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On the Vacuum Polarization Density Caused by an External Field

Christian Hainzl

Abstract. We consider an external potential, $-\lambda\varphi$, due to one or more nuclei. Following the Dirac picture such a potential polarizes the vacuum. The polarization density, $\rho^\lambda_{\rm vac}$, as derived in physics literature, after a well-known renormalization procedure, depends decisively on the strength of λ. For small λ, more precisely as long as the lowest eigenvalue, $e_1(\lambda)$, of the corresponding Dirac operator stays in the gap of the essential spectrum, the integral over the density $\rho^\lambda_{\rm vac}$ vanishes. In other words the vacuum stays neutral. But as soon as $e_1(\lambda)$ dives into the lower continuum the vacuum gets spontaneously charged with charge $2e$. Global charge conservation implies that two positrons were emitted out of the vacuum, this is, a large enough external potential can produce electron-positron pairs.

We give a rigorous proof of that phenomenon.

1 Introduction

In 1934 Dirac and Heisenberg realized that accepting the Dirac picture of electrons filling up the negative energy states, called vacuum, consequently implies that a charged nucleus thrown into the vacuum causes a redistribution of the Dirac sea, an effect denoted as *vacuum polarization.* Uehling and Serber in 1935 [27, 25], long before standard renormalization procedure, demonstrated that such an indicated production of *virtual* electron-positron pairs give rise to a modification of the Coulomb potential and thus causes energy shifts of bound electrons.

Concerning the traditional Lamb shift, known as the splitting of the $2s_{1/2}$- and $2p_{1/2}$-state in hydrogen, this effect only accounts for about 2.5 percent. However the Uehling potential represents the dominating radiative correction in muonic atoms which emphasizes the importance of vacuum polarization (VP). Notice, whereas interaction with a photon field can be treated non-relativistically there is *no* non-relativistic equivalence for VP. It is a purely relativistic effect.

Within the framework of QED, VP is treated by means of perturbation theory as developed by Dyson, Feynman, and Schwinger.

Only recently Hainzl and Siedentop demonstrated in [11] that the effective one-particle Hamiltonian obtained from VP can be handled non-perturbatively and gives rise to a self-adjoint operator. The effective potential we gain is in fact the same as the physicists obtain after mass and charge renormalization (neglecting photon terms) and use to calculate the hyperfine structure of bound states. We refer to [20, Section 4] for a nice review concerning the influence of VP on the Lamb shift of heavy atoms.

The main goal of the present paper is to study the vacuum polarization density caused by an external field, i.e., by one or more nuclei. As foreseen by physicists, e.g., [10, 9], the behavior of the density turns out to depend on the lowest eigenvalue of the corresponding Dirac operator. As long as this eigenvalue stays isolated the integral over the density vanishes, that means the vacuum stays neutral. But as soon as that eigenvalue touches the lower continuum the vacuum gets spontaneously charged, i.e., an electron, more precisely two electrons due to degeneracy of the "ground state", are trapped in the vacuum and two positrons are emitted. In other words large fields can produce electron-positron pairs. Such a situation can be realized by heavy ion collision.

1.1 Model

The free Dirac operator is given by

$$D^0 := \boldsymbol{\alpha} \cdot \frac{1}{i} \nabla + \beta \tag{1}$$

in which $\boldsymbol{\alpha}, \beta$ denote the 4×4 Dirac matrices. The underlying Hilbert space is given by $\mathfrak{H} = L^2(\Gamma)$ with $\Gamma = \mathbb{R}^3 \times \{1, 2, 3, 4\}$. We pick units in which the electron mass is equal to one. We regard the case of one, or more, smeared nuclei with density $n \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, non-negative, and assuming $\int_{\mathbb{R}^3} n = 1$.

We remark that it is an experimental fact that the nucleus cannot shrink to a point. In fact a point nucleus creates instability if one includes polarization effects, as shown in [11, Section 3.5].

The corresponding electric potential reads

$$\varphi = |\cdot|^{-1} * n. \tag{2}$$

and the operator to be studied is given by

$$D^{\lambda\varphi} := D^0 - \lambda\varphi, \tag{3}$$

where $\lambda \geq 0$ is a parameter and can be thought of as αZ, α the fine structure constant, $e := -\sqrt{\alpha}$ the charge of an electron, and $-Ze$ the charge of the nucleus (nuclei). In the following we want to allow any value of λ.

Due the smearing out of the Coulomb singularity the case of large values of λ does not influence the behavior of the essential spectrum as well as the self-adjointness as it would be the case of the Coulomb potential. The following Lemma is well known, e.g., Weidmann [28, Theorem 10.37].

Lemma 1 *Let* $\varphi = |\cdot|^{-1} * n$, $n \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$*, non-negative. Then,* $\forall \lambda \geq 0$, $D^{\lambda\varphi} = D^0 - \lambda\varphi$ *is self-adjoint with domain* $H^1(\Gamma)$ *and the essential spectrum of* $D^{\lambda\varphi}$ *is given by*

$$\sigma_{\text{ess}}(D^{\lambda\varphi}) = (-\infty, -1] \cup [1, \infty). \tag{4}$$

Throughout the paper we will denote the spectrum of $D^{\lambda\varphi}$ by $\sigma(D^{\lambda\varphi})$ and $e_i(\lambda)$ as the corresponding eigenvalues.

The following is well known: For fixed λ there is an infinite number of eigenvalues which accumulate at 1 and each $e_i(\lambda)$ depends continuously on λ. For small values of λ all eigenvalues stay in the gap $(-1, 1)$ of the essential spectrum. However, for each i one finds a λ_i such that for $\lim_{\lambda \to \lambda_i} e_i(\lambda) = -1$, i.e., the eigenvalue $e_i(\lambda)$ dives into the lower continuum. We do not at all discuss what happens to the eigenvalues after reaching the continuum. In fact one knows from [2] that below -1 there are no embedded eigenvalues. However, the behavior of the eigenvalues after reaching $(-\infty, -1]$ won't play any role. Our theorems only depend on the *number of eigenvalues, counting multiplicity, that vanish in the lower continuum.* Namely, due to our assumption $\varphi \geq 0$, all eigenvalues are monotonously decreasing (this is a consequence of [23, Theorem XII.13] and the fact that each eigenvalue has non-positive derivative). That means they will not reappear after having reached -1. The fact that each eigenvalue reaches -1 for a large enough parameter can be seen by, e.g., a Theorem of Dolbeaut-Esteban-Séré [3].

We will see that whenever an eigenvalue dives into the "sea of occupied states", i.e., $(\infty, -1]$, a specific number of $e^- e^+$ pairs are created depending on the degeneracy of the dived eigenvalue.

1.2 Vacuum polarization density

As already mentioned above, according to Dirac the vacuum consists of electrons occupying the negative energy states of the free Dirac operator. If one puts a nucleus into the vacuum, then the electrons rearrange and one ends up with virtual electron-positron pairs. In other words the vacuum gets polarized, see e.g., [9, page 257], for a picture describing this phenomenon, and [11] for a "mathematical" derivation of the vacuum polarization density, which follows the idea of the early papers in QED [5, 12, 29, 17, 8]. For a review about the old-fashioned way of QED we refer to [18].

The operator describing this polarization effect is given by

$$Q^{\lambda\varphi} := P_-^{\lambda\varphi} - P_-^0, \tag{5}$$

where

$$P_-^{\lambda\varphi} := \chi_{(-\infty,-1]}(D^{\lambda\varphi}). \tag{6}$$

Physically speaking we project onto the occupied states of the Dirac sea.

Remark 1 Notice, in the case that the lowest eigenvalue of $D^{\lambda\varphi}$, $e_1(\lambda)$, is strictly positive, our definition is equivalent to [11, Equation (12)], apart from a minus sign which is chosen to adapt to the definition in the physics literature.

Usually the first idea to define a density via $Q^{\lambda\varphi}$ would simply be taking the diagonal of the Kernel. Unfortunately, the operator $Q^{\lambda\varphi}$ is not trace class. The question how to extract from $Q^{\lambda\varphi}$ a physically meaningful density was first posed in the thirties by Dirac [4, 5] and Heisenberg [12] and in more recent literature this procedure is known as *charge renormalization* (see e.g., [8, 6]). As in [11] we use Cauchy's formula to express the $Q^{\lambda\varphi}$ in terms of the respective resolvents (Kato [14], Section VI,5, Lemma 5.6)

$$\begin{aligned} Q^{\lambda\varphi} = P_-^{\lambda\varphi} - P_-^0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \left(\frac{1}{D^0-\gamma+i\eta} - \frac{1}{D^{\lambda\varphi}-\gamma+i\eta}\right) \\ &:= P_\gamma^{\lambda\varphi} - P_\gamma^0, \end{aligned} \tag{7}$$

where

$$P_\gamma^{\lambda\varphi} := \frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{D^{\lambda\varphi}-\gamma+i\eta}, \tag{8}$$

with $-1 < \gamma < e_j(\lambda)$, $e_j(\lambda)$ being the lowest *isolated* eigenvalue of $D^{\lambda\varphi}$. Notice that the second equality in (7) is a consequence of the fact that $\frac{1}{D^0-z}$ is holomorphic with respect to z in the complex strip between $(-1,1)$ and $\frac{1}{D^{\lambda\varphi}-z}$ between $(-1, e_j(\lambda))$.

We decompose $Q^{\lambda\varphi}$ into 4 terms:

$$Q^{\lambda\varphi} = \lambda Q_1 + \lambda^2 Q_2 + \lambda^3 Q_3 + \lambda^4 Q_4^\lambda, \tag{9}$$

where

$$\begin{aligned} Q_1 &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}, \\ Q_2 &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}, \\ Q_3 &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}, \\ Q_4^\lambda &:= \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^{\lambda\varphi}-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}. \end{aligned} \tag{10}$$

The first three terms we consider by means of its Fourier representation. A simple variable transform $i\eta \to i\eta + \gamma$ does not change the kernel of the operators $\hat{Q}_1$ to $\hat{Q}_3$ which is the reason why we suppressed the γ in the denominator. The first term is treated in detail in [11, Section 3.2]. There, by a well known renormalization procedure following Weisskopf [29] and Pauli and Rose [21], we extracted the corresponding physical density

$$\rho_1^\lambda(x) := e\lambda\mathcal{F}^{-1}\left[\frac{4\pi\hat{n}(k)C(k)}{|k|^2}\right](x), \tag{11}$$

where (see [11, Equation (21)])

$$
\begin{aligned}
C(k)/k^2 &= \frac{1}{2}\int_0^1 dx(1-x^2)\log[1+k^2(1-x^2)/4] \\
&= \frac{1}{3}\left[(1-\frac{2}{k^2})\sqrt{1+\frac{4}{k^2}}\log\frac{\sqrt{1+4/k^2}+1}{\sqrt{1+4/k^2}-1}+\frac{4}{k^2}-\frac{5}{3}\right],
\end{aligned}
\tag{12}
$$

which was first explicitly written down by Uehling [27] and Serber [25] and later by Schwinger [24] and others (see also [13, 16, 10]). Observe in [11, Eq. (52)] that renormalization consists of subtracting an operator with infinite diagonal from Q_1. From what remains one defines, in (11), the diagonal corresponding to Q_1. This subtraction reflects the main difficulty concerning the proof of our main Theorem.

The second and third term in (10) have a well-defined integrable diagonal when using the Fourier representation. Additionally the density corresponding to Q_2 vanishes, either through integration over η or due to the fact that the Dirac matrices are traceless. Quite generally, if we expand $\mathrm{tr}_{\mathbb{C}^4} Q_4^\lambda$ into an infinite sum, each term with an even number of φ vanishes.

The density corresponding to Q_3 is given by

$$
\rho_3^\lambda(x) := e\lambda^3(2\pi)^{-3}\int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \sum_{\sigma=1}^{4} e^{i(p-q)\cdot x}\hat{Q}_3(p,\sigma;q,\sigma) \tag{13}
$$

where $\hat{Q}_3$ denotes the kernel of the Fourier representation

$$
\begin{aligned}
\hat{Q}_3(p,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \int_{\mathbb{R}^3} dp_1 \int_{\mathbb{R}^3} dp_2 (D_p+i\eta)^{-1}\circ\hat{\varphi}(p-p_1)\circ(D_{p_1}+i\eta)^{-1} \\
\circ\hat{\varphi}(p_1-p_2)\circ(D_{p_2}+i\eta)^{-1}\circ\hat{\varphi}(p_2-q)\circ(D_q+i\eta)^{-1}
\end{aligned}
\tag{14}
$$

with $D_r := \boldsymbol{\alpha}\cdot r+\beta$. Since Q_3 might not be trace class, we define, for simplicity, $\int \rho_3^\lambda(x)dx := e\lambda^3 \int \mathrm{tr}_{\mathbb{C}^4}\hat{Q}_3(p,p)dp$.

The operator Q_4^λ will be shown to be trace class in Lemma 3, so we can define ρ_4^λ quite general via the diagonal of $\lambda^4 \mathrm{tr}_{\mathbb{C}^4} Q_4^\lambda$,

$$
\rho_4^\lambda(x) := e\lambda^4 \mathrm{tr}_{\mathbb{C}^4} Q_4^\lambda(x,x) \tag{15}
$$

Therefore the *renormalized* density reads

$$
\rho_{\mathrm{vac}}^\lambda(x) := \rho_1^\lambda(x)+\rho_3^\lambda(x)+\rho_4^\lambda(x). \tag{16}
$$

Before formulating our main theorem it is necessary to introduce the counting function $d(\lambda)$ which counts the number of eigenvalues which dived in the lower continuum for parameters smaller equal λ.

$$
d(\bar{\lambda}) := \{\#\text{eigenvalues, with multiplicity, that reached } -1 \text{ for parameters } \lambda \le \bar{\lambda}\}. \tag{17}
$$

Theorem 1 *Let $n \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and non-negative, $\varphi = n * \frac{1}{|\cdot|}$. Then*

$$\int_{\mathbb{R}^3} \rho^\lambda_{\text{vac}}(x)dx = ed(\lambda). \tag{18}$$

Theorem 1 exactly reflects the picture which is presented by physicists, e.g., Greiner et al. [9, 10]:

As long as the external potential, respectively λ, is so weak that the lowest eigenvalue, $e_1(\lambda)$, of $D^{\lambda\varphi}$ is in the gap $(-1, 1)$ the vacuum stays neutral (and consists only of *virtual* electron-positron pairs). As soon as the lowest eigenvalue dives into the essential spectrum, $(-\infty, -1]$, i.e., the sea of occupied states, the vacuum immediately gets charged with charge $2e$ (assuming that the ground state energy of $D^{\lambda\varphi}$ is twice degenerate, due to the spin). This can be interpreted in the following way: when the unoccupied bound state dives in the sea of occupied states it traps two electrons which stay in the potential well of the nucleus (nuclei). Due to Dirac's picture two "holes" emerge which are repelled and emitted as positrons out of the vacuum. *Consequently we end up with real electron-positron, e^-e^+, pairs.*

This effect of spontaneously emitted positrons is verified in experiment by collision of heavy nuclei, which when approaching each other create an effective field strong enough to let the lowest eigenvalue dive into the continuum (see [22]).

Remark 2 In more recent physics literature, compare, e.g., [9, Equation (7.23)] or [20, Equation (230)], the VP-density is "formally" denoted as the diagonal of the operator

$$\frac{e}{2}\text{tr}_{\mathbb{C}^4}[P^{\lambda\varphi}_- - P^{\lambda\varphi}_+], \tag{19}$$

with $P^{\lambda\varphi}_+ := 1 - P^{\lambda\varphi}_-$. Since $\text{tr}_{\mathbb{C}^4}[P^0_+ - P^0_-] = 0$ and $-P^{\lambda\varphi}_+ + P^0_+ = P^{\lambda\varphi}_- - P^0_-$ we see that (19) coincides with our initial operator $e\,\text{tr}_{\mathbb{C}^4}[P^{\lambda\varphi}_- - P^0_-]$.

The proof of Theorem 1 will mainly be based on two ingredients: A work of Avron, Seiler, and Simon [1] concerning the index of pairs of projectors (see also [7]) and arguments of Kato [14].

The proof of Theorem 1 will be given in Section 3. In Section 2 we show that for $\text{tr}[P^{\lambda\varphi}_- - P^0_-]^{2m+1}$, $m \geq 1$, a result similar to (18) holds.

2 Result on $\text{tr}[P^{\lambda\varphi}_- - P^0_-]^{2m+1}$, with $m \geq 1$

Recall that the vacuum polarization is in fact described by the operator $Q^{\lambda\varphi} = P^{\lambda\varphi}_- - P^0_-$. Renormalization is inevitable, since that operator is not trace class. Nevertheless, due to Klaus and Scharf [15] it is at least an Hilbert-Schmidt operator. Due to [1] (in fact this follows already from Effros [7]) the traces of $\left(Q^{\lambda\varphi}\right)^{2m+1}$, $m \geq 1$, are equal. Therefore it is self-evident to ask for their behavior.

Theorem 2 *Let $n \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, non-negative, $\varphi = n * \frac{1}{|\cdot|}$. Then, $\forall m \geq 1$,*

$$\mathrm{tr}[P_-^{\lambda\varphi} - P_-^0]^{2m+1} = d(\lambda), \tag{20}$$

where $d(\lambda)$ is defined as in (17).

Proof. Notice that, since $\hat{\varphi}(k) = \hat{n}(k)\frac{4\pi}{k^2}$,

$$\int_{\mathbb{R}^3} dk \frac{k^2 \log(2+|k|)|\hat{\varphi}(k)|^2}{1+|k|} \leq \||\hat{n}|^2\|_p \left\| \frac{\log(2+|\cdot|)}{|\cdot|^2(1+|\cdot|)} \right\|_q . \tag{21}$$

Take $q = \frac{4}{3}$, $p = 4$, then the second term on the right-hand side is finite, as well as by Hausdorff-Young inequality

$$\||\hat{n}|^2\|_4 = \|\hat{n}\|_8^2 \leq C_{8/7}^3 \|n\|_{8/7}^2 < \infty. \tag{22}$$

Therefore the potential φ is regular in the sense of Klaus and Scharf [15], cf. [19, Equation (1.7)], namely the operator $Q^{\lambda\varphi} \in \mathfrak{S}_2(\mathfrak{H})$, i.e., $Q^{\lambda\varphi}$ is a Hilbert-Schmidt operator. Consequently $Q^{\lambda\varphi} \in \mathfrak{S}_\mathfrak{m}(\mathfrak{H})$ for any $m \geq 2$.

To prove the Theorem we first look at the set of all $\lambda \geq 0$ such that the lowest eigenvalue, $e_1(\lambda)$, corresponding to $D^{\lambda\varphi}$ fulfills

$$e_1(\lambda) > -1. \tag{23}$$

This is an open set so that we can always find a γ, with $-1 < \gamma < e_1(\lambda)$ and

$$Q^{\lambda\varphi} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \left(\frac{1}{D^0 - \gamma + i\eta} - \frac{1}{D^{\lambda\varphi} - \gamma + i\eta} \right) = P_\gamma^{\lambda\varphi} - P_\gamma^0 \tag{24}$$

We are going to show that for $m \geq 1$

$$\mathrm{tr}[P_\gamma^{\lambda\varphi} - P_\gamma^0]^{2m+1} = 0 \tag{25}$$

on the set $\{\lambda | e_1(\lambda) > \gamma\}$. Since γ can be chosen arbitrarily close to -1 we infer that

$$\mathrm{tr}[P_-^{\lambda\varphi} - P_-^0]^{2m+1} = 0 \tag{26}$$

on $\{\lambda | e_1(\lambda) > -1\}$.

To this aim we recall some results from Avron, Seiler, and Simon [1] (see also [7]) concerning the index of pairs of projections:

Regard the family of orthogonal projections $P_\gamma^{\lambda\varphi}$, $\lambda \geq 0$. Since $P_\gamma^{\lambda\varphi} - P_\gamma^{\mu\varphi} \in \mathfrak{S}_2(\mathfrak{H})$, [1, Proposition 3.2] implies that all pairs $(P_\gamma^{\lambda\varphi}, P_\gamma^{\mu\varphi})$ are Fredholm. Combining [1, Theorem 3.1] and [1, Theorem 4.1] we obtain that for $m, l \geq 1$

$$\begin{aligned} \mathrm{tr}\left[P_\gamma^{\lambda\varphi} - P_\gamma^{\mu\varphi}\right]^{2m+1} &= \mathrm{tr}\left[P_\gamma^{\lambda\varphi} - P_\gamma^{\mu\varphi}\right]^{2l+1} = \mathrm{ind}(P_\gamma^{\lambda\varphi}, P_\gamma^{\mu\varphi}) \\ &= \dim(\mathrm{Ker} P_\gamma^{\mu\varphi} \cap \mathrm{Ran} P_\gamma^{\lambda\varphi}) - \dim(\mathrm{Ker} P_\gamma^{\lambda\varphi} \cap \mathrm{Ran} P_\gamma^{\mu\varphi}) \end{aligned} \tag{27}$$

is an integer.

Remark 3 More generally, a pair (P, Q) of orthogonal projections is called *Fredholm*, if the operator $T = QP$, as an operator from $\mathrm{Ran}P \to \mathrm{Ran}Q$, is Fredholm. The corresponding index $\mathrm{ind}(P, Q)$ is defined as

$$\mathrm{ind}(P,Q) := \mathrm{ind}\, T = \dim(\mathrm{Ker}T) - \dim(\mathrm{Ran}T)^{\perp}. \tag{28}$$

Next we come back to the proof of (25). Observe that on $\{\lambda|e_1(\lambda) > \gamma\}$ $P_\gamma^{\lambda\varphi}$ is a continuous family with respect to the operator norm. Namely, using (8) and expanding the resolvent we get

$$\begin{aligned}\|P_\gamma^{\lambda\varphi} - P_\gamma^{\mu\varphi}\| &\le |\lambda-\mu|\|\varphi\| \int_{-\infty}^{\infty} d\eta \|\frac{1}{D^{\mu\varphi} - \gamma + i\eta}\|\|\frac{1}{D^{\lambda\varphi} - \gamma + i\eta}\| \\ &\le |\lambda-\mu|\|\varphi\| \int_{-\infty}^{\infty} d\eta \frac{1}{(\delta_1^2+\eta^2)^{1/2}(\delta_2^2+\eta^2)^{1/2}},\end{aligned} \tag{29}$$

with $\delta_1 := \min\{\gamma+1, e_1(\mu)-\gamma\}$ and $\delta_2 := \min\{\gamma+1, e_1(\lambda)-\gamma\}$. Notice that since $e_1(\lambda)$ is continuous the integral in the right-hand side of (29) can be bounded uniformly on a small enough closed neighborhood of each λ in $\{\lambda|e_1(\lambda) > \gamma\}$.

Due to (27), $\|P_\gamma^{\lambda\varphi} - P_\gamma^{\mu\varphi}\| < 1$ implies $\mathrm{ind}(P_\gamma^{\lambda\varphi}, P_\gamma^{\mu\varphi}) = 0$. Using [1, Theorem 3.4 (c)],

$$\mathrm{ind}(P_\gamma^{\lambda\varphi}, P_\gamma^0) = \mathrm{ind}(P_\gamma^{\lambda\varphi}, P_\gamma^{\mu\varphi}) + \mathrm{ind}(P_\gamma^{\mu\varphi}, P_\gamma^0), \tag{30}$$

the continuity of $P_\gamma^{\lambda\varphi}$ immediately gives that $\mathrm{ind}(P_\gamma^{\lambda\varphi}, P_\gamma^0) = 0$ on the whole set $\{\lambda|e_1(\lambda) > \gamma\}$. Together with (27) we arrive at (25).

Summarizing, the argument given above was based on the fact that on the set $\{\lambda|e_1(\lambda) > -1\}$, $P_\gamma^{\lambda\varphi}$ can be *continuously deformed* into P_γ^0. Throughout the rest of the paper we will repeat this argument several times.

In the following we consider the case that an eigenvalue has dived into the lower continuum. We know that there are no eigenvalues below -1. However, for notational simplification we treat them as if they stay embedded.

Fix now $\bar\lambda$ such that $e_1(\bar\lambda) \le -1$ and $e_2(\bar\lambda) > -1$, and γ with $-1 < \gamma < e_2(\bar\lambda)$. Additionally we choose a $\lambda' < \bar\lambda$ such that $-1 < e_1(\lambda') < \gamma$ and a γ' with $-1 < \gamma' < e_1(\lambda')$. We know

$$\mathrm{tr}\left(Q^{\bar\lambda\varphi}\right)^{2m+1} = \mathrm{tr}[P_\gamma^{\bar\lambda\varphi} - P_\gamma^0]^{2m+1} = \mathrm{ind}(P_\gamma^{\bar\lambda\varphi}, P_\gamma^0). \tag{31}$$

Due to [1, Theorem 3.4 (c)]

$$\mathrm{ind}(P_\gamma^{\bar\lambda\varphi}, P_\gamma^0) = \mathrm{ind}(P_\gamma^{\bar\lambda\varphi}, P_\gamma^{\lambda'\varphi}) + \mathrm{ind}(P_\gamma^{\lambda'\varphi}, P_{\gamma'}^{\lambda'\varphi}) + \mathrm{ind}(P_{\gamma'}^{\lambda'\varphi}, P_{\gamma'}^0). \tag{32}$$

The first and third term in the right-hand side in (32) vanish which can be seen by repeating the argument given above. Namely due to our choice of parameters $P_\gamma^{\bar\lambda\varphi}$ can be continuously deformed into $P_\gamma^{\lambda'\varphi}$. As well $P_{\gamma'}^{\lambda'\varphi}$ can be continuously deformed into $P_{\gamma'}^0$, which equals P_γ^0.

Concerning the second term in the right-hand side of (32) we note that by Cauchy's formula we obtain

$$P_{\gamma}^{\lambda'\varphi} - P_{\gamma'}^{\lambda'\varphi} = P_{e_1(\lambda')} \tag{33}$$

where $P_{e_1(\lambda')}$ is the projector on the eigenspace corresponding to the eigenvalue $e_1(\lambda')$. Consequently

$$\operatorname{ind}(P_{\gamma}^{\lambda'\varphi}, P_{\gamma'}^{\lambda'\varphi}) = \operatorname{tr}[P_{e_1(\lambda')}]. \tag{34}$$

By means of our definition (17) of $d(\lambda)$, obviously $\operatorname{tr}[P_{e_1(\lambda')}] = d(\bar{\lambda})$, whence

$$\operatorname{tr}\left(Q^{\bar{\lambda}\varphi}\right)^{2m+1} = d(\bar{\lambda}). \tag{35}$$

Repeating this argument whenever an eigenvalue dives into the lower continuum, $(-\infty, -1]$, we arrive at the statement of the theorem.

Notice, due to continuity in λ the argument works no matter how many eigenvalues "meet" at -1. □

3 Proof of Theorem 1

Summarizing the proof of Theorem 2, we exploited the fact that $P_-^{\lambda\varphi}$ build a continuous family of projectors on the non-connected intervals

$$[0, \lambda_1) \cup (\lambda_1, \lambda_2) \cup \cdots \cup (\lambda_i, \lambda_{i+1}) \ldots \tag{36}$$

where λ_i denotes parameters where an eigenvalue reaches -1. As long as λ, μ belong to a connected interval the index of the corresponding projection vanishes,

$$\operatorname{ind}(P_-^{\lambda\varphi}, P_-^{\mu\varphi}) = 0, \tag{37}$$

but if λ moves to a different not connected interval the index jumps by an integer value.

In order to prove Theorem 1 we first recall the definition of the density

$$\rho_{\text{vac}}^{\lambda}(x) = \rho_1^{\lambda}(x) + \rho_3^{\lambda}(x) + \rho_4^{\lambda}(x), \tag{38}$$

the terms on the right-hand side being defined in (11), (13), (15). By means of our explicit choice of ρ_1^{λ} via Fourier transform $\hat{\rho}_1^{\lambda}(k) = e\lambda 4\pi\hat{n}\frac{C(k)}{k^2}$ and the fact that $\lim_{|k|\to 0} \frac{C(k)}{k^2} = 0$ we immediately obtain

$$\int_{\mathbb{R}^3} \rho_1^{\lambda}(x)dx = \hat{\rho}_1^{\lambda}(0) = 0. \tag{39}$$

Therefore, our goal in the following will be to show that for all λ

$$\int_{\mathbb{R}^3} \rho_3^{\lambda}(x)dx = 0, \qquad \int_{\mathbb{R}^3} \rho_4^{\lambda}(x)dx = ed(\lambda). \tag{40}$$

It still remains to show that Q_4^{λ} is trace class which works analogously to [11, Lemma 3].

Lemma 2

$$\operatorname{tr}|Q_4^\lambda| = \|Q_4^\lambda\|_1 \le C^\mu \|\varphi\|_4^4, \tag{41}$$

with an appropriate constant C^μ depending on $\mu := \min\{\gamma+1, e_i(\lambda)-\gamma\}$, $e_i(\lambda)$ denoting the lowest isolated eigenvalue of $D^{\lambda\varphi}$.

Proof. Let $e_i(\lambda)$ be the lowest isolated eigenvalue of $D^{\lambda\varphi}$, then as usually, we choose a γ with $-1<\gamma<e_i(\lambda)$. Using (10) we obtain (apart from a factor $\frac{1}{2\pi}$)

$$\begin{aligned}
&\|Q_4^\lambda\|_1 \\
&\le \int_{-\infty}^{\infty} d\eta \left\| \frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^{\lambda\varphi}-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\right\|_1 \\
&\le \int_{-\infty}^{\infty} d\eta \left\| \frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\right\|_1 \\
&\qquad\qquad \times \left\|(D^0-\gamma+i\eta)\frac{1}{D^{\lambda\varphi}-\gamma+i\eta}\right\|,
\end{aligned} \tag{42}$$

with $\|(D^0-\gamma+i\eta)\frac{1}{D^{\lambda\varphi}-\gamma+i\eta}\| \le 1+\lambda\|\varphi\frac{1}{D^{\lambda\varphi}-\gamma}\|$ which depends on μ. Moreover, with $\|(D^0+i\eta)\frac{1}{D^0-\gamma+i\eta}\| \le 1+\mu^{-1}$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\eta \left\| \frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}\right\|_1 \\
&\quad\le \int_{-\infty}^{\infty} d\eta \left\| \frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\right\|_1 (1+\mu^{-1})^5 \\
&\quad\le \int_{-\infty}^{\infty} d\eta \left\|\varphi\frac{1}{D^0+i\eta}\right\|_4^3 \left\|\varphi\frac{1}{D^0+i\eta}\frac{1}{D^0+i\eta}\right\|_4 (1+\mu^{-1})^5.
\end{aligned} \tag{43}$$

Applying an inequality of Simon [26, Theorem 4.1],

$$\|f(x)g(-i\nabla)\|_4 \le (2\pi)^{-3/4}\|f\|_4\|g\|_4, \tag{44}$$

to the factors in (43), gives

$$\begin{aligned}
\|\varphi\frac{1}{D^0+i\eta}\|_4 &\le \frac{1}{2^{1/4}\pi^{3/4}}\|\varphi\|_4\|1/\sqrt{|\cdot|^2+1+\eta^2}\|_4 \\
\|\varphi\frac{1}{D^0+i\eta}\frac{1}{D^0+i\eta}\|_4 &\le \frac{1}{2^{1/4}\pi^{3/4}}\|\varphi\|_4\|1/(|\cdot|^2+1+\eta^2)\|_4.
\end{aligned} \tag{45}$$

Putting all together and evaluating the integrals (cf. [11, Lemma 3]) we arrive at

$$\|Q_4^\lambda\|_1 \le C^\mu\|\varphi\|_4^4, \tag{46}$$

with an appropriate C^μ. □

In the following we will proceed analogously to the proof of Theorem 2. We will circumvent the problem that $P_-^{\lambda\varphi} - P_-^0$ is not trace class by defining a family of trace class operators K^ε converging strongly to φ. We define K^ε via its Fourier representation

$$\hat{K}^\varepsilon(p,q) := f_\varepsilon(p)\hat{\varphi}_\varepsilon(p-q)f_\varepsilon(q), \tag{47}$$

with

$$f_\varepsilon(p) := \chi(1/\varepsilon - |p|), \qquad \varphi_\varepsilon(x) := \varphi(x)\chi(1/\varepsilon - |x|), \tag{48}$$

χ denoting the Heaviside step function. Obviously $f_\varepsilon \to 1$ and $\varphi_\varepsilon \to \varphi$ pointwise when $\varepsilon \to 0$.

The family of operators

$$D^{\lambda K^\varepsilon} := D^0 - \lambda K^\varepsilon \tag{49}$$

turn out to converge strongly to $D^{\lambda\varphi}$. For convenience we define $Q^{\lambda \mathrm{K}^\varepsilon}$ via an appropriate γ chosen corresponding to $D^{\lambda\varphi}$,

$$Q^{\lambda \mathrm{K}^\varepsilon} := P_\gamma^{\lambda \mathrm{K}^\varepsilon} - P_\gamma^0, \tag{50}$$

leaving off the subscript γ, since it will not cause ambiguities. Furthermore $P_\gamma^{\lambda \mathrm{K}^\varepsilon} - P_\gamma^0$ will be trace class so we can repeat the arguments given in the proof of Theorem 2. Since we already removed the "bad" part of $P_-^{\lambda\varphi} - P_-^0$ by charge renormalization (in [11]), that is the part of Q_1 which prevents $P_\gamma^{\lambda\varphi} - P_\gamma^0$ from being trace class, it suffices to show that Q_3^ε (respectively $Q_4^{\lambda,\varepsilon}$) converge (in trace norm) to Q_3 (respectively Q_4^λ). Q_3^ε and $Q_4^{\lambda,\varepsilon}$ are terms we obtain by expanding (50).

Recall $\int \rho_3^\lambda(x)dx = e\lambda^3 \int_{\mathbb{R}^3} \mathrm{tr}_{\mathbb{C}^4}\hat{Q}_3(p,p)dp$.

First we state a few useful properties of K^ε.

Lemma 3 (a) *For all $\varepsilon > 0$, K^ε is trace class and $K^\varepsilon \geq 0$. Moreover, $\sigma_{\mathrm{ess}}(D^{\lambda K^\varepsilon}) = (-\infty,-1] \cup [1,\infty)$.*
(b) *$D^{\lambda K^\varepsilon} \to D^{\lambda\varphi}$ strongly as $\varepsilon \to 0$.*
(c) *$P_\gamma^{\lambda \mathrm{K}^\varepsilon} - P_\gamma^0$ is trace class for all $\varepsilon > 0$ if $\gamma \notin \sigma(D^{\lambda K^\varepsilon})$.*

Proof. (a) The fact that K^ε is trace class is a direct consequence of Lemma 5. In Fourier representation we can decompose

$$\hat{K}^\varepsilon(p,q) = L_\varepsilon^* L_\varepsilon(p,q), \tag{51}$$

with $L_\varepsilon(p,p') = f_\varepsilon(p)h_\varepsilon(p-p')$, $h_\varepsilon(p) = (2\pi)^{-3/4}\widehat{\sqrt{\varphi_\varepsilon}}(p)$. By our choice of f_ε and φ_ε

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |L_\varepsilon(p,q)|^2 dpdq < \infty \tag{52}$$

whence K^ε is trace class. Equation (51) immediately implies $K^\varepsilon \geq 0$. The compactness of K^ε yields $\sigma_{\mathrm{ess}}(D^{\lambda K^\varepsilon}) = (-\infty,-1] \cup [1,\infty)$ by Weyl's Theorem.

(b) For $\psi \in H^1(\Gamma)$

$$\lim_{\varepsilon\to 0} \|[D^{\lambda K^\varepsilon} - D^{\lambda\varphi}]\psi\| = \lambda \lim_{\varepsilon\to 0} \|[K^\varepsilon - \varphi]\psi\| = 0, \tag{53}$$

since $\hat{K^\varepsilon}^* \hat{K}^\varepsilon \to \hat\varphi * \hat\varphi$ in the sense of distributions, and these operators are bounded.
(c) Let $e_i^\varepsilon(\lambda)$ be the lowest isolated eigenvalue of $D^{\lambda K^\varepsilon}$. Then, with $-1 < \gamma < e_i^\varepsilon(\lambda)$,

$$\begin{aligned} P_\gamma^{\lambda \mathrm{K}^\varepsilon} - P_\gamma^0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \left(\frac{1}{D^0-\gamma+i\eta} - \frac{1}{D^{\lambda K^\varepsilon}-\gamma+i\eta}\right) \\ &= \lambda \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0-\gamma+i\eta} K^\varepsilon \frac{1}{D^{\lambda K^\varepsilon}-\gamma+i\eta}. \end{aligned} \tag{54}$$

Consequently

$$\|P_\gamma^{\lambda \mathrm{K}^\varepsilon} - P_\gamma^0\|_1 \le \lambda \|K^\varepsilon\|_1 \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{\left((1+\gamma)^2+\eta^2\right)^{1/2}\left(\bar\delta^2+\eta^2\right)^{1/2}}, \tag{55}$$

with $\bar\delta := \min\{\gamma+1, e_i^\varepsilon(\lambda)-\gamma\}$ using $\|\frac{1}{D^{\lambda K^\varepsilon}-\gamma+i\eta}\| \le (\bar\delta^2+\eta^2)^{-1/2}$ and $\|\frac{1}{D^0-\gamma+i\eta}\| \le \left((1+\gamma)^2+\eta^2\right)^{-1/2}$. □

Let us fix an arbitrary λ such that $e_1(\lambda) > -1$ and γ with $-1 < \gamma < e_1(\lambda)$. Since $D^{\lambda K^\varepsilon} \to D^{\lambda\varphi}$ strongly, a Theorem of Kato [14, VIII-5, Theorem 5.1] tells us that $\sigma(D^{\lambda K^\varepsilon})$ is asymptotically concentrated in any open set containing $\sigma(D^{\lambda\varphi})$. Thus we can find a δ small enough such that $\gamma < e_1(\lambda) - \delta$ and a corresponding ε_0 such that for all $\varepsilon \le \varepsilon_0$, $\sigma(D^{\lambda K^\varepsilon})$ is concentrated in a δ-neighborhood of $\sigma(D^{\lambda\varphi})$, in particular $e_i^\varepsilon(\lambda) > e_1(\lambda) - \delta$ for each eigenvalue $e_i^\varepsilon(\lambda)$ of $D^{\lambda K^\varepsilon}$.

Thus we are able to guarantee that $P_\gamma^{\lambda \mathrm{K}^\varepsilon}$ can be continuously deformed into P_γ^0. Therefore we can argue analogously to the proof of Theorem 2 combined with the trace class property of $P_\gamma^{\lambda \mathrm{K}^\varepsilon} - P_\gamma^0$ to obtain

$$\mathrm{tr}[P_\gamma^{\lambda \mathrm{K}^\varepsilon} - P_\gamma^0] = \mathrm{ind}(P_\gamma^{\lambda \mathrm{K}^\varepsilon}, P_\gamma^0) = 0. \tag{56}$$

Expanding the resolvent this implies

$$\begin{aligned} 0 = \lambda\,\mathrm{tr}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0+i\eta}K^\varepsilon\frac{1}{D^0+i\eta} + \lambda^2\,\mathrm{tr}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0+i\eta}K^\varepsilon\frac{1}{D^0+i\eta}K^\varepsilon\frac{1}{D^0+i\eta} \\ + \lambda^3\,\mathrm{tr}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0+i\eta}K^\varepsilon\frac{1}{D^0+i\eta}K^\varepsilon\frac{1}{D^0+i\eta}K^\varepsilon\frac{1}{D^0+i\eta} + \\ \lambda^4\,\mathrm{tr}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0-\gamma+i\eta}K^\varepsilon\frac{1}{D^0-\gamma+i\eta}K^\varepsilon \times \\ \times \frac{1}{D^{\lambda K^\varepsilon}-\gamma+i\eta}K^\varepsilon\frac{1}{D^0-\gamma+i\eta}K^\varepsilon\frac{1}{D^0-\gamma+i\eta} \\ := \lambda\,\mathrm{tr}\,Q_1^\varepsilon + \lambda^2\,\mathrm{tr}\,Q_2^\varepsilon + \lambda^3\,\mathrm{tr}\,Q_3^\varepsilon + \lambda^4\,\mathrm{tr}\,Q_4^{\lambda,\varepsilon}. \end{aligned} \tag{57}$$

Observe, this holds in particular in a small neighborhood of 0. Thus, since the fourth term on the right-hand side is of order $O(\lambda^4)$, each term in (57) vanishes separately. In particular we have:

$$\operatorname{tr} Q_3^\varepsilon = 0, \quad \operatorname{tr} Q_4^{\lambda,\varepsilon} = 0, \tag{58}$$

the latter one on the set $\{\lambda | e_1(\lambda) > -1\}$.

Assume for a moment we have already shown

$$e\lambda^3 \lim_{\varepsilon\to 0} \operatorname{tr} Q_3^\varepsilon = \int_{\mathbb{R}^3} \rho_3^\lambda(x)dx, \quad e\lambda^4 \lim_{\varepsilon\to 0} \operatorname{tr} Q_4^{\lambda,\varepsilon} = \int_{\mathbb{R}^3} \rho_4^\lambda(x)dx, \tag{59}$$

then by (58) obviously $\int_{\mathbb{R}^3} \rho_3^\lambda(x)dx = 0$ and $\int_{\mathbb{R}^3} \rho_4^\lambda(x)dx = 0$ whence Theorem 1 on $\{\lambda | e_1(\lambda) > -1\}$.

In order to prove (59) we formulate an auxiliary Lemma:

Lemma 4 (a) *There exists a non-negative function $g \in L^1(\mathbb{R}^3)$, such that*

$$\left| \operatorname{tr}_{\mathbb{C}^4} \hat{Q}_3^\varepsilon(p,p) \right| \le g(p) \tag{60}$$

uniformly in ε.

(b) *Let $e_i(\lambda)$ be the lowest isolated eigenvalue of $D^{\lambda\varphi}$. Fix γ and $\bar{\delta}$ with $-1+\bar{\delta} < \gamma < e_i(\lambda) - \bar{\delta}$. Furthermore fix ε_0 such that for all $\varepsilon \le \varepsilon_0$, $\sigma(D^{\lambda K^\varepsilon})$ is in a $\bar{\delta}$-neighborhood of $\sigma(D^{\lambda\varphi})$. Then*

$$\|Q_4^{\lambda,\varepsilon}\|_1 \le C^\mu \|\varphi\|_4^4 \tag{61}$$

uniformly in $\varepsilon \le \varepsilon_0$, where C^μ is an appropriate constant depending on $\mu := \min\{\gamma + 1 - \bar{\delta}, e_i(\lambda) - \bar{\delta} - \gamma\}$, and

$$\lim_{\varepsilon\to 0} \operatorname{tr} Q_4^{\lambda,\varepsilon} = \operatorname{tr} Q_4^\lambda. \tag{62}$$

Proof. (a) We will proceed similarly to [11, Lemma 4]. For completeness we will repeat some parts of the proof. The "eigenfunctions" of the free Dirac operator in momentum space are

$$u_\tau(p) := \begin{cases} \frac{1}{N_+(p)} \begin{pmatrix} \sigma \cdot p\, \mathbf{e}_\tau \\ -(1 - E(p))\mathbf{e}_\tau \end{pmatrix} & \tau = 1, 2, \\ \frac{1}{N_-(p)} \begin{pmatrix} \sigma \cdot p\, \mathbf{e}_\tau \\ -(1 + E(p))\mathbf{e}_\tau \end{pmatrix} & \tau = 3, 4 \end{cases} \tag{63}$$

with $\mathbf{e}_\tau := (1,0)^t$ for $\tau = 1,3$ and $\mathbf{e}_\tau := (0,1)^t$ for $\tau = 2,4$ and

$$N_+(p) = \sqrt{2E(p)(E(p)-1)}, \quad N_-(p) = \sqrt{2E(p)(E(p)+1)}. \tag{64}$$

The indices 1 and 2 refer to positive "eigenvalue" $E(p)$ and the indices 3 and 4 to negative $-E(p)$. Using Plancherel's theorem we get

$$\begin{aligned}\mathrm{tr}_{\mathbb{C}^4}\hat{Q}_3^\varepsilon(p,p) = \sum_{\tau_0=1}^{4}\langle u_{\tau_0}(p)|\hat{Q}_3^\varepsilon|u_{\tau_0}(p)\rangle = \frac{1}{(2\pi)^4}\int_{\mathbb{R}^3}dp_1\int_{\mathbb{R}^3}dp_2\sum_{\tau_0,\tau_1,\tau_2,=1}^{4} \\ \times\langle u_{\tau_0}(p)|\hat{K}^\varepsilon|u_{\tau_1}(p_1)\rangle\langle u_{\tau_1}(p_1)|\hat{K}^\varepsilon|u_{\tau_2}(p_2)\rangle\langle u_{\tau_2}(p_2)|\hat{K}^\varepsilon|u_{\tau_0}(p)\rangle \\ \times\int_{-\infty}^{\infty}d\eta\frac{1}{(ia_{\tau_0}E(p)-\eta)(ia_{\tau_1}E(p_1)-\eta)(ia_{\tau_2}E(p_2)-\eta)(ia_{\tau_0}E(p)-\eta)},\end{aligned} \tag{65}$$

with $a_\tau = 1$ for $\tau = 1, 2$ and $a_\tau = -1$ for $\tau = 3, 4$. The integral over η is seen to vanish by Cauchy's theorem, if all four a_{τ_j} have the same sign. In fact we only treat one case. The others then work analogously.

Set

$$a_{\tau_2} = -1, \;\; a_{\tau_0} = a_{\tau_1} = 1. \tag{66}$$

Using $f_\varepsilon \leq 1$ the first factor in (65) can be estimated by

$$\begin{aligned}&\sum_{\tau_0=1,2}\langle u_{\tau_0}(p)|\hat{K}^\varepsilon|u_{\tau_1}(p_1)\rangle\sum_{\tau_1=1,2}\langle u_{\tau_1}(p_1)|\hat{K}^\varepsilon|u_{\tau_2}(p_2)\rangle\sum_{\tau_2=3,4}\langle u_{\tau_2}(p_2)|\hat{K}^\varepsilon|u_{\tau_0}(p)\rangle \\ &\leq|\hat{\varphi}(p-p_1)\hat{\varphi}(p_1-p_2)\hat{\varphi}(p_2-p)|\Big|\mathrm{tr}_{\mathbb{C}^2}\Big[\frac{\sigma\cdot p\sigma\cdot p_1+(1-E(p))(1-E(p_1))}{N_-(p_2)^2N_+(p)^2N_+(p_1)^2} \\ &\times\left[\sigma\cdot p_1\sigma\cdot p_2+(1-E(p_1))(1+E(p_2))\right]\left[\sigma\cdot p_2\sigma\cdot p+(1+E(p_2))(1-E(p))\right]\Big]\Big| \\ &\leq \mathrm{c}|\hat{\varphi}(p-p_1)\hat{\varphi}(p_1-p_2)\hat{\varphi}(p_2-p)|\frac{|p\cdot p_2-(E(p_2)-1)(1+E(p))|+|p\wedge p_2|}{N_-(p_2)N_+(p)}.\end{aligned} \tag{67}$$

(c being a generic constant.) Since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\eta\frac{1}{(iE(p)-\eta)(iE(p_1)-\eta)(-iE(p_2)-\eta)(iE(p)-\eta)} = \frac{1}{2(E(p)+E(p_1))^2E(p)} \tag{68}$$

our term of interest (65) is bounded by a constant times

$$\int_{\mathbb{R}^3}dp_1\int_{\mathbb{R}^3}dp_2|\hat{\varphi}(p-p_1)\hat{\varphi}(p_1-p_2)\hat{\varphi}(p_2-p)| \times\frac{|p\cdot p_2-(E(p_2)-1)(1+E(p))|+|p\wedge p_2|}{2N_-(p_2)N_+(p)(E(p)+E(p_1))^2E(p)}. \tag{69}$$

Substituting $p_2 \to p_2 + p$, $p_1 \to p_1 + p_2 + p$ we get

$$|(69)| \leq \int_{\mathbb{R}^3} dp_1 \int_{\mathbb{R}^3} dp_2 |\hat{\varphi}(p_1+p_2)\hat{\varphi}(p_1)\hat{\varphi}(p_2)| \\ \times \frac{|p\cdot(p_2+p) - (E(p_2+p)-1)(1+E(p))| + |p \wedge (p_2+p)|}{2N_-(p_2+p)N_+(p)(E(p)+E(p_1+p_2+p))^2 E(p)}. \quad (70)$$

Since

$$|p\cdot(p_2+p) - (E(p_2+p)-1)(1+E(p))| + |p\wedge p_2| \leq 4|p||p_2|$$

we obtain as an upper bound the function

$$\bar{g}(p) := \int_{\mathbb{R}^3} dp_1 \int_{\mathbb{R}^3} dp_2 |\hat{\varphi}(p_1+p_2)\hat{\varphi}(p_1)\hat{\varphi}(p_2)||p_2| \frac{1}{N_-(p_2+p)E(p)^3}, \quad (71)$$

which is obviously in $L^1(\mathbb{R}^3)$, whence (a) is proven.
(b) Analogously to (42) and (43) we get (apart from a constant)

$$\|Q_4^{\lambda,\varepsilon}\|_1 \leq \int_{-\infty}^{\infty} d\eta \\ \times \left\| \frac{1}{D^0-\gamma+i\eta} K^\varepsilon \frac{1}{D^0-\gamma+i\eta} K^\varepsilon \frac{1}{D^{\lambda K^\varepsilon}-\gamma+i\eta} K^\varepsilon \frac{1}{D^0-\gamma+i\eta} K^\varepsilon \frac{1}{D^0-\gamma+i\eta} \right\|_1 \\ \leq \int_{-\infty}^{\infty} d\eta \left\| \frac{1}{D^0+i\eta} K^\varepsilon \frac{1}{D^0+i\eta} K^\varepsilon \frac{1}{D^0+i\eta} K^\varepsilon \frac{1}{D^0+i\eta} K^\varepsilon \frac{1}{D^0+i\eta} \right\|_1 \\ \times \left(1 + \left\| K^\varepsilon \frac{1}{D^{\lambda K^\varepsilon}-\gamma} \right\|\right)(1+\mu^{-1})^5. \quad (72)$$

The first term in the third line is trace class, so we can evaluate it in Fourier representation. Since $f_\varepsilon \leq 1$ we are in the situation of Lemma 2 and end up with

$$\|Q_4^{\lambda,\varepsilon}\|_1 \leq \mathrm{c}\left(1 + \|K^\varepsilon\|\mu^{-1}\right)(1+\mu^{-1})^5 \|\varphi\|_4^4 \quad (73)$$

which implies (61) since $\|K^\varepsilon\|$ is uniformly bounded.

In order to prove (62) it suffices to show

$$\|Q_5^{\lambda,\varepsilon} - Q_5^{\lambda}\|_1 \to_{\varepsilon\to 0} 0,$$

since $\operatorname{tr} Q_4 = 0$, with

$$Q_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}\varphi\frac{1}{D^0+i\eta}. \quad (74)$$

In fact, due to Lemma 2, Q_4 is trace class and using the Fourier representation one easily sees that its trace vanishes. Indeed each operator Q_{2n} with an even number of potentials has vanishing trace, which is well known as Furry's Theorem. For convenience we denote

$$\begin{aligned} Q_5^{\lambda,\varepsilon} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta N_\eta^\varepsilon M_\eta^\varepsilon M_\eta^\varepsilon M_\eta^\varepsilon M_\eta^\varepsilon \\ Q_5^{\lambda} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta N_\eta M_\eta M_\eta M_\eta M_\eta, \end{aligned} \tag{75}$$

where $N_\eta^\varepsilon = \frac{1}{D^{\lambda K^\varepsilon}-\gamma+i\eta} K^\varepsilon \frac{1}{D^0-\gamma+i\eta}$, $M_\eta^\varepsilon = K^\varepsilon \frac{1}{D^0-\gamma+i\eta}$, $N_\eta = \frac{1}{D^{\lambda\varphi}-\gamma+i\eta}\varphi\frac{1}{D^0-\gamma+i\eta}$ and $M_\eta = \varphi\frac{1}{D^0-\gamma+i\eta}$. A straightforward calculation gives

$$\begin{aligned} \|Q_5^{\lambda,\varepsilon} - Q_5^{\lambda}\|_1 \le \sup_\eta \|M_\eta^\varepsilon - M_\eta\|_4 \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \Big[\|N_\eta^\varepsilon M_\eta^\varepsilon M_\eta^\varepsilon M_\eta^\varepsilon\|_{4/3} \\ + \|N_\eta^\varepsilon M_\eta^\varepsilon M_\eta^\varepsilon M_\eta\|_{4/3} + \|N_\eta^\varepsilon M_\eta^\varepsilon M_\eta M_\eta\|_{4/3} + \|N_\eta^\varepsilon M_\eta M_\eta M_\eta\|_{4/3}\Big] \\ + \sup_\eta \|N_\eta^\varepsilon - N_\eta\|_\infty \frac{1}{2\pi}\int_{-\infty}^{\infty} d\eta \|M_\eta M_\eta M_\eta M_\eta\|_1. \end{aligned} \tag{76}$$

Notice, since trace ideals fulfill $\mathfrak{S}_p \subset \mathfrak{S}_1$, for $p > 1$, we can easily estimate both integrals in (76) in analogy to (61) and (41).

Obviously

$$\|M_\eta^\varepsilon - M_\eta\|_4 \le \Big\| \frac{1}{\sqrt{p^2+1}}[\hat{\varphi} - \hat{K}^\varepsilon] \Big\|_4 := \Big(\int dp dq f_\varepsilon(p,q) \Big)^{1/4}. \tag{77}$$

Recalling the definition of K^ε in (47), we see that $f_\varepsilon(p,q) \to_{\varepsilon\to 0} 0$ pointwise, as well as

$$f_\varepsilon(p,q) \le \frac{1}{p^2+1}\big[|\hat{\varphi}| * |\hat{\varphi}|\big]^2(p-q)\frac{1}{q^2+1} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3),$$

which implies, by dominated convergence theorem, that

$$\sup_\eta \|M_\eta^\varepsilon - M_\eta\|_4 \to_{\varepsilon\to 0} 0.$$

Notice that

$$\frac{1}{D^{\lambda K^\varepsilon} - \gamma + i\eta} \to \frac{1}{D^{\lambda\varphi} - \gamma + i\eta} \quad \text{strongly} \tag{78}$$

and

$$K^\varepsilon \frac{1}{D^0 - \gamma + i\eta} \to \varphi \frac{1}{D^0 - \gamma + i\eta} \quad \text{in } \mathfrak{S}_4, \tag{79}$$

both uniformly in η. Together with the fact that $\varphi\frac{1}{D^0-\gamma+i\eta}$ is compact (even in $\mathfrak{S}_4$), i.e., it can be approximated in norm by a finite rank operator, we conclude

$$\sup_\eta \|N_\eta^\varepsilon - N_\eta\|_\infty \to_{\varepsilon\to 0} 0$$

which yields (62). □

Now we are ready to prove (59). Obviously, due to our definition (47) and (48),

$$\mathrm{tr}_{\mathbb{C}^4}\hat{Q}_3^{\varepsilon}(p,p) \to \mathrm{tr}_{\mathbb{C}^4}\hat{Q}_3(p,p) \tag{80}$$

pointwise as $\varepsilon \to 0$. By means of Lemma 4 (a) and the dominated convergence theorem

$$e\lambda^3 \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} \mathrm{tr}_{\mathbb{C}^4}\hat{Q}_3^{\varepsilon}(p,p)dp = e\lambda^3\int_{\mathbb{R}^3}\mathrm{tr}_{\mathbb{C}^4}\hat{Q}_3(p,p)dp = \int_{\mathbb{R}^3}\rho_3^{\lambda}(x)dx. \tag{81}$$

By means of Lemma 4 (b) we obtain

$$e\lambda^4 \lim_{\varepsilon\to 0}\mathrm{tr}\, Q_4^{\lambda,\varepsilon} = e\lambda^4\,\mathrm{tr}\, Q_4^{\lambda} = \int_{\mathbb{R}^3}\rho_4^{\lambda}(x)dx, \tag{82}$$

whence (59).

Fix again $\bar\lambda$ such that $e_1(\bar\lambda) \le -1$ and $e_2(\bar\lambda) > -1$. (As in the previous section we use the notation $e_1(\bar\lambda)$ for convenience. The argument works whatever happens to the eigenvalue after reaching the lower continuum. Let us remark, that for our results only the features of the eigenvalues before "diving" play a role.) We can find a $\delta > 0$ small enough such that the following holds: We can choose a γ with $-1+\delta < \gamma < e_2(\bar\lambda) - \delta$. Additionally we choose a $\lambda' < \bar\lambda$ such that $-1+\delta < e_1(\lambda') < \gamma - \delta$ and a γ' with $-1+\delta < \gamma' < e_1(\lambda') - \delta$. Moreover we find, due to Kato [14, VII-5, Theorem 5.1], an ε_0 such that for all $\varepsilon \le \varepsilon_0$, $\sigma(D^{\bar\lambda \mathrm{K}^{\varepsilon}})$ is in a δ-neighborhood of $\sigma(D^{\bar\lambda\varphi})$ as well as $\sigma(D^{\lambda'\mathrm{K}^{\varepsilon}})$ in a δ-neighborhood of $\sigma(D^{\lambda'\varphi})$. We can write

$$\mathrm{tr}[P_{\gamma}^{\bar\lambda\mathrm{K}^{\varepsilon}} - P_{\gamma}^{0}] = \mathrm{tr}[P_{\gamma}^{\bar\lambda\mathrm{K}^{\varepsilon}} - P_{\gamma}^{\lambda'\mathrm{K}^{\varepsilon}}] + \mathrm{tr}[P_{\gamma}^{\lambda'\mathrm{K}^{\varepsilon}} - P_{\gamma'}^{\lambda'\mathrm{K}^{\varepsilon}}] + \mathrm{tr}[P_{\gamma'}^{\lambda'\mathrm{K}^{\varepsilon}} - P_{\gamma}^{0}]. \tag{83}$$

By our choice of parameters $P_{\gamma}^{\bar\lambda\mathrm{K}^{\varepsilon}}$ can be continuously deformed into $P_{\gamma}^{\lambda'\mathrm{K}^{\varepsilon}}$, as well as $P_{\gamma'}^{\lambda'\mathrm{K}^{\varepsilon}}$ into $P_{\gamma'}^{0}$, which equals P_{γ}^{0}. Consequently the first and the third term on the right-hand side of (83) vanish. Due to Cauchy's formula

$$\mathrm{tr}[P_{\gamma}^{\lambda'\mathrm{K}^{\varepsilon}} - P_{\gamma'}^{\lambda'\mathrm{K}^{\varepsilon}}] = \mathrm{tr}[P_{e_1(\lambda')}], \tag{84}$$

due to the fact that by our choice of parameters the eigenspace of the set $\{e_j^{\varepsilon}(\lambda')|\gamma' < e_j^{\varepsilon}(\lambda') < \gamma\}$ has the same dimension as the eigenspace of $e_1(\lambda')$. Recall $P_{e_1(\lambda')}$ denotes the projector on the eigenspace corresponding to $e_1(\lambda')$. By definition (17)

$$\mathrm{tr}[P_{e_1(\lambda')}] = d(\bar\lambda). \tag{85}$$

Whence

$$\mathrm{tr}[P_{\gamma}^{\bar\lambda\mathrm{K}^{\varepsilon}} - P_{\gamma}^{0}] = d(\bar\lambda). \tag{86}$$

Expanding the left-hand side as in (57) and using the fact that we already know that the first three terms vanish we see

$$\bar\lambda^4\,\mathrm{tr}\, Q_4^{\bar\lambda,\varepsilon} = d(\bar\lambda). \tag{87}$$

By means of Lemma 4 (b) $\lim_{\varepsilon\to 0} \operatorname{tr} Q_4^{\bar\lambda,\varepsilon} = \operatorname{tr} Q_4^{\bar\lambda}$, so we infer

$$\int_{\mathbb{R}^3} \rho_4^{\bar\lambda}(x)dx = ed(\bar\lambda). \tag{88}$$

Repeating this argument whenever an $e_i(\lambda)$ dives into $(-\infty,-1]$ we arrive at the theorem.

A Criterion for a kernel to be trace class

It is well known that given an integral operator via a kernel $K(x,y)$, the fact that $\int_{\mathbb{R}^n} dxK(x,x) < \infty$ does not at all guarantee that K is trace class.

For a specific class of kernels we give a sufficient condition for the corresponding operator to be trace class.

Lemma 5 *Let*

$$K(x,y) = f_1(x)g(x-y)f_2(y), \tag{89}$$

with $f_1, f_2 \in L^2(\mathbb{R}^n)$ *and* $\hat g \in L^1(\mathbb{R}^n)$. *Then* K *is trace class.*

Proof. We can write

$$\hat g(k) = |\hat g(k)|^{1/2}|\hat g(k)|^{1/2}\operatorname{sgn}(\hat g(k)). \tag{90}$$

Define

$$h_1(x) := (2\pi)^{-3/2}\mathcal{F}^{-1}[|\hat g|^{1/2}](x), \quad h_2(x) := \mathcal{F}^{-1}[|\hat g|^{1/2}\operatorname{sgn}(\hat g)](x), \tag{91}$$

such that

$$g = \mathcal{F}^{-1}[\hat g] = (2\pi)^{-3/2}\mathcal{F}^{-1}[|\hat g|^{1/2}] * \mathcal{F}^{-1}[|\hat g|^{1/2}\operatorname{sgn}(\hat g)] = h_1 * h_2. \tag{92}$$

Therefore

$$K(x,y) = \int_{\mathbb{R}^n} dzL^1(x,z)L^2(z,y), \tag{93}$$

with

$$L^1(x,z) = f_1(x)h_1(x-z) \quad L^2(z,y) = h_2(z-y)f_2(y). \tag{94}$$

Observe

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} dxdz|L^j(x,z)|^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} dxdz|f_j(x)|^2|h_j(z)|^2 = \|f_j\|_2^2\|\hat g\|_1 < \infty \tag{95}$$

for $j = 1, 2$, which implies the Lemma.

□

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Christian Hainzl
CEREMADE
Université Paris-Dauphine
Maréchal de Tassigny
F-75775 Paris
France

and

Laboratoire de Mathématiques
Paris-Sud-Bât. 425
F-91405 Orsay Cedex
France
email: hainzl@ceremade.dauphine.fr

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