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C^*-Independence, Product States and Commutation

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Abstract. Let D be a unital C^*-algebra generated by C^*-subalgebras A and B possessing the unit of D. Motivated by the commutation problem of C^*-independent algebras arising in quantum field theory, the interplay between commutation phenomena, product type extensions of pairs of states and tensor product structure is studied. Roos's theorem [11] is generalized in showing that the following conditions are equivalent: (i) every pair of states on A and B extends to an uncoupled product state on D; (ii) there is a representation π of D such that $\pi(A)$ and $\pi(B)$ commute and π is faithful on both A and B; (iii) $A \otimes_{\min} B$ is canonically isomorphic to a quotient of D.

The main results involve unique common extensions of pairs of states. One consequence of a general theorem proved is that, in conjunction with the unique product state extension property, the existence of a faithful family of product states forces commutation. Another is that if D is simple and has the unique product extension property across A and B then the latter C^*-algebras must commute and D be their minimal tensor product.

1 Introduction

One approach to the concept of independence in quantum field theory, in general terms, has that two subsystems of observables may be deemed independent if each can be prepared in any of its states without having regard to the other. In a formulation originating in [5], when observables are realized as hermitian elements in a C^*-algebra this translates into the notion of C^*-independence. Commutation of the arising observable algebras is not a precondition of C^*-independence nor it is a mathematical necessity, as many counterexamples show (see [12, p. 205]). The commutation question was first considered in [9, 10] and more lately in [3] where a natural commutation conjecture involving product states was negated and a request for appropriate sufficient conditions issued.

We investigate commutation phenomena occurring in C^*-independence. At the same time we attempt to throw light upon the role of product states. Forewarned by [3, III] that faithful product states can exist alongside absence of nontrivial commutation across C^*-independent algebras, we proceed indirectly. The introduction of the notion of *uncoupled* product states allows us access to tensor products as likewise (and more immediately) does a condition which we call *faithful independent commutation.* Roos's theorem [11] is generalized and minimal tensor product of C^*-algebras is characterized in terms of state extension conditions alone, without assuming mutual commutativity. Our main results are gathered in the third and final section of the paper where we study unique common extension properties. Amongst other things we deduce that if a simple C^*-algebra has the unique product state extension property across a pair of generating C^*-subalgebras A and B, then A and B commute and D is canonically isomorphic to the minimal tensor product of A and B. In general, the unique product state extension property together with the existence of a faithful family of product states is sufficient to compel commutation.

We use [2] and [13] as general references for C^*-algebras, the latter for tensor products particularly. Let A be a C^*-algebra, let $S(A)$ be its state space and $P(A)$ its set of pure states. Given $\varphi \in S(A)$ and $a \in A$ such that $\varphi(a\,a^*) = 1$ we use φ_a to denote the transformed state of A given by

$$\varphi_a(x) = \varphi(a\,x\,a^*)\,, \qquad x \in A\,.$$

We recall [2, 2.4.10] that if $\varphi \in S(A)$ then $\ker \pi_\varphi$ is the largest ideal in $\ker \varphi$, where π_φ is the associated GNS representation. If J is an ideal of A, J^0 denotes its annihilator in A^*.

Let A and B be unital C^*-algebras and let β be a C^*-norm on the algebraic tensor product $A \otimes B$. The completion with respect to β is written $A \otimes_\beta B$. We may regard A and B as C^*-subalgebras of $A \otimes_\beta B$. If $\varrho \in S(A)$ and $\tau \in S(B)$ the tensor product $\varrho \otimes \tau \in A^* \otimes B^*$ has unique extension to a state on $A \otimes_{\min} B$, and on $A \otimes_\beta B$ via the canonical $*$-homomorphism onto $A \otimes_{\min} B$. By custom we continue to denote the resulting (tensor product) extension in $S(A \otimes_\beta B)$ by $\varrho \otimes \tau$, regardless of β.

Now let A and B be C^*-subalgebras of a unital C^*-algebra D such that A and B generate D as a C^*-algebra and both contain the identity of D.

1.1 Definition. A and B are said to be C^*-*independent* in D if ϱ and τ have a common extension in $S(D)$ for all $(\varrho, \tau) \in S(A) \times S(B)$.

Thus (when identified with their canonical images) A and B are C^*-independent in $A \otimes_\beta B$ for any C^*-norm β on $A \otimes B$. Conversely [11], in a result referred to as Roos's theorem, if A and B commute (i.e., $a\,b = b\,a$, for all $a \in A$ and $b \in B$) and are C^*-independent in D then the natural map is a $*$-isomorphism from $A \otimes B$ onto the $*$-subalgebra of D generated by A and B.

Recent papers that discuss various forms of independence in operator algebras include [3, 4] and [6, 7]. The comprehensive survey [12] is a main source of information on both mathematical and physical aspects. For a more recent development we refer the reader to [8, Chapter 11]. The reader is directed to the references contained in these works for an extensive literature. The related question of the existence and uniqueness of common extensions of a given pair of states is investigated in [1] in the significant case of subsystems of a Fermion system, with decisive results [1, Theorem 4, Theorem 5].

2 Uncoupled product states

Throughout this section D denotes a unital C^*-algebra and A and B denote C^*-subalgebras of D containing the identity of D the union of which generates D. *It is not assumed that A and B commute.*

If $S \subset S(D)$ we shall employ the contraction $S|A = \{\varphi|A \mid \varphi \in S(D)\}$ (with the corresponding meaning for $S|B$).

2.1 Definition. A state φ of D is said to be

(a) a *product state across* A, B if $\varphi(a\,b) = \varphi(a)\,\varphi(b)$ whenever $a \in A$ and $b \in B$;

(b) an *uncoupled product state across* A, B if

$$\varphi\Big(\prod_{i=1}^{n} a_i\, b_i\Big) = \varphi\Big(\prod_{i=1}^{n} a_i\Big)\,\varphi\Big(\prod_{i=1}^{n} b_i\Big)\,,$$

whenever $a_1, \ldots, a_n \in A\,, b_1, \ldots, b_n \in B\,.$

The definition (a) is symmetric about A and B (states are hermitian). Since 1's can be inserted willy-nilly without effect, the value of an uncoupled product state at $\prod_{i=1}^{n} a_i\, b_i$ ($a_i \in A, b_i \in B$) is the same at any rearrangement of the product that leaves unaltered the relative order of the a_i's, and the relative order of the b_i's.

As is clear, product states and uncoupled product states are one and the same if A and B commute, and they coincide with the tensor product states when D is a C^*-tensor product of A and B.

Let $\Delta(A,B)$ and $\Delta_u(A,B)$, respectively, be the (possibly empty) sets of all product states across A, B and uncoupled product states across A, B. Given $\varrho \in S(A)$ we shall write

$$E(\varrho) = \{\varphi \in S(D) \mid \varphi|A = \varrho\}\,.$$

Similarly, $E(\tau)$ shall denote the set of extensions of τ in $S(D)$ whenever $\tau \in S(B)$. Routine verifications show that $\Delta(A,B)$ and $\Delta_u(A,B)$ are weak*-compact and that $E(\varrho)$, $E(\varrho) \cap \Delta(A,B)$ and $E(\varrho) \cap \Delta_u(A,B)$ are convex and weak*-compact. (These facts concerning $\Delta_u(A,B)$ are also visibly true from those of $\Delta(A,B)$ via Lemma 2.5 below.)

Let $\varrho \in S(A)$ and $\tau \in S(B)$. If ϱ and τ have a common extension in $\Delta_u(A,B)$ we shall denote it $\varrho \wedge \tau$ (by the definition there can be at most one such extension). Thus,

$$\tau \in E(\varrho) \cap \Delta_u(A,B)|B \iff \varrho \in E(\tau) \cap \Delta_u(A,B)|A \iff \varrho \wedge \tau \text{ exists.}$$

Further, if $\tau = \sum_{i=1}^{n} \lambda_i\, \tau_i$ (convex sum) where $\tau_i \in S(B)$ such that $\varrho \wedge \tau_i$ exists for each i, then $\varrho \wedge \tau$ exists and equals $\sum_{i=1}^{n} \lambda_i\,(\varrho \wedge \tau_i)\,.$

2.2 Lemma.

(a) *Let* $\varrho \in S(A)$. *Then*

(i) *the restriction map,* $E(\varrho) \cap \Delta(A, B) \to E(\varrho) \cap \Delta(A, B)|B$ *is affine and weak***-continuous;*

(ii) *the restriction map,* $E(\varrho) \cap \Delta_u(A, B) \to E(\varrho) \cap \Delta_u(A, B)|B$ *is an affine weak***-homeomorphism.*

(b) *The map* $\Delta_u(A, B)|A \times \Delta_u(A, B)|B \to \Delta_u(A, B)$, *given by* $(\varrho, \tau) \to \varrho \wedge \tau$, *is a weak***-homeomorphism and is affine in each variable.*

Proof. (a) Affineness and (weak*-) continuity are apparent in both (i) and (ii). Further, in (ii), the map is a bijection and so a homeomorphism by compactness. (b) To see continuity let $(\varrho_\alpha, \tau_\alpha) \to (\varrho, \tau)$ in the product weak*-topology on the domain. By compactness of $\Delta_u(A, B)$, some subnet $\varrho_\beta \wedge \tau_\beta$ of $\varrho_\alpha \wedge \tau_\alpha$ has weak*-limit φ in $\Delta_u(A, B)$. Since φ must agree with ϱ on A and τ on B we have $\varrho_\beta \wedge \tau_\beta \to \varrho \wedge \tau$, establishing continuity. The remainder is clear. □

2.3 Definition. D is said to have the *C^*-product property* (respectively, the *C^*-uncoupled product property) across* A, B if ϱ and τ have a common product state extension (respectively a common uncoupled product state extension) for all $(\varrho, \tau) \in S(A) \times S(B)$.

The C^*-product property was introduced in [4].

2.4 Lemma. *Let* ϱ *and* τ *have common extension in* $\Delta(A, B)$ *(respectively, in* $\Delta_u(A, B)$*) for all* $(\varrho, \tau) \in P(A) \times P(B)$. *Then* D *has the* C^**-product property property (respectively, the* C^**-uncoupled product property) across* A, B.

Proof. Let $\varrho \in P(A)$. The assumption implies that $E(\varrho) \cap \Delta(A, B)|B$ contains $P(B)$ and so contains $S(B)$ by the Krein-Milman theorem. For any $\tau \in S(B)$ it follows that $E(\tau) \cap \Delta(A, B)|B$ contains $P(A)$ and therefore contains $S(A)$ whence D has the C^*-product property across A, B. A similar argument proves the statement in parentheses. □

In the next result and later $J(A, B)$ denotes the norm closed ideal of D generated by the set of A-B commutators, $\{a\, b - b\, a \mid a \in A, b \in B\}$.

We shall refer to the elements of the form $\prod_{i=1}^n a_i\, b_i$, where $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$ as the *A-B generators* of D.

2.5 Lemma. $\Delta_u(A, B) = \Delta(A, B) \cap J(A, B)^0$.

Proof. Suppose that φ is a product state across A, B vanishing on J where $J = J(A, B)$, and let $\overline{\varphi}$ be the induced state on D/J. Let $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$. Since the images of A and B pairwise commute in D/J we see that

$$\varphi\Big(\prod_{i=1}^n a_i\, b_i\Big) = \overline{\varphi}\Big(\prod_{i=1}^n a_i \cdot \prod_{i=1}^n b_i + J\Big) = \varphi\Big(\prod_{i=1}^n a_i\Big) \cdot \varphi\Big(\prod_{i=1}^n b_i\Big)$$

so that φ is uncoupled. Conversely, suppose that φ is an uncoupled product state across A, B. If $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ where $n \geq 2$ we have, by definition

$$\begin{aligned}&\varphi(a_1\, b_1 \cdots a_{i-1}\, b_{i-1}\, (a_i\, b_i)\, a_{i+1}\, b_{i+1} \cdots a_n\, b_n)\\ &\quad = \varphi(a_1\, b_1 \cdots, a_{i-1}\, b_{i-1}\, (b_i\, a_i)\, a_{i+1}\, b_{i+1} \cdots a_n\, b_n)\,.\end{aligned}$$

Thus, if $a \in A, b \in B$ and x, y are A-B generators, we have

$$\varphi(x\, a\, b\, y) = \varphi(x\, b\, a\, y)\,,$$

an equation that continues to hold for all x and y in D (since D is the norm closed linear span of its A-B generators and φ is linear and continuous). Therefore, φ vanishes on all elements of the form

$$x\,(a\,b - b\,a)\,y\,, \text{ where } x, y \in D \text{ and } a \in A, b \in B\,.$$

Since the norm closed linear span of all such elements is exactly the ideal $J(A, B)$, φ must vanish on $J(A, B)$. □

We shall now introduce a modified form of commutation.

2.6 Definition. A and B are said to *faithfully independently commute* if there is a $*$-homomorphism π on D for which the following conditions hold:

(a) π is faithful on A and B;

(b) $\pi(A)$ and $\pi(B)$ commute and are C^*-independent in $\pi(D)$.

The above considerations combine to give the following generalized and extended form of Roos's theorem.

2.7 Theorem. *The following conditions are equivalent:*

(a) *ϱ and τ have common uncoupled state extension across A, B for all $(\varrho, \tau) \in P(A) \times P(B)$.*

(b) *D has the C^*-uncoupled product property across A, B.*

(c) *A and B faithfully independently commute.*

(d) *There is a (unique) C^*-norm β on $A \otimes B$ and a $*$-isomorphism $A \otimes_\beta B \to D/J(A, B)$ sending $a \otimes b \to a\,b + J(A, B)$.*

(e) *There is a (unique) norm closed ideal J of D and a $*$-isomorphism $A \otimes_{\min} B \to D/J$, sending $a \otimes b \to a\,b + J$.*

Proof. (a) $\Rightarrow$ (b) This was given in Lemma 2.4.

(b)$\Rightarrow$ (c), (d) Assume (b). Let $\varrho \in S(A)$ and $\tau \in S(B)$. By assumption and Lemma 2.5 ϱ and τ have a common extension in D vanishing on $J(A,B)$, implying that ϱ and τ vanish on $A \cap J(A,B)$ and $B \cap J(A,B)$, respectively. It follows that

$$A \cap J(A,B) = B \cap J(A,B) = \{0\}\,.$$

So the quotient map, $\pi\colon D \to D/J(A,B)$, is faithful on A and B. Since, visibly, $\pi(A)$ and $\pi(B)$ are C^*-independent in $\pi(D)$, this proves (c). Moreover, since $\pi \otimes \pi$ is a $*$-algebra isomorphism from $A \otimes B$ onto $\pi(A) \otimes \pi(B)$ and the latter, by Roos's theorem, embeds as a $*$-subalgebra of $\pi(D)$ via $\pi(a) \otimes \pi(b) \to \pi(a\,b)$, the norm on $\pi(D)$ pulls back to a C^*-norm on $A \otimes B$ giving (d).

(d)$\Rightarrow$ (e) This is immediate from the fact that, canonically, $A \otimes_{\min} B$ is a quotient of $A \otimes_\beta B$.

(e)$\Rightarrow$ (a). This is clear. $\square$

In the sequel we shall denote by π_ϱ the GNS representation associated with a given state ϱ.

2.8 Lemma. *Let ϱ and τ have common extension in $\Delta_u(A,B)$. Then $\ker \pi_\varrho = A \cap \ker \pi_\varphi$ and $\ker \pi_\tau = B \cap \ker \pi_\varphi$, where $\varphi = \varrho \wedge \tau$.*

Proof. Since φ extends ϱ the ideal $A \cap \ker \pi_\varphi$ is contained in $\ker \varrho$ and so is contained in $\ker \pi_\varrho$. On the other hand, the norm closed ideal J of D generated by $\ker \pi_\varrho$ is the norm closed linear span of all elements of the form

$$y = a_1\, b_1 \cdots a_n\, b_n\, a\, c_1\, d_1 \cdots c_m\, d_m$$

where the a_i and c_j belong to A, the b_i and d_j belong to B and a lies in $\ker \pi_\varrho$. For any such element y we have

$$\varphi(y) = \varrho(a_1 \cdots a_n\, a\, c_1 \cdots c_m)\, \tau(b_1 \cdots b_n\, d_1 \cdots d_m) = 0\,.$$

Hence, J is contained in $\ker \pi_\varphi$ so that $\ker \pi_\varrho$ is contained in $A \cap \ker \pi_\varphi$ as required. $\square$

2.9 Proposition. *There is a set S of uncoupled product states across A, B such that $\{\pi_\varphi \mid \varphi \in S\}$ is faithful on D if, and only if, there is a $*$-isomorphism $A \otimes_{\min} B \to D$ sending $a \otimes b \to a\,b$.*

Proof. Suppose that $\{\pi_\varphi \mid \varphi \in S\}$ is faithful on D for some subset S of $\Delta_u(A,B)$. It follows from Lemma 2.5 that $J(A,B) \subset \ker \varphi$ for each $\varphi \in S$. Hence, A and B pairwise commute. Now let $(\sigma, \omega) \in P(A) \times P(B)$. We have

$$S = \{\varrho \wedge \tau \mid \varrho \in S|A\,, \tau \in S|B\}\,.$$

Put

$$S_1 = \{\varrho_a \mid \varrho \in S|A\,, \varrho(a\,a^*) = 1, a \in A\}$$

and

$$S_2 = \{\tau_b \mid \tau \in S|B\,, \tau(b\,b^*) = 1, b \in B\}\,.$$

By Lemma 2.8 $\{\pi_\varrho \mid \varrho \in S|A\}$ and $\{\pi_\tau \mid \tau \in S|B\}$ are faithful on A and B, respectively. It follows from [2, 2.4.8 (ii) and 3.4.2 (ii)] that σ and ω lie in the respective weak*-closures of S_1 and S_2. Further, given typical elements ϱ_a and τ_b of S_1 and S_2, we see that $\varrho_a \wedge \tau_b$ exists and equals $(\varrho \wedge \tau)_{a\,b}$. Now Lemma 2.2 implies that $\sigma \wedge \omega$ exists. Hence, by the faithfulness assumption together with Theorem 2.7 (a)$\Rightarrow$ (e) (or Roos's theorem, itself) $A \otimes_{\min} B$ is $*$-isomorphic to D as claimed.

If D is isomorphic to $A \otimes_{\min} B$ in the above way, then the family $\{\pi_{\varrho\otimes\tau} \mid \varrho \in P(A), \tau \in P(B)\}$ is well known to be faithful on $A \otimes_{\min} B$. □

2.10 Corollary. *Let $\varrho \in S(A)$ and $\tau \in S(B)$. Then ϱ and τ have a common extension to an uncoupled product state across A, B if, and only if, there is a $*$-isomorphism $\pi_\varrho(A) \otimes_{\min} \pi_\tau(B) \to \pi_\varphi(D)$ sending $\pi_\varrho(a) \otimes \pi_\tau(b) \to \pi_\varphi(a\,b)$, where $\varphi = \varrho \wedge \tau$.*

Proof. Suppose that $\varphi = \varrho \wedge \tau$ exists. Via Lemma 2.8 there is a $*$-homomorphism between $\pi_\varrho(A) \otimes_{\min} \pi_\tau(B)$ and $\pi_\varphi(A) \otimes_{\min} \pi_\varphi(B)$ sending $\pi_\varrho(a) \otimes \pi_\tau(b)$ to $\pi_\varphi(a) \otimes \pi_\varphi(b)$. But the induced state $\overline{\varphi}$ on $\pi_\varphi(D)$, given by $\overline{\varphi}(\pi(x)) = \varphi(x)$, is an uncoupled product state across $\pi_\varphi(A), \pi_\varphi(B)$ and $\pi_{\overline{\varphi}}$ is faithful on $\pi_\varphi(D)$. Hence, the required $*$-isomorphism is produced by Proposition 2.9 together with the above remark. The converse is clear. □

2.11 Remarks. (a) The C^*-uncoupled product property across A, B is implemented by conditional expectations onto A and B as follows. Given ϱ in $S(A)$ and τ in $S(B)$ consider the composition, Q_ϱ,

$$D \to D/J \to A \otimes_{\min} B \to B$$

where the first map is the quotient homomorphism, the second is the inverse of the one given by Theorem 2.7 (e) and the third is the projection that sends $a \otimes b \to \varrho(a)\,b$ [13, IV, 4.25]; and let Q_τ be the corresponding composition finishing at A. Chasing the maps, we see that Q_ϱ and Q_τ are norm one projections onto B and A, respectively, and that

$$\tau\,Q_\varrho = \varrho\,Q\tau = \varrho \wedge \tau\,.$$

(b) If D has a faithful uncoupled product state across A, B then, as soon as it is observed, by Lemma 2.5, that A and B must pairwise commute the conclusion that $D \simeq A \otimes_{\min} B$ is seen directly from [12, Corollary 3.5] and [3, Proposition 12]. The same conclusion cannot be drawn if it is known only that there is a faithful *product* state even when D is finite-dimensional and both A and B are abelian and C^*-independent [3, III]. Evidently, $\Delta_u(A,B) \neq \Delta(A,B)$ in that case. Nevertheless, if A and B are abelian and C^*-independent in D, then D always has the C^*-uncoupled product property across A, B (see Proposition 2.13 below).

(c) The commutator ideal of D, the norm closed ideal of D generated by all commutators $x\,y - y\,x$ as x and y range over D, clearly contains $J(A,B)$. If there be any at all, the pure states of D vanishing on the commutator ideal are precisely the multiplicative states of D. Trivially, the latter are examples of uncoupled product states.

2.12 Lemma. *Let $\varphi \in S(D)$.*

(a) *If $\varphi|A$ is multiplicative, then $\varphi(a\,x) = \varphi(a)\,\varphi(x)$ for all $a \in A$ and $x \in D$.*

(b) *If $\varphi|A$ and $\varphi|B$ are multiplicative then φ is multiplicative and is the unique common extension of $\varphi|A$ and $\varphi|B$ in $S(D)$.*

Proof. (a) Let φ be multiplicative on A and let a and x be self-adjoint elements of A and D, respectively. With $\alpha = \varphi(a)$ the Cauchy-Schwarz inequality gives

$$|\varphi((a - \alpha\,1)\,x)|^2 \leq \varphi((a - \alpha\,1)^2)\,\varphi(x^2) = (\varphi(a - \alpha\,1))^2\,\varphi(x^2) = 0$$

so that $\varphi(a\,x) = \varphi(a)\,\varphi(x)$, from which the general case follows.

(b) Let φ be multiplicative on both A and B and let $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$ where $n \geq 2$. Iterating part (a) (symmetrized),

$$\begin{aligned} \varphi\left(\prod_{i=1}^n a_i\,b_i\right) &= \varphi(a_1)\,\varphi\left(b_1 \prod_{i=2}^n a_i\,b_i\right) = \varphi(a_1)\,\varphi(b_1)\,\varphi\left(\prod_{i=2}^n a_i\,b_i\right) \\ &= \cdots = \prod_{i=1}^n \varphi(a_i)\,\varphi(b_i)\,. \end{aligned}$$

So for any A-B generators $x_1, \ldots, x_n$, $y_1, \ldots, y_m$ we have $\varphi(x_i\,y_j) = \varphi(x_i)\,\varphi(y_j)$ and therefore that $\varphi\Big(\sum_{i=1}^n x_i \sum_{j=1}^m y_j\Big) = \varphi\Big(\sum_{i=1}^n x_i\Big)\,\varphi\Big(\sum_{j=1}^m y_j\Big)$ implying that φ is multiplicative on D. $\square$

2.13 Proposition.

(a) *Let A be abelian. Then A and B are C^*-independent in D if, and only if, D has the C^*-product property across A, B.*

(b) *Let A and B be abelian. Then the following conditions are equivalent.*

(i) *A and B are C^*-independent in D.*

(ii) *D has the C^*-uncoupled product property across A, B.*

(iii) *A and B faithfully independently commute.*

(iv) *There is a $*$-isomorphism $A \otimes_{\min} B \to D/J$ sending $a \otimes b \to a\,b + J$, where J is the commutator ideal of D.*

Proof. (a) If $\varrho \in P(A)$ and $\tau \in S(B)$ have common extension φ in $S(D)$ then, since ϱ is multiplicative, φ is a product state across A, B by Lemma 2.12 (a). Hence, if A and B are C^*-independent in D, then the latter has the required C^*-product property by Lemma 2.4. The converse is clear.

(b) Since A and B are abelian, $D/J(A,B)$ is abelian. Therefore $J(A,B)$ contains, and so equals, the commutator ideal of D. By Theorem 2.7 it is enough to prove (i)$\Rightarrow$(ii). Assume (i) and let $(\varrho, \tau) \in P(A) \times P(B)$. Since ϱ and τ are multiplicative, the assumption together with Lemma 2.12 (b) implies that ϱ and τ have common extension to a multiplicative state on D, and (b) is now immediate from Theorem 2.7. $\square$

We remark that in the light of Remark 2.11 (b) neither of the conditions in Proposition 2.13 (b) imply that A and B commute.

3 Unique common extensions

If K is a compact convex set let $A(K)$ denote the continuous (real) affine functions on K. If K is the state space of a unital C^*-algebra A the evaluation map $A_{sa} \to A(K)$ $(a \to \hat{a})$ is an order isomorphism and linear isometry. The same is true of the evaluation map $W_{sa} \to A^b(S)$ when W is a von Neumann algebra, S is the normal state space of W and $A^b(S)$ denotes the bounded (real) affine functions on S.

In all that follows D continues to denote a unital C^*-algebra and A and B denote C^*-subalgebras the union of which generates D and such that $1 \in A \cap B$. As before, we do not assume that A and B commute.

In order to formulate solutions to the the commutation problem we study two natural unique common extension properties. The first characterizes faithful independent commutation and determines the set of corresponding uncoupled product states amongst likely weak*-closed sets of states. The underlying idea of the proof below is to employ convexity to construct implementing conditional expectations (see Remarks 2.11 (a)).

3.1 Theorem. *Let Δ be a weak*-closed subset of $S(A) \times S(B)$. Then the following statements are equivalent:*

(a) *ϱ and τ have unique common extension in Δ and $\Delta \cap E(\varrho)$, $\Delta \cap E(\tau)$ are convex, for all $(\varrho, \tau) \in S(A) \times S(B)$.*

(b) *$\Delta = \Delta_u(A, B)$ and A and B faithfully independently commute.*

Proof. (b)⇒ (a) This is immediate from Theorem 2.7 and remarks following Definition 2.1.

(a)⇒ (b). Assume (a). By the uniqueness condition it is enough to show that Δ is contained in $\Delta_u(A,B)$. If $\varrho \in S(A)$ and $\tau \in S(B)$, let $\varphi_{\varrho,\tau}$ be their unique common extension in Δ. Now fix $\varrho \in S(A)$ and consider the set

$$K = E(\varrho) \cap \Delta\,.$$

We have

$$K = \{\varphi_{\varrho,\tau} \mid \tau \in S(B)\} \text{ and } K|B = S(B)\,.$$

The restriction map, $r\colon\ K \to S(B)$, is an affine homeomorphism with inverse $\alpha\colon\ S(B) \to K$ given by $\alpha(\tau) = \varphi_{\varrho,\tau}$. Consider now the complex linear extension, $Q_\varrho\colon\ D \to B$, of the composition

$$D_{sa} \to A(S(D)) \xrightarrow{\beta} A(K) \xrightarrow{\alpha^*} A(S(B)) \to B_{sa}\,,$$

where the first and last isometries are the appropriate evaluation map and inverse, respectively, and β is the restriction map, $f \to f|K$. Letting $x \in D_{sa}$ and $\tau \in S(B)$ we see that $Q_\varrho(x)\hat{} = (\hat{x}|K) \circ \alpha$ and deduce that

$$\tau\, Q_\varrho = \varphi_{\varrho,\tau}\,.$$

In particular, for $b \in B$ we have $\tau\, Q_\varrho(b) = \tau(b)$ for all $\tau \in S(B)$ so that $Q_\varrho(b) = b$. Hence, Q_ϱ is a norm one projection onto B. In addition, for $a \in A$ we have $Q_\varrho(a) = \varrho(a)\,1$, since $\tau\, Q_\varrho(a) = \varrho(a) = \tau(\varrho(a)\,1)$ for all $\tau \in S(B)$.

The upshot is that for all $\varrho \in S(A)$ there is a surjective norm one projection, $Q_\varrho\colon\ D \to B$, such that

$$\tau\, Q_\varrho = \varphi_{\varrho,\tau} \quad \text{for all} \quad \tau \in S(B), \quad \text{and} \quad Q_\varrho(a) = \varrho(a)\,1 \quad \text{for all} \quad a \in A\,.$$

By symmetry, for all $\tau \in S(B)$ there is a surjective norm one projection, $Q_\tau\colon\ D \to A$, such that

$$\varrho\, Q_\tau = \varphi_{\varrho,\tau} \quad \text{for all} \quad \varrho \in S(A) \quad \text{and} \quad Q_\tau(b) = \tau(b)\,1 \quad \text{for all } b \in B\,.$$

We claim that for all $\varrho \in S(A)$ and all $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_{n-1} \in B$, where $n \geq 2$, we have

$$Q_\varrho(a_1\, b_1 \cdots a_{n-1}\, b_{n-1} a_n) = \varrho(a_1 \cdots a_n)\, b_1 \cdots b_{n-1}\,.$$

Observe first that if $\varrho \in S(A)$, $\tau \in S(B)$ and $a_1, a_2 \in A$ and $b_1 \in B$ then, since $Q_\tau\colon\ D \to A$ is a conditional expectation, we have

$$Q_\tau(a_1\, b_1\, a_2) = a_1\, Q_\tau(b_1)\, a_2 = \tau(b_1)\, a_1\, a_2$$

so that

$$\tau\, Q_\varrho(a_1\, b_1\, a_2) = \varrho\, Q_\tau(a_1\, b_1, a_2) = \tau[\varrho(a_1\, a_2)\, b_1]$$

which, being valid for all τ in $S(B)$, proves that

$$Q_\varrho(a_1\, b_1\, a_2) = \varrho(a_1\, a_2)\, b_1\,.$$

Now fix $n \geq 2$ and suppose that for all $\varrho \in S(A)$ we have

$$Q_\varrho(a_1\, b_1 \cdots a_{k-1}\, b_{k-1}\, a_k) = \varrho(a_1 \cdots a_k)\, b_1 \cdots b_{k-1}$$

whenever the a_i are in A, the b_j are in B and $2 \leq k \leq n$. Take any $a_1, a_2 \ldots$ in A and $b_1, b_2 \ldots$ in B. For $\varrho \in S(A), \tau \in S(B)$ and $n \geq 2$ we have

$$\begin{aligned} \varrho\, Q_\tau(b_1\, a_1 \cdots b_{n-1}\, a_{n-1}\, b_n) &= \tau\, Q_\varrho(b_1\, a_1 \cdots b_{n-1}\, a_{n-1}\, b_n) \\ &= \tau\, [b_1\, Q_\varrho(a_1\, b_2 \cdots b_{n-1}\, a_{n-1})\, b_n] \\ &= \tau\, [b_1\, (\varrho(a_1 \cdots a_{n-1})\, b_2 \cdots b_{n-1})\, b_n] \\ &= \varrho\, [\tau(b_1 \cdots b_n)\, a_1 \cdots a_{n-1}]\,, \end{aligned}$$

giving

$$Q_\tau(b_1\, a_1 \cdots b_{n-1}\, a_{n-1}\, b_n) = \tau(b_1 \cdots b_n)\, a_1 \cdots a_{n-1}$$

and, in turn,

$$\begin{aligned} \tau\, Q_\varrho(a_1\, b_1 \cdots a_n\, b_n\, a_{n+1}) &= \varrho\, Q_\tau(a_1\, b_1 \cdots a_n\, b_n\, a_{n+1}) \\ &= \varrho\, [a_1\, Q_\tau(b_1\, a_2 \cdots a_n\, b_n)\, a_{n+1}] \\ &= \varrho\, [a_1\, (\tau(b_1 \cdots b_n)\, a_2 \cdots a_n)\, a_{n+1}] \\ &= \tau\, [\varrho(a_1 \cdots a_n\, a_{n+1})\, b_1 \cdots b_n] \end{aligned}$$

so that

$$Q_\varrho(a_1\, b_1 \cdots a_n\, b_n\, a_{n+1}) = \varrho(a_1 \cdots a_{n+1})\, b_1 \cdots b_n\,,$$

thereby proving the claim.

Hence, if $\varrho \in S(A), \tau \in S(B)$ and $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$, where $n \geq 2$, we have

$$Q_\varrho(a_1\, b_1 \cdots a_n\, b_n) = Q_\varrho(a_1\, b_1 \cdots a_n)\, b_n = \varrho(a_1 \cdots a_n)\, b_1 \cdots b_n$$

giving

$$\varphi_{\varrho,\tau}(a_1\, b_1 \cdots a_n\, b_n) = \tau\, Q_\varrho(a_1\, b_1 \cdots a_n\, b_n) = \varrho(a_1 \cdots a_n)\, \tau(b_1 \cdots b_n)\,,$$

so that $\varphi_{\varrho,\tau}$ is an uncoupled product state across A, B. Therefore, Δ is contained in $\Delta_u(A, B)$, as required. □

3.2 Definition. D is said to have the *unique C^*-product property across A, B* if ϱ and τ have unique common extension in $\Delta(A, B)$, for all $(\varrho, \tau) \in S(A) \times S(B)$.

3.3 Theorem. *D has the unique C^*-product property across A, B if, and only if, $\Delta(A,B) = \Delta_u(A,B)$ and A and B faithfully independently commute.*

Proof. This is immediate from Theorem 3.1 on putting $\Delta = \Delta(A,B)$. □

3.4 Theorem. *The following statements are equivalent:*

(a) *D has the unique C^*-product property across A, B and has a faithful family of GNS representations associated with product states across A, B.*

(b) *There is a $*$-isomorphism $A \otimes_{\min} B \to D$ sending $a \otimes b \to a\,b$.*

Proof. Combine Proposition 2.9 with Theorem 3.3. □

One immediate consequence of Theorem 3.4 is that if D has the unique C^*-product property across A, B and has a faithful family of product states across A, B then $A \otimes_{\min} B$ identifies with D as in Theorem 3.4 (b). This answers a question raised in [3, 12].

Since any $*$-representation of a simple C^*-algebra is faithful, Theorem 3.4 provides the following answer to the commutation question when D is simple.

3.5 Theorem. *If D is simple and has the unique C^*-product property across A, B then there is a $*$-isomorphism $A \otimes_{\min} B \to D$ sending $a \otimes b \to a\,b$.*

Identifying D with its image in D^{**}, and A^{**} and B^{**} with the respective weak*-closures of A and B in D^{**}, we have that D^{**} is generated as a von Neumann algebra by A^{**} and B^{**}. Thus, if A and B are C^*-independent in D, normal states ϱ on A^{**} and τ on B^{**} have a common extension to a normal state on B^{**} (i.e., A^{**} and B^{**} are *W^*-independent* in D^{**}). Therefore, by [3, Proposition 3], [4, Theorem2.12], and [6, Theorem 2.5] we have the following.

3.6 Lemma. *The following statements are equivalent:*

(a) *A and B are C^*-independent in D.*

(b) *$\| a\,b \| = \| a \| \, \| b \|$ for all $a \in A^{**}$ and $b \in B^{**}$.*

Given a C^*-algebra C we use z_C to denote the central projection of C^{**} such that $C^{**} z_C$ is the atomic part of C^{**}, and we recall that multiplication by z_C is faithful on C. The atomic states of C are those states ϱ for which $\varrho(z_C) = 1$.

3.7 Theorem. *The following statements are equivalent:*

(a) *ϱ and τ have unique common extension in $S(D)$ for all $(\varrho, \tau) \in P(A) \times S(B)$.*

(b) *$A^{**} z_A$ and B^{**} commute and A and B faithfully independently commute.*

(c) *$A^{**} z_A$ and B^{**} commute and A and B are C^*-independent in D.*

(d) *z_A is a central projection in D^{**} and there is a $*$-isomorphism $A \otimes_\beta B \to D\,z_A$ sending $a \otimes b \to a\,b\,z_A$, for some C^*-norm β on $A \otimes B$.*

Proof. (a)$\Rightarrow$ (b) Assume (a). If $\varrho \in P(A)$ and $\tau \in S(B)$ let $\varphi_{\varrho,\tau}$ denote their unique common extension in $S(D)$.

Let e be a minimal projection in A^{**}. Let $\varrho \in P(A)$ such that $\varrho(e) = 1$. The weak*-compact set (actually a face, in this case) of extensions of ϱ in $S(D)$ is given by

$$E(\varrho) = \{\varphi_{\varrho,\tau} \mid \tau \in S(B)\} = \{\varphi \in S(D) \mid \varphi(e) = 1\}$$

and is affinely homeomorphic to $S(B)$ via the restriction map

$$r\colon\; E(\varrho) \to S(B)$$

and may be identified with the normal state space, $S_n(e\,D^{**}\,e)$, of $e\,D^{**}\,e$. The induced surjective positive linear isometry

$$A^b(S(B)) \to A^b(S_n(e\,D^{**}\,e)) \qquad (\hat{b} \to \hat{b} \circ r\,, \quad b \in B^{**}_{sa})$$

in turn induces a surjective positive linear isometry and hence, *Jordan* isomorphism, $\psi\colon\; B^{**}_{sa} \to (e\,D^{**}\,e)_{sa}$, where $\psi(b) = e\,b\,e$ for all $b \in B^{**}_{sa}$, the latter equalities, for $b \in B^{**}_{sa}$, following from the evaluations

$$\hat{b} \circ r\,(\varphi_{\varrho,\tau}) = \tau(b) = \varphi_{\varrho,\tau}(e\,b\,e) = (e\,b\,e)\hat{}\,(\varphi_{\varrho,\tau})\,,$$

for all $\tau \in S(B)$.

Since ψ is a Jordan homomorphism, for each projection f in B^{**}, $e\,f\,e$ is also a projection, giving $e\,f = f\,e$. Hence, e commutes with all elements of B^{**}. Therefore, since $A^{**}\,z_A$ is the weak* closed linear span of the minimal projections of A^{**}, $A^{**}\,z_A$ and B^{**} commute.

Using Lemma 3.6 in the second equality below we have

$$\|(a\,z_A)(b\,z_A)\| = \|(a\,z_A)\,b\| = \|a\,z_A\| \cdot \|b\| = \|a\,z_A\| \cdot \|b\,z_A\|$$

giving that $A\,z_A$ and $B\,z_A$ are C^*-independent, by further use of Lemma 3.6. Moreover, the previous identity yields that $\|b\| = \|b\,z_A\|$ and hence z_A acts faithfully on B. In other words A and B faithfully independently commute.

(b)$\Rightarrow$(c) This is clear.

(c)$\Rightarrow$ (d) Assume (c). Then z_A is a central projection of D^{**}, and $A^{**}\,z_A$ and $B^{**}\,z_A$ are commuting C^*-subalgebras of $D\,z_A$. As above, it follows from Lemma 3.6 that $A\,z_A$ and $B\,z_A$ are C^*-independent in $D\,z_A$ and z_A acts faithfully on both A and B. Roos's theorem now gives (d).

(d)$\Rightarrow$(a) Let φ be a common extension of $\varrho \in P(A)$ and $\tau \in S(B)$. Then $\varphi(z_A) = 1$ and the induced state $\overline{\varphi}$ on $D\,z_A$, given by $\overline{\varphi}(x\,z_A) = \varphi(x)$ pulls back to a state ψ on $A \otimes_\beta B$ restricting to ϱ on A and τ on B so that $\psi = \varrho \otimes \tau$. It follows that $\varphi = \varrho \wedge \tau$. □

3.8 Corollary. *Let ϱ and τ have unique common extension in $S(D)$ for all $(\varrho, \tau) \in P(A) \times S(B)$. Then A and B commute if any one of the following statements is true.*

(a) *z_A acts faithfully by multiplication on D.*

(b) *Each pure state of D restricts to an atomic state of A.*

(c) *All irreducible $*$-representations of D are finite-dimensional.*

Proof. (a) This is immediate from Theorem 3.7 (a)$\Rightarrow$ (d).

(b) Given (b), for all $\varphi \in P(D)$ we have $\varphi(z_A) = 1$ so that the support projection $s(\varphi) \leq z_A$ and hence that $z_D \leq z_A$. Therefore A and B commute by (a).

(c) This follows from (b) since, in case of (c), if $\varphi \in P(D)$ with restriction $\varrho \in S(A)$ then $\pi_\varrho(A)$ is finite-dimensional because $\pi_\varphi(D)$ is, implying that ϱ is atomic. $\square$

We close with a final observation that, in the light of Theorem 3.7 and Corollary 3.8, improves Corollary 3.3 in the case when A is abelian.

3.9 Proposition. *If A is abelian and D has the unique C^*-product property across A, B then ϱ and τ have unique common extension in $S(D)$ for all $(\varrho, \tau) \in P(A) \times S(B)$.*

Proof. By Lemma 2.12 (a) if A is abelian then every extension in $S(D)$ of a pure state of A is a product state across A, B. $\square$

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