t)$. Moreover, denoting the coordinate map w.r.t. this basis by $K^{\mathcal{B}} : \mathrm{sl}(2,\mathbb{R}) \to \mathbb{R}^3$ and the standard scalar product in $\mathbb{R}^3$ also by $\langle \vec{x}\,|\,\vec{y}\,\rangle$ for $\vec{x},\vec{y} \in \mathbb{R}^3$, we have

$$q(t,s) \;=\; \langle K^{\mathcal{B}}(t)|\,\Gamma_{2,1}\,|K^{\mathcal{B}}(s)\rangle\;, \qquad \Gamma_{2,1} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\,.$$

Finally, let us set $\mathrm{Ad}^{\mathcal{B}}_T = K^{\mathcal{B}}\mathrm{Ad}_T(K^{\mathcal{B}})^{-1}$. This means $(\mathrm{Ad}^{\mathcal{B}}_T)_{j,k} = \langle b_j|\mathrm{Ad}_T(b_k)\rangle$. Note that $J = b_3 \in \mathrm{SL}(2,\mathbb{R})$ and that $\mathrm{Ad}^{\mathcal{B}}_J = -\Gamma_{2,1}$. Thus $T^*JT = J$ implies $(\mathrm{Ad}^{\mathcal{B}}_T)^*\Gamma_{2,1}\mathrm{Ad}^{\mathcal{B}}_T = \Gamma_{2,1}$. Hence $\mathrm{Ad}^{\mathcal{B}}_T$ is an element of the Lorentz group $\mathrm{SO}(2,1)$ given by all $A \in \mathrm{Mat}_{3\times3}(\mathbb{R})$ satisfying $A^*\Gamma_{2,1}A = \Gamma_{2,1}$. As $\mathrm{Ad}^{\mathcal{B}}_{\mathbf{1}} = \mathrm{Ad}^{\mathcal{B}}_{-\mathbf{1}} = \mathbf{1}$, the adjoint representation gives an isomorphism $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SO}(2,1)$. Important for the sequel is that the eigenvalues μ_1,μ_2,μ_3 of $A \in \mathrm{SO}(2,1)$ satisfy $\mu_1\mu_2\mu_3 = \pm1$. One of these eigenvalues, say μ_3, must be real, while the other two may either be real as well or be complex conjugates of each other.

Next it follows from a short calculation that for $T = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$

$$\mathrm{Ad}^{\mathcal{B}}_T \;=\; \begin{pmatrix} \hat{a}\hat{d}+\hat{b}\hat{c} & \hat{d}\hat{b}-\hat{a}\hat{c} & \hat{a}\hat{c}+\hat{b}\hat{d} \\ \hat{d}\hat{c}-\hat{a}\hat{b} & \frac{1}{2}(\hat{d}^2-\hat{b}^2-\hat{c}^2+\hat{a}^2) & \frac{1}{2}(\hat{d}^2-\hat{b}^2+\hat{c}^2-\hat{a}^2) \\ \hat{d}\hat{c}+\hat{a}\hat{b} & \frac{1}{2}(\hat{d}^2+\hat{b}^2-\hat{c}^2-\hat{a}^2) & \frac{1}{2}(\hat{d}^2+\hat{b}^2+\hat{c}^2+\hat{a}^2) \end{pmatrix}\,.$$

In particular, for a rotation R_η by η we get

$$\mathrm{Ad}^{\mathcal{B}}_{R_\eta} \;=\; \begin{pmatrix} \cos(2\eta) & -\sin(2\eta) & 0 \\ \sin(2\eta) & \cos(2\eta) & 0 \\ 0 & 0 & 1 \end{pmatrix}\,.$$

The eigenvalues are $e^{2\imath\eta}, e^{-2\imath\eta}, 1$ with respective eigenvectors $\vec{v}_1 = (\vec{e}_1 - \imath\vec{e}_2)/\sqrt{2}, \vec{v}_2 = (\vec{e}_1 + \imath\vec{e}_2)/\sqrt{2}, \vec{v}_3 = \vec{e}_3$.

We will also need the adjoint representation of the normal form (8):

$$\mathrm{Ad}_{R\,\exp(\lambda P+\lambda^2 Q)} \;=\; \mathrm{Ad}_R\left(\mathbf{1} \;+\; \lambda\,\mathrm{ad}_P \;+\; \lambda^2\,\mathrm{ad}_Q \;+\; \frac{\lambda^2}{2}\,(\mathrm{ad}_P)^2 \;+\; \mathcal{O}(\lambda^3)\right)\,, \tag{27}$$

where $\mathrm{ad}_t(s) = [t,s]$ for $t,s \in \mathrm{sl}(2,\mathbb{R})$. Let us write out more explicit formulas in the representation w.r.t. the basis $\mathcal{B}$. For $P = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{sl}(2,\mathbb{R})$,

$$\mathrm{ad}_P^{\mathcal{B}} \;=\; \begin{pmatrix} 0 & b-c & b+c \\ c-b & 0 & -2a \\ b+c & -2a & 0 \end{pmatrix} \;, \tag{28}$$

and

$$(\mathrm{ad}_P^{\mathcal{B}})^2 \;=\; \begin{pmatrix} 4bc & -2a(b+c) & 2a(c-b) \\ -2a(b+c) & 4a^2-(c-b)^2 & c^2-b^2 \\ 2a(b-c) & b^2-c^2 & (b+c)^2+4a^2 \end{pmatrix} \;. \tag{29}$$

Finally let $\mathcal{P} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ be the permutation operator $\mathcal{P}\phi\otimes\psi = \psi\otimes\phi$. It can readily be checked that

$$\mathcal{P} \;=\; \frac{1}{2}\left(\mathbf{1}\otimes\mathbf{1} \;-\; \sum_{j=1}^{3} \det(b_j)\; b_j \otimes b_j\right) \;.$$

Multiplying this identity from the left by $T \otimes J$ and from the right by $\mathbf{1} \otimes T^{-1}J$, one gets for $T \in \mathrm{SL}(2,\mathbb{R})$,

$$T \otimes T^t\, \mathcal{P} \;=\; \frac{1}{2}\left(-J \otimes J \;+\; \sum_{j,k=1}^{3} \det(b_j)\; (\mathrm{Ad}_T^{\mathcal{B}})_{k,j}\; b_k J \otimes J b_j\right) \;. \tag{30}$$

This is useful for the calculation of the Landauer conductance because

$$\|Tw\|^2 \;=\; \langle w|\, \mathrm{Tr}_1(T \otimes T^t\, \mathcal{P})|w\rangle \;,$$

where Tr_1 is the partial trace over the first component of $\mathbb{C}^2 \otimes \mathbb{C}^2$. Replacing (30), the first term vanishes because $\mathrm{Tr}(J) = 0$. Moreover, $\mathrm{Tr}(b_k J) = -2\,\delta_{k,3}$ so that

$$\|Tw\|^2 \;=\; \sum_{j=1}^{3} (\mathrm{Ad}_T^{\mathcal{B}})_{3,j}\,(-1)\det(b_j)\,\langle w|Jb_j|w\rangle \;.$$

Let us define a map $g : \mathbb{C}^2 \to \mathbb{R}^3$ by

$$g(w) = -\begin{pmatrix} \det(b_1)\,\langle w|Jb_1|w\rangle \\ \det(b_2)\,\langle w|Jb_2|w\rangle \\ \det(b_3)\,\langle w|Jb_3|w\rangle \end{pmatrix} = \begin{pmatrix} w_1\overline{w_2} + \overline{w_1}w_2 \\ |w_2|^2 - |w_1|^2 \\ |w_1|^2 + |w_2|^2 \end{pmatrix} \;, \qquad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \;.$$

Then one has

$$\|Tw\|^2 \;=\; \langle \vec{e}_3|\,\mathrm{Ad}_T^{\mathcal{B}}\,|g(w)\rangle \;. \tag{31}$$

Since $g(v) = \vec{e}_3$, this implies in particular

$$\|Tv\|^2 \;=\; \langle \vec{e}_3|\,\mathrm{Ad}_T^{\mathcal{B}}\,|\vec{e}_3\rangle \;. \tag{32}$$

8 Calculation of the averaged Landauer resistance

The Landauer resistance $\rho_\lambda(N)$ of a system of length N is defined by

$$\rho_\lambda(N) \;=\; \mathbf{E}\,\left(\mathrm{Tr}\left(|\mathcal{T}_\lambda(N)|^2\right)\right) \;=\; 2\,\mathbf{E}\,\left(\|\mathcal{T}_\lambda(N)v\|^2\right)\;,$$

with v as in (9). Because $|\mathcal{T}_\lambda(N)|^2$ is positive and has unit determinant, one has $\rho_\lambda(N) \geq 2$ for all N. Using the identity (32) and then the representation property $\mathrm{Ad}^{\mathcal{B}}_{ST} = \mathrm{Ad}^{\mathcal{B}}_S \mathrm{Ad}^{\mathcal{B}}_T$ iteratively, the expectation value appearing in the Landauer resistance can readily be calculated:

$$\rho_\lambda(N) \;=\; 2\,\langle \vec{e}_3|\,(\mathbf{E}\,(\mathrm{Ad}^{\mathcal{B}}_{T_\sigma}))^N\,|\vec{e}_3\rangle\;,$$

so that $\hat{\gamma}(\lambda) = \lim_{N\to\infty} \log(\rho_\lambda(N))/(2N)$. Replacing the normal form (8) (recall that the phases disappear right away in the definition of the Landauer conductance):

$$\rho_\lambda(N) \;=\; 2\,\langle(\mathrm{Ad}^{\mathcal{B}}_{M^{-1}})^t\,\vec{e}_3|\,\left(\mathbf{E}\,\mathrm{Ad}^{\mathcal{B}}_{R_\sigma\,\exp(\lambda P_\sigma+\lambda^2 Q_\sigma+\mathcal{O}(\lambda^3))}\right)^N\,|\mathrm{Ad}^{\mathcal{B}}_M\vec{e}_3\rangle\;.$$

Next we need to do (non-degenerate, in λ) perturbation theory of the eigenvalues μ_1,μ_2,μ_3 of $\mathbf{E}\,\mathrm{Ad}^{\mathcal{B}}_{R_\sigma\,\exp(\lambda P_\sigma+\lambda^2 Q_\sigma+\mathcal{O}(\lambda^3))}$, which according to (27) is up to $\mathcal{O}(\lambda^3)$ given by

$$\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma}) \;+\; \lambda\,\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma}\mathrm{ad}^{\mathcal{B}}_{P_\sigma}) \;+\; \lambda^2\,\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma}\mathrm{ad}^{\mathcal{B}}_{Q_\sigma}) \;+\; \frac{\lambda^2}{2}\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma}(\mathrm{ad}^{\mathcal{B}}_{P_\sigma})^2)\;,$$

Let us note that the eigenvalues of $\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_{\eta_\sigma}})$ are $\mu_1 = \mathbf{E}(e^{2\imath\eta_\sigma})$, $\mu_2 = \mathbf{E}(e^{-2\imath\eta_\sigma})$, $\mu_3 = 1$ with the eigenvectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ respectively. Therefore, unless η_σ is independent of σ, the matrix $\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_{\eta_\sigma}})$ is not an element of $\mathrm{SO}(2,1)$ and two of its (complex conjugate) eigenvalues are strictly within the unit circle.

We first focus on the eigenvalue μ_3. Because $\langle\vec{e}_3|\mathrm{ad}^{\mathcal{B}}_P|\vec{e}_3\rangle = 0$, one has $\mu_3 = 1 + \mathcal{O}(\lambda^2)$ and the eigenvector is $\vec{v}_3 + \mathcal{O}(\lambda^2)$. Second order perturbation theory now shows

$$\begin{aligned}\mu_3 \;&=\; 1 \;+\; \frac{\lambda^2}{2}\,\langle\vec{v}_3|\,\mathbf{E}\,(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma}(\mathrm{ad}^{\mathcal{B}}_{P_\sigma})^2)|\vec{v}_3\rangle\\ &\quad+\;\lambda^2\,\langle\vec{v}_3|\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma}\mathrm{ad}^{\mathcal{B}}_{P_\sigma})\frac{1}{\mathbf{1}-\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma})}\,(\mathbf{1}-|\vec{v}_3\rangle\langle\vec{v}_3|)\\ &\quad\mathbf{E}(\mathrm{Ad}^{\mathcal{B}}_{R_\sigma}\mathrm{ad}^{\mathcal{B}}_{P_\sigma})|\vec{v}_3\rangle \;+\; \mathcal{O}(\lambda^3)\;.\end{aligned}$$

Recalling $(b+c)^2+4a^2 = 4\,|\beta_\sigma|^2$ and using $\langle\vec{v}_3|\mathrm{ad}^{\mathcal{B}}_{P_\sigma}|\vec{v}_1\rangle = \frac{1}{\sqrt{2}}(b+c+2\imath a) = \imath\sqrt{2}\,\beta_\sigma$, we deduce

$$\mu_3 \;=\; 1 \;+\; 2\,\lambda^2\,\left[\mathbf{E}(|\beta_\sigma|^2) \;+\; 2\,\Re e\left(\frac{\mathbf{E}(\beta_\sigma)\,\mathbf{E}(\overline{\beta_\sigma}\,e^{2\imath\eta_\sigma})}{1-\mathbf{E}(e^{2\imath\eta_\sigma})}\right)\right] \;+\; \mathcal{O}(\lambda^3)\;. \qquad (33)$$

Now let us analyze the eigenvalues μ_1 and μ_2. If $|\,\mathbf{E}(e^{2\imath\eta_\sigma})| < 1$ (as for the dimer model), they are strictly within the unit disc and remain there also for λ sufficiently small. As $\rho_\lambda(N) \geq 2$, we conclude that $\mu_3 \geq 1$ (this also implies $D \geq 0$, like in Proposition 1, because otherwise $\rho_\lambda(N) \to 0$ for $N \to \infty$). It follows that the scaling exponent of the Landauer resistance defined in (5) is given solely by μ_3, namely $\hat{\gamma}(\lambda) = \frac{1}{2}(\mu_3 - 1) + \mathcal{O}(\lambda^3)$ so that, when comparing with (22), $\hat{\gamma}(\lambda) = 2\,\gamma(\lambda) + \mathcal{O}(\lambda^3)$ as claimed in Theorem 1.

In the case where $\eta_\sigma = \eta$ independently of σ (as in the Anderson model), one has $D \geq 0$. First order perturbation theory (in λ) shows that $\mu_{1,2}$ move along the the unit circle, while in second order they lie inside of the unit circle. Thus the same argument as above applies to deduce Theorem 1.

9 Higher orders for the Anderson model

This section serves two purposes: it provides an example to which the general theory applies and we moreover outline (algebra is left to the reader) how it can be extended to calculate the next higher order in perturbation theory. The implications of the results have already been discussed in the introduction.

The one-dimensional Anderson model is a random Jacobi matrix given by the finite difference equation

$$-\psi_{n+1} + \lambda\, v_n\, \psi_n - \psi_{n-1} \;=\; E\,\psi_n \;.$$

Here $|E| < 2$ is a fixed energy and v_n are centered real i.i.d. random variables with finite moments. This equation is rewritten as usual using transfer matrices:

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} \;=\; T_{\lambda,n} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \;, \qquad T_{\lambda,n} \;=\; \begin{pmatrix} \lambda v_n - E & -1 \\ 1 & 0 \end{pmatrix} \;.$$

As above, we also write v_σ for one of the random variables such that $v_n = v_{\sigma_n}$ and $T_{\lambda,\sigma_n} = T_{\lambda,n}$. For the basis change to the normal form of the transfer matrix $T_{\lambda,\sigma}$, let us introduce

$$E \;=\; -2\cos(k) \;, \qquad M \;=\; \frac{1}{\sqrt{\sin(k)}} \begin{pmatrix} \sin(k) & 0 \\ -\cos(k) & 1 \end{pmatrix} \;.$$

It is then a matter of computation to verify

$$M T_{\lambda,\sigma} M^{-1} \;=\; R_k(1 + \lambda\, P_\sigma) \;, \qquad P_\sigma \;=\; -\,\frac{v_\sigma}{\sin(k)} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;.$$

Comparing with (8), we see that the rotation R_k by the angle k is not random in this example, that P_σ is nilpotent so that $\exp(\lambda P_\sigma) = 1 + \lambda P_\sigma$ and that $Q_\sigma = 0$. Furthermore

$$\beta_\sigma \;=\; \frac{\imath\, v_\sigma}{2\,\sin(k)} \;, \qquad \langle v|P_\sigma|v\rangle \;=\; -\beta_\sigma \;.$$

One has $\beta_\sigma^2 = -|\beta_\sigma|^2$, $\beta_\sigma^4 = |\beta_\sigma|^4$ and

$$\langle e_\theta|\tilde{P}_\sigma|e_\theta\rangle \;=\; \beta_\sigma\,(e^{2\imath\theta} - e^{-2\imath\theta})\;, \qquad \langle e_\theta|\,|P_\sigma|^2|e_\theta\rangle \;=\; |\beta_\sigma|^2\,(2 + e^{2\imath\theta} + e^{-2\imath\theta})\;. \tag{34}$$

As $\mathbf{E}(\beta_\sigma) = 0$, one can immediately deduce from (22) the well-known formula [Tho, PF, Luc] for the Lyapunov exponent, namely $\gamma(\lambda) = \frac{1}{2}\lambda^2\,\mathbf{E}(|\beta_\sigma|^2) + \mathcal{O}(\lambda^3) = \lambda^2\,\frac{\mathbf{E}(|v_\sigma|^2)}{8\sin^2(k)} + \mathcal{O}(\lambda^3)$ as long as $e^{2\imath jk} \neq 1$ for $j = 1, 2$ (the latter condition excludes band edges and the Kappus-Wegner anomaly at the band center $E = 0$).

In order to calculate the 4th order of the Lyapunov exponent and the variance, we need higher-order expansions than (21) and (18). We will assume $e^{2\imath jk} \neq 1$ for $j = 1, \ldots, 4$. After some algebra,

$$\begin{aligned} e^{2\imath\mathcal{S}_{\lambda,\sigma}(\theta)} \;&=\; \frac{\langle v|R_k(1+\lambda P_\sigma)|e_\theta\rangle}{\langle \overline{v}|R_k(1+\lambda P_\sigma)|e_\theta\rangle} \\ &=\; e^{2\imath(\theta+k)}\Big[1 \,-\, \lambda\beta_\sigma(e^{2\imath\theta} + 2 + e^{-2\imath\theta}) \\ &\qquad +\lambda^2\beta_\sigma^2(e^{4\imath\theta} + 3e^{2\imath\theta} + 3 + e^{-2\imath\theta}) + \mathcal{O}(\lambda^3)\Big]\;, \end{aligned} \tag{35}$$

and, using (34),

$$\begin{aligned} \gamma_{\lambda,\sigma}(\theta) \;&=\; \log(\|(1+\lambda P_\sigma)e_\theta\|) \\ &=\; \Re e\left[\lambda\beta_\sigma e^{2\imath\theta} - \lambda^2\beta_\sigma^2\left(\frac{1}{2}e^{4\imath\theta} + e^{2\imath\theta} + \frac{1}{2}\right)\right. \\ &\qquad +\lambda^3\beta_\sigma^3\left(\frac{1}{3}e^{6\imath\theta} + e^{4\imath\theta} + e^{2\imath\theta}\right) \\ &\qquad \left. +\lambda^4\,\beta_\sigma^4\left(-\frac{1}{4}e^{8\imath\theta} - e^{6\imath\theta} - \frac{3}{2}e^{4\imath\theta} - e^{2\imath\theta} - \frac{1}{4}\right) \;+\; \mathcal{O}(\lambda^5)\;\right] \end{aligned} \tag{36}$$

Using an argument similar to Lemma 1, we deduce from (35) and its square that

$$\mathbf{E}_\nu(e^{2\imath\theta}) \;=\; \frac{\lambda^2\mathbf{E}(|\beta_\sigma|^2)}{1 - e^{-2\imath k}} \;+\; \mathcal{O}(\lambda^3)\;, \qquad \mathbf{E}_\nu(e^{4\imath\theta}) \;=\; \frac{\lambda^2\mathbf{E}(|\beta_\sigma|^2)}{1 - e^{-4\imath k}} \;+\; \mathcal{O}(\lambda^3)\;.$$

Moreover, $\mathbf{E}_\nu(e^{6\imath\theta}) = \mathcal{O}(\lambda^2)$ and $\mathbf{E}_\nu(e^{8\imath\theta}) = \mathcal{O}(\lambda^2)$. Following the argument of Section 4, we therefore obtain from (36) and $\Re e(1 - e^{\imath\varphi})^{-1} = \frac{1}{2}$

$$\gamma(\lambda) \;=\; \frac{1}{2}\,\lambda^2\,\mathbf{E}(|\beta_\sigma|^2) \;+\; \lambda^4\,\left(\frac{3}{4}\,\mathbf{E}(|\beta_\sigma|^2)^2 \;-\; \frac{1}{4}\,\mathbf{E}(|\beta_\sigma|^4)\right) \;+\; \mathcal{O}(\lambda^5)\;.$$

This coincides with the fourth-order contribution obtained in [Luc] by using complex energy Dyson-Schmidt variables.

In order to calculate the variance according to formula (23), one first needs to go through the arguments of Section 6. One finds

$$\mathbf{E}_2\,\sum_{m=1}^{\infty}\left(e^{2\imath\theta_m} - \mathbf{E}_\nu(e^{2\imath\theta})\right) \;=\; \frac{e^{2\imath\theta_0}}{e^{-2\imath k} - 1}\,\left(1 - \lambda\beta_1(e^{2\imath\theta_0} + 2 + e^{-2\imath\theta_0})\right) \;+\; \mathcal{O}(\lambda^2)\;,$$

a similar expression for the correlation sum of $e^{4\imath\theta}$, and that $\mathbf{E}_2 \sum_{m=2}^{\infty}(\gamma_m - \gamma)$ is up to $\mathcal{O}(\lambda^4)$ equal to

$$\lambda^2\,\mathbf{E}(|\beta_\sigma|^2)\,\Re e\left[\frac{e^{2\imath\theta_0}(1-\lambda\beta_1(e^{2\imath\theta_0}+2+e^{-2\imath\theta_0}))}{e^{-2\imath k}-1} \right.$$
$$\left. +\frac{1}{2}\,\frac{e^{4\imath\theta_0}(1-2\lambda\beta_1(e^{2\imath\theta_0}+2+e^{-2\imath\theta_0}))}{e^{-4\imath k}-1}\right] .$$

Finally, after having squared (36),

$$\mathbf{E}_\nu(\gamma_1^2)-\gamma^2 \;=\; \frac{1}{2}\,\lambda^2\,\mathbf{E}(|\beta_\sigma|^2)\;-\;\frac{1}{8}\,\lambda^4\,\mathbf{E}(|\beta_\sigma|^4)\;-\;\frac{1}{2}\,\lambda^4\,\mathbf{E}(|\beta_\sigma|^2)^2\;+\;\mathcal{O}(\lambda^5)\;,$$

and

$$\mathbf{E}_\nu\left(2\,\gamma_1\,\mathbf{E}_2\,\sum_{m=2}^{\infty}(\gamma_m-\gamma)\right)\;=\;\frac{7}{8}\,\lambda^4\,\mathbf{E}(|\beta_\sigma|^2)^2\;+\;\mathcal{O}(\lambda^5)\;.$$

Combining these results according to (23), we obtain

$$\sigma(\lambda)\;=\;\frac{1}{2}\,\lambda^2\,\mathbf{E}(|\beta_\sigma|^2)\;+\;\lambda^4\left(\frac{3}{8}\,\mathbf{E}(|\beta_\sigma|^2)^2\;-\;\frac{1}{8}\,\mathbf{E}(|\beta_\sigma|^4)\right)\;+\;\mathcal{O}(\lambda^5)\;.$$

Therefore we see that Lyapunov exponent and variance are only equal to lowest order in perturbation theory. Finally let us also argue that there cannot exist a universal analytic function f such that $\sigma = f(\gamma)$. Indeed, if $f(x) = f_1 x + f_2 x^2 + \mathcal{O}(x^3)$, then $\sigma(\lambda) = f(\gamma(\lambda))$ for the Anderson model implies in order λ^2 that $f_1 = 1$, but in order λ^4 there is already a problem due to the prefactors of $\mathbf{E}(|\beta_\sigma|^4)$.

Acknowledgments. This work profited from financial support of the SFB 288. H. S.-B. would like to thank T. Kottos for pointing out references [DLA] and [ST].

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Communicated by Yosi Avron
submitted 15/03/04, accepted 23/04/04