Ann. Henri Poincaré 5 (2004) 1013 – 1039

1424-0637/04/061013-27
DOI 10.1007/s00023-004-0188-2

Annales Henri Poincaré

A Direct Proof of the Nekhoroshev Theorem for Nearly Integrable Symplectic Maps

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Abstract. We provide the direct proof of the Nekhoroshev theorem on the stability of nearly integrable analytic symplectic maps. Specifically, we prove the stability of the actions for a number of iterations which grows exponentially with an inverse power of the norm of the perturbation by conjugating the generating function of the map to suitable normal forms with exponentially small remainder.

1 Introduction

The stability of the actions of nearly integrable analytic symplectic maps for a number of iterations which grows exponentially with an inverse power of the norm of the perturbation was conjectured by N.N. Nekhoroshev already in his 1977 article (see [7], Section 2.2).

Up to now, direct proofs exist only for specific situations (isochronous systems and neighborhoods of elliptic equilibrium points; see [1], [2] and [10]). For the general situation of nearly integrable symplectic maps, with hypotheses which extend in a natural way those of Nekhoroshev theorem, an indirect proof has been provided by Kuksin in [5], where the exponential stability is proved by showing the existence of a quasi-integrable non-autonomous analytic Hamiltonian system interpolating the map (see also [4], [6]). Kuksin's article is based essentially on a constructive version of Grauert analytic embedding theorem, nevertheless it seems not so straightforward to recover (even up to a small order) the interpolating Hamiltonian of a given map, so we think that a direct proof which provides also explicit algorithms for the construction of the normal forms would be welcome.

In this paper we provide such a direct proof, obtaining an exponential stability result which is independent of the one obtained by Kuksin in [5] and by Kuksin–Poschël in [6] (see points i–ii and iii below). Precisely, we consider symplectic maps which are generated by a function:

$$S(I,\varphi) = I \cdot \varphi + h(I) + \varepsilon f(I,\varphi) \tag{1}$$

defined for I in an open set $B \subset \mathbb{R}^n$ and $\varphi \in \mathbb{T}^n$; h and f are analytic functions; ε is a small parameter. The function S generates (implicitly) the map $\mathcal{C} : (I,\varphi) \mapsto$

(I', φ') through the equations:

$$\begin{aligned} I &= I' + \varepsilon \frac{\partial f}{\partial \varphi}(I', \varphi) \\ \varphi' &= \varphi + \frac{\partial h}{\partial I}(I') + \varepsilon \frac{\partial f}{\partial I}(I', \varphi). \end{aligned} \tag{2}$$

For $\varepsilon = 0$ the map is integrable: the actions do not change, while the angles at any iteration rotate by an angle $\omega(I)$, with $\omega = \frac{\partial h}{\partial I}$. For $\varepsilon \neq 0$, in general the problem of the stability of the actions arises. The Nekhoroshev theorem for maps then can be stated as follows (conjectured in [7]; first proof by interpolation in [5]):

Theorem 1 *If h is convex, there exist positive constants $\varepsilon_0, a, b, t_0, d_0$ such that for any $\varepsilon < \varepsilon_0$, and for any $I(0), \varphi(0) \in B' \times \mathbb{T}^n$, with $B' = \{I \in B : \mathrm{dist}(I, \partial B) > 2d_0 \varepsilon^a\}$, denoting: $(I(t), \varphi(t)) = \mathcal{C}^t(I(0), \varphi(0))$, it is:*

$$|I(t) - I(0)| \leq d_0 \ \varepsilon^a \tag{3}$$

for any $t \in \mathbb{Z}$ satisfying:

$$|t| \leq t_0 \exp\left(\frac{\varepsilon_0}{\varepsilon}\right)^b \quad . \tag{4}$$

The above theorem is different from the one conjectured in [7], Section 2.2 because the ‘P-steepness’ hypothesis on h used in Nekhoroshev’s paper is here replaced by the stronger convexity hypothesis. Here, we refer to the convex situation which is a non-trivial case of P-steepness sufficient to illustrate Nekhoroshev theorem (see [3]). Nevertheless, the result extends also to the P-steepness case.

We remark however that the convexity hypothesis cannot be replaced by the weaker quasi-convexity hypothesis, which is commonly used in the Hamiltonian case. Indeed, quasi-convexity of h is not allowed (for the generic steep case, Nekhoroshev replaced the steepness condition, valid for the Hamiltonian case, with the ‘P-steepness’ condition), as can be easily seen by the trivial counter-example $S(I, \varphi) = I\varphi + 2\pi I + \varepsilon \cos(\varphi)$.

The exponential stability result which is proved in this paper is independent from the one stated in [5] (and also in [6]) for the following reasons:

i) the critical parameter ε_0 appearing in Kuksin’s result is necessarily smaller than the critical ε_* for which the analytic Hamiltonian interpolation can be proved to exist. Such a problem here does not exist.

ii) within Kuksin’s technique it is necessarily: $\varepsilon_0 \to 0$ for $\sup_{I \in B} |\omega(I)| \to \infty$, because it can be shown that the analyticity radius with respect to time of the interpolating Hamiltonian goes to 0 as a positive power of $1/|\omega(I)|$. A similar limitation (which is not natural in the Nekhoroshev theorem) is absent in our proof.

iii) Kuksin's technique, using the Nekhoroshev theorem for quasi-convex Hamiltonian systems, allows to provide the stability exponents $a = b = 1/(2(n+1))$ as well as $a = 1/4+b, b = 1/(4(n+1))$ (for other possible exponents see [8], 'Theorem 1*'). Instead, with the direct proof, using the original geometric construction by Nekhoroshev, we will obtain a worse stability exponent $b \sim 1/n^2$. More precisely, for any $n \geq 2$, we obtain:

$$b = \frac{2-6a}{3n^2+3n-2} \quad , \quad a \in \left(0 \, , \frac{1}{3} \right) \quad ,$$

which include:

$$b = \frac{1}{6n^2+6n-4} \quad , \quad a = \frac{1}{4} \quad .$$

The article is structured as follows: in Section 2 we illustrate the basic idea allowing the construction of resonant normal forms for symplectic maps. In Section 3 we define the geometry of resonances in the action domain. In Sections 4, 5 and 6 we provide the technical details for the construction of the normal forms. In Section 7 we conclude the proof.

2 Normal forms for symplectic maps via generating functions

Following [1] and [10], we look for near to the identity symplectic transformations $\Phi : (I,\varphi) \mapsto (I',\varphi')$ generated by a function $\tilde{\chi}(I,\varphi) = I \cdot \varphi + \varepsilon\chi(I,\varphi)$ through the equations:

$$\begin{aligned} I &= I' + \varepsilon \frac{\partial \chi}{\partial \varphi}(I',\varphi) \\ \varphi' &= \varphi + \varepsilon \frac{\partial \chi}{\partial I}(I',\varphi) \quad , \end{aligned} \tag{5}$$

such that the conjugate map $\mathcal{C}' = \Phi^{-1} \circ \mathcal{C} \circ \Phi$ is 'more integrable' than $\mathcal{C}$. This means that $\mathcal{C}'$ is generated by a function $S'(I,\varphi) = I \cdot \varphi + h(I) + \varepsilon f'(I,\varphi)$, with f' smaller than f (except for a possible resonant term). We remark that if ε is suitably small, then $\mathcal{C}'$ can be also generated by some function S'. The specific form of S' is obtained by suitably choosing the function χ.

The composition of these symplectic maps satisfies the following lemma:

Lemma 1 *Let S be as in* (1)*, let χ be defined and analytic in $B \times \mathbb{T}^n$, and denote by $\mathcal{C}$ and Φ the symplectic maps defined implicitly by* (2) *and* (5) *respectively. If ε is small the transformation $\mathcal{C}' = \Phi^{-1} \circ \mathcal{C} \circ \Phi$ is generated by:*

$$S' = I \cdot \varphi + h(I) + \varepsilon f(I,\varphi) + \varepsilon[\chi(I,\varphi) - \chi(I,\varphi+\omega(I))] + \varepsilon^2 f'(I,\varphi) \tag{6}$$

where $\omega(I) = \frac{\partial h}{\partial I}(I)$, and the remainder f' is real analytic and bounded uniformly in ε in $B \times \mathbb{T}^n$.

The lemma is proved, with detailed estimates, in Section 5.

Lemma 1 concerns the composition of the symplectic map generated by S, with any near to the identity symplectic map, written in the form (5). The normal forms of S are instead obtained by means of specific choices of χ, depending on the resonance properties of the domain. For example, the non-resonant normal form is obtained when the term of order ε of S' does not depend on the angles. This happens if the function χ is chosen as

$$f(I,\varphi)+[\chi(I,\varphi)-\chi(I,\varphi+\omega(I))]=g(I) \quad , \tag{7}$$

where $g(I)$ is analytic in B. The above equation is the 'homological equation' for symplectic maps, which can be solved by Fourier series. Denoting:

$$f=\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi} \quad , \quad \chi=\sum_{k\in\mathbb{Z}^n}\chi_k(I)e^{ik\cdot\varphi} \quad ,$$

χ and g satisfy eq. (7) if it is:

$$\begin{aligned} f_0(I)&=g(I)\\ f_k(I)+\chi_k(I)[1-e^{ik\cdot\omega(I)}]&=0 \qquad , \text{ if } \; k\neq 0 \quad . \end{aligned} \tag{8}$$

A formal solution is therefore given by:

$$g=f_0 \quad , \quad \chi=-\sum_{k\in\mathbb{Z}^n\backslash 0}\frac{f_k(I)}{1-e^{ik\cdot\omega(I)}}e^{ik\cdot\varphi} \quad . \tag{9}$$

In general, as for the Hamiltonian case, the above solution is only formal, in the sense that χ is not defined in any open subset of B, because the set of I where for some $k\in\mathbb{Z}^n\backslash 0$ it is:

$$1-e^{ik\cdot\omega(I)}=0$$

is dense in B. However, an analytic solution can be found by restricting suitably the spectrum of f and the domain of definition of χ (as it is done in the proof of the Nekhoroshev theorem for Hamiltonian systems).

Remark. In the Hamiltonian case, an integrable Hamiltonian $h(I)$ produces the small denominators $k\cdot\omega(I)$, with $k\in\mathbb{Z}^n\backslash 0$, and therefore the resonances are given by the equations $k\cdot\omega(I)=0$, with $k\in\mathbb{Z}^n\backslash 0$. Instead, in the case of symplectic maps, the small denominators are: $1-e^{ik\cdot\omega(I)}$, with $k\in\mathbb{Z}^n\backslash 0$, and therefore the resonances are given by the equations:

$$k\cdot\omega(I)=2k_0\pi \quad , \tag{10}$$

with $k\in\mathbb{Z}^n\backslash 0$ and $k_0\in\mathbb{Z}$. These kinds of denominators, which appeared already in the proof of the isochronous case (see [1], [10]) are consistent with the representation of the n degrees of freedom integrable map generated by $I\cdot\varphi+h(I)$ as the 2π-time flow of the $n+1$ degrees of freedom Hamiltonian system with Hamilton function $\frac{h(I)}{2\pi}+I_{n+1}$, whose resonances are given by eq. (10).

3 The geometry of resonances

As explained in the previous section, the geometry of resonances refers to the frequency vector $\Omega = (\Omega_0, \ldots, \Omega_n) \in \mathbb{R}^{n+1}$ defined by $\Omega_0 = 2\pi$, and $\Omega_i = \omega_i(I)$ for any $i = 1, \ldots, n$. We adapt to maps the original geometric construction of the Nekhoroshev paper [7] (see also [3] for the convex case), introducing some differences with respect to the Hamiltonian case (see the remarks below).

As it is usual, the construction of resonant domains is parameterized by a positive number $K > 0$ and by n positive parameters:

$$0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n < \pi \quad . \tag{11}$$

More precisely, for any choice of these parameters, we define the following structures:

- *K-lattices*: for any $K > 0$, we first consider the set of all integer lattices $\Lambda \subseteq \mathbb{Z}^n$ which are generated by $d \leq n$ independent integer vectors $k^{(i)}$, $i \leq d$, with order $\left|k^{(i)}\right| = \sum_{j=1}^n \left|k_j^{(i)}\right| \leq K$; then, among these lattices, we do not consider those which are properly contained in other of these lattices of the same dimension. The remaining lattices will be called K-lattices.

- *resonant manifolds*: for any K-lattice Λ we define its resonant manifold:

$$\begin{aligned} \mathcal{R}_\Lambda \quad = \quad & \{I \in B : \ \forall k \in \Lambda \ \text{ there exists } \ k_0 \in \mathbb{Z} : \\ & k \cdot \omega(I) + 2\pi k_0 = 0\} \quad . \end{aligned} \tag{12}$$

- *resonant zones*: For any d-dimensional K-lattice Λ, $1 \leq d \leq n$, its resonant zone is

$$\begin{aligned} \mathcal{Z}_\Lambda \quad = \quad & \{I \in B : \ \forall k \in \Lambda \ \text{ with } \ |k| \leq K \ \text{ there exists } \ k_0 \in \mathbb{Z} : \\ & |k \cdot \omega(I) + 2\pi k_0| \leq \|k\| \, \alpha_d\} \ , \end{aligned} \tag{13}$$

  where $\| \ \|$ denotes the Euclidean norm of a vector.

- *resonant blocks*: for any d-dimensional K-lattice Λ, $1 \leq d \leq n-1$, its resonant block is

$$D_\Lambda = \{I \in \mathcal{Z}_\Lambda \text{ such that } I \notin \mathcal{Z}_{\Lambda'} \text{ for any } K\text{−lattice } \Lambda', \text{with } \dim\Lambda' = d+1\} \tag{14}$$

  while the non-resonant block, corresponding to $\Lambda = \{0\}$, is:

$$D_0 = \{I \in B : \ \ I \notin \mathcal{Z}_{\Lambda'} \text{ for any } K\text{−lattice } \Lambda', \text{with } \dim\Lambda' = 1\} \tag{15}$$

  and the completely resonant block is:

$$D_{\mathbb{Z}^n} = \mathcal{Z}_{\mathbb{Z}^n} \quad . \tag{16}$$

- *cylinders*: For any $I \in \mathbb{R}^n$ and $\delta > 0$ denote

$$\Xi(I,\delta) = \{\overline{I} \in \mathbb{R}^n : \quad \|I - \overline{I}\| \leq \delta\} \quad .$$

For any Λ with $1 \leq \dim\Lambda \leq n$, we denote by $I + \langle\Lambda\rangle$ the plane through I spanned by the lattice Λ. We associate to each $I \in D_\Lambda$ the *cylinder of radius* δ, which we denote by $\mathcal{C}_{\Lambda,\delta}(I)$, defined as the connected component containing I of the set:

$$\left(\bigcup_{I' \in I + \langle\Lambda\rangle} \Xi(I',\delta) \right) \cap \mathcal{Z}_\Lambda \ .$$

The intersection of $\mathcal{C}_{\Lambda,\delta}(I)$ with the border of $\mathcal{Z}_\Lambda$ gives the cylinder lateral walls.

- the *extended block* is

$$D^{\rm ext}_{\Lambda,\delta} = \bigcup_{I \in D_\Lambda} \mathcal{C}_{\Lambda,\delta}(I) \quad .$$

Remarks. *I)* In the definition of the resonant manifolds and zones we have considered resonances $k \cdot \omega(I) + 2\pi k_0 = 0$, with $k_0 \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, with the norm of k bounded by K, while in the analogous Hamiltonian construction the norm of the complete resonant vectors (k_0, k) is bounded. In the case of maps any harmonic of the perturbation labeled by some $k \in \mathbb{Z}^n$ produces resonances of the same width with any vector $k_0 \in \mathbb{Z}$, and therefore it is necessary to consider all these resonances. Nevertheless, the restriction on the norm of vector k is sufficient to control the density and avoid the overlapping of all these resonances.
II) Though the resonances are defined with respect to the $n+1$-dimensional frequency vector Ω, we need to define only n parameters α_i (and not $n+1$), because the $n+1$-dimensional resonance does not exist.

The original confinement argument of the Nekhoroshev theorem resides mainly on two facts: first, the parameters entering the geometric construction are such that the extended block of a given lattice Λ does not intersect the resonant zones of any lattice of the same dimension of Λ; second, in an exponentially long time any motion with initial action in a given cylinder can leave it only through its lateral walls. In the rest of the section we focus our attention on the first point, while the second will be considered in Section 7.

Through this paper, we will use the following notations. We denote by $\varrho_I, \varrho_\varphi$ the analyticity radii such that h and f are analytic in $B_{\varrho_I} \times \mathbb{T}^n_{\varrho_\varphi}$; we denote by M a Lipschitz constant of $\omega(I) = \partial h/\partial I$ in the set B_{ϱ_I} and by M_0 a positive constant such that:

$$\left| \frac{\partial^2 h}{\partial I^2}(I) u \cdot u \right| \geq M_0 u \cdot u \tag{17}$$

for any $u \in \mathbb{R}^n$ and any $I \in B$. The hypotheses of Theorem 1 (analyticity of h and f and convexity of h) guarantee that such constants are strictly positive, and we take the freedom to choose M_0 and M in such a way that: $M_0 < M$.

We now prove the following technical lemmas about the resonant domains:

Lemma 2 *Let $h : B \to \mathbb{R}$ convex, and let M, M_0 be defined as above. Consider any K-lattices Λ, Λ' with $d = \dim\Lambda = \dim\Lambda'$. If δ_Λ and the α_i satisfy:*

$$\delta_\Lambda \leq \frac{2}{M} dK^{d-1}\alpha_d \quad , \quad \alpha_{d+1} \geq 6K\frac{M}{M_0} dK^{d-1}\alpha_d \tag{18}$$

then it is $(D^{\rm ext}_{\Lambda,\delta_\Lambda})_{\delta_\Lambda} \cap (\mathcal{Z}_{\Lambda'}) = \emptyset$, where $(D^{\rm ext}_{\Lambda,\delta_\Lambda})_{\delta_\Lambda} = \cup_{I \in D^{\rm ext}_{\Lambda,\delta_\Lambda}} \Xi(I, \delta_\Lambda)$.

Before proving Lemma 2 we estimate the small denominators in any resonant block and the diameters of the cylinders.

Lemma 3 *For any K-lattice Λ with $\dim \Lambda = d \in [1, n-1]$ and for any $I \in D_\Lambda$ it is:*

$$|k \cdot \omega(I) + 2\pi k_0| > \|k\|\, \alpha_{d+1} \tag{19}$$

for any $k \in \mathbb{Z}^n \backslash \Lambda$ with $|k| \leq K$ and for any $k_0 \in \mathbb{Z}$.

For any $I \in D_0$ it is:

$$|k \cdot \omega + 2\pi k_0| > \alpha_1 \tag{20}$$

for any $k \in \mathbb{Z}^n \backslash 0$ with $|k| \leq K$ and for any $k_0 \in \mathbb{Z}$.

The proof of Lemma 3 follows directly from the definition of the resonant zones and blocks.

Lemma 4 *For any K-lattice Λ with $\dim \Lambda = d \in [1, n]$ and $\delta \leq \frac{2}{5M} dK^{d-1}\alpha_d$, if $I \in D_\Lambda$ and $I' \in \mathcal{C}_{\Lambda,\delta}(I)$ then it is:*

$$\|I - I'\| \leq \frac{3}{M_0} dK^{d-1}\alpha_d \ . \tag{21}$$

Proof of Lemma 4. Let $\hat{I} \in I + \langle\Lambda\rangle$ (we recall that $I + \langle\Lambda\rangle$ denotes the plane through I spanned by Λ) such that $I' \in \Xi(\hat{I}, \delta)$. The following inequalities hold:

$$\begin{aligned} &M_0\|I - I'\|^2 \leq |(\omega(I) - \omega(I')) \cdot (I - I')| \\ &\leq |(\omega(I) - \omega(I')) \cdot (I - \hat{I})| + |(\omega(I) - \omega(I')) \cdot (\hat{I} - I')| \\ &\leq |P_\Lambda(\omega(I) - \omega(I')) \cdot (I - \hat{I})| + M\, \|I - I'\|\, \delta \quad , \end{aligned} \tag{22}$$

where we denote by $P_\Lambda v$ the orthogonal projection of a vector $v \in \mathbb{R}^n$ over the real space spanned by the lattice Λ. Let $k^{(1)}, \ldots, k^{(d)} \in \mathbb{Z}^n$ be a K-base of Λ. Therefore, since $I, I' \in \mathcal{Z}_\Lambda$, for any i there exist n_i, n'_i such that: $\left|\omega(I) \cdot k^{(i)} + 2\pi n_i\right| \leq \alpha_d$ and $\left|\omega(I') \cdot k^{(i)} + 2\pi n'_i\right| \leq \alpha_d$. Moreover, one can choose $n_i = n'_i$: if we consider any arc

$I(s) \in \mathcal{C}_{\Lambda,\delta}(I)$, $s \in [0,1]$, with $I(0) = I, I(1) = I'$, it is $\left|\omega(I(s)) \cdot k^{(i)} + 2\pi n_i\right| \leq \alpha_d$ for any $s \in [0,1]$. In fact, let us suppose that there exists $s_0 \in (0,1)$, and a neighborhood U of s_0 such that for any $s_1, s_2 \in U$ with $s_1 \leq s_0 < s_2$ it is $\left|\omega(I(s_1)) \cdot k^{(i)} + 2\pi n_i\right| \leq \alpha_d$ and $\left|\omega(I(s_2)) \cdot k^{(i)} + 2\pi n_i\right| > \alpha_d$. In such a case, there exists also an integer $\hat{n} \neq n_i$ such that $\left|\omega(I(s_2)) \cdot k^{(i)} + 2\pi \hat{n}\right| \leq \alpha_d$. It would be also:

$$\alpha_d \geq \left|\omega(I(s_1)) \cdot k^{(i)} + 2\pi n_i\right| \geq 2\pi \left|\hat{n} - n_i\right| - KM \left\|I(s_1) - I(s_2)\right\| - \alpha_d \quad , \tag{23}$$

which is in contrast with $\alpha_d < \pi$ as soon as $|s_1 - s_2|$ tends to zero. Therefore, for any $k^{(i)}$ it is: $\left|\omega(I') \cdot k^{(i)} + 2\pi n_i\right| \leq \alpha_d$, from which it follows also:

$$\begin{aligned} &\left|P_\Lambda(\omega(I) - \omega(I')) \cdot k^{(i)}\right| = \left|(\omega(I) - \omega(I')) \cdot k^{(i)}\right| \\ &= \left|\omega(I) \cdot k^{(i)} + 2\pi n_i - \omega(I') \cdot k^{(i)} - 2\pi n_i\right| \leq 2\alpha_d \end{aligned} \tag{24}$$

From "technical Lemma 1" of [3] it follows:

$$\|P_\Lambda(\omega(I) - \omega(I'))\| \leq 2dK^{d-1}\alpha_d \quad . \tag{25}$$

From (22) follows also:

$$M_0\|I - I'\|^2 \leq 2dK^{d-1}\alpha_d(\|I - I'\| + \delta) + M \|I - I'\| \delta \tag{26}$$

from which follows (21).

Proof of Lemma 2. Since $I \in (D^{\rm ext}_{\Lambda,\delta_\Lambda})_{\delta_\Lambda}$, there exists $I' \in D_\Lambda$ with $\|I - I'\| \leq \frac{3}{M_0} dK^{d-1}\alpha_d + \delta_\Lambda$. The lattice Λ' contains at least a vector $k \in \mathbb{Z}^n \backslash \Lambda$ with $|k| \leq K$. From Lemma 3, for any $n \in \mathbb{Z}$ it is

$$|\omega(I) \cdot k + 2\pi n| \geq \|k\| \alpha_{d+1} - \|k\| M \frac{3}{M_0} dK^{d-1}\alpha_d - \|k\| M\delta_\Lambda \geq \|k\| \alpha_d \quad . \tag{27}$$

This inequality implies that $I \notin \mathcal{Z}_{\Lambda'}$. In fact, if $I \in \mathcal{Z}_{\Lambda'}$, for any $k \in \Lambda'$ with $|k| \leq K$ there exists n such that: $|\omega(I) \cdot k + 2\pi n| \leq \|k\| \alpha_d$, and this is in contrast with (27).

4 The resonant normal forms

In this section we construct the resonant normal form for any resonant domain. The construction follows closely that of Hamiltonian systems, except that the homological equation is replaced by equation (7), as explained in Section 2.

We first fix some notation. For any real domain $D' \subseteq \mathbb{R}^n$ and $\sigma > 0$ we denote by

$$D'_\sigma = \bigcup_{x \in D'} \{x' \in \mathbb{C}^n : \ \ \|x - x'\| \leq \sigma\} \tag{28}$$

its complex extension of radius σ; we denote by

$$\mathbb{T}^n_\sigma = \{\varphi \in (\mathbb{C}/2\pi\mathbb{Z})^n : \quad |\mathrm{Im}\,\varphi_i| \le \sigma \quad \text{for} \quad \text{any} \quad i \le k\} \tag{29}$$

the complex extension of $\mathbb{T}^n$ of radius σ. We will handle real domains of the form $D' \times \mathbb{T}^n$ with $D' \subseteq \mathbb{R}^n$; for any function $u : D' \times \mathbb{T}^n \to \mathbb{C}$, for we consider the Fourier decomposition

$$u = \sum_{k\in\mathbb{Z}^n} u_k(I) e^{ik\cdot\varphi} \quad , \tag{30}$$

and, for any $\Lambda \subseteq \mathbb{Z}^n$, the Fourier projection $\Pi_\Lambda u = \sum_{k\in\Lambda} u_k(I)e^{ik\cdot\varphi}$ and for any $K > 0$ we define the cut-off projection: $T_K u = \sum_{|k|\le K} u_k(I)e^{ik\cdot\varphi}$. For any 'extension vector' $\sigma = (\sigma_I, \sigma_\varphi)$ (extension vectors will be always considered with positive entries, and inequalities on extension vectors are intended as inequalities on the entries) we denote by $|u|_\sigma$ the sup-norm in the domain ${D'}_{\sigma_I} \times \mathbb{T}^n_{\sigma_\varphi}$.

Lemma 5 *Let $K \ge 2$ be such that: $\frac{K}{\ln K} \ge \frac{32n}{\varrho_\varphi}$, and define*

$$N = \frac{1}{32n}\varrho_\varphi \frac{K}{\ln K} \quad . \tag{31}$$

Let the function

$$W = I' \cdot \varphi + h(I') + \varepsilon f(I', \varphi) \tag{32}$$

be analytic in $B_{\varrho_I} \times \mathbb{T}^n_{\varrho_\varphi}$. Let $\Lambda \subseteq \mathbb{Z}^{n+1}$ be a K-lattice and let $\alpha \in (0,1)$, $r_I \in (0, \frac{\varrho_I}{2}]$, and $D' \subseteq B$ be such that for any $I' \in D'_{r_I}$ it is:

$$\left|1 - e^{ik\cdot\omega(I')}\right| \ge \frac{\alpha}{2} \tag{33}$$

for any $k \in \Lambda\backslash 0$ and $|k| \le K$.

If r, N and K satisfy:

$$\begin{array}{l} r_\varphi = \frac{\varrho_\varphi}{2} \quad , \quad r_I \le \frac{\varrho_\varphi}{2^7 MN} \\ K \ge \frac{6}{\varrho_\varphi} \ln\left(\frac{2^{2n+7} n^{n-1}}{\varrho_\varphi^n}\right) \end{array} \tag{34}$$

and it is:

$$\varepsilon n\, |f|_\varrho \le \Gamma \frac{1}{2^{2n+18}} \frac{\alpha^2 r_I r_\varphi}{nN} \quad , \tag{35}$$

with:

$$\Gamma \le \min\left\{1, \frac{2\varrho_\varphi}{M\varrho_I n}, \frac{2}{\alpha N}\right\} \quad , \tag{36}$$

there exists a symplectic map: $\Phi : D'_{r_I/2} \times \mathbb{T}^n_{r_\varphi/2} \to D'_{r_I} \times \mathbb{T}^n_{r_\varphi}$, which is a diffeomorphism on its image, such that, denoting by $\mathcal{C}$ the symplectic map generated by W, its pull-back $\mathcal{C}' = \Phi^{-1} \circ \mathcal{C} \circ \Phi : D'_{\frac{r_I}{2}} \times \mathbb{T}^n_{\frac{r_\varphi}{2}} \to D'_{r_I} \times \mathbb{T}^n_{r_\varphi}$ is generated by the function:

$$W'(I', \varphi) = I' \cdot \varphi + h(I') + \varepsilon u(I', \varphi) + \varepsilon \mathcal{R}(I', \varphi) \quad , \tag{37}$$

where $\mathcal{R}$ and u are analytic in $D'_{\frac{r_I}{2}} \times \mathbb{T}^n_{\frac{r_\varphi}{2}}$, $u = \Pi_\Lambda u$, $|u - \Pi_\Lambda T_K f|_{\frac{r}{2}} \leq 2\,|f|_\varrho$ and

$$|\mathcal{R}|_{\frac{r}{2}} \leq \frac{1}{4^N}\frac{|f|_\varrho}{N} \quad . \tag{38}$$

Denoting $(I',\varphi') = \Phi(I,\varphi)$, with $(I,\varphi) \in D'_{r_I/2} \times \mathbb{T}^n_{r_\varphi/2}$, it is:

$$|I'_i - I_i| \leq \frac{r_I}{2^{11}} \tag{39}$$

for any $i \leq n$.

5 On the conjugation of nearly integrable symplectic maps with near to the identity symplectic map

In this section we compute the pull-back of nearly integrable symplectic maps with respect to near to the identity symplectic maps, so as to prove Lemma 1. Consider the generating function:

$$S(I',\varphi) = I' \cdot \varphi + k(I') + \varepsilon w(I',\varphi) \quad , \tag{40}$$

where k and w are analytic in $B' \times \mathbb{T}^n$, with $B' \subseteq \mathbb{R}^n$ open set, $\varepsilon \in \mathbb{R}$ (in the following it will be useful to consider both choices $k = 0$ and $k = h$). If ε is suitably small, the following equations:

$$\begin{aligned} I &= I' + \varepsilon \frac{\partial w}{\partial \varphi}(I',\varphi) \\ \varphi' &= \varphi + \frac{\partial k}{\partial I'}(I') + \varepsilon \frac{\partial w}{\partial I}(I',\varphi) \quad , \end{aligned} \tag{41}$$

define implicitly a symplectic diffeomorphism: $\mathcal{C}(I,\varphi) = (I',\varphi')$.

We consider then a near to the identity symplectic map $\Phi : (I_0,\varphi_0) \mapsto (I,\varphi)$ generated by $\tilde\chi(I,\varphi_0) = I\cdot\varphi_0 + \varepsilon\chi(I,\varphi_0)$, where χ is analytic in a domain $D' \times \mathbb{T}^n$, with $D' \subseteq B'$. We want to compute the generating function for the map $\mathcal{C}' = \Phi^{-1} \circ \mathcal{C} \circ \Phi$. It is convenient to refer to the following diagram:

$$\begin{array}{ccc} (I,\varphi) & \xrightarrow{\ \mathcal{C}\ } & (I',\varphi') \\ \Phi\big\uparrow & & \big\uparrow\Phi \\ (I_0,\varphi_0) & \xrightarrow{\ \mathcal{C}'\ } & (I'_0,\varphi'_0). \end{array} \tag{42}$$

Lemma 6 *Let S,χ, $\mathcal{C}$, $\mathcal{C}'$ and Φ as above. If ε is suitably small, the transformation $\mathcal{C}'$ is generated by:*

$$\begin{aligned} S' \;=\; & I\cdot\varphi_0 - I\cdot\varphi + I'_0\cdot\varphi'_0 - I'\cdot\varphi'_0 + I'\cdot\varphi + \\ & k(I') + \varepsilon w(I',\varphi) + \varepsilon[\chi(I,\varphi_0) - \chi(I',\varphi'_0)] \quad , \end{aligned} \tag{43}$$

where $I,\varphi,I',\varphi',I_0,\varphi'_0$ are functions of the independent variables I'_0,φ_0.

Proof of Lemma 6. Because of equations (41) the standard Lie condition reads:

$$(I - I') \cdot d\varphi + (\varphi' - \varphi) \cdot dI' = d(k(I') + \varepsilon w(I', \varphi)) \quad , \tag{44}$$

and it is defined globally on $B' \times \mathbb{T}^n$. Similarly, to compute the generating function S' it is sufficient to compute the differential form $(I_0 - I_0') \cdot d\varphi_0 + (\varphi_0' - \varphi_0) \cdot dI_0'$, because it is:

$$dS'(I_0', \varphi_0) = d(I_0' \cdot \varphi_0) + (I_0 - I_0') \cdot d\varphi_0 + (\varphi_0' - \varphi_0) \cdot dI_0' \quad . \tag{45}$$

With reference to diagram (42), it is:

$$\begin{aligned} dS(I', \varphi) &= d(I' \cdot \varphi) + (I - I') \cdot d\varphi + (\varphi' - \varphi) \cdot dI' \\ d\tilde{\chi}(I, \varphi_0) &= d(I \cdot \varphi_0) + (I_0 - I) \cdot d\varphi_0 + (\varphi - \varphi_0) \cdot dI \\ d\tilde{\chi}(I', \varphi_0') &= d(I' \cdot \varphi_0') + (I_0' - I') \cdot d\varphi_0' + (\varphi' - \varphi_0') \cdot dI' \quad , \end{aligned} \tag{46}$$

and therefore:

$$\begin{aligned} d(S(I', \varphi) + \tilde{\chi}(I, \varphi_0) - \tilde{\chi}(I', \varphi_0')) = {} & d(I_0' \cdot \varphi_0 + I \cdot \varphi - I_0' \cdot \varphi_0') \\ & + (I_0 - I_0') \cdot d\varphi_0 + (\varphi_0' - \varphi_0) \cdot dI_0' \quad , \end{aligned} \tag{47}$$

so that the new generating function is:

$$\begin{aligned} S' &= I_0' \cdot \varphi_0' - I \cdot \varphi + S(I', \varphi) + \tilde{\chi}(I, \varphi_0) - \tilde{\chi}(I', \varphi_0') \\ &= I_0' \cdot \varphi_0' - I \cdot \varphi + I' \cdot \varphi + I \cdot \varphi_0 - I' \cdot \varphi_0' + k(I') \\ &\quad + \varepsilon w(I', \varphi) + \varepsilon[\chi(I, \varphi_0) - \chi(I', \varphi_0')] \quad , \end{aligned} \tag{48}$$

where $I, \varphi, I', \varphi', I_0, \varphi_0'$ are functions of the independent variables I_0', φ_0.

Lemma 6 provides the analytic expression of the generating function of $\mathcal{C}'$. In the following lemma we provide estimates for $\mathcal{C}'$ on convenient domains. To fix notations, for any set $V' \subseteq \mathbb{R}^n$ and any $\sigma = (\sigma_I, \sigma_\varphi)$ we will denote $V_\sigma = V'_{\sigma_I} \times \mathbb{T}^n_{\sigma_\varphi}$.

Lemma 7 *Consider the generating function:*

$$W(I', \varphi) = I' \cdot \varphi + k(I') + \varepsilon w(I', \varphi) \quad , \tag{49}$$

where $w : B'_{\tilde{\varrho}_I} \times \mathbb{T}^n_{\tilde{\varrho}_\varphi} \to \mathbb{C}$, *and* $k : B'_{\tilde{\varrho}_I} \to \mathbb{C}$ *are analytic. Let* Δ_I, Δ_φ *defined by:*

$$\Delta_I = \frac{\varepsilon n}{4} \max_i \left| \frac{\partial w}{\partial \varphi_i} \right|_{\tilde{\varrho}} \quad , \Delta_\varphi = \frac{\varepsilon n}{4} \max_i \left| \frac{\partial w}{\partial I_i} \right|_{\tilde{\varrho}} \tag{50}$$

satisfy the following inequalities:

$$\frac{\Delta_\varphi}{2} + \mathcal{M}\tilde{\varrho}_I < \tilde{\varrho}_\varphi \quad , \quad \Delta_I < \tilde{\varrho}_I \tag{51}$$

where $\mathcal{M}$ *is a Lipschitz constant for* $\frac{\partial k}{\partial I'}$ *on* $B'_{\tilde\varrho_I}$. *If:*

$$\varepsilon n \max_{i,j} \left| \frac{\partial^2 w}{\partial\varphi_i \partial I'_j} \right|_{\tilde\varrho} \leq \frac{1}{2} \quad , \tag{52}$$

then the equations:

$$\begin{aligned} I &= I' + \varepsilon \frac{\partial w}{\partial \varphi}(I',\varphi) \\ \varphi' &= \varphi + \frac{\partial k}{\partial I'}(I') + \varepsilon \frac{\partial w}{\partial I'}(I',\varphi) \end{aligned} \tag{53}$$

define implicitly the analytic symplectic diffeomorphism: $\mathcal{C}(I,\varphi) = (I',\varphi')$ *such that:*

- *for any* $(I,\varphi) \in B'_{\tilde\varrho_I - \Delta_I} \times \mathbb{T}^n_{\tilde\varrho_\varphi}$ *there exists* $I' \in B'_{\tilde\varrho_I}$ *such that:*

  $$I = I' + \varepsilon \frac{\partial w}{\partial \varphi}(I',\varphi) \quad , \tag{54}$$

  and therefore $B'_{\tilde\varrho_I - \Delta_I} \times \mathbb{T}^n_{\tilde\varrho_\varphi}$ *is in the domain of* $\mathcal{C}$*;*

- *for any* $(I,\varphi) \in B'_{\tilde\varrho_I - \Delta_I} \times \mathbb{T}^n_{\tilde\varrho_\varphi - \Delta_\varphi}$ *it is:*

  $$\begin{array}{c} \max_i |I'_i - I_i| \leq \frac{\Delta_I}{2} \\ \max_i \left|\varphi'_i - \varphi_i - \frac{\partial k}{\partial I'_i}(I')\right| \leq \frac{\Delta_\varphi}{2} \\ \max_i |\mathrm{Im}\,\varphi'_i - \mathrm{Im}\,\varphi_i| \leq \frac{\Delta_\varphi}{2} + \mathcal{M}\tilde\varrho_I \end{array} \quad , \tag{55}$$

  so that $\mathcal{C}(B_{\tilde\varrho-\Delta}) \subseteq B'_{\tilde\varrho_I - \frac{\Delta_I}{2}} \times \mathbb{T}^n_{\tilde\varrho_\varphi - \frac{\Delta_\varphi}{2} + \mathcal{M}\tilde\varrho_I}$*;*

- *for any* $(I',\varphi') \in B'_{\tilde\varrho_I} \times \mathbb{T}^n_{\tilde\varrho_\varphi - \Delta_\varphi}$ *there exists* $\varphi \in \mathbb{T}^n_{\tilde\varrho_\varphi}$ *such that:*

  $$\varphi' = \varphi + \frac{\partial k}{\partial I'}(I') + \varepsilon \frac{\partial w}{\partial I'}(I',\varphi) \quad , \tag{56}$$

  and therefore $B'_{\tilde\varrho_I} \times \mathbb{T}^n_{\tilde\varrho_\varphi - \Delta_\varphi}$ *is in the domain of* $\mathcal{C}^{-1}$*;*

- *for any* $(I',\varphi') \in B'_{\tilde\varrho_I - \Delta_I} \times \mathbb{T}^n_{\tilde\varrho_\varphi - \Delta_\varphi}$, *denoting* $(I,\varphi) = \mathcal{C}^{-1}(I',\varphi')$ *it is:*

  $$\begin{array}{c} \max_i |I'_i - I_i| \leq \frac{\Delta_I}{2} \\ \max_i \left|\varphi'_i - \varphi_i - \frac{\partial k}{\partial I'_i}(I')\right| \leq \frac{\Delta_\varphi}{2} \\ \max_i |\mathrm{Im}\,\varphi'_i - \mathrm{Im}\,\varphi_i| \leq \frac{\Delta_\varphi}{2} + \mathcal{M}\tilde\varrho_I \end{array} \quad , \tag{57}$$

  so that $\mathcal{C}^{-1}(B_{\tilde\varrho-\Delta}) \subseteq B'_{\tilde\varrho_I - \frac{\Delta_I}{2}} \times \mathbb{T}^n_{\tilde\varrho_\varphi - \frac{\Delta_\varphi}{2} + \mathcal{M}\tilde\varrho_I}$.

Proof of Lemma 7. We first prove that equations (53) can be inverted to define $\mathcal{C}$ in $B'_{\tilde{\varrho}_I-\Delta_I}\times\mathbb{T}^n_{\tilde{\varrho}_\varphi}$. Let us fix $\varphi\in\mathbb{T}^n_{\tilde{\varrho}_\varphi}$, and denote $B_\varphi=\cup_{I'\in B'_{\tilde{\varrho}_I}}\{I'+\varepsilon\frac{\partial w}{\partial\varphi}(I',\varphi)\}$. The map:

$$\begin{array}{ll} B'_{\tilde{\varrho}_I} & \longrightarrow B_\varphi \\ I' & \longmapsto I=I'+\varepsilon\dfrac{\partial w}{\partial\varphi}(I',\varphi) \end{array} \tag{58}$$

is injective. In fact, if there exist $I'_1,I'_2\in B'_{\tilde{\varrho}_I}$ such that:

$$I'_1+\varepsilon\frac{\partial w}{\partial\varphi}(I'_1,\varphi)=I'_2+\varepsilon\frac{\partial w}{\partial\varphi}(I'_2,\varphi)\quad, \tag{59}$$

then it is trivially:

$$\|I'_1-I'_2\|\leq\varepsilon n\max_{i,j}\left|\frac{\partial^2 w}{\partial\varphi_i\partial I'_j}\right|_{\tilde{\varrho}}\|I'_1-I'_2\|\quad. \tag{60}$$

If $I'_1\neq I'_2$, the above inequality is in contrast with eq. (52). This means that we can define a map $u:\mathcal{B}\to B'_{\tilde{\varrho}_I}$, where

$$\mathcal{B}=\{(\varphi,I)\ \text{ such that }\ \varphi\in\mathbb{T}^n_{\tilde{\varrho}_\varphi}\ \text{ and }\ I\in B_\varphi\}\quad,$$

such that:

$$I=u(I,\varphi)+\varepsilon\frac{\partial w}{\partial\varphi}(u(I,\varphi),\varphi)\quad. \tag{61}$$

The map u is analytic. To prove this we use the local inversion theorem for holomorphic maps. Indeed, for any $(I'_0,\varphi_0)\in B'_{\tilde{\varrho}_I}\times\mathbb{T}^n_{\tilde{\varrho}_\varphi}$, the Jacobian matrix of $I'+\varepsilon\frac{\partial w}{\partial\varphi}(I',\varphi)$ is:

$$J_{ik}=\delta_{ik}+\varepsilon\frac{\partial^2 w}{\partial I_k\varphi_i}(I',\varphi)\quad, \tag{62}$$

which is non-singular because by eq. (52) it follows: $\varepsilon n\left|\frac{\partial^2 w}{\partial\varphi_i\partial I'_j}\right|_{\tilde{\varrho}}<1$. This proves that u is analytic in a suitable neighborhood of $(I'_0+\varepsilon\frac{\partial w}{\partial\varphi}(I'_0,\varphi_0),\varphi_0)$, for any $(I'_0,\varphi_0)\in B'_{\tilde{\varrho}_I}\times\mathbb{T}^n_{\tilde{\varrho}_\varphi}$, and therefore u is analytic in $\mathcal{B}$.

The second equation of (53) defines φ' as a function of I',φ, and therefore it is possible to define φ' as a function of I,φ, analytic in $\mathcal{B}$. This allows one to define the analytic map $\mathcal{C}:\mathcal{B}\to B'_{\tilde{\varrho}_I}\times\mathbb{T}^n_{\tilde{\varrho}_\varphi}$:

$$\begin{array}{rcl} I' & = & u(I,\varphi) \\ \varphi' & = & \varphi+\dfrac{\partial k}{\partial I'}(u(I,\varphi))+\varepsilon\dfrac{\partial w}{\partial I'}(u(I,\varphi),\varphi)\quad. \end{array} \tag{63}$$

Now, we prove: $B'_{\tilde{\varrho}_I-\Delta_I}\times\mathbb{T}^n_{\tilde{\varrho}_\varphi}\subseteq\mathcal{B}$, so that $B'_{\tilde{\varrho}_I-\Delta_I}\times\mathbb{T}^n_{\tilde{\varrho}_\varphi}$ is in the domain of $\mathcal{C}$. Specifically, for any $(I,\varphi)\in B'_{\tilde{\varrho}_I-\Delta_I}\times\mathbb{T}^n_{\tilde{\varrho}_\varphi}$ we prove that there exists $I'\in B'_{\tilde{\varrho}_I}$

such that: $I = I' + \varepsilon \frac{\partial w}{\partial \varphi}(I', \varphi)$. Let us consider the map $y : B'_{\tilde{\varrho}_I} \to \mathbb{C}^n$ such that $y(I') = I - \varepsilon \frac{\partial w}{\partial \varphi}(I', \varphi)$. We prove that y has a fixed point $\overline{I} \in B'_{\tilde{\varrho}_I}$. We consider the sequence $I_j = y^j(I)$. Using (52) and (50) one easily proves: $\|I_j - I\| \le \Delta_I/2$ for any $j \in \mathbb{N}$, so that all I_j are in the domain of y, and moreover I_j is a Cauchy sequence, so that its limit $\overline{I}$ exists, it is $\overline{I} \in B'_{\tilde{\varrho}_I}$, and it is a fixed point of y. This proves $B'_{\tilde{\varrho}_I - \Delta_I} \times \mathbb{T}^n_{\tilde{\varrho}_\varphi} \subseteq \mathcal{B}$.

In a very similar way we prove that: $B'_{\tilde{\varrho}_I} \times \mathbb{T}^n_{\tilde{\varrho}_\varphi - \Delta_\varphi}$ is in the domain of $\mathcal{C}^{-1}$. Specifically, for any $(I', \varphi') \in B'_{\tilde{\varrho}_I} \times \mathbb{T}^n_{\tilde{\varrho}_\varphi - \Delta_\varphi}$ we prove that there exists $\varphi \in \mathbb{T}^n_{\tilde{\varrho}_\varphi}$ such that: $\varphi' = \varphi + \frac{\partial k}{\partial I'}(I') + \varepsilon \frac{\partial w}{\partial I'}(I', \varphi)$. Let us consider the map: $z : \mathbb{T}^n_{\tilde{\varrho}_\varphi} \to \mathbb{C}^n$, such that $z(\varphi) = \varphi' - \frac{\partial k}{\partial I'}(I') - \varepsilon \frac{\partial w}{\partial I'}(I', \varphi)$. We prove that z has a fixed point $\overline{\varphi} \in \mathbb{T}^n_{\tilde{\varrho}_\varphi - \Delta_\varphi}$. We consider the sequence $\varphi_j = z^j(\varphi' - \frac{\partial k}{\partial I'}(I'))$. Using (52) and (50) one proves: $\|\varphi_j - \varphi_0\| \le \Delta_\varphi/2$ for any $j \in \mathbb{N}$, so that all φ_j are in the domain of z, and moreover φ_j is a Cauchy sequence, so that its limit $\overline{\varphi}$ exists, it is $\overline{\varphi} \in \mathbb{T}^n_{\tilde{\varrho}_\varphi}$, and it is a fixed point of z. eqs. (55), (57) immediately follow from eq. (50).

Lemma 8 *Consider the generating functions:*

$$\begin{aligned} W(I', \varphi) &= I' \cdot \varphi + h(I') + \varepsilon w(I', \varphi) \\ \tilde{\chi}(I', \varphi) &= I' \cdot \varphi + \varepsilon \chi(I', \varphi) \end{aligned} \tag{64}$$

where $w : B'_{\tilde{\varrho}_I} \times \mathbb{T}^n_{\tilde{\varrho}_\varphi} \to \mathbb{C}$, $\chi : B'_{\tilde{\varrho}_I} \times \mathbb{T}^n_{\tilde{\varrho}_\varphi} \to \mathbb{C}$ *and* $h : B'_{\tilde{\varrho}_I} \to \mathbb{C}$ *are analytic and satisfy the estimates:*

$$\varepsilon n \max_{i,j} \left| \frac{\partial^2 w}{\partial \varphi_i \partial I'_j} \right|_{\tilde{\varrho}} \le \frac{1}{2} \quad , \quad \varepsilon n \max_{i,j} \left| \frac{\partial^2 \chi}{\partial \varphi_i \partial I'_j} \right|_{\tilde{\varrho} - \delta} \le \frac{1}{2} \tag{65}$$

with some $\delta < \tilde{\varrho}$. *Let now* Δ *and* ζ *be defined by:*

$$\begin{aligned} \Delta_I &= \frac{\varepsilon n}{4} \max_i \left| \frac{\partial w}{\partial \varphi_i} \right|_{\tilde{\varrho}} \quad , \quad \Delta_\varphi = \frac{\varepsilon n}{4} \max_i \left| \frac{\partial w}{\partial I_i} \right|_{\tilde{\varrho}} \\ \zeta_I &= \frac{\varepsilon n}{4} \max_i \left| \frac{\partial \chi}{\partial \varphi_i} \right|_{\tilde{\varrho} - \delta} \quad , \quad \zeta_\varphi = \frac{\varepsilon n}{4} \max_i \left| \frac{\partial \chi}{\partial I_i} \right|_{\tilde{\varrho} - \delta} \end{aligned} \tag{66}$$

satisfying:

$$\zeta < \tilde{\varrho} - \delta \quad , \quad \frac{\Delta_\varphi}{2} + \zeta_\varphi + \mathcal{M} \tilde{\varrho}_I \le \delta_\varphi \quad , \quad \Delta_I + \zeta_I \le \delta_I \tag{67}$$

where $\mathcal{M}$ *is a Lipschitz constant for* $\frac{\partial h}{\partial I'}$ *on* $B'_{\tilde{\varrho}_I}$. *Then, denoting by* $\mathcal{C}$ *the canonical transformation generated by* W, *with* Φ *the canonical transformation generated by* $\tilde{\chi}$ *and with* $\mathcal{C}' = \Phi^{-1} \circ \mathcal{C} \circ \Phi$, *the maps* $\mathcal{C}$ *and* $\mathcal{C}^{-1}$ *are analytic and symplectic diffeomorphisms from* $B_{\tilde{\varrho} - \Delta}$ *on their image;* Φ *and* Φ^{-1} *are analytic and symplectic diffeomorphisms from* $B_{\tilde{\varrho} - \delta - \zeta}$ *on their image;* $\mathcal{C}'$ *and* ${\mathcal{C}'}^{-1}$ *are analytic and symplectic diffeomorphisms from* $B_{\tilde{\varrho} - 2\delta - \Delta - 2\zeta}$ *on their image and it is:*

$$\mathcal{C}'(B_{\tilde{\varrho} - 2\delta - \Delta - 2\zeta}) \subseteq B_{\tilde{\varrho} - \delta - \Delta - 2\zeta} \quad , \quad {\mathcal{C}'}^{-1}(B_{\tilde{\varrho} - 2\delta - \Delta - 2\zeta}) \subseteq B_{\tilde{\varrho} - \delta - \Delta - 2\zeta} \quad . \tag{68}$$

Let now η *satisfying:*

$$2\delta + \Delta + 2\zeta + 2\eta < \tilde{\varrho} \quad , \quad \Delta_I + \zeta_I \leq \frac{\eta_I}{2n} \quad . \tag{69}$$

The map $\mathcal{C}'$ *can be generated by the function:*

$$\begin{aligned} W'(I',\varphi) &= I'\cdot\varphi + h(I') + \varepsilon(w(I_0',\varphi_0) + \chi(I_0',\varphi_0) - \chi(I_0',\varphi_0+\omega(I_0'))) \\ &+ \varepsilon w'(I',\varphi) \end{aligned} \tag{70}$$

where w' *is analytic in* $B_{\tilde{\varrho}-2\delta-\Delta-2\zeta-2\eta}$ *and satisfies the following estimate:*

$$|w'|_{\tilde{\varrho}-2\delta-\Delta-2\zeta-2\eta} \leq \varepsilon^{-1}\Big(\frac{9}{4}\mathcal{M}n\zeta_I^2 + 4\Delta_I\zeta_\varphi + 4\Delta_\varphi\zeta_I + 8\zeta_I\zeta_\varphi\Big) \quad . \tag{71}$$

Proof of Lemma 8. It is convenient to refer to the following diagram:

$$\begin{array}{ccc} (I,\varphi) & \xrightarrow{\ \mathcal{C}\ } & (I',\varphi') \\ {\scriptstyle\Phi}\big\uparrow & & \big\uparrow{\scriptstyle\Phi} \\ (I_0,\varphi_0) & \xrightarrow{\ \mathcal{C}'\ } & (I_0',\varphi_0') \end{array} \quad . \tag{72}$$

First, we prove that $\mathcal{C}'$ can be generated by a function $W'(I_0',\varphi_0)$. We denote:

$$I_0' = I_0 + u(I_0,\varphi_0) \quad . \tag{73}$$

Because:

$$\frac{\Delta_\varphi}{2} + \zeta_\varphi + \mathcal{M}\tilde{\varrho}_I < \delta_\varphi \quad , \quad \frac{\Delta_I}{2} + \zeta_I \leq \delta_I \quad , \tag{74}$$

for any $(I_0,\varphi_0) \in B_{\tilde{\varrho}-\delta-\Delta-2\zeta}$ it is $(I,\varphi),(I',\varphi'),(I_0',\varphi_0') \in B'_{\tilde{\varrho}_I-\delta_I-\frac{\Delta_I}{2}-\zeta_I} \times \mathbb{T}^n_{\tilde{\varrho}_\varphi-\delta_\varphi-\frac{\Delta_\varphi}{2}-\zeta_\varphi+\mathcal{M}\tilde{\varrho}_I}$, and in particular the functions u_j are analytic in $B_{\tilde{\varrho}-\delta-\Delta-2\zeta}$. For Lemma 7, they satisfy the estimates:

$$|u_j|_{\tilde{\varrho}-2\delta-\Delta-2\zeta} \leq \frac{\Delta_I}{2} + \zeta_I \quad , \tag{75}$$

and therefore from Cauchy estimates it follows:

$$\left|\frac{\partial u_j}{\partial I_{0k}}\right|_{\tilde{\varrho}-2\delta-\Delta-2\zeta-\eta} \leq \frac{\Delta_I}{2\eta_I} + \frac{\zeta_I}{\eta_I} \quad , \tag{76}$$

and because $\Delta_I + \zeta_I \leq \frac{\eta_I}{2n}$ the inversion theorem for holomorphic functions allows to define functions $\tilde{u}_j$ such that:

$$I_0 = I_0' + \tilde{u}(I_0',\varphi_0) \quad , \tag{77}$$

which are analytic in:

$$A = \bigcup_{\varphi_0 \in \mathbb{T}^n_{\tilde\varrho_\varphi - 2\delta_\varphi - \Delta_\varphi - 2\zeta_\varphi - \eta_\varphi}} \{ \bigcup_{I_0 \in B'_{\tilde\varrho_I - 2\delta_I - \Delta_I - 2\zeta_I - \eta_I}} (I_0 + u(I_0,\varphi_0), \varphi_0)\} \quad . \tag{78}$$

Moreover, it is $B_{\tilde\varrho - 2\delta - \Delta - 2\zeta - 2\eta} \subseteq A$: for any $(I'_0, \varphi_0) \in B'_{\tilde\varrho_I - 2\delta_I - \Delta_I - 2\zeta_I - 2\eta_I} \times \mathbb{T}^n_{\tilde\varrho_\varphi - 2\delta_\varphi - \Delta_\varphi - 2\zeta_\varphi - \eta_\varphi}$ we prove that there exists $I_0 \in B'_{\tilde\varrho_I - 2\delta_I - \Delta_I - 2\zeta_I - \eta_I}$ such that: $I'_0 = I_0 + \varepsilon u(I_0, \varphi_0)$. Let us consider the map: $y : B'_{\tilde\varrho_I - 2\delta_I - \Delta_I - 2\zeta_I - \eta_I} \rightarrow \mathbb{C}^n$, such that $y(I) = I'_0 - \varepsilon u(I, \varphi_0)$. We prove that y has a fixed point $\overline{I} \in B'_{\tilde\varrho_I - 2\delta_I - \Delta_I - 2\zeta_I - \eta_I}$, and consequently $I_0 = \overline{I} \in B'_{\tilde\varrho_I - 2\delta_I - \Delta_I - 2\zeta_I - \eta_I}$. We consider the sequence $I_j = y^j(I'_0)$. Using the inequality:

$$\Delta_I + \zeta_I \leq \frac{\eta_I}{2n} \quad , \tag{79}$$

one easily proves that I_j is a Cauchy sequence and $\|I_j - I'_0\| \leq \eta_I$ for any $j \in \mathbb{N}$. Therefore, its limit $\overline{I}$ exists, it is in $B'_{\tilde\varrho_I - 2\delta_I - \Delta_I - 2\zeta_I - \eta_I}$ and it is a fixed point of y.

Because I_0 is an analytic function of I'_0, φ_0, we can find a function $W'(I'_0, \varphi_0)$ satisfying:

$$dW'(I'_0, \varphi_0) = I_0 \cdot d\varphi_0 + \varphi'_0 \cdot dI'_0 = d(I'_0 \cdot \varphi_0) + (I_0 - I'_0) \cdot d\varphi_0 + (\varphi'_0 - \varphi_0) \cdot dI'_0 \tag{80}$$

which defines implicitly the map $\mathcal{C}'$ on a suitable domain. We work out a more explicit expression for W'.

For any $(I'_0, \varphi_0) \in B_{\tilde\varrho - 2\delta - \Delta - 2\zeta - 2\eta}$, it is $I_0 = I'_0 + \tilde u(I'_0, \varphi_0) \in B_{\tilde\varrho - 2\delta - \Delta - 2\zeta - \eta}$, and then $(I, \varphi), (I', \varphi'), \varphi'_0$ are analytic functions of I'_0, φ_0. Applying Lemma 6 to the generating maps W and $\tilde\chi$ we obtain the new generating function:

$$\begin{aligned} W' \;\; = \;\; & I'_0 \cdot \varphi_0 + h(I') + \varepsilon(w(I', \varphi) + \chi(I, \varphi_0) - \chi(I', \varphi'_0)) \\ & + (I'_0 \cdot \varphi'_0 - I \cdot \varphi + I' \cdot \varphi + I \cdot \varphi_0 - I' \cdot \varphi'_0 - I'_0 \cdot \varphi_0) \quad , \end{aligned} \tag{81}$$

where I, φ, I', φ', I_0, φ'_0 are functions of the independent variables $(I'_0, \varphi_0) \in B_{\tilde\varrho - 2\delta - \Delta - 2\zeta - 2\eta}$. We need to give more explicit expression to (81). We observe that it is:

$$\begin{aligned} & I'_0 \cdot \varphi'_0 - I \cdot \varphi + I' \cdot \varphi + I \cdot \varphi_0 - I' \cdot \varphi'_0 - I'_0 \cdot \varphi_0 = I'_0 \cdot (\varphi'_0 - \varphi_0) \\ & + I \cdot (\varphi_0 - \varphi) + I' \cdot (\varphi - \varphi'_0) = (I'_0 - I') \cdot (\varphi'_0 - \varphi) + (I'_0 - I) \cdot (\varphi - \varphi_0) \\ & = \varepsilon \tfrac{\partial \chi}{\partial \varphi}(I', \varphi'_0) \cdot \Big(- \varepsilon \tfrac{\partial \chi}{\partial I'}(I', \varphi'_0) + \omega(I') + \varepsilon \tfrac{\partial w}{\partial I'}(I', \varphi) \Big) \\ & + \Big(\varepsilon \tfrac{\partial \chi}{\partial \varphi}(I', \varphi'_0) - \varepsilon \tfrac{\partial w}{\partial \varphi}(I', \varphi) \Big) \cdot \varepsilon \tfrac{\partial \chi}{\partial I'}(I, \varphi_0) \end{aligned} \tag{82}$$

and therefore it is:

$$\begin{aligned} W' \;\; = \;\; & I'_0 \cdot \varphi_0 + h(I') + \varepsilon(w(I', \varphi) + \chi(I, \varphi_0) - \chi(I', \varphi'_0)) \\ & + \varepsilon \omega(I') \cdot \frac{\partial \chi}{\partial \varphi}(I', \varphi'_0) + \varepsilon^2 \Big(\frac{\partial \chi}{\partial \varphi}(I', \varphi'_0) \cdot \Big(- \frac{\partial \chi}{\partial I'}(I', \varphi'_0) + \frac{\partial w}{\partial I'}(I', \varphi) \Big) \\ & - \Big(- \frac{\partial \chi}{\partial \varphi}(I', \varphi'_0) + \frac{\partial w}{\partial \varphi}(I', \varphi) \Big) \cdot \frac{\partial \chi}{\partial I'}(I, \varphi_0) \Big) \end{aligned}$$

$$\begin{aligned} &= I'_0 \cdot \varphi_0 + h(I'_0) + \varepsilon[w(I'_0,\varphi_0) + \chi(I'_0,\varphi_0) - \chi(I'_0,\varphi_0+\omega(I'_0))] \\ &+ \varepsilon w'(I'_0,\varphi_0) \ , \end{aligned} \tag{83}$$

where:

$$\begin{aligned} w' &= \varepsilon^{-1}\left[h(I') - h(I'_0) + \varepsilon\omega(I')\cdot\frac{\partial\chi}{\partial\varphi}(I',\varphi'_0)\right] + [w(I',\varphi) - w(I'_0,\varphi_0)] \\ &+[\chi(I,\varphi_0) - \chi(I'_0,\varphi_0)] + [\chi(I'_0,\varphi_0+\omega(I'_0)) - \chi(I',\varphi'_0)] \\ &+\varepsilon\Big(\frac{\partial\chi}{\partial\varphi}(I',\varphi'_0)\cdot\Big(-\frac{\partial\chi}{\partial I'}(I',\varphi'_0) + \frac{\partial w}{\partial I'}(I',\varphi)\Big) \\ &-\Big(-\frac{\partial\chi}{\partial\varphi}(I',\varphi'_0) + \frac{\partial w}{\partial\varphi}(I',\varphi)\Big)\cdot\frac{\partial\chi}{\partial I'}(I,\varphi_0)\Big) \end{aligned} \tag{84}$$

We now provide the estimates for the different contributions to w' on the set $I'_0,\varphi_0 \in B_{\tilde\varrho-2\delta-\Delta-2\zeta-2\eta}$. First of all, we recall that the functions $\tilde u_j$ defined in (77) are analytic in $B_{\tilde\varrho-2\delta-\Delta-2\zeta}$. Therefore, for any $I'_0,\varphi_0 \in B_{\tilde\varrho-2\delta-\Delta-2\zeta-2\eta}$ it is $(I_0,\varphi_0) \in B_{\tilde\varrho-2\delta-\Delta-2\zeta-\eta}$, and then, by analyticity of $\mathcal{C}'$ in $B_{\tilde\varrho-2\delta-\Delta-2\zeta-\eta}$, all $I,\varphi,I',\varphi',I_0,\varphi'_0$ are analytic functions of $I'_0,\varphi_0 \in B_{\tilde\varrho-2\delta-\Delta-2\zeta-2\eta}$. Moreover, it is:

$$\begin{aligned} I_0,I',I &\in B_{\tilde\varrho_I-\delta_I-\Delta_I-2\zeta_I-\eta_I} \\ \varphi_0,\varphi',\varphi &\in \mathbb{T}^n_{\tilde\varrho_\varphi-\delta_\varphi-\Delta_\varphi-2\zeta_\varphi-\eta_\varphi} \ . \end{aligned} \tag{85}$$

The estimate of the Taylor remainder of h around I' provides:

$$\begin{aligned} \left|h(I') - h(I'_0) + \varepsilon\omega(I')\cdot\frac{\partial\chi}{\partial\varphi}(I',\varphi'_0)\right| &= |h(I'_0) - h(I') - \omega(I')\cdot(I'-I'_0)| \\ &\le \mathcal{M}\,\|I'-I'_0\|^2 \le \frac{1}{4}\mathcal{M}n\zeta_I^2 \ , \end{aligned} \tag{86}$$

while the other terms are estimated by:

$$\begin{aligned} |w(I',\varphi) - w(I'_0,\varphi_0)| &\le 2\varepsilon^{-1}(\Delta_I\zeta_\varphi + \Delta_\varphi\zeta_I) \\ |\chi(I,\varphi_0) - \chi(I'_0,\varphi_0)| &\le 2\varepsilon^{-1}(\Delta_I + \zeta_I)\zeta_\varphi \\ |\chi(I'_0,\varphi'_0) - \chi(I',\varphi'_0)| &\le 2\varepsilon^{-1}\zeta_I\zeta_\varphi \\ \left|\chi(I'_0,\varphi'_0) - \chi(I'_0,\varphi_0 + \frac{\partial h}{\partial I}(I'_0))\right| &\le 4\varepsilon^{-1}(\zeta_\varphi + \frac{\Delta_\varphi}{2} + \mathcal{M}n\frac{\zeta_I}{2})\zeta_I \ , \end{aligned} \tag{87}$$

and finally it is:

$$\begin{aligned} &\varepsilon\Big|\Big(\tfrac{\partial\chi}{\partial\varphi}(I',\varphi'_0)\cdot\Big(-\tfrac{\partial\chi}{\partial I'}(I',\varphi'_0) + \tfrac{\partial(u+v)}{\partial I'}(I',\varphi)\Big) \\ &-\Big(-\tfrac{\partial\chi}{\partial\varphi}(I',\varphi'_0) + \tfrac{\partial(u+v)}{\partial\varphi}(I',\varphi)\Big)\cdot\tfrac{\partial\chi}{\partial I'}(I,\varphi_0)\Big)\Big| \\ &\le (\varepsilon n)^{-1}(2\zeta_I\zeta_\varphi + \zeta_I\Delta_\varphi + \zeta_\varphi\Delta_I) \ . \end{aligned} \tag{88}$$

Therefore, collecting all estimates, w' satisfies the following estimate:

$$|w'|_{\tilde\varrho-2\delta-\Delta-2\zeta-2\eta} \le \varepsilon^{-1}\Big(\frac{9}{4}\mathcal{M}n\zeta_I^2 + 4\Delta_I\zeta_\varphi + 4\Delta_\varphi\zeta_I + 8\zeta_I\zeta_\varphi\Big) \ . \tag{89}$$

6 Proof of Lemma 5

In this section, for any $\delta = (\delta_I, \delta_\varphi)$ we denote $D_\delta = D'_{\delta_I} \times \mathbb{T}^n_{\delta_\varphi}$.

As usual in perturbation theory, we construct the transformation Φ as the composition of many near to the identity symplectic maps, each of which reduces the norm of the remainder by a suitable factor. More precisely, we construct the canonical transformations $\Phi_1, \ldots \Phi_N$, with N defined as in (31):

$$N = \frac{1}{16n} r_\varphi \frac{K}{\ln K} \quad , \tag{90}$$

(we remark that because $K > 2$ it is $N \leq \frac{1}{8n} r_\varphi K$) and the extension vectors: $r^0, r^1, \ldots, r^N$ with $r_0 = r$ and

$$r^i = \frac{3}{4} r - (i-1)\sigma \quad , \quad i = 1, \ldots, N \tag{91}$$

where:

$$\sigma = \frac{3r}{16N} \quad , \tag{92}$$

such that: $\Phi_i(D_{r^i}) \subseteq D_{r^{i-1}}$ is a symplectic diffeomorphism on its image, and it is also $\Phi_i^{-1}(D_{r^i}) \subseteq D_{r^{i-1}}$; denoting $\mathcal{C}_i = \Psi_i^{-1} \circ \mathcal{C} \circ \Psi_i$, where $\Psi_i = \Phi_1 \circ \ldots \Phi_i$, the map:

$$\mathcal{C}_i : D_{r^i} \longrightarrow D_r \quad , \tag{93}$$

is a diffeomorphism on its image and it is generated by a function $W_i : D_{r^i} \to \mathbb{C}$ of the form:

$$W_i(I', \varphi) = I' \cdot \varphi + h(I') + \varepsilon u^i(I', \varphi) + \varepsilon v^i(I', \varphi) \tag{94}$$

where $u^0 = 0$, $v^0 = f$, $u^i = u^{i-1} + \Pi_\Lambda T_K v^{i-1}$ and v^i satisfies the estimate:

$$\left| v^i \right|_{r^i} \leq \frac{1}{4^i} \frac{|f|_\varrho}{N} \tag{95}$$

for any $i \geq 1$. If $\Phi_1, \ldots, \Phi_N$ exist, then the map Φ of Lemma 5 is $\Phi = \Phi_1 \circ \ldots \circ \Phi_N$. Indeed, it is: $\Phi : D_{\frac{r}{2}} \to D_r$; Φ is a symplectic diffeomorphism on its image; $\mathcal{C}' = \Phi^{-1} \circ \mathcal{C} \circ \Phi$ is generated by W' of the form (37) where

$$u = \sum_{j=0}^{N} u^j \quad , \quad \mathcal{R} = v^N \quad , \tag{96}$$

and therefore it is:

$$\begin{aligned} |u - \Pi_\Lambda T_K f|_{\frac{r}{2}} &\leq \sum_{j=1}^{N} \left| v^j \right|_{r^j} \leq \sum_{j=1}^{N} \frac{|f|_\varrho}{4^j} \leq 2\,|f|_\varrho \\ |\mathcal{R}|_{\frac{r}{2}} &\leq \frac{1}{4^N} \frac{|f|_\varrho}{N} \quad . \end{aligned} \tag{97}$$

6.1 Existence of Φ_1

We apply Lemma 8 to the generating functions W and $I' \cdot \varphi + \varepsilon\chi_1$ where χ_1 is defined by the Fourier expansion:

$$\chi_1 = - \sum_{k\in\mathbb{Z}^n\setminus 0,|k|\le K} \frac{f_k(I')}{1-e^{ik\cdot\omega(I')}} e^{ik\cdot\varphi} \quad . \tag{98}$$

With the notations of Lemma 8, we set $\tilde{\varrho} = r^0 = r$, and $\delta = \eta = \frac{r}{32}$. χ_1 is analytic in D_r because of (33), and its norm can be estimated following Rüssmann ([9]):

$$|\chi_1|_r \le \frac{2^{n+2}}{\alpha} |f|_r \quad , \tag{99}$$

and therefore by Cauchy estimates we get:

$$\begin{aligned}
\varepsilon n \left|\frac{\partial^2 f}{\partial\varphi_i \partial I'_j}\right|_r &\le \frac{4\varepsilon n}{\varrho_I \varrho_\varphi} |f|_\varrho \\
\varepsilon n \left|\frac{\partial^2 \chi_1}{\partial\varphi_i \partial I'_j}\right|_{r-\delta} &\le \frac{\varepsilon n 2^{n+12}}{\alpha r_I r_\varphi} |f|_\varrho \\
\Delta_I = \varepsilon n \max_i \left|\frac{\partial f}{\partial\varphi_i}\right|_r &\le \frac{\varepsilon n}{2\varrho_\varphi} |f|_\varrho \\
\Delta_\varphi = \varepsilon n \max_i \left|\frac{\partial f}{\partial I_i}\right|_r &\le \frac{\varepsilon n}{2\varrho_I} |f|_\varrho \\
\zeta_I = \varepsilon n \max_i \left|\frac{\partial \chi_1}{\partial\varphi_i}\right|_{r-\delta} &\le \frac{\varepsilon n 2^{n+5}}{r_\varphi} |f|_\varrho \\
\zeta_\varphi = \varepsilon n \max_i \left|\frac{\partial \chi_1}{\partial I_i}\right|_{r-\delta} &\le \frac{\varepsilon n 2^{n+5}}{r_I} |f|_\varrho \quad .
\end{aligned} \tag{100}$$

The hypotheses of Lemma 5 allow to apply Lemma 8 which proves that the canonical transformation Φ_1 generated by $I'\cdot\varphi + \varepsilon\chi_1(I',\varphi)$ maps D_{r^1}, with $r^1 = \frac{3}{4}r$, into D_r, it is a symplectic diffeomorphism on its image, and it is also $\Phi_i^{-1}(D_{r^1}) \subseteq D_{r^0}$; denoting $\mathcal{C}_1 = \Phi_1^{-1} \circ \mathcal{C} \circ \Phi_1$, the map:

$$\mathcal{C}_1 : D_{r^1} \longrightarrow D_r \quad , \tag{101}$$

is well defined, is a diffeomorphism on its image and it is generated by a function $W_1 : D_{r^1} \to \mathbb{C}$ of the form:

$$W_1(I',\varphi) = I'\cdot\varphi + h(I') + \varepsilon\tilde{u}^1(I',\varphi) + \varepsilon w^1(I',\varphi) \tag{102}$$

where:

$$\tilde{u}^1(I',\varphi) = f(I',\varphi) + \chi_1(I',\varphi) - \chi_1(I',\varphi + \omega(I')) \tag{103}$$

and:

$$\left|w^1\right|_{r^1} \le \varepsilon^{-1}\Big(\frac{9}{4}Mn\zeta_I^2 + 4\Delta_I\zeta_\varphi + 4\Delta_\varphi\zeta_I + 8\zeta_I\zeta_\varphi\Big) \quad . \tag{104}$$

Using the hypotheses of Lemma 5 one proves: $\left|w^1\right|_{r^1} \le (1/8 + 1/32)\frac{|f|_\varrho}{N}$. From the definition of χ_1 we obtain:

$$\tilde{u}^1(I',\varphi) = \Pi_\Lambda T_K f(I',\varphi) + (1 - T_K) f(I',\varphi) \quad , \tag{105}$$

and therefore, if we define: $u^1 = \Pi_\Lambda T_K f(I',\varphi)$ and $v^1 = w^1 + (1-T_K)f(I',\varphi)$, by estimating the term $(1-T_K)f(I',\varphi)$ as in [3], also using (34), we obtain:

$$|(1-T_K)f|_{r^1} \le |(1-T_K)f|_{\frac{\varrho}{2}} \le \frac{n^{n-1}2^{2n+2}}{\varrho_\varphi^n} e^{-K\frac{\varrho_\varphi}{4}} |f|_\varrho \le \frac{|f|_\varrho}{32N} \quad , \tag{106}$$

so that:

$$\left|v^1\right|_{r^1} \le \frac{|f|_\varrho}{4N} \quad . \tag{107}$$

6.2 Iteration

We now suppose that $\Phi_1,\ldots,\Phi_i$ exist and apply Lemma 8 to the generating functions W_i and $I'\cdot\varphi + \varepsilon\chi_{i+1}$ where χ_{i+1} is defined by the Fourier expansion:

$$\chi_{i+1} = -\sum_{k\in\mathbb{Z}^n\backslash 0, |k|\le K} \frac{v_k^i(I')}{1-e^{ik\cdot\omega(I')}} e^{ik\cdot\varphi} \quad . \tag{108}$$

Setting $\xi = r/(32N)$, we apply Lemma 8 with $\tilde\varrho = r^i - \xi$ and $\delta = \eta = \xi$. The function χ_{i+1} is analytic in D_{r^i} because of (33), and its norm can be estimated following Rüssmann ([9]):

$$|\chi_{i+1}|_{r_i} \le \frac{2^{n+2}}{\alpha} |v_i|_{r^i} \quad , \tag{109}$$

and therefore by Cauchy estimates we get:

$$\begin{aligned}
\varepsilon n \left|\frac{\partial^2 v^i}{\partial\varphi_i \partial I'_j}\right|_{r^i-\xi} &\le \frac{\varepsilon n}{\xi_I \xi_\varphi} \left|v^i\right|_{r^i} \\
\varepsilon n \left|\frac{\partial^2 \chi_{i+1}}{\partial\varphi_i \partial I'_j}\right|_{r^i-2\xi} &\le \frac{\varepsilon n 2^n}{\alpha \xi_I \xi_\varphi} \left|v^i\right|_{r^i} \\
\Delta_I = \frac{\varepsilon n}{4} \max_i \left|\frac{\partial v^i}{\partial\varphi_i}\right|_{r^i-\xi} &\le \frac{8\varepsilon n N}{r_\varphi} \left|v^i\right|_{r^i} \\
\Delta_\varphi = \frac{\varepsilon n}{4} \max_i \left|\frac{\partial v^i}{\partial I_i}\right|_{r^i-\xi} &\le \frac{8\varepsilon n N}{r_I} \left|v^i\right|_{r^i} \\
\zeta_I = \varepsilon n \max_i \left|\frac{\partial \chi_{i+1}}{\partial\varphi_i}\right|_{r^i-2\xi} &\le \frac{2^{n+5}\varepsilon n N}{\alpha r_\varphi} \left|v^i\right|_{r^i} \\
\zeta_\varphi = \varepsilon n \max_i \left|\frac{\partial \chi_{i+1}}{\partial I_i}\right|_{r^i-2\xi} &\le \frac{2^{n+5}\varepsilon n N}{\alpha r_I} \left|v^i\right|_{r^i} \quad .
\end{aligned} \tag{110}$$

The hypotheses of Lemma 5 allow to apply Lemma 8 which proves that the canonical transformation Φ_{i+1} generated by $I'\cdot\varphi + \varepsilon\chi_{i+1}(I',\varphi)$ maps $D_{r^i-6\xi} \subseteq D_{r^{i+1}}$ into D_{r^i}, it is a symplectic diffeomorphism on its image, and it is also $\Phi_{i+1}^{-1}(D_{r^{i+1}}) \subseteq D_{r^i}$; denoting $\mathcal{C}_{i+1} = \Phi_{i+1}^{-1} \circ \mathcal{C}_i \circ \Phi_{i+1}$, the map:

$$\mathcal{C}_{i+1} : D_{r^{i+1}} \longrightarrow D_r \quad , \tag{111}$$

is well defined, is a diffeomorphism on its image and it is generated by a function $W_{i+1} : D_{r^{i+1}} \to \mathbb{C}$ of the form:

$$W_{i+1}(I', \varphi) = I' \cdot \varphi + h(I') + \varepsilon \tilde{u}^{i+1}(I', \varphi) + \varepsilon w^{i+1}(I', \varphi) \tag{112}$$

where:

$$\tilde{u}^{i+1}(I', \varphi) = v^i(I', \varphi) + \chi_{i+1}(I', \varphi) - \chi_{i+1}(I', \varphi + \omega(I')) \tag{113}$$

and:

$$\left| w^{i+1} \right|_{r^{i+1}} \leq \varepsilon^{-1} \Big(\frac{9}{4} M n \zeta_I^2 + 4\Delta_I \zeta_\varphi + 4\Delta_\varphi \zeta_I + 8\zeta_I \zeta_\varphi \Big) \quad . \tag{114}$$

Using the hypotheses of Lemma 5 one proves: $\left| w^{i+1} \right|_{r^{i+1}} \leq \frac{|v^i|_{r^i}}{8}$. From the definition of χ_{i+1} we obtain:

$$\tilde{u}^{i+1}(I', \varphi) = \Pi_\Lambda T_K v^i(I', \varphi) + (1 - T_K) v^i(I', \varphi) \quad . \tag{115}$$

Therefore, if we define: $u^{i+1} = u^i + \Pi_\Lambda T_K v^i(I', \varphi)$ and $v^{i+1} = w^{i+1} + (1 - T_K) v^i(I', \varphi)$, estimating the term $(1 - T_K) v^i$ as in [3] and using (34), we obtain:

$$\left| (1 - T_K) v^i \right|_{r^{i+1}} \leq \frac{n^{n-1} 2^{n+2}}{6^n \xi_\varphi^n} e^{-3K\xi_\varphi} \left| v^i \right|_{r^i} \leq \frac{1}{8} \left| v^i \right|_{r^i} \quad , \tag{116}$$

so that:

$$\left| v^{i+1} \right|_{r^{i+1}} \leq \frac{|f|_\varrho}{4^{i+1} N} \quad . \tag{117}$$

7 Proof of the theorem

Through this section, we complete the proof of Theorem 1 and we compute the constants $a, b, \varepsilon_0, d_0, t_0$ appearing in the statement as a function of $n, \varrho_I, \varrho_\varphi$, $M, M_0, |f|_\varrho$, diam$B$ (we recall that $\varrho_I, \varrho_\varphi$ denote analyticity radii such that h and f are analytic in $B_{\varrho_I} \times \mathbb{T}^n_{\varrho_\varphi}$; M denotes a Lipschitz constant of $\omega(I) = \partial h / \partial I$ in the set B_{ϱ_I} and M_0 denotes a convexity constant for h in B (see equation 17)).

For any positive K and α_1 we fix the parameters $\alpha_j, \delta_j, r_j = (r_I^j, r_\varphi^j)$ as follows:

$$\begin{aligned}
\alpha_j &= j! \Big(2^4 \frac{M}{M_0} \Big)^{j-1} K^{\frac{1}{2} j(j-1)} \alpha_1 \\
\delta_j &= \frac{1}{4M} j K^{j-1} \alpha_j \\
r_I^j &= \frac{\alpha_j}{2MK} \\
r_\varphi^j &= \frac{\varrho_\varphi}{2}
\end{aligned} \tag{118}$$

for any $j = 1, \ldots, n$, and moreover:

$$r_I^0 = \frac{\alpha_1}{8MK} \quad , \quad r_\varphi^j = \frac{\varrho_\varphi}{2} \quad . \tag{119}$$

We consider only those K, α_1 such that it is also:

$$\alpha_n < \pi \quad , \tag{120}$$

and let N be defined as in (31). With such choices, the hypotheses of Lemma 2, Lemma 3 and Lemma 4 are satisfied, so that we deduce that for any K-lattice Λ with $\dim\Lambda = j \in \{1, \dots, n\}$, and for any $I \in D_\Lambda$ and $I' \in \mathcal{C}_{\Lambda,\delta_j}(I)$, it is:

$$\|I - I'\| \le \frac{3}{M_0} j K^{j-1} \alpha_j \quad . \tag{121}$$

In view of the construction of the normal forms described in Lemma 5 in any of the resonant extended blocks, except for the completely resonant one, we remark that using (120) and (121) in the case $j \in \{1, \dots, n-1\}$ one proves that if $I \in (D^{\rm ext}_{\Lambda,\delta_j})_{r^j_I}$, then it is: $\left|1 - e^{ik\cdot\omega(I)}\right| \ge \alpha_{j+1}/8$ for any $k \in \mathbb{Z}^n \backslash \Lambda$ and $|k| \le K$; if $I \in (D_0)_{r^0_I}$, then it is: $\left|1 - e^{ik\cdot\omega(I)}\right| \ge \alpha_1/8$ for any $k \in \mathbb{Z}^n \backslash 0$ and $|k| \le K$.

Then, for any Λ with $j = \dim\Lambda \le n-1$ we apply Lemma 5 with $D' = D^{\rm ext}_{\Lambda,\delta_j}$, $r = r^j$ and $\alpha = \alpha_{j+1}/4$; while we apply Lemma 5 in the non-resonant block D_0 with $r = r^0$ and $\alpha = \alpha_1/4$.

It is possible to apply Lemma 5 to all these sets if the parameters satisfy the following inequalities:

$$\begin{aligned} K &\ge \max\left\{2, K_*, \tfrac{6}{\varrho_\varphi} \ln\left(\tfrac{2^{2n+7} n^{n-1}}{\varrho_\varphi^n}\right)\right\} \\ \alpha_n &\le \min\left\{4, 2^{4+n} n \tfrac{M^2}{M_0} \varrho_I, 2^{n+1} n^3 \tfrac{M}{M_0}\right\} \\ \varepsilon n \left|f\right|_\varrho &\le C \frac{\alpha_n^3}{K^{\frac{3n^2+4-3n}{2}}} \end{aligned} \tag{122}$$

where K_*, C are defined by:

$$\begin{aligned} \tfrac{1}{32n} \varrho_\varphi \tfrac{K_*}{\ln K_*} &= 1 \\ C = \frac{1}{2^{2n+10} M n!^3 \left(2^4 \frac{M}{M_0}\right)^{3(n-1)}} &\min\left\{1, \tfrac{2\varrho_\varphi}{nM\varrho_I}, \tfrac{16n}{\varrho_\varphi \alpha_n}\left(2^4 \tfrac{M}{M_0}\right)^{n-1} K^{\frac{n^2-n-2}{2}}\right\} \end{aligned} \tag{123}$$

To compute constants, we assume $n \ge 2$, so that $C \ge C_0$ with:

$$C_0 = \frac{1}{2^{2n+10} M n!^3 \left(2^4 \frac{M}{M_0}\right)^{3(n-1)}} \min\left\{1, \frac{2\varrho_\varphi}{nM\varrho_I}, \frac{4n}{\varrho_\varphi}\left(2^4 \frac{M}{M_0}\right)^{n-1}\right\} \quad . \tag{124}$$

7.1 Stability of D_0

We now consider a motion $(I_t, \varphi_t) = \mathcal{C}^t(I_0, \varphi_0)$, $t \in \mathbb{Z}$, with $I_0 \in D_0$. Because of Lemma 5, there exists a symplectic map

$$\begin{aligned} \Phi : D_{0,r^0_I/2} \times \mathbb{T}_{r^0_\varphi/2} &\longrightarrow D_{0,r^0_I} \times \mathbb{T}_{r^0_\varphi} \\ (I, \varphi) &\longmapsto (I', \varphi') \end{aligned} \tag{125}$$

which conjugates $\mathcal{C}$ to the map $\mathcal{C}'$ generated by the analytic function:

$$W'(I',\varphi) = I'\cdot\varphi + h(I') + \varepsilon u(I') + \varepsilon\mathcal{R}(I',\varphi) \tag{126}$$

with $|\mathcal{R}|_{r_0/2} \le \frac{1}{4^N}\frac{|f|_\varrho}{N}$. Therefore, denoting $(I_t,\varphi_t) = \Phi(I'_t,\varphi'_t)$, it is: $|I_t - I_0| \le |I'_t - I_t| + |I'_t - I'_0| + |I'_0 - I_0|$ with:

$$\begin{aligned} |I'_0 - I_0| &\le \frac{r_I^0}{2^{11}} \\ |I'_t - I_t| &\le \frac{r_I^0}{2^{11}} \\ |I'_t - I'_0| &\le \varepsilon\,|t|\,\frac{2}{r_{\varphi,0}}\,|\mathcal{R}|_{\frac{r_0}{2}} \le \varepsilon\,|t|\,\frac{4}{\varrho_\varphi}\frac{1}{4^N}\frac{|f|_\varrho}{N} \quad , \end{aligned} \tag{127}$$

for those t such that $I'_t \in D'_{\frac{r_I^0}{2}}$, and therefore it is $|I_t - I_0| \le \frac{r_I^0}{4}$ for any t satisfying:

$$\varepsilon\,|t|\,\frac{4}{\varrho_\varphi}\frac{1}{4^N}\frac{|f|_\varrho}{N} \le \frac{r_I^0}{8} \quad . \tag{128}$$

7.2 Stability of D_Λ

We now consider a K-lattice Λ with $j = \dim\Lambda \in \{1,\dots,n-1\}$, and a motion $(I_t,\varphi_t) = \mathcal{C}^t(I_0,\varphi_0)$ with $I_0 \in D_\Lambda$. We set $D' = D^{\mathrm{ext}}_{\Lambda,\delta_j}$. Because of Lemma 5 there exists a symplectic map

$$\begin{aligned} \Phi : D'_{r_I^j/2}\times\mathbb{T}_{r_\varphi^j/2} &\longrightarrow D'_{r_I^j}\times\mathbb{T}_{r_{\varphi^j}} \\ (I,\varphi) &\longmapsto (I',\varphi') \end{aligned} \tag{129}$$

which conjugates $\mathcal{C}$ to the map $\mathcal{C}'$ generated by the analytic function:

$$W'(I',\varphi) = I'\cdot\varphi + h(I') + \varepsilon u(I',\varphi) + \varepsilon\mathcal{R}(I',\varphi) \tag{130}$$

with $|u|_{r^j/2} \le 2\,|f|_\varrho$, $\Pi_\Lambda u = u$ and $|\mathcal{R}|_{r^j/2} \le \frac{1}{4^N}\frac{|f|_\varrho}{N}$.

To be definite we consider positive t (a very similar argument apply to negative t), and we prove that I_t satisfies one of the two following statements:

- $I_t \in \mathcal{C}_{\Lambda,\delta_j}(I_0)$ for any t satisfying

$$\varepsilon n\,|t|\,\frac{4}{\varrho_\varphi}\frac{1}{4^N}\frac{|f|_\varrho}{N} \le \frac{r_I^j}{16} \quad ; \tag{131}$$

- there exists t_0 satisfying (131) such that $I_t \in \mathcal{C}_{\Lambda,\delta_j}(I_0)$ for any t with $|t| \le t_0$, and $I_{t_0+1} \in D_{\Lambda'}$, with Λ' a suitable K-lattice of dimension strictly smaller than j.

Let t_0 satisfying (131) and also $I_t \in \mathcal{C}_{\Lambda,\delta_j}(I_0)$ for any positive $t \leq t_0$. Actually, it is also: $I_{t_0} \in \mathcal{C}_{\Lambda,\delta_j/4}(I_0)$. In fact, it is $I_{t_0} - I_0 = (I'_{t_0} - I_{t_0}) + (I'_0 - I_0) + (I'_{t_0} - I'_0)$, where $\left|I_{t_0} - I'_{t_0}\right| \leq r_I^j/2^{11} \leq \delta_j/16$, $|I_0 - I'_0| \leq r_I^j/16 \leq \delta_j/16$, and:

$$I'_{t_0} - I'_0 = \varepsilon \frac{\partial u}{\partial \varphi} + \varepsilon \frac{\partial \mathcal{R}}{\partial \varphi} = \lambda + x \tag{132}$$

where $\lambda = \varepsilon \frac{\partial u}{\partial \varphi} \in < \Lambda >$, and satisfies:

$$\|\lambda\| = \varepsilon \left\| \frac{\partial u}{\partial \varphi} \right\| \leq \frac{4\varepsilon n}{\varrho_\varphi} |f|_\varrho \leq \frac{r_I^j}{16} \tag{133}$$

while x can be in any direction but satisfies:

$$\|x\| = \varepsilon \left\| \frac{\partial \mathcal{R}}{\partial \varphi} \right\| \leq \frac{4\varepsilon n}{\varrho_\varphi} |t_0| \frac{|f|_\varrho}{N 4^N} \leq \frac{r_I^j}{16} \leq \frac{\delta_j}{16} \quad . \tag{134}$$

But I_{t_0+1} is in a ball of radius $r_I^j/4 + \delta_j/16$ from I'_{t_0}, and therefore it is in $\mathcal{C}_{\Lambda,\delta_j}(I_0)$, otherwise, because of Lemma 2, it is not in a resonant zone related to a j-dimensional K-lattice. Therefore, it is in a resonant block $D_{\Lambda'}$ with $\dim \Lambda' \leq j-1$.

7.3 Stability of all motions

From the two previous subsections, it is clear that for a generic initial condition, which is in any of the resonant or non resonant blocks, the actions cannot move more than n times the dimension of the completely resonant cylinder, plus the stability radius of the non resonant domain, estimated by the quantity:

$$d \leq \frac{3}{M_0} n^2 K^{n-1} \alpha_n + \frac{r_{I,0}}{4} \leq \frac{4}{M_0} n^2 K^{n-1} \alpha_n \quad , \tag{135}$$

in a number of iterations t satisfying:

$$|t| \leq \frac{N}{\varepsilon n} 4^N \frac{r_I^0 \varrho_\varphi}{2^7 |f|_\varrho} \quad , \tag{136}$$

provided that in the meanwhile they do not leave the action domain B. But such an escape cannot occur if the initial datum is chosen at a distance from the border of B strictly larger than d.

7.4 Choice of the parameters

The stability arguments shown above work if the parameters K, α_n are suitably chosen. First of all, we set:

$$\alpha_n = \frac{\pi}{2} \varepsilon^{\frac{1}{2} - \gamma} \tag{137}$$

with $0 \leq \gamma \leq 1/2$, so that $\alpha_i < \pi$ for any $i = 1, \ldots, n$, when $\varepsilon < 2$, and in particular it is:

$$\alpha_1 = \frac{\pi}{2n!\left(2^4 \frac{M}{M_0}\right)^{n-1} K^{\frac{n(n-1)}{2}}} \varepsilon^{\frac{1}{2}-\gamma} \quad . \tag{138}$$

Then, the third estimate of (122), assuming $n \geq 2$, is satisfied if:

$$K \leq \left(\frac{C_0 \pi^3}{8n\,|f|_\varrho}\right)^{\frac{2}{3n^2-3n+4}} \frac{1}{\varepsilon^{\frac{6\gamma-1}{3n^2-3n+4}}} \quad . \tag{139}$$

Therefore, we can set:

$$K = \left(\frac{\varepsilon_*}{\varepsilon}\right)^b \tag{140}$$

where:

$$b = \frac{6\gamma - 1}{3n^2 - 3n + 4} \quad , \quad \varepsilon_* = \left(\frac{C_0 \pi^3}{8n\,|f|_\varrho}\right)^{\frac{2}{6\gamma-1}} \quad . \tag{141}$$

The first two equations of (122) and the condition $\varepsilon < 2$ are therefore satisfied by imposing $\varepsilon \leq \tilde{\varepsilon}_0$, with

$$\tilde{\varepsilon}_0 = \min \left\{ 2\ , \varepsilon_* \left(\max\left\{2, K_*, \frac{6}{\varrho_\varphi} \ln\left(\frac{2^{2n+7} n^{n-1}}{\varrho_\varphi^n}\right)\right\}\right)^{\frac{1}{b}} , \left(\frac{2}{\pi} \min\left\{4, 2^{4+n} n \frac{M^2}{M_0} \varrho_I, 2^{n+1} n^3 \frac{M}{M_0}\right\}\right)^{\frac{2}{1-2\gamma}} \right\} \quad . \tag{142}$$

Then, up to a number of iterations smaller than:

$$T_* = \frac{\varrho_\varphi^2}{2^{16} \varepsilon n^2 M\,|f|_\varrho} e^{\frac{\varrho_\varphi}{2^6 n}\left(\frac{\varepsilon_*}{\varepsilon}\right)^b} \geq \frac{\varrho_\varphi^2}{2^{18} n^2 M\,|f|_\varrho} e^{\frac{\varrho_\varphi}{2^6 n}\left(\frac{\varepsilon_*}{\varepsilon}\right)^b} \tag{143}$$

the actions cannot move by a quantity larger than:

$$\Delta = \frac{2n^2 \varepsilon_*^{b(n-1)} \pi}{M_0} \varepsilon^a \tag{144}$$

where $a = \frac{1}{2} - \gamma - b(n-1)$. The theorem is proved if we can find a $\gamma \in (1/6, 1/2)$ such that $a, b > 0$. It is sufficient to choose:

$$b = \frac{2 - 6a}{3n^2 + 3n - 2} \tag{145}$$

for any $a \in \left(0, \frac{1}{3}\right)$. In fact, for any choice of a in this interval it is $a, b > 0$ and also:

$$\gamma = \frac{n^2(3 - 6a) + n(6a - 1) + 2 - 8a}{6n^2 + 6n - 4} \in \left(\frac{1}{6}, \frac{1}{2}\right) \tag{146}$$

for any $n \geq 2$.

Therefore, for any $a \in \left(0, \frac{1}{3}\right)$, let b be defined as in (145), ε_* be defined as in (141) (in formula 141 the constants γ, C_0 are as in 146,124), $\tilde{\varepsilon}_0$ be defined as in (142) (in formula 142 the constants γ, K_* are are as in 146,123 first line), and finally let d_0, t_0 be defined by:

$$d_0 = \frac{2n^2 \varepsilon_*^{b(n-1)} \pi}{M_0} \quad , \quad t_0 = \frac{\varrho_\varphi^2}{2^{18} n^2 M \left| f \right|_\varrho} \quad . \tag{147}$$

All the constants $a, b, \varepsilon_*, \tilde{\varepsilon}_0, d_0, t_0$ turn out to be defined as functions of $n, \varrho_I, \varrho_\varphi$, $M, M_0, |f|_\varrho$. Then, for any initial datum $(I_0, \varphi_0) \in B \times \mathbb{T}^n$ with $\mathrm{dist}(I_0, \partial B) \geq 2d_0 \varepsilon^a$ and for any $\varepsilon \leq \tilde{\varepsilon}_0$ it is:

$$|I_t - I_0| \leq d_0 \varepsilon^a \tag{148}$$

up to a number of iterations $t \in \mathbb{Z}$ such that:

$$|t| \leq t_0 e^{\frac{\varrho_\varphi}{2^6 n} \left(\frac{\varepsilon_*}{\varepsilon}\right)^b} \quad . \tag{149}$$

Therefore, if the diameter of the action domain B is large that $2d_0$, the theorem is proved on a non-empty set of initial conditions setting:

$$\varepsilon_0 = \min \left\{ \tilde{\varepsilon}_0, \left(\frac{\varrho_\varphi}{2^6 n} \right)^{\frac{1}{b}} \varepsilon_* \right\} \quad ,$$

otherwise the theorem is proved by adding the additional constraint on ε: $2d_0 \varepsilon \leq \mathrm{diam} B$, so that, in any case, the theorem is proved on a non-empty set of initial conditions setting:

$$\varepsilon_0 = \min \left\{ \frac{\mathrm{diam} B}{2d_0}, \tilde{\varepsilon}_0, \left(\frac{\varrho_\varphi}{2^6 n} \right)^{\frac{1}{b}} \varepsilon_* \right\} \quad .$$

References

[1] A. Bazzani, Normal forms for symplectic maps of $\mathbf{R}^{2n}$. Unpublished note, (1987).

[2] A. Bazzani, S. Marmi and S. Turchetti, Nekhoroshev estimate for isochronous nonresonant symplectic maps. *Celest. Mech. and Dynamical Astronomy* **4**, 333–359 (1990).

[3] G. Benettin, L. Galgani and A. Giorgilli, A proof of Nekhoroshev's theorem for the stability times in nearly integrable Hamiltonian systems. *Cel. Mech.* **37**, 1 (1985).

[4] R. Douady, Application du théorème de tores invariantes, Thèse 3ème cycle, Université Paris VII (1982).

[5] S.B. Kuksin, On the inclusion of an almost integrable analytic symplectomorphism into a Hamiltonian flow, *Russian journal of Mathematical Physics* **1**, 2, 191–207 (1993).

[6] S. Kuksin and J. Pöschel, On the inclusion of analytic symplectic maps in analytic Hamiltonian flows and its applications, Seminar on Dynamical Systems (St. Petersburg, 1991), 96–116, Progr. Nonlinear Differential Equations Appl., 12, Birkhäuser, Basel (1994).

[7] N.N. Nekhoroshev, Exponential estimates of the stability time of near-integrable Hamiltonian systems, *Russ. Math. Surveys* **32**, 1–65 (1977).

[8] J. Pöschel, Nekhoroshev estimates for quasi-convex Hamiltonian systems, *Math. Z.* **213**, 187 (1993).

[9] H. Rüssmann, On optimal estimates for the solutions of linear partial differential equations of first order with constant coefficients on the torus, Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974), pp. 598–624. *Lecture Notes in Phys.* **38** (1975).

[10] G. Turchetti and A. Bazzani, Normal forms for symplectic maps. Advances in nonlinear dynamics and stochastic processes, II (Rome, 1986), 25–36, World Sci. Publishing, Singapore, (1987).

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Communicated by Eduard Zehnder
submitted 16/06/03, accepted 31/03/04