

## Schrödinger Operators on Lattices. The Efimov Effect and Discrete Spectrum Asymptotics

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**Abstract.** The Hamiltonian of a system of three quantum mechanical particles moving on the three-dimensional lattice  $\mathbb{Z}^3$  and interacting via zero-range attractive potentials is considered. For the two-particle energy operator  $h(k)$ , with  $k \in \mathbb{T}^3 = (-\pi, \pi]^3$  the two-particle quasi-momentum, the existence of a unique positive eigenvalue below the bottom of the continuous spectrum of  $h(k)$  for  $k \neq 0$  is proven, provided that  $h(0)$  has a zero energy resonance. The location of the essential and discrete spectra of the three-particle discrete Schrödinger operator  $H(K)$ ,  $K \in \mathbb{T}^3$  being the three-particle quasi-momentum, is studied. The existence of infinitely many eigenvalues of  $H(0)$  is proven. It is found that for the number  $N(0, z)$  of eigenvalues of  $H(0)$  lying below  $z < 0$  the following limit exists

$$\lim_{z \rightarrow 0^-} \frac{N(0, z)}{|\log |z||} = \mathcal{U}_0$$

with  $\mathcal{U}_0 > 0$ . Moreover, for all sufficiently small nonzero values of the three-particle quasi-momentum  $K$  the finiteness of the number  $N(K, \tau_{ess}(K))$  of eigenvalues of  $H(K)$  below the essential spectrum is established and the asymptotics for the number  $N(K, 0)$  of eigenvalues lying below zero is given.

### 1 Introduction

One of the remarkable results in the spectral analysis for continuous three-particle Schrödinger operators is the Efimov effect: if in a system of three-particles, interacting by means of short-range pair potentials none of the three two-particle subsystems has bound states with negative energy, but at least two of them have a resonance with zero energy, then this three-particle system has an infinite number of three-particle bound states with negative energy, accumulating at zero.

This effect was first discovered by Efimov [5]. Since then this problem has been studied in many physics journals and books [1, 3, 7]. A rigorous mathematical proof of the existence of Efimov's effect was originally carried out in [25] by Yafaev and then in [20, 22, 23, 24]. Efimov's effect was further studied in [2, 4, 10, 11, 14, 15, 16, 18, 19].

Denote by  $N(z)$ ,  $z < 0$  the number of eigenvalues of the Hamiltonian below  $z < 0$ . The growth of  $N(z)$  has been studied by S. Albeverio, R. Høegh-Krohn, and T.T. Wu in [1] for the symmetric case. Namely, the authors of [1] have first found (without proofs) the exponential asymptotics of eigenvalues corresponding to spherically symmetric bound states.

This result is consistent with the lower bound

$$\liminf_{z \rightarrow 0} \frac{N(z)}{|\log |z||} > 0$$

established in [23] without any symmetry assumptions.

The main result obtained by Sobolev [22] is the limit

$$\lim_{z \rightarrow 0} |\log |z||^{-1} N(z) = \mathcal{U}_0, \quad (1.1)$$

where the coefficient  $\mathcal{U}_0$  does not depend on the potentials  $v_\alpha$  and is a positive function of the ratios  $m_1/m_2, m_2/m_3$  of the masses of the three-particles.

In [2] the Fredholm determinant asymptotics of convolution operators on large finite intervals with rational symbols having real zeros are studied, as well as the connection with the Efimov effect.

In models of solid state physics [8, 18, 19, 21] and also in lattice field theory [9, 17] discrete Schrödinger operators are considered, which are lattice analogs of the continuous three-particle Schrödinger operator. The presence of Efimov's effect for these operators was demonstrated at the physical level of rigor without a mathematical proof for a system of three identical quantum particles in [18, 19].

Although the energy operator of a system of three-particles on lattice is bounded and the perturbation operator in the pair problem is a compact operator, the study of spectral properties of energy operators of systems of two and three particles on a lattice is more complex than in the continuous case.

In the continuous case [6] (see also [7, 21]) the energy of the center-of-mass motion can be separated out from the total Hamiltonian, that is, the energy operator can be split into a sum of a center-of-mass motion and a relative kinetic energy. So that the three-particle "bound states" are eigenvectors of the relative kinetic energy operator. Therefore Efimov's effect either exists or does not exist for all values of the total momentum simultaneously.

In lattice terms the "center-of-mass separation" corresponds to a realization of the Hamiltonians as a "fibered operator", that is, as the "direct integral of a family of operators"  $H(K)$  depending on the values of the total quasi-momentum  $K \in \mathbb{T}^3 = (-\pi, \pi]^3$  (see [8, 21]). In this case a "bound state" is an eigenvector of the operator  $H(K)$  for some  $K \in \mathbb{T}^3$ . Typically, this eigenvector depends continuously on  $K$ . Therefore, Efimov's effect may exist only for some values of  $K \in \mathbb{T}^3$  (see [11]).

In [10] was stated the existence infinitely many bound states (Efimov's effect) for the discrete three-particle Schrödinger operators associated with a system of three arbitrary quantum particles moving on three dimensional lattice and interacting via zero-range attractive pairs potentials. In this work only a sketch of proof of results has been given.

In [11] the existence of Efimov's effect for a system of three identical quantum particles (bosons) on a three-dimensional lattice interacting via zero-range attractive pair potentials has been proven, in the case, where all three two-particle subsystems have resonances at the bottom of the three-particle continuum.

In [12, 13] the finiteness of the number of bound states was proven, in the cases, where either none of the two-particle subsystems or only one of the two-particle subsystems have a zero energy resonance.

In [15] (a detailed proof is in [16]) for the difference operator on a lattice associated with a system of three identical particles interacting via zero-range attractive pair potentials under the assumption that all two-particle subsystems have resonance at the bottom of the three-particle continuum the following results have been established:

- 1) for the zero value of the total quasi-momentum ( $K = 0$ ) there are infinitely many eigenvalues lying below the bottom and accumulating at the bottom of essential spectrum (Efimov's effect).
- 2) for all  $K \in U_\delta^0(0) = \{K \in \mathbb{T}^3 : 0 < |K| < \delta\}$ ,  $\delta > 0$  sufficiently small, the three-particle operator has a finite number of eigenvalues below the bottom of essential spectrum.

The results are quite surprising and clearly put in evidence the difference between the continuum and discussed cases.

In the present work we consider a system of three arbitrary quantum particles on the three-dimensional lattice  $\mathbb{Z}^3$  interacting via zero-range pair attractive potentials.

Let us denote by  $\tau_{ess}(K)$  the bottom of essential spectrum of the three-particle discrete Schrödinger operator  $H(K)$ ,  $K \in \mathbb{T}^3$  and by  $N(K, z)$  the number of eigenvalues lying below  $z \leq \tau_{ess}(K)$ .

The main results of the present paper are as follows:

- (i) for the two-particle energy operator  $h(k)$  on the three-dimensional lattice  $\mathbb{Z}^3$ ,  $k$  being the two-particle quasi-momentum, we prove the existence of a unique positive eigenvalue below the bottom of the continuous spectrum of  $h(k)$ ,  $k \neq 0$  for the nontrivial values of the quasi-momentum  $k$ , provided that the two-particle Hamiltonian  $h(0)$  corresponding to the zero value of  $k$  has a zero energy resonance.
- (ii) we establish the location of the essential spectrum of the discrete three-particle operator  $H(K)$ . The infinitely many eigenvalues of the three-particle discrete Schrödinger operator arise from the existence of resonances of the two-particle operators at the bottom of three-particle continuum. Therefore we obtain a lower bound for the location of discrete spectrum of  $H(K)$  in terms of zero-range interaction potentials.
- (iii) for the number  $N(0, z)$  we obtain the limit result

$$\lim_{z \rightarrow -0} \frac{N(0, z)}{|\log |z||} = \mathcal{U}_0, \quad (0 < \mathcal{U}_0 < \infty).$$

- (iv) for any  $K \in U_\delta^0(0)$  we prove the finiteness of  $N(K, \tau_{ess}(K))$  and establish the following limit result

$$\lim_{|K| \rightarrow 0} \frac{N(K, 0)}{|\log |K||} = 2\mathcal{U}_0.$$

We remark that whereas the result (iii) is similar to that of continuous case and, the results (i) and (iv) are surprising and characteristic for the lattice systems, in fact they do not have any analogues in the continuous case.

The plan of the paper is as follows:

Section 1 is an introduction to the whole paper.

In Section 2 the Hamiltonians of systems of two- and three-particles in coordinate and momentum representations are described as bounded self-adjoint operators in the corresponding Hilbert spaces.

In Section 3 we introduce the total quasi-momentum and decompose the energy operators into von Neumann direct integrals, choosing relative coordinate systems.

In Section 4 we state the main results of the paper.

In Section 5 we study spectral properties of the two-particle discrete Schrödinger operator  $h(k)$ ,  $k \in \mathbb{T}^3$  on the three-dimensional lattice  $\mathbb{Z}^3$ . We prove the existence of unique positive eigenvalue below the bottom of the continuous spectrum of  $h(k)$  (Theorem 5.4) and obtain an asymptotics for the Fredholm's determinant associated with  $h(k)$ .

In Section 6 we introduce the "channel operators" and describe its spectrum by the spectrum of the two-particle discrete Schrödinger operators. Applying a Faddeev type system of integral equations we establish the location of the essential spectrum (Theorem 4.3). We obtain a lower bound for the location of discrete spectrum of  $H(K)$  lying below the bottom of the essential spectrum (see Theorem 4.4). We prove the finiteness of eigenvalues below the bottom of the essential spectrum of  $H(K)$  for  $K \in U_\delta^0(0)$  (Theorem 4.6).

In Section 7 we follow closely A. Sobolev method to derive the asymptotics for the number of eigenvalues of  $H(K)$  (Theorem 4.7).

Throughout the paper we adopt the following conventions: For each  $\delta > 0$  the notation  $U_\delta(0) = \{K \in \mathbb{T}^3 : |K| < \delta\}$  stands for a  $\delta$ -neighborhood of the origin and  $U_\delta^0(0) = U_\delta(0) \setminus \{0\}$  for a punctured  $\delta$ -neighborhood. The subscript  $\alpha$  (and also  $\beta$  and  $\gamma$ ) always equal to 1 or 2 or 3 and  $\alpha \neq \beta, \beta \neq \gamma, \gamma \neq \alpha$ .

## 2 Energy operators for two and three arbitrary particles on a lattice in the coordinate and momentum representations

Let  $\mathbb{Z}^\nu - \nu$ -dimensional lattice.

The free Hamiltonian  $\widehat{H}_0$  of a system of three quantum mechanical particles on the three-dimensional lattice  $\mathbb{Z}^3$  in the coordinate representation is usually associated with the following bounded self-adjoint operator on the Hilbert space  $\ell_2((\mathbb{Z}^3)^3)$ :

$$\widehat{H}_0 = \frac{1}{2m_1} \Delta_{x_1} + \frac{1}{2m_2} \Delta_{x_2} + \frac{1}{2m_3} \Delta_{x_3}, \quad (2.1)$$

with  $\Delta_{x_1} = \Delta \otimes I \otimes I$ ,  $\Delta_{x_2} = I \otimes \Delta \otimes I$  and  $\Delta_{x_3} = I \otimes I \otimes \Delta$ , where  $m_\alpha > 0, \alpha = 1, 2, 3$  are different numbers, having the meaning of a mass of the particle  $\alpha$ .

The Laplacian  $\Delta$  is a difference operator which describes the transport of a particle from one side to another, i.e.,

$$(\Delta\hat{\psi})(x) = \sum_{|s|=1} [\hat{\psi}(x) - \hat{\psi}(x+s)], \quad \hat{\psi} \in \ell_2(\mathbb{Z}^3).$$

The three-particle Hamiltonian  $\hat{H}$  of the quantum-mechanical three-particles systems with two-particle interactions  $\hat{v}_{\beta\gamma}, \beta, \gamma = 1, 2, 3$  in the coordinate representation is a bounded perturbation of the free Hamiltonian  $\hat{H}_0$

$$\hat{H} = \hat{H}_0 - \hat{V}_1 - \hat{V}_2 - \hat{V}_3, \tag{2.2}$$

where  $\hat{V}_\alpha, \alpha = 1, 2, 3$  are multiplication operators on  $\ell_2((\mathbb{Z}^3)^3)$

$$(\hat{V}_\alpha\hat{\psi})(x_1, x_2, x_3) = \hat{v}_{\beta\gamma}(x_\beta - x_\gamma)\hat{\psi}(x_1, x_2, x_3) = \mu_\alpha\delta_{x_\beta x_\gamma}\hat{\psi}(x_1, x_2, x_3),$$

$$\hat{\psi} \in \ell_2((\mathbb{Z}^3)^3).$$

Here  $\mu_\alpha > 0$  interaction energy of particles  $\beta$  and  $\gamma$ ,  $\delta_{x_\beta x_\gamma}$  is the Kronecker delta.

It is clear that the three-particle Hamiltonian (2.2) is a bounded self-adjoint operator on the Hilbert space  $\ell_2((\mathbb{Z}^3)^3)$ .

Similarly as we introduced  $\hat{H}$ , we shall introduce the corresponding two-particle Hamiltonians  $\hat{h}_\alpha, \alpha = 1, 2, 3$  as bounded self-adjoint operators on the Hilbert space  $\ell_2((\mathbb{Z}^3)^2)$

$$\hat{h}_\alpha = \hat{h}_\alpha^0 - \hat{v}_\alpha,$$

where

$$\hat{h}_\alpha^0 = \frac{1}{2m_\beta}\Delta_{x_\beta} + \frac{1}{2m_\gamma}\Delta_{x_\gamma},$$

with  $\Delta_{x_\beta} = \Delta \otimes I, \Delta_{x_\gamma} = I \otimes \Delta$  and

$$(\hat{v}_\alpha\hat{\varphi})(x_\beta, x_\gamma) = \mu_\alpha\delta_{x_\beta x_\gamma}\hat{\varphi}(x_\beta, x_\gamma), \quad \hat{\varphi} \in \ell_2((\mathbb{Z}^3)^2).$$

Let us rewrite our operators in the momentum representation. Let  $\mathcal{F}_m : L_2((\mathbb{T}^3)^m) \rightarrow \ell_2((\mathbb{Z}^3)^m)$  denote the standard Fourier transform, where  $(\mathbb{T}^3)^m, m \in \mathbb{N}$  denotes the Cartesian  $m$ th power of the set  $\mathbb{T}^3 = (-\pi, \pi]^3$ .

**Remark 2.1** *The operations addition and multiplication by real numbers of elements of  $\mathbb{T}^3 \subset \mathbb{R}^3$  should be regarded as operations on  $\mathbb{R}^3$  modulo  $(2\pi\mathbb{Z}^1)^3$ . For example, let*

$$a = \left(\frac{2\pi}{3}, -\frac{3\pi}{4}, \frac{11\pi}{12}\right) \quad \text{and} \quad b = \left(\frac{2\pi}{3}, -\frac{\pi}{2}, \frac{5\pi}{6}\right) \in \mathbb{T}^3$$

then

$$a + b = \left(-\frac{2\pi}{3}, \frac{3\pi}{4}, -\frac{\pi}{4}\right) \in \mathbb{T}^3, \quad 12a = (0, \pi, \pi) \in \mathbb{T}^3.$$

The three- resp. two-particle Hamiltonians (in the momentum representation) are given by the bounded self-adjoint operators on the Hilbert spaces  $L_2((\mathbb{T}^3)^3)$  resp.  $L_2((\mathbb{T}^3)^2)$  as follows

$$H = \mathcal{F}_3^{-1} \widehat{H} \mathcal{F}_3$$

resp.

$$h_\alpha = \mathcal{F}_2^{-1} \widehat{h}_\alpha \mathcal{F}_2, \quad \alpha = 1, 2, 3.$$

One has

$$H = H_0 - V_1 - V_2 - V_3,$$

where

$$H_0 = \widehat{\Delta}_{k_1} + \widehat{\Delta}_{k_2} + \widehat{\Delta}_{k_3},$$

with  $\widehat{\Delta}_{k_1} = \widehat{\Delta}_1 \otimes I \otimes I$ ,  $\widehat{\Delta}_{k_2} = I \otimes \widehat{\Delta}_2 \otimes I$  and  $\widehat{\Delta}_{k_3} = I \otimes I \otimes \widehat{\Delta}_3$  and  $\widehat{\Delta}_\alpha$ ,  $\alpha = 1, 2, 3$  is the multiplication operator by the function  $\varepsilon_\alpha(k)$

$$(\widehat{\Delta}_\alpha f)(k) = \varepsilon_\alpha(k) f(k), \quad f \in L_2(\mathbb{T}^3).$$

The functions  $\varepsilon_\alpha$ ,  $\alpha = 1, 2, 3$  defined above are of the form

$$\varepsilon_\alpha(p) = \frac{1}{m_\alpha} \varepsilon(p), \quad \varepsilon(p) = \sum_{i=1}^3 (1 - \cos p^{(i)}), \quad p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{R}^3$$

and  $V_\alpha$ ,  $\alpha = 1, 2, 3$  are integral operators of convolution type

$$\begin{aligned} & (V_\alpha f)(k_1, k_2, k_3) \\ &= \frac{\mu_\alpha}{(2\pi)^3} \int_{(\mathbb{T}^3)^3} \delta(k_\alpha - k'_\alpha) \delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_1, k'_2, k'_3) dk'_1 dk'_2 dk'_3, \\ & f \in L_2((\mathbb{T}^3)^3), \end{aligned}$$

where  $\delta(k)$  denotes the Dirac delta-function.

For the two-particle Hamiltonians  $h_\alpha$ ,  $\alpha = 1, 2, 3$  we have:

$$h_\alpha = h_\alpha^0 - v_\alpha,$$

where

$$h_\alpha^0 = \widehat{\Delta}_{k_\beta} + \widehat{\Delta}_{k_\gamma},$$

with  $\widehat{\Delta}_{k_\beta} = \widehat{\Delta}_\beta \otimes I$ ,  $\widehat{\Delta}_{k_\gamma} = I \otimes \widehat{\Delta}_\gamma$  and

$$(v_\alpha f)(k_\beta, k_\gamma) = \frac{\mu_\alpha}{(2\pi)^3} \int_{(\mathbb{T}^3)^2} \delta(k_\beta + k_\gamma - k'_\beta - k'_\gamma) f(k'_\beta, k'_\gamma) dk'_\beta dk'_\gamma, \quad f \in L_2((\mathbb{T}^3)^2).$$

### 3 Decomposition of the energy operators into von Neumann direct integrals. Quasimomentum and coordinate systems

Given  $m \in \mathbb{N}$ , denote by  $\hat{U}_s^m, s \in \mathbb{Z}^3$  the unitary operators on the Hilbert space  $\ell_2((\mathbb{Z}^3)^m)$  defined as:

$$(\hat{U}_s^m f)(n_1, n_2, \dots, n_m) = f(n_1 + s, n_2 + s, \dots, n_m + s), \quad f \in \ell_2((\mathbb{Z}^3)^m).$$

We easily see that

$$\hat{U}_{s+p}^m = \hat{U}_s^m \hat{U}_p^m, \quad s, p \in \mathbb{Z}^3,$$

that is,  $\hat{U}_s^m, s \in \mathbb{Z}^3$  is the unitary representation of the Abelian group  $\mathbb{Z}^3$ .

Via the Fourier transform  $\mathcal{F}_m$  the unitary representation of  $\mathbb{Z}^3$  in  $\ell_2((\mathbb{Z}^3)^m)$  induces the representation of the group  $\mathbb{Z}^3$  in the Hilbert space  $L_2((\mathbb{T}^3)^m)$  by unitary (multiplication) operators  $U_s^m = \mathcal{F}_m^{-1} \hat{U}_s^m \mathcal{F}_m, s \in \mathbb{Z}^3$  given by:

$$(U_s^m f)(k_1, k_2, \dots, k_m) = \exp(-i(s, k_1 + k_2 + \dots + k_m)) f(k_1, k_2, \dots, k_m), \quad (3.1)$$

$$f \in L_2((\mathbb{T}^3)^m).$$

For any  $K \in \mathbb{T}^3$  we define  $\mathbb{F}_K^m$  as follows

$$\mathbb{F}_K^m = \{(k_1, \dots, k_{m-1}, K - k_1 - \dots - k_{m-1}) \in (\mathbb{T}^3)^m : k_1, k_2, \dots, k_{m-1} \in \mathbb{T}^3, K - k_1 - \dots - k_{m-1} \in \mathbb{T}^3\}.$$

Decomposing the Hilbert space  $L_2((\mathbb{T}^3)^m)$  into the direct integral

$$L_2((\mathbb{T}^3)^m) = \int_{K \in \mathbb{T}^3} \oplus L_2(\mathbb{F}_K^m) dK$$

we obtain the corresponding decomposition of the unitary representation  $U_s^m, s \in \mathbb{Z}^3$  into the direct integral

$$U_s^m = \int_{K \in \mathbb{T}^3} \oplus U_s(K) dK,$$

where

$$U_s(K) = \exp(-i(s, K)) I \quad \text{on} \quad L_2(\mathbb{F}_K^m)$$

and  $I = I_{L_2(\mathbb{F}_K^m)}$  denotes the identity operator on the Hilbert space  $L_2(\mathbb{F}_K^m)$ .

The above Hamiltonians  $\hat{H}$  and  $\hat{h}_\alpha, \alpha = 1, 2, 3$  obviously commute with the groups of translations  $\hat{U}_s^3$  and  $\hat{U}_s^2, s \in \mathbb{Z}^3$ , respectively, that is,

$$\hat{U}_s^3 \hat{H} = \hat{H} \hat{U}_s^3, \quad s \in \mathbb{Z}^3$$

and

$$\hat{U}_s^2 \hat{h}_\alpha = \hat{h}_\alpha \hat{U}_s^2, \quad s \in \mathbb{Z}^3, \quad \alpha = 1, 2, 3.$$

Correspondingly, the Hamiltonians  $H$  and  $h_\alpha$ ,  $\alpha = 1, 2, 3$  (in the momentum representation) commute with the groups  $U_s^m$ ,  $s \in \mathbb{Z}^3$  given by (3.1) for  $m = 3$  and  $m = 2$ , respectively.

Hence, the operators  $H$  and  $h_\alpha$ ,  $\alpha = 1, 2, 3$ , can be decomposed into the direct integrals

$$H = \int_{K \in \mathbb{T}^3} \oplus \tilde{H}(K) dK \quad \text{and} \quad h_\alpha = \int_{k \in \mathbb{T}^3} \oplus \tilde{h}_\alpha(k) dk, \quad \alpha = 1, 2, 3,$$

with respect to the decompositions

$$L_2((\mathbb{T}^3)^3) = \int_{K \in \mathbb{T}^3} \oplus L_2(\mathbb{F}_K^3) dK \quad \text{and} \quad L_2((\mathbb{T}^3)^2) = \int_{k \in \mathbb{T}^3} \oplus L_2(\mathbb{F}_k^2) dk,$$

respectively.

Given a cyclic permutation  $\alpha\beta\gamma$  of 123 we introduce the mappings

$$\pi_{\alpha\beta}^{(3)} : (\mathbb{T}^3)^3 \rightarrow (\mathbb{T}^3)^2, \quad \pi_{\alpha\beta}^{(3)}((k_\alpha, k_\beta, k_\gamma)) = (k_\alpha, k_\beta)$$

and

$$\pi_\alpha^{(2)} : (\mathbb{T}^3)^2 \rightarrow \mathbb{T}^3, \quad \pi_\alpha^{(2)}((k_\beta, k_\gamma)) = k_\beta.$$

Denote by  $\pi_K^{(3)}$ ,  $K \in \mathbb{T}^3$  resp.  $\pi_k^{(2)}$ ,  $k \in \mathbb{T}^3$  the restriction of  $\pi_{\alpha\beta}^{(3)}$  resp.  $\pi_\alpha^{(2)}$  onto  $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3$  resp.  $\mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$ , that is,

$$\pi_K^{(3)} = \pi_{\alpha\beta}^{(3)}|_{\mathbb{F}_K^3} \quad \text{and} \quad \pi_k^{(2)} = \pi_\alpha^{(2)}|_{\mathbb{F}_k^2}. \tag{3.2}$$

At this point it is useful to remark that  $\mathbb{F}_K^3$ ,  $K \in \mathbb{T}^3$  and  $\mathbb{F}_k^2$ ,  $k \in \mathbb{T}^3$  are six- and three-dimensional manifolds isomorphic to  $(\mathbb{T}^3)^2$  and  $\mathbb{T}^3$ , respectively.

**Lemma 3.1** *The mappings  $\pi_K^{(3)}$ ,  $K \in \mathbb{T}^3$  and  $\pi_k^{(2)}$ ,  $k \in \mathbb{T}^3$  are bijective from  $\mathbb{F}_K^3 \subset (\mathbb{T}^3)^3$  and  $\mathbb{F}_k^2 \subset (\mathbb{T}^3)^2$  onto  $(\mathbb{T}^3)^2$  and  $\mathbb{T}^3$  with the inverse mappings given by*

$$(\pi_K^{(3)})^{-1}(k_\alpha, k_\beta) = (k_\alpha, k_\beta, K - k_\alpha - k_\beta)$$

and

$$(\pi_k^{(2)})^{-1}(k_\beta) = (k_\beta, k - k_\beta). \quad \square$$

Let

$$U_K : L_2(\mathbb{F}_K^3) \rightarrow L_2((\mathbb{T}^3)^2), \quad U_K f = f \circ (\pi_K^{(3)})^{-1}, \quad K \in \mathbb{T}^3,$$

and

$$u_k : L_2(\mathbb{F}_k^2) \rightarrow L_2(\mathbb{T}^3), \quad u_k g = g \circ (\pi_k^{(2)})^{-1}, \quad k \in \mathbb{T}^3,$$



where  $\pi_K^{(3)}$  and  $\pi_k^{(2)}$  are defined by (3.2). Then  $U_K$  and  $u_k$  are unitary operators and

$$H(K) = U_K \tilde{H}(K) U_K^{-1}, \quad h_\alpha(k) = u_k \tilde{h}_\alpha(k) u_k^{-1}, \quad \alpha = 1, 2, 3.$$

The operator  $H(K)$ ,  $K \in \mathbb{T}^3$  has form

$$H(K) = H_0(K) - V_1 - V_2 - V_3.$$

In the coordinates  $(k_\alpha, k_\beta)$  the operators  $H_0(K)$  and  $V_\alpha$  are defined on the Hilbert space  $L_2((\mathbb{T}^3)^2)$  by

$$(H_0(K)f)(k_\alpha, k_\beta) = E_{\alpha\beta}(K; k_\alpha, k_\beta) f(k_\alpha, k_\beta), \quad f \in L_2((\mathbb{T}^3)^2),$$

$$(V_\alpha f)(k_\alpha, k_\beta) = \frac{\mu_\alpha}{(2\pi)^3} \int_{\mathbb{T}^3} f(k_\alpha, k'_\beta) dk'_\beta, \quad f \in L_2((\mathbb{T}^3)^2),$$

where

$$E_{\alpha\beta}(K; k_\alpha, k_\beta) = \varepsilon_\alpha(k_\alpha) + \varepsilon_\beta(k_\beta) + \varepsilon_\gamma(K - k_\alpha - k_\beta).$$

The operator  $h_\alpha(k)$ ,  $k \in \mathbb{T}^3$ ,  $\alpha = 1, 2, 3$  has form

$$h_\alpha(k) = h_\alpha^0(k) - v_\alpha,$$

where

$$(h_\alpha^0(k)f)(k_\beta) = E_k^{(\alpha)}(k_\beta) f(k_\beta), \quad f \in L_2(\mathbb{T}^3),$$

$$(v_\alpha f)(k_\beta) = \frac{\mu_\alpha}{(2\pi)^3} \int_{\mathbb{T}^3} f(k'_\beta) dk'_\beta, \quad f \in L_2(\mathbb{T}^3)$$

and

$$E_k^{(\alpha)}(k_\beta) = \varepsilon_\beta(k_\beta) + \varepsilon_\gamma(k - k_\beta). \tag{3.3}$$

### 4 Statement of the main results

For each  $K \in \mathbb{T}^3$  the minimum and the maximum taken over  $(p, q)$  of the function  $E_{\alpha\beta}(K; p, q)$  are independent of  $\alpha$ ,  $\beta = 1, 2, 3$ . We set:

$$E_{\min}(K) \equiv \min_{p,q} E_{\alpha\beta}(K, p, q), \quad E_{\max}(K) \equiv \max_{p,q} E_{\alpha\beta}(K, p, q).$$

**Definition 4.1** *The operator  $h_\alpha(0)$  is said to have a zero energy resonance if the equation*

$$\frac{\mu_\alpha m_\beta m_\gamma}{(2\pi)^3} \int_{\mathbb{T}^3} (\varepsilon(q'))^{-1} \varphi(q') dq' = \varphi(q), \quad m_{\beta\gamma} \equiv \frac{m_\beta m_\gamma}{m_\beta + m_\gamma}$$

*has a nonzero solution  $\varphi$  in the Banach space  $C(\mathbb{T}^3)$ . Without loss of generality we can always normalize  $\varphi$  so that  $\varphi(0) = 1$ .*

Let the operator  $h_\alpha(0)$  have a zero energy resonance. Then the function

$$\psi(q) = (\varepsilon(q))^{-1}$$

is a solution (up to a constant factor) of the Schrödinger equation  $h_\alpha(0)f = 0$  and  $\psi$  belongs to  $L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$ .

Set

$$\mu_\alpha^0 = (2\pi)^3 m_{\beta\gamma}^{-1} \left( \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq \right)^{-1}. \tag{4.1}$$

**Hypothesis 4.2** *We assume that  $\mu_\alpha = \mu_\alpha^0$ ,  $\mu_\beta = \mu_\beta^0$  and  $\mu_\gamma \leq \mu_\gamma^0$ .*

The main results of the paper are given in the following theorems.

**Theorem 4.3** *For the essential spectrum  $\sigma_{ess}(H(K))$  of  $H(K)$  the following equality*

$$\sigma_{ess}(H(K)) = \cup_{\alpha=1}^3 \cup_{p \in \mathbb{T}^3} \{ \sigma_d(h_\alpha(K-p)) + \varepsilon_\alpha(p) \} \cup [E_{\min}(K), E_{\max}(K)],$$

*holds, where  $\sigma_d(h_\alpha(k))$  is the discrete spectrum of the operator  $h_\alpha(k)$ ,  $k \in \mathbb{T}^3$ .*

Denote by  $\tau_s(K)$  the bottom of the spectrum of the self-adjoint bounded operator  $H(K)$ , that is,

$$\tau_s(K) = \inf_{\|f\|=1} (H(K)f, f).$$

We set:

$$\tau_s^\gamma(K) \equiv \inf_{\|f\|=1} [(H_0(K)f, f) - (V_\gamma f, f)], \gamma = 1, 2, 3. \tag{4.2}$$

As in the introduction, let  $N(K, z)$  denote the number of eigenvalues of the operator  $H(K)$ ,  $K \in \mathbb{T}^3$  below  $z \leq \tau_{ess}(K)$ , where  $\tau_{ess}(K) \equiv \inf \sigma_{ess}(H(K))$  is the bottom of the essential spectrum of  $H(K)$ .

**Theorem 4.4** *Assume Hypothesis 4.2. Then for all  $K \in \mathbb{T}^3$  the inequality*

$$\tau_s^\alpha(K) - \mu_\beta^0 - \mu_\gamma \leq \tau_s(K)$$

*holds.*

Theorem 4.4 yields the following

**Corollary 4.5** *Assume Hypothesis 4.2. All eigenvalues of the operator  $H(K)$ ,  $K \in \mathbb{T}^3$  below the bottom of  $\tau_{ess}(K)$  belong to the interval  $[\tau_s^\alpha(K) - \mu_\beta^0 - \mu_\gamma, \tau_{ess}(K)]$ .*

**Theorem 4.6** *Assume Hypothesis 4.2. Then for all  $K \in U_\delta^0(0)$ ,  $\delta > 0$  sufficiently small, the operator  $H(K)$  has a finite number of eigenvalues below the bottom of the essential spectrum of  $H(K)$ .*

**Theorem 4.7** *Assume Hypothesis 4.2. Then the operator  $H(0)$  has infinitely many eigenvalues below the bottom of the essential spectrum and the functions  $N(0, z)$  and  $N(K, 0)$  obey the relations*

$$\lim_{z \rightarrow -0} \frac{N(0, z)}{|\log |z||} = \lim_{|K| \rightarrow 0} \frac{N(K, 0)}{2|\log |K||} = \mathcal{U}_0 \quad (0 < \mathcal{U}_0 < \infty). \tag{4.3}$$

**Remark 4.8** *The constant  $\mathcal{U}_0$  does not depend on the interaction energies  $\mu_\alpha$ ,  $\alpha = 1, 2, 3$  and is given as a positive function depending only on the ratios  $\frac{m_\beta}{m_\alpha}$ ,  $\alpha \neq \beta$ ,  $\alpha, \beta = 1, 2, 3$  between the masses.*

### 5 Spectral properties of the two-particle operator $h_\alpha(k)$

In this section we study the spectral properties of the two-particle discrete Schrödinger operator  $h_\alpha(k)$ ,  $k \in \mathbb{T}^3$ .

We consider the family of the self-adjoint operators  $h_\alpha(k)$ ,  $k \in \mathbb{T}^3$  on the Hilbert space  $L_2(\mathbb{T}^3)$

$$h_\alpha(k) = h_\alpha^0(k) - \mu_\alpha v. \tag{5.1}$$

The nonperturbed operator  $h_\alpha^0(k)$  on  $L_2(\mathbb{T}^3)$  is multiplication operator by the function  $E_k^{(\alpha)}(p)$

$$(h_\alpha^0(k)f)(p) = E_k^{(\alpha)}(p)f(p), \quad f \in L_2(\mathbb{T}^3),$$

where  $E_k^{(\alpha)}(p)$  is defined in (3.3). The perturbation  $v$  is an integral operator of rank one

$$(vf)(p) = (2\pi)^{-3} \int_{\mathbb{T}^3} f(q) dq, \quad f \in L_2(\mathbb{T}^3).$$

Therefore by the Weyl theorem the continuous spectrum  $\sigma_{\text{cont}}(h_\alpha(k))$  of the operator  $h_\alpha(k)$ ,  $k \in \mathbb{T}^3$  coincides with the spectrum  $\sigma(h_\alpha^0(k))$  of  $h_\alpha^0(k)$ . More specifically,

$$\sigma_{\text{cont}}(h_\alpha(k)) = [E_{\min}^{(\alpha)}(k), E_{\max}^{(\alpha)}(k)],$$

where

$$E_{\min}^{(\alpha)}(k) \equiv \min_{p \in \mathbb{T}^3} E_k^{(\alpha)}(p), \quad E_{\max}^{(\alpha)}(k) \equiv \max_{p \in \mathbb{T}^3} E_k^{(\alpha)}(p). \tag{5.2}$$

Set

$$l_{\beta\gamma} = \frac{m_\gamma}{m_\beta + m_\gamma}, \quad \beta, \gamma = 1, 2, 3, \quad \beta \neq \gamma.$$

**Lemma 5.1** *There exist an odd and analytic function  $p_\alpha : \mathbb{T}^3 \rightarrow \mathbb{T}^3$  such that for any  $k \in \mathbb{T}^3$  the point  $p_\alpha(k)$  is a unique nondegenerate minimum of the function  $E_k^{(\alpha)}(p)$  and*

$$p_\alpha(k) = l_{\gamma\beta}k + O(|k|^3) \text{ as } k \rightarrow 0. \tag{5.3}$$

*Proof.* The function  $E_k^{(\alpha)}(p)$  can be rewritten in the form

$$E_k^{(\alpha)}(p) = 3m_{\beta\gamma}^{-1} - \sum_{j=1}^3 (a_\alpha(k^{(j)}) \cos p^{(j)} + b_\alpha(k^{(j)}) \sin p^{(j)}), \tag{5.4}$$

where the coefficients  $a_\alpha(k^{(j)})$  and  $b_\alpha(k^{(j)})$  are given by

$$a_\alpha(k^{(j)}) = m_\beta^{-1} + m_\gamma^{-1} \cos k^{(j)}, \quad b_\alpha(k^{(j)}) = m_\gamma^{-1} \sin k^{(j)}. \tag{5.5}$$

The equality (5.4) implies the following representation for  $E_k^{(\alpha)}(p)$

$$E_k^{(\alpha)}(p) = 3m_{\beta\gamma}^{-1} - \sum_{j=1}^3 r_\alpha(k^{(j)}) \cos(p^{(j)} - p_\alpha(k^{(j)})), \tag{5.6}$$

where

$$r_\alpha(k^{(j)}) = \sqrt{a_\alpha^2(k^{(j)}) + b_\alpha^2(k^{(j)})}, \quad p_\alpha(k^{(j)}) = \arcsin \frac{b_\alpha(k^{(j)})}{r_\alpha(k^{(j)})}, \quad k^{(j)} \in (-\pi, \pi].$$

Taking into account (5.5), we have that the vector function

$$p_\alpha : \mathbb{T}^3 \rightarrow \mathbb{T}^3, \quad p_\alpha = p_\alpha(k^{(1)}, k^{(2)}, k^{(3)}) = (p_\alpha(k^{(1)}), p_\alpha(k^{(2)}), p_\alpha(k^{(3)})) \in \mathbb{T}^3$$

is odd regular and it is the minimum point of  $E_k^{(\alpha)}(p)$ . One has, as easily seen from the definition

$$p_\alpha(k) = l_{\gamma\beta} k + O(|k|^3) \text{ as } k \rightarrow 0. \quad \square$$

Let  $\mathbf{C}$  be the complex plane. For any  $k \in \mathbb{T}^3$  and  $z \in \mathbf{C} \setminus \sigma_{\text{cont}}(h_\alpha(k))$  we define a function (the Fredholm determinant associated with the operator  $h_\alpha(k)$ )

$$\Delta_\alpha(k, z) = 1 - \mu_\alpha(2\pi)^{-3} \int_{\mathbb{T}^3} (E_k^{(\alpha)}(q) - z)^{-1} dq.$$

Note that the function  $\Delta_\alpha(k, z)$  is real-analytic in  $\mathbb{T}^3 \times (\mathbf{C} \setminus \sigma_{\text{cont}}(h_\alpha(k)))$

The following lemma is a simple consequence of the Birman-Schwinger principle and the Fredholm theorem.

**Lemma 5.2** *Let  $k \in \mathbb{T}^3$ . The point  $z \in \mathbf{C} \setminus \sigma_{\text{cont}}(h_\alpha(k))$  is an eigenvalue of the operator  $h_\alpha(k)$  if and only if*

$$\Delta_\alpha(k, z) = 0. \quad \square$$

**Lemma 5.3** *The following statements are equivalent:*

- (i) *the operator  $h_\alpha(0)$  has a zero energy resonance;*
- (ii)  $\Delta_\alpha(0, 0) = 0$ ;
- (iii)  $\mu_\alpha = \mu_\alpha^0$ .

*Proof.* Let the operator  $h_\alpha(0)$  has a zero energy resonance for some  $\mu_\alpha > 0$ . Then by Definition 4.1 the equation

$$\varphi(p) = \mu_\alpha m_{\beta\gamma} (2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} \varphi(q) dq$$

has a simple solution in  $C(\mathbb{T}^3)$  and the solution  $\varphi(q)$  is equal to 1 (up to a constant factor). Therefore we see that

$$1 = \mu_\alpha m_{\beta\gamma} (2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq$$

and hence

$$\Delta_\alpha(0, 0) = 1 - \mu_\alpha m_{\beta\gamma} (2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq = 0$$

and so  $\mu_\alpha = \mu_\alpha^0$ .

Let for some  $\mu_\alpha > 0$  the equality

$$\Delta_\alpha(0, 0) = 1 - \mu_\alpha m_{\beta\gamma} (2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq = 0$$

holds and consequently  $\mu_\alpha = \mu_\alpha^0$ . Then only the function  $\varphi(q) \equiv \text{constant} \in C(\mathbb{T}^3)$  is a solution of the equation

$$\varphi(p) = \mu_\alpha m_{\beta\gamma} (2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} \varphi(q) dq,$$

that is, the operator  $h_\alpha(0)$  has a zero energy resonance. □

**Theorem 5.4** *Let the operator  $h_\alpha(0)$  have a zero energy resonance. Then for all  $k \in \mathbb{T}^3$ ,  $k \neq 0$  the operator  $h_\alpha(k)$  has a unique simple eigenvalue  $z_\alpha(k)$  below the bottom of the continuous spectrum of  $h_\alpha(k)$ . Moreover  $z_\alpha(k)$  is even on  $\mathbb{T}^3$  and  $z_\alpha(k) > 0$  for  $k \neq 0$ .*

*Proof.* By Lemma 5.3

$$\Delta_\alpha(0, 0) = 1 - \mu_\alpha^0 m_{\beta\gamma} (2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon(q))^{-1} dq = 0$$

and hence it is easy to see that for any  $z < 0$  the inequality  $\Delta_\alpha(0, z) > 0$  holds. By Lemma 5.2 the operator  $h_\alpha(0)$  has no negative eigenvalues. Since  $p = p_\alpha(k)$  is the nondegenerate minimum of the function  $E_k^{(\alpha)}(p)$  we define  $\Delta_\alpha(k, E_{\min}^{(\alpha)}(k))$  as

$$\Delta_\alpha(k, E_{\min}^{(\alpha)}(k)) = 1 - \mu_\alpha^0 (2\pi)^{-3} \int_{\mathbb{T}^3} (E_k^{(\alpha)}(q) - E_{\min}^{(\alpha)}(k))^{-1} dq.$$

By dominated convergence theorem we have

$$\lim_{z \rightarrow E_{\min}^{(\alpha)}(k)} \Delta_\alpha(k, z) = \Delta_\alpha(k, E_{\min}^{(\alpha)}(k)).$$

For all  $k \neq 0, q \neq 0$  the inequality

$$E_k^{(\alpha)}(q + p_\alpha(k)) - E_{\min}^{(\alpha)}(k) < E_0(q)$$

holds and hence we obtain the following inequality

$$\Delta_\alpha(k, E_{\min}^{(\alpha)}(k)) < \Delta_\alpha(0, 0) = 0, k \neq 0. \tag{5.7}$$

For each  $k \in \mathbb{T}^3$  the function  $\Delta_\alpha(k, \cdot)$  is monotone decreasing on  $(-\infty, E_{\min}^{(\alpha)}(k)]$  and  $\Delta_\alpha(k, z) \rightarrow 1$  as  $z \rightarrow -\infty$ . Then by virtue of (5.7) there is a number  $z_\alpha(k) \in (-\infty, E_{\min}^{(\alpha)}(k))$  such that  $\Delta_\alpha(k, z_\alpha(k)) = 0$ . By Lemma 5.2 for any nonzero  $k \in \mathbb{T}^3$  the operator  $h_\alpha(k)$  has an eigenvalue below  $E_{\min}^{(\alpha)}(k)$ . For any  $k \in \mathbb{T}^3$  and  $z \in (-\infty, E_{\min}^{(\alpha)}(k))$  the equality  $\Delta_\alpha(-k, z) = \Delta_\alpha(k, z)$  holds and hence  $z_\alpha(k)$  is even.

Let us prove the positivity of the eigenvalue  $z_\alpha(k), k \neq 0$ . First we verify, for all  $k \in \mathbb{T}^3, k \neq 0$ , the inequality

$$\Delta_\alpha(k, 0) > 0. \tag{5.8}$$

Applying the definition of  $\mu_\alpha^0$  by (4.1) we have

$$\Delta_\alpha(k, 0) = \mu_\alpha^0(2\pi)^{-3} \int_{\mathbb{T}^3} \frac{\varepsilon_\gamma(k - p) - \varepsilon_\gamma(p)}{E_0^{(\alpha)}(p)E_k^{(\alpha)}(p)} dp. \tag{5.9}$$

Making a change of variables  $q = \frac{k}{2} - p$  in (5.9) and using the equality  $\Delta_\alpha(k, 0) = \Delta_\alpha(-k, 0)$  it is easy to show that

$$\begin{aligned} \Delta_\alpha(k, 0) &= \frac{\Delta_\alpha(k, 0) + \Delta_\alpha(-k, 0)}{2} \\ &= \frac{\mu_\alpha^0}{2}(2\pi)^{-3} \int_{\mathbb{T}^3} (\varepsilon_\gamma(\frac{k}{2} + p) - \varepsilon_\gamma(\frac{k}{2} - p))(\varepsilon_\beta(\frac{k}{2} + p) - \varepsilon_\beta(\frac{k}{2} - p))F(k, p)dp, \end{aligned}$$

where

$$F(k, p) = \frac{E_0^{(\alpha)}(\frac{k}{2} + p) + E_0^{(\alpha)}(\frac{k}{2} - p)}{E_0^{(\alpha)}(\frac{k}{2} + p)E_0^{(\alpha)}(\frac{k}{2} - p)E_k^{(\alpha)}(\frac{k}{2} + p)E_k^{(\alpha)}(\frac{k}{2} - p)} > 0.$$

A simple computation shows that

$$(\varepsilon_\gamma(\frac{k}{2} + p) - \varepsilon_\gamma(\frac{k}{2} - p))(\varepsilon_\beta(\frac{k}{2} + p) - \varepsilon_\beta(\frac{k}{2} - p)) = \frac{4}{m_\beta m_\gamma} \left( \sum_{i=1}^3 \cos \frac{k^{(i)}}{2} \cos p^{(i)} \right)^2 \geq 0.$$

Thus the inequality (5.8) is proven.

For any  $k \in \mathbb{T}^3$  the function  $\Delta_\alpha(k, \cdot)$  is monotone decreasing and the inequalities

$$\Delta_\alpha(k, 0) > \Delta_\alpha(k, z_\alpha(k)) = 0 > \Delta_\alpha(k, E_{\min}^{(\alpha)}(k)), \quad k \neq 0$$

hold. Therefore the eigenvalue  $z_\alpha(k)$  of the operator  $h_\alpha(k)$  belongs to interval  $(0, E_{\min}^{(\alpha)}(k))$ .  $\square$

The following decomposition is important for the proof of the asymptotics (4.3).

**Lemma 5.5** *Let  $\mu_\alpha = \mu_\alpha^0, \alpha = 1, 2, 3$ . Then for any  $k \in U_\delta(0), \delta > 0$  sufficiently small, and  $z \leq E_{\min}^{(\alpha)}(k)$  the following decomposition holds:*

$$\Delta_\alpha(k, z) = \frac{\mu_\alpha^0 m_{\beta\gamma}^{3/2}}{\sqrt{2\pi}} \left[ E_{\min}^{(\alpha)}(k) - z \right]^{\frac{1}{2}} + \Delta_\alpha^{(20)}(E_{\min}^{(\alpha)}(k) - z) + \Delta_\alpha^{(02)}(k, z),$$

where  $\Delta_\alpha^{(20)}(E_{\min}^{(\alpha)}(k) - z) = O(E_{\min}^{(\alpha)}(k) - z)$  as  $z \rightarrow E_{\min}^{(\alpha)}(k)$  and  $\Delta_\alpha^{(02)}(k, z) = O(|k|^2)$  as  $k \rightarrow 0$ .

*Proof.* Let

$$E_\alpha(k, p) = E_k^{(\alpha)}(p + p_\alpha(k)) - E_{\min}^{(\alpha)}(k),$$

where  $p_\alpha(k) \in \mathbb{T}^3$  is the minimum point of the function  $E_k^{(\alpha)}(p)$ , that is,  $E_{\min}^{(\alpha)}(k) = E_k^{(\alpha)}(p_\alpha(k))$ . Then using (5.6) we conclude

$$E_\alpha(k, p) = \sum_{j=1}^3 r_\alpha(k^{(j)})(1 - \cos p^{(j)}).$$

We define the function  $\tilde{\Delta}_\alpha(k, w)$  on  $\mathbb{T}^3 \times \mathbf{C}_+$  by  $\tilde{\Delta}_\alpha(k, w) = \Delta_\alpha(k, E_{\min}^{(\alpha)}(k) - w^2)$ , where  $\mathbf{C}_+ = \{z \in \mathbf{C} : \text{Re} z > 0\}$ . The function  $\tilde{\Delta}_\alpha(k, w)$  represented as

$$\begin{aligned} \tilde{\Delta}_\alpha(k, w) &= 1 - \mu_\alpha(2\pi)^{-3} \int_{\mathbb{T}^3} \frac{dp}{E_\alpha(k, p) + w^2} \\ &= 1 - \mu_\alpha(2\pi)^{-3} \int_{\mathbb{T}^3} \frac{dp}{\sum_{j=1}^3 r_\alpha(k^{(j)})(1 - \cos p^{(j)}) + w^2}. \end{aligned}$$

Let  $V_\delta(0)$  be the complex  $\delta$ -neighborhood of the point  $w = 0 \in \mathbf{C}$ . Denote by  $\Delta_\alpha^*(k, w)$  the analytic continuation of the function  $\tilde{\Delta}_\alpha(k, w)$  to the region  $\mathbb{T}^3 \times (\mathbf{C}_+ \cup V_\delta(0))$ . This function is even in  $k \in \mathbb{T}^3$ .

Therefore

$$\Delta_\alpha^*(k, w) = \Delta_\alpha^*(k, w) + \tilde{\Delta}_\alpha^{(20)}(k, w),$$

where  $\tilde{\Delta}_\alpha^{(20)}(k, w) = O(|k|^2)$  uniformly in  $w \in \mathbf{C}_+$  as  $k \rightarrow 0$ . Taylor series expansion gives

$$\Delta_\alpha^*(k, w) = \tilde{\Delta}_\alpha^{(01)}(0, 0)w + \tilde{\Delta}_\alpha^{(02)}(0, w)w^2$$

where  $\tilde{\Delta}_\alpha^{(02)}(0, w) = O(1)$  as  $w \rightarrow 0$ . Simple computation shows that

$$\frac{\partial \Delta_\alpha^*(0, 0)}{\partial w} = \tilde{\Delta}_\alpha^{(01)}(0, 0) = \frac{\mu_\alpha^0 m_{\beta\gamma}^{3/2}}{\sqrt{2\pi}} \neq 0. \tag{5.10}$$

□

**Corollary 5.6** *The function  $z_\alpha(k) = E_{\min}^{(\alpha)}(k) - w_\alpha^2(k)$  is real-analytic in  $U_\delta(0)$ , where  $w_\alpha(k)$  is a unique simple solution of the equation  $\tilde{\Delta}_\alpha(k, w) = 0$  and  $w_\alpha(k) = O(|k|^2)$  as  $k \rightarrow 0$ .*

*Proof.* Since  $\tilde{\Delta}_\alpha(0, 0) = 0$  and the inequality (5.10) holds the equation  $\tilde{\Delta}_\alpha(k, w) = 0$  has a unique simple solution  $w_\alpha(k)$ ,  $k \in U_\delta(0)$  and it is real-analytic in  $U_\delta(0)$ . Taking into account that the function  $\tilde{\Delta}_\alpha(k, w)$  is even in  $k \in U_\delta(0)$ ,  $\delta > 0$  and  $w_\alpha(0) = 0$  we have that  $w_\alpha(k) = O(|k|^2)$ . Therefore the function  $z_\alpha(k) = E_{\min}^{(\alpha)}(k) - w_\alpha^2(k)$  is real-analytic in  $U_\delta(0)$ . □

**Lemma 5.7** *Let  $\mu_\alpha = \mu_\alpha^0$  for some  $\alpha = 1, 2, 3$ . Then for any  $k \in U_\delta^0(0)$  there exists a number  $\delta(k) > 0$  such that, for all  $z \in V_{\delta(k)}(z_\alpha(k))$ , where  $V_{\delta(k)}(z_\alpha(k))$  is the  $\delta(k)$ -neighborhood of the point  $z_\alpha(k)$ , the following representation holds*

$$\Delta_\alpha(k, z) = C_1(k)(z - z_\alpha(k))\hat{\Delta}_\alpha(k, z).$$

Here  $C_1(k) \neq 0$  and  $\hat{\Delta}_\alpha(k, z)$  is continuous in  $V_{\delta(k)}(z_\alpha(k))$  and  $\hat{\Delta}_\alpha(k, z_\alpha(k)) \neq 0$ .

*Proof.* Since  $z_\alpha(k) < E_{\min}^{(\alpha)}(k)$ ,  $k \neq 0$  the function  $\Delta_\alpha(k, z)$  can be expanded as follows

$$\Delta_\alpha(k, z) = \sum_{n=1}^{\infty} C_n(k)(z - z_\alpha(k))^n, \quad z \in V_{\delta(k)}(z_\alpha(k)),$$

where

$$C_1(k) = \frac{\mu_\alpha^0 m_{\beta\gamma}^{3/2}}{\sqrt{2\pi}} \frac{1}{2\sqrt{E_{\min}^{(\alpha)}(k) - z_\alpha(k)}} \neq 0, \quad k \neq 0.$$

Therefore  $\hat{\Delta}_\alpha(k, z)$  is continuous in  $V_{\delta(k)}(z_\alpha(k))$ . Since  $z_\alpha(k)$ ,  $k \neq 0$  is a unique simple solution of the equation  $\Delta_\alpha(k, z) = 0$ ,  $z \leq E_{\min}^{(\alpha)}(k)$ , we have  $\hat{\Delta}_\alpha(k, z_\alpha(k)) \neq 0$ . □

### 6 Spectrum of the operator $H(K)$

The “channel operator”  $H_\alpha(K)$ ,  $K \in \mathbb{T}^3$  acts in the Hilbert space  $L_2((\mathbb{T}^3)^2)$  as

$$H_\alpha(K) = H_0(K) - V_\alpha. \tag{6.1}$$



The decomposition of the space  $L_2((\mathbb{T}^3)^2)$  into the direct integral

$$L_2((\mathbb{T}^3)^2) = \int_{p \in \mathbb{T}^3} \oplus L_2(\mathbb{T}^3) dp$$

yields for the operator  $H_\alpha(K)$  the decomposition into the direct integral

$$H_\alpha(K) = \int_{p \in \mathbb{T}^3} \oplus H_\alpha(K, p) dp.$$

The fiber operator  $H_\alpha(K, p)$  acts in the Hilbert space  $L_2(\mathbb{T}^3)$  and has the form

$$H_\alpha(K, p) = h_\alpha(K - p) + \varepsilon_\alpha(p)I,$$

where  $I$  is identity operator and  $h_\alpha(k)$  is the two-particle operator defined by (5.1). The representation of the operator  $H_\alpha(K, p)$  implies the equality

$$\sigma(H_\alpha(K, p)) = \sigma_d(h_\alpha(K - p)) \cup [E_{\min}^{(\alpha)}(K - p), E_{\max}^{(\alpha)}(K - p)] + \varepsilon_\alpha(p),$$

where  $\sigma_d(h_\alpha(k))$  is the discrete spectrum of the operator  $h_\alpha(k)$ . The theorem (see, e.g., [21]) on the spectrum of decomposable operators and above-obtained structure for the spectrum of  $H_\alpha(K, p)$  gives

**Lemma 6.1** *The equality holds*

$$\sigma(H_\alpha(K)) = \cup_{p \in \mathbb{T}^3} \{\sigma_d(h_\alpha(K - p) + \varepsilon_\alpha(p))\} \cup [E_{\min}(K), E_{\max}(K)].$$

**Lemma 6.2** *Let  $\mu_\alpha = \mu_\alpha^0$  for some  $\alpha = 1, 2, 3$ . Then for any  $K \in U_\delta^0(0)$ ,  $\delta > 0$  sufficiently small, the following inequality*

$$\tau_s^\alpha(K) < E_{\min}(K)$$

holds, where  $\tau_s^\alpha(K)$  is defined in (4.2).

*Proof.* By Theorem 5.4 for each  $K \in \mathbb{T}^3$  and  $p \in \mathbb{T}^3$ ,  $p \neq K$  the operator  $h_\alpha(K - p)$  has an unique simple positive eigenvalue  $z_\alpha(K - p)$  below the bottom of  $\sigma_{cont}(h_\alpha(K - p))$ . Therefore using Lemma 6.1 for the spectrum  $\sigma(H_\alpha(K))$  of the operator  $H_\alpha(K)$  we conclude that

$$\begin{aligned} \tau_s^\alpha(K) &= \inf_{p \in \mathbb{T}^3} [\varepsilon_\alpha(p) + \sigma(h_\alpha(K - p))] \\ &= \min_{p \in \mathbb{T}^3} [\varepsilon_\alpha(p) + z_\alpha(K - p)]. \end{aligned}$$

By Theorem 5.4 for each  $K \in U_\delta^0(0)$  and  $p \neq K$  the inequality

$$\varepsilon_\alpha(p) + z_\alpha(K - p) < E_{\min}^{(\alpha)}(K - p) + \varepsilon_\alpha(p)$$

holds.

On the other hand, by computing partial derivatives, it is easy to see that for any  $K \in U_\delta^0(0)$  the point  $p = K$  cannot be a minimum point for the function  $E_{\min}^{(\alpha)}(K - p) + \varepsilon_\alpha(p)$ . Therefore  $\tau_s^\alpha(K) < E_{\min}(K)$  holds.  $\square$

*Proof of Theorem 4.4.* One can check that

$$\sup_{\|f\|=1} (V_\alpha f, f) = \mu_\alpha, \quad \alpha = 1, 2, 3.$$

We obtain

$$\begin{aligned} (H(K)f, f) &= (H_0(K)f, f) - (V_\alpha f, f) - (V_\beta f, f) - (V_\gamma f, f) \\ &= (H_\alpha(K)f, f) - (V_\beta f, f) - (V_\gamma f, f) \geq (H_\alpha(K)f, f) - (\mu_\beta^0 + \mu_\gamma)\|f\|^2. \end{aligned}$$

Thus

$$\inf_{\|f\|=1} (H(K)f, f) \geq \inf_{\|f\|=1} (H_\alpha(K)f, f) - \mu_\beta^0 - \mu_\gamma.$$

The definition (4.2) of  $\tau_s^\alpha(K)$  imply that

$$\tau_s^\alpha(K) - \mu_\beta^0 - \mu_\gamma \leq \inf_{\|f\|=1} (H(K)f, f) = \tau_s(K) < \tau_{ess}(K). \quad \square$$

Let  $W_\alpha(K, z), \alpha = 1, 2, 3$  be the operators on  $L_2((\mathbb{T}^3)^2)$  defined as

$$W_\alpha(K, z) = I + V_\alpha^{\frac{1}{2}} R_\alpha(K, z) V_\alpha^{\frac{1}{2}},$$

where  $R_\alpha(K, z), \alpha = 1, 2, 3$  are the resolvents of  $H_\alpha(K), \alpha = 1, 2, 3$ . One can check that

$$W_\alpha(K, z) = (I - V_\alpha^{\frac{1}{2}} R_0(K, z) V_\alpha^{\frac{1}{2}})^{-1},$$

where  $R_0(K, z)$  the resolvent of the operator  $H_0(K)$ .

For  $z < \tau_{ess}(K), \tau_{ess}(K) = \inf \sigma_{ess}(H(K))$  the operators  $W_\alpha(K, z), \alpha = 1, 2, 3$  are positive.

Denote by  $\mathcal{L} = L_2^{(3)}((\mathbb{T}^3)^2)$  the Hilbert space of vector functions  $w$  with components  $w_\alpha \in L_2((\mathbb{T}^3)^2), \alpha = 1, 2, 3$ .

Let

$$\mathbf{T}(K, z), \quad z < \tau_{ess}(K)$$

be the operator on  $\mathcal{L}$  with the entries

$$\begin{aligned} \mathbf{T}_{\alpha\alpha}(K, z) &= 0, \\ \mathbf{T}_{\alpha\beta}(K, z) &= W_\alpha^{\frac{1}{2}}(K, z) V_\alpha^{\frac{1}{2}} R_0(K, z) V_\beta^{\frac{1}{2}} W_\beta^{\frac{1}{2}}(K, z). \end{aligned}$$

For any bounded self-adjoint operator  $A$  acting in the Hilbert space  $\mathcal{H}$  not having any essential spectrum on the right of the point  $z$  we denote by  $\mathcal{H}_A(z)$  the subspace such that  $(Af, f) > z(f, f)$  for any  $f \in \mathcal{H}_A(z)$  and set  $n(z, A) = \sup_{\mathcal{H}_A(z)} \dim \mathcal{H}_A(z)$ . By the definition of  $N(K, z)$  we have

$$N(K, z) = n(-z, -H(K)), \quad -z > -\tau_{ess}(K).$$

The following lemma is a realization of the well-known Birman-Schwinger principle for the three-particle Schrödinger operators on a lattice (see [22, 24]).

**Lemma 6.3** For  $z < \tau_{ess}(K)$  the operator  $\mathbf{T}(K, z)$  is compact and continuous in  $z$  and

$$N(K, z) = n(1, \mathbf{T}(K, z)).$$

*Proof.* First verify the equality

$$N(K, z) = n(1, R_0^{\frac{1}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z)), \quad V = V_1 + V_2 + V_3. \quad (6.2)$$

Assume that  $u \in \mathcal{H}_{-H(K)}(-z)$ , that is,  $((H_0(K) - z)u, u) < (Vu, u)$ . Then

$$(y, y) < (R_0^{\frac{1}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z)y, y), \quad y = (H_0(K) - z)^{\frac{1}{2}}u.$$

Thus  $N(K, z) \leq n(1, R_0^{\frac{1}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z))$ . Reversing the argument we get the opposite inequality, which proves (6.2).

Now we use the following well-known fact.

**Proposition 6.4** Let  $T_1, T_2$  be bounded operators. If  $z \neq 0$  is an eigenvalue of  $T_1T_2$  then  $z$  is an eigenvalue for  $T_2T_1$  as well of the same algebraic and geometric multiplicities.

Using Proposition 6.4 we get

$$n(1, R_0^{\frac{1}{2}}(K, z)VR_0^{\frac{1}{2}}(K, z)) = n(1, \mathbf{M}(K, z)),$$

where  $\mathbf{M}(K, z)$  the operator on  $\mathcal{L}$  with the entries

$$M_{\alpha\beta} = V_\alpha^{\frac{1}{2}}R_0(K, z)V_\beta^{\frac{1}{2}}, \quad \alpha, \beta = 1, 2, 3.$$

Let us check that

$$n(1, \mathbf{M}(K, z)) = n(1, \mathbf{T}(K, z)).$$

We shall show that for any  $u \in \mathcal{H}_{\mathbf{M}(K, z)}(1)$  there exists  $y \in \mathcal{H}_{\mathbf{T}(K, z)}(1)$  such that  $(y, y) < (\mathbf{T}(K, z)y, y)$ . Let  $u \in \mathcal{H}_{\mathbf{M}(K, z)}(1)$ , that is,

$$\sum_{\alpha=1}^3 (u_\alpha, u_\alpha) < \sum_{\alpha, \beta=1}^3 (V_\alpha^{\frac{1}{2}}R_0(K, z)V_\beta^{\frac{1}{2}}u_\beta, u_\alpha)$$

and hence

$$\sum_{\alpha=1}^3 ((I - V_\alpha^{\frac{1}{2}}R_0(K, z)V_\alpha^{\frac{1}{2}})u_\alpha, u_\alpha) < \sum_{\beta \neq \alpha=1}^3 (V_\alpha^{\frac{1}{2}}R_0(K, z)V_\beta^{\frac{1}{2}}u_\beta, u_\alpha).$$

Denoting by  $y_\alpha = (I - V_\alpha^{\frac{1}{2}}R_0(K, z)V_\alpha^{\frac{1}{2}})^{\frac{1}{2}}u_\alpha$  we have

$$\sum_{\alpha=1}^3 (y_\alpha, y_\alpha) < \sum_{\beta \neq \alpha=1}^3 (W_\alpha^{\frac{1}{2}}(K, z)V_\alpha^{\frac{1}{2}}R_0(K, z)V_\beta^{\frac{1}{2}}W_\beta^{\frac{1}{2}}(K, z)y_\beta, y_\alpha),$$

that is,  $(y, y) \leq (\mathbf{T}(K, z)y, y)$ . Thus  $n(1, \mathbf{M}(K, z)) \leq n(1, \mathbf{T}(K, z))$ .

By the same way one can check  $n(1, \mathbf{T}(K, z)) \leq n(1, \mathbf{M}(K, z))$ . □

Set

$$\Sigma(K) := \cup_{\alpha=1}^3 \sigma(H_\alpha(K)).$$

Denote by  $L_2^{(3)}(\mathbb{T}^3)$  the space of vector functions  $w = (w_1, w_2, w_3)$ ,  $w_\alpha \in L_2(\mathbb{T}^3)$ ,  $\alpha = 1, 2, 3$  and define compact operator  $T(K, z)$ ,  $z \in \mathbf{C} \setminus \Sigma(K)$  on  $L_2^{(3)}(\mathbb{T}^3)$  with the entries

$$T_{\alpha\alpha}(K, z) = 0,$$

$$(T_{\alpha\beta}(K, z)w_\beta)(k_\alpha) = \sqrt{\mu_\alpha\mu_\beta}(2\pi)^{-3} \int_{\mathbb{T}^3} \frac{\Delta_\alpha^{-\frac{1}{2}}(K, k_\alpha, z)\Delta_\beta^{-\frac{1}{2}}(K, k_\beta, z)}{E_{\alpha\beta}(K; k_\alpha, k_\beta) - z} w_\beta(k_\beta) dk_\beta,$$

$$w \in L_2^{(3)}(\mathbb{T}^3),$$

where

$$\Delta_\alpha(K, k_\alpha, z) := \Delta_\alpha(K - k_\alpha, z - \varepsilon_\alpha(k_\alpha)).$$

Now we show that the numbers of eigenvalues greater than 1 of the operators  $\mathbf{T}(K, z)$  and  $T(K, z)$  are coincide.

Let

$$\Psi = \text{diag}\{\Psi_1, \Psi_2, \Psi_3\} : L_2^{(3)}((\mathbb{T}^3)^2) \rightarrow L_2^{(3)}(\mathbb{T}^3)$$

be the operator with the entries

$$(\Psi_\alpha f)(k_\alpha) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} f(k_\alpha, q) dq, \quad \alpha = 1, 2, 3 \tag{6.3}$$

and  $\Psi^* = \text{diag}\{\Psi_1^*, \Psi_2^*, \Psi_3^*\}$  its adjoint.

**Lemma 6.5** *The following equalities*

$$\mathbf{T}(K, z) = \Psi^* T(K, z) \Psi \quad \text{and} \quad n(1, \mathbf{T}(K, z)) = n(1, T(K, z))$$

hold.

*Proof.* One can easily check that the equalities

$$\Psi_\alpha f = (2\pi)^{\frac{3}{2}} \mu_\alpha^{-\frac{1}{2}} V_\alpha^{1/2} f \quad \text{and} \quad V_\alpha^{1/2} W_\alpha^{1/2} f = \Delta_\alpha^{-\frac{1}{2}}(K, k_\alpha, z) V_\alpha^{1/2} f \tag{6.4}$$

hold. The equalities (6.4) imply the first equality of Lemma 6.5. By Proposition 6.4 we have

$$n(1, \mathbf{T}(K, z)) = n(1, \Psi^* T(K, z) \Psi) = n(1, T(K, z) \Psi \Psi^*) = n(1, T(K, z)). \quad \square$$

Now we establish a location of the essential spectrum of  $H(K)$ . For any  $K \in \mathbb{T}^3$  and  $z \in \mathbf{C} \setminus \Sigma(K)$  the kernels of the operators  $T_{\alpha\beta}(K, z)$ ,  $\alpha, \beta = 1, 2, 3$  are continuous functions on  $(\mathbb{T}^3)^2$ . Therefore the Fredholm determinant  $D_K(z)$  of the operator  $I - T(K, z)$ , where  $I$  is the identity operator in  $L_2^{(3)}(\mathbb{T}^3)$ , exists and is a

real-analytic function on  $\mathbf{C} \setminus \Sigma(K)$ . The following theorem is a lattice analog of the well-known Faddeev's result for the three-particle Schrödinger operators with the zero-range interactions and can be proved similarly to that of the identical particle case (see [12]).

**Theorem 6.6** *For any  $K \in \mathbb{T}^3$  the number  $z \in \mathbf{C} \setminus \Sigma(K)$  is an eigenvalue of the operator  $H(K)$  if and only if the number 1 is eigenvalue of  $T(K, z)$ .  $\square$*

According to Fredholm's theorem the following lemma holds.

**Lemma 6.7** *The number  $z \in \mathbf{C} \setminus \Sigma(K)$  is an eigenvalue of the operator  $H(K)$  if and only if*

$$D_K(z) = 0. \quad \square$$

*Proof of Theorem 4.3.* By the definition of the essential spectrum, it is easy to show that  $\Sigma(K) \subset \sigma_{ess}(H(K))$ . Since the function  $D_K(z)$  is analytic in  $\mathbf{C} \setminus \Sigma(K)$  by Lemma 6.7 we conclude that the set

$$\sigma(H(K)) \setminus \Sigma(K) = \{z : D_K(z) = 0\}$$

is discrete. Thus

$$\sigma(H(K)) \setminus \Sigma(K) \subset \sigma(H(K)) \setminus \sigma_{ess}(H(K)).$$

Therefore the inclusion  $\sigma_{ess}(H(K)) \subset \Sigma(K)$  holds.  $\square$

Now we are going to prove the finiteness of  $N(K, \tau_{ess}(K))$  for  $K \in U_\delta^0(0)$ ,  $\delta > 0$  sufficiently small. First we shall prove that the operator  $T(K, \tau_{ess}(K))$  belongs to the Hilbert-Schmidt class.

The point  $p = 0$  is the nondegenerate minimum of the functions  $\varepsilon_\alpha(p)$  and  $z_\alpha(p)$  (see Corollary 5.6) and hence  $p = 0$  is the nondegenerate minimum of  $Z_\alpha(0, p)$  defined by

$$Z_\alpha(K, p) := \varepsilon_\alpha(p) + z_\alpha(K - p).$$

By the definition of  $\varepsilon_\alpha(p)$  we have

$$\varepsilon_\alpha(p) = \frac{1}{2m_\alpha} p^2 + O(|p|^4) \text{ as } p \rightarrow 0. \quad (6.5)$$

Using the definition of  $E_{\min}^{(\alpha)}(k)$ , asymptotics (5.3) and (5.4) we obtain that

$$E_{\min}^{(\alpha)}(k) = \frac{1}{2(m_\beta + m_\gamma)} k^2 + O(|k|^4) \text{ as } k \rightarrow 0. \quad (6.6)$$

Corollary 5.6 and simple computations give

$$\left( \frac{\partial^2 Z_\alpha(0, 0)}{\partial p^{(i)} \partial p^{(j)}} \right)_{i,j=1}^3 = \frac{n_\alpha}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where  $n_\alpha \equiv \frac{M}{m_\alpha(m_\beta + m_\gamma)}$ .

Therefore for all  $K \in U_\delta^0(0)$  at the nondegenerate minimum point  $p_\alpha^Z(K) \in U_\delta^0(0)$  of the function  $Z_\alpha(K, p)$ ,  $K \in U_\delta^0(0)$  the inequality

$$B(K) = \left( \frac{\partial^2 Z_\alpha}{\partial p^{(i)} \partial p^{(j)}}(K, p_\alpha^Z(K)) \right)_{i,j=1}^3 > 0$$

holds. Hence the asymptotics

$$Z_\alpha(K, p) = \tau_s^\alpha(K) + (B(K)(p - p_\alpha^Z(K)), p - p_\alpha^Z(K)) + o(|p - p_\alpha^Z(K)|^2) \text{ as } |p - p_\alpha^Z(K)| \rightarrow 0 \quad (6.7)$$

is valid, where  $\tau_s^\alpha(K) = Z_\alpha(K, p_\alpha^Z(K))$ . From Lemma 5.7 we conclude that for all  $K \in U_\delta^0(0)$ ,  $p \in U_{\delta(K)}(p_\alpha^Z(K))$  the equality

$$\Delta_\alpha(K, p, \tau_s^\alpha(K)) = (Z_\alpha(K, p) - \tau_s^\alpha(K)) \hat{\Delta}_\alpha(K, p, \tau_s^\alpha(K)) \quad (6.8)$$

holds, where  $\hat{\Delta}_\alpha(K, p_\alpha^Z(K), \tau_s^\alpha(K)) \neq 0$ . Putting (6.7) into (6.8) we get the following

**Lemma 6.8** *Let  $\mu_\alpha = \mu_\alpha^0, \alpha = 1, 2, 3$ . Then for any  $K \in U_\delta^0(0), \delta = \delta(K)$  sufficiently small, there are positive nonzero constants  $c$  and  $C$  depending on  $K$  and  $U_{\delta(K)}(p_\alpha^Z(K))$  such that for all  $p \in U_{\delta(K)}(p_\alpha^Z(K))$  the following inequalities*

$$c|p - p_\alpha^Z(K)|^2 \leq \Delta_\alpha(K, p, \tau_s^\alpha(K)) \leq C|p - p_\alpha^Z(K)|^2 \quad (6.9)$$

hold. □

**Lemma 6.9** *Let  $\mu_\alpha \leq \mu_\alpha^0$  for all  $\alpha = 1, 2, 3$ . Then for any  $K \in U_\delta^0(0), \delta > 0$  sufficiently small, the operator  $T(K, \tau_{ess}(K))$  belongs to the Hilbert-Schmidt class.*

*Proof.* As we shall see that it is sufficient to prove Lemma 6.9 in the case  $\mu_\alpha = \mu_\alpha^0$  for all  $\alpha = 1, 2, 3$ . By Lemma 6.2 we have

$$\tau_{ess}(K) = \min_\alpha \tau_s^\alpha(K) < E_{\min}(K), K \in U_\delta^0(0). \quad (6.10)$$

The operator  $h_\alpha(0)$  has a zero energy resonance. By Theorem 5.4 the operator  $h_\alpha(k), k \in \mathbb{T}^3, k \neq 0$  has a unique eigenvalue  $z_\alpha(k), z_\alpha(k) < E_{\min}^{(\alpha)}(k)$ .

Since  $\tau_s^\alpha(K) = \min_{p \in \mathbb{T}^3} Z_\alpha(K, p)$  the function  $Z_\alpha(K, p)$  has a unique minimum and hence for all  $p \in \mathbb{T}^3 \setminus U_\delta(p_\alpha^Z(K))$  we obtain

$$\Delta_\alpha(K, p, \tau_s^\alpha(K)) \geq C > 0. \quad (6.11)$$

According to Lemma 6.2 for all  $p_\alpha, p_\beta \in \mathbb{T}^3$  and  $K \in U_\delta^0(0)$  the inequality

$$E_{\alpha\beta}(K; p_\alpha, p_\beta) - \tau_s^\alpha(K) \geq E_{\min}(K) - \tau_s^\alpha(K) > 0 \quad (6.12)$$

holds. Using (6.9), (6.11) and taking into account (6.12) we can make certain that for all  $K \in U_\delta^0(0)$  and  $p_\alpha \in U_\delta(p_\alpha^Z(K))$ ,  $p_\beta \in U_\delta(p_\beta^Z(K))$  the modules of the kernels  $T_{\alpha\beta}(K, \tau_{ess}(K); p_\alpha, p_\beta)$  of the integral operators  $T_{\alpha\beta}(K, \tau_{ess}(K))$  can be estimated by

$$\frac{C_0(K)}{|p_\alpha - p_\alpha^Z(K)||p_\beta - p_\beta^Z(K)|} + C_1,$$

where  $C_0(K)$  and  $C_1$  some constants. Taking into account (6.10) we conclude that

$$T_{\alpha\beta}(K, \tau_{ess}(K)), \alpha, \beta = 1, 2, 3$$

are Hilbert-Schmidt operators. Thus,  $T(K, \tau_{ess}(K))$  belongs to the Hilbert-Schmidt class. □

Now we shall prove the finiteness of  $N(K, \tau_{ess}(K))$  (Theorem 4.6).

**Theorem 6.10** *Assume Hypothesis 4.2. Then for the number  $N(K, \tau_{ess}(K))$  the relation*

$$N(K, \tau_{ess}(K)) \leq \lim_{\gamma \rightarrow 0} n(1 - \gamma, T(K, \tau_{ess}(K)))$$

*holds.*

*Proof.* By Lemmas 6.3 and 6.5 we have

$$N(K, z) = n(1, T(K, z)) \text{ as } z < \tau_{ess}(K)$$

and by Lemma 6.9 for any  $\gamma \in [0, 1)$  the number  $n(1 - \gamma, T(K, \tau_{ess}(K)))$ ,  $K \in U_\delta^0(0)$  is finite. Then according to the Weyl inequality

$$n(\lambda_1 + \lambda_2, A_1 + A_2) \leq n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

for all  $z < \tau_{ess}(K)$  and  $\gamma \in (0, 1)$  we have

$$N(K, z) = n(1, T(K, z)) \leq n(1 - \gamma, T(K, \tau_{ess}(K))) + n(\gamma, T(K, z) - T(K, \tau_{ess}(K))).$$

Since  $T(K, z)$  is continuous from the left up to  $z = \tau_{ess}(K)$ ,  $K \in U_\delta^0(0)$ , we obtain

$$\lim_{z \rightarrow \tau_{ess}(K)} N(K, z) = N(K, \tau_{ess}(K)) \leq n(1 - \gamma, T(K, \tau_{ess}(K))) \text{ for all } \gamma \in (0, 1)$$

and so

$$N(K, \tau_{ess}(K)) \leq \lim_{\gamma \rightarrow 0} n(1 - \gamma, T(K, \tau_{ess}(K))). \quad \square$$

### 7 Asymptotics for the number of eigenvalues of the operator $H(K)$

We recall that in this section we closely follow A. Sobolev’s method to derive the asymptotics for the number of eigenvalues of  $H(K)$  (Theorem 4.7). As we shall see, the discrete spectrum asymptotics of the operator  $T(K, z)$  as  $|K| \rightarrow 0$  and  $z \rightarrow -0$  is determined by the integral operator

$$\mathbf{S}_r, \mathbf{r} = 1/2 \left| \log \left( \frac{|K|^2}{2M} + |z| \right) \right|$$

in

$$L_2((0, \mathbf{r}) \times \sigma^{(3)}), \sigma = L_2(\mathbf{S}^2),$$

with the kernel  $S_{\alpha\beta}(x - x'; \langle \xi, \eta \rangle)$ ,  $\xi, \eta \in \mathbf{S}^2$ ,  $\mathbf{S}^2$  is the unit sphere in  $\mathbb{R}^3$ , where

$$S_{\alpha\alpha}(x; t) = 0, \quad S_{\alpha\beta}(x; t) = (2\pi)^{-2} \frac{u_{\alpha\beta}}{\cosh(x + r_{\alpha\beta}) + s_{\alpha\beta}t} \tag{7.1}$$

and

$$u_{\alpha\beta} = k_{\alpha\beta} \left( \frac{m_{\beta\gamma}^{-1} m_{\alpha\gamma}^{-1}}{n_\alpha n_\beta} \right)^{\frac{1}{4}}, \quad r_{\alpha\beta} = \frac{1}{2} \log \frac{m_{\beta\gamma}}{m_{\alpha\gamma}}, \quad s_{\alpha\beta} = \frac{(m_{\alpha\gamma} m_{\beta\gamma})^{\frac{1}{2}}}{m_\gamma},$$

$k_{\alpha\beta}$  being such that  $k_{\alpha\beta} = 1$  if both subsystems  $\alpha$  and  $\beta$  have zero resonances, otherwise  $k_{\alpha\beta} = 0$ . The eigenvalues asymptotics for the operator  $\mathbf{S}_r$  have been studied in detail by Sobolev [22], by employing an argument used in the calculation of the canonical distribution of Toeplitz operators. We here summarize some results obtained in [22].

**Lemma 7.1** *The following equality*

$$\lim_{r \rightarrow \infty} \frac{1}{2} r^{-1} n(\lambda, \mathbf{S}_r) = \mathcal{U}(\lambda)$$

holds, where the function  $\mathcal{U}(\lambda)$  is continuous in  $\lambda > 0$  and  $\mathcal{U}_0$  in (4.3) defined as  $\mathcal{U}_0 = \mathcal{U}(1)$ .

**Lemma 7.2** *Let  $A(z) = A_0(z) + A_1(z)$ , where  $A_0$  ( $A_1$ ) is compact and continuous in  $z < 0$  ( $z \leq 0$ ). Assume that for some function  $f(\cdot)$ ,  $f(z) \rightarrow 0$ ,  $z \rightarrow -0$  the limit*

$$\lim_{z \rightarrow -0} f(z) n(\lambda, A_0(z)) = l(\lambda),$$

exists and is continuous in  $\lambda > 0$ . Then the same limit exists for  $A(z)$  and

$$\lim_{z \rightarrow -0} f(z) n(\lambda, A(z)) = l(\lambda). \quad \square$$

Now we are going to reduce the study of the asymptotics for the operator  $T(K, z)$  to that of the asymptotics of  $\mathbf{S}_r$ .



By (6.5) we obtain

$$E_{\alpha\beta}(K; l_\alpha K - p, l_\beta K - q) = \frac{p^2}{2m_{\alpha\gamma}} + \frac{(p, q)}{m_\gamma} + \frac{q^2}{2m_{\beta\gamma}} + \frac{K^2}{2M} + O(|K|^4 + |p|^4 + |q|^4) \text{ as } K, p, q \rightarrow 0, \quad (7.2)$$

where  $m_\alpha$  is the mass of the particle  $\alpha$  and

$$l_\alpha \equiv \frac{m_\alpha}{M}, \quad M \equiv m_1 + m_2 + m_3, \quad \alpha = 1, 2, 3.$$

From Lemma 5.5 and (6.6) we easily receive the following

**Lemma 7.3** *For any  $K \in U_\delta(0)$  and  $z \in [-\delta, 0]$  we have*

$$\Delta_\alpha(K, l_\alpha K - p, z) = \frac{\mu_\alpha^0 m_{\beta\gamma}^{3/2}}{2\pi} \left[ n_\alpha p^2 + \frac{K^2}{M} - 2z \right]^{\frac{1}{2}} + O(|K|^2 + |p|^2 + |z|) \text{ as } K, p, z \rightarrow 0. \quad (7.3)$$

□

The following theorem is basic for the proof of the asymptotics (4.3).

**Theorem 7.4** *The equality*

$$\lim_{\frac{|K|^2 + |z| \rightarrow 0}{M}} \frac{n(1, T(K, z))}{|\log(\frac{|K|^2}{M} + |z|)|} = \lim_{r \rightarrow \infty} \frac{1}{2} r^{-1} n(1, \mathbf{S}_r)$$

holds.

**Remark 7.5** *Since  $\mathcal{U}(\cdot)$  is continuous in  $\lambda$ , according to Lemma 7.2 a compact and continuous up to  $z = 0$  perturbations of the operator  $A_0(z)$ , do not contribute to the asymptotics (4.3). During the proof of Theorem 7.4 we use this fact without further comments. First we prove Theorem 7.4 under the condition that all two-particle operators have zero energy resonances, that is, in the case where  $\mu_1 = \mu_1^0, \mu_2 = \mu_2^0$  and  $\mu_3 = \mu_3^0$ . The case where only two operators  $h_\alpha(0)$  and  $h_\beta(0)$  have zero energy resonance can be proven similarly.*

*Proof of Theorem 7.4.* Let  $\tilde{T}(K, z), z \in \mathbf{C} \setminus \sigma_{ess}(H(K)), K \in U_\delta(0)$  be operator on  $L_2^{(3)}(\mathbb{T}^3)$  with the entries

$$\begin{aligned} \tilde{T}_{\alpha\alpha}(K, z) &= 0, \\ (\tilde{T}_{\alpha\beta}(K, z)w_\beta)(p_\alpha) &= \sqrt{\mu_\alpha \mu_\beta} (2\pi)^{-3} \int_{\mathbb{T}^3} \frac{\Delta_\alpha^{-\frac{1}{2}}(K, l_\alpha K - p_\alpha, z) \Delta_\beta^{-\frac{1}{2}}(K, l_\beta K - p_\beta, z)}{E_{\alpha\beta}(K; l_\alpha K - p_\alpha, l_\beta K - p_\beta) - z} w_\beta(p_\beta) dp_\beta, \end{aligned}$$

$w \in L_2^{(3)}(\mathbb{T}^3)$ .

The operators  $T(K, z)$ ,  $K \in U_\delta(0)$  and  $\tilde{T}(K, z)$ ,  $K \in U_\delta(0)$  are unitarily equal and the equivalence is performed by the unitary dilation

$$\mathbf{Y}_K = \text{diag}\{Y_K^{(1)}, Y_K^{(2)}, Y_K^{(3)}\} : L_2^{(3)}((\mathbb{T}^3)^2) \rightarrow L_2^{(3)}((\mathbb{T}^3)^2),$$

$$(Y_K^{(\alpha)} f)(p_\alpha) = f(l_\alpha K - p_\alpha).$$

Let  $T(\delta, \frac{K^2}{2M} + |z|)$  be an operator on  $L_2^{(3)}(\mathbb{T}^3)$  with the entries

$$T_{\alpha\alpha}(\delta; \frac{K^2}{2M} + |z|) = 0,$$

$$(T_{\alpha\beta}(\delta; \frac{K^2}{2M} + |z|)w)(p)$$

$$= D_{\alpha\beta} \int_{\mathbb{T}^3} \frac{\chi_\delta(p)\chi_\delta(q)(n_\alpha p^2 + 2(\frac{K^2}{2M} + |z|))^{-1/4}(n_\beta q^2 + 2(\frac{K^2}{2M} + |z|))^{-1/4}}{\frac{q^2}{m_{\beta\gamma}} + \frac{2}{m_\gamma}(p, q) + \frac{p^2}{m_{\alpha\gamma}} + 2(\frac{K^2}{2M} + |z|)} w(q) dq,$$

where

$$D_{\alpha\beta} = \frac{m_{\alpha\gamma}^{-\frac{3}{4}} m_{\beta\gamma}^{-\frac{3}{4}}}{2\pi^2}, \quad \alpha, \beta, \gamma = 1, 2, 3, \quad \alpha \neq \beta \neq \gamma$$

and  $\chi_\delta(\cdot)$  is the characteristic function of  $U_\delta(0) = \{p : |p| < \delta\}$ .

**Lemma 7.6** *The operator  $\tilde{T}(K, z) - T(\delta; \frac{K^2}{2M} + |z|)$  belongs to the Hilbert-Schmidt class and is continuous in  $K \in U_\delta(0)$  and  $z \leq 0$ .*

*Proof.* Applying asymptotics (7.2) and (7.3) one can estimate the kernel of the operator  $\tilde{T}_{\alpha\beta}(K, z) - T_{\alpha\beta}(\delta; \frac{K^2}{2M} + |z|)$  by

$$C[(p^2 + q^2)^{-1} + |p|^{-\frac{1}{2}}(p^2 + q^2)^{-1} + (|q|^{-\frac{1}{2}}(p^2 + q^2)^{-1} + 1)]$$

and hence the operator  $\tilde{T}_{\alpha\beta}(K, z) - T_{\alpha\beta}(\delta; \frac{K^2}{2M} + |z|)$  belongs to the Hilbert-Schmidt class for all  $K \in U_\delta(0)$  and  $z \leq 0$ . In combination with the continuity of the kernel of the operator in  $K \in U_\delta(0)$  and  $z < 0$  this gives the continuity of  $\tilde{T}(K, z) - T(\delta; \frac{K^2}{2M} + |z|)$  in  $K \in U_\delta(0)$  and  $z \leq 0$ . □

The space of vector functions  $w = (w_1, w_2, w_3)$  with coordinates having support in  $U_\delta(0)$  is an invariant subspace for the operator  $T(\delta, \frac{K^2}{2M} + |z|)$ .

Denote by  $\mathcal{L}_\delta$  the space of vector functions  $w = (w_1, w_2, w_3)$ ,  $w_\alpha \in L_2(U_\delta(0))$ , that is,

$$\mathcal{L}_\delta = \oplus_{\alpha=1}^3 L_2(U_\delta(0)).$$

Let  $T_0(\delta, \frac{K^2}{2M} + |z|)$  be the restriction of the operator  $T(\delta, \frac{K^2}{2M} + |z|)$  to the invariant subspace  $\mathcal{L}_\delta$ . One verifies that the operator  $T_0(\delta, \frac{K^2}{2M} + |z|)$  is unitarily equivalent

to the operator  $T_1(\delta, \frac{K^2}{2M} + |z|)$  with entries

$$T_{\alpha\alpha}^{(1)}(\delta; \frac{K^2}{2M} + |z|) = 0,$$

$$(T_{\alpha\beta}^{(1)}(\delta; \frac{K^2}{2M} + |z|)w)(p) = D_{\alpha\beta} \int_{U_r(0)} \frac{(n_\alpha p^2 + 2)^{-1/4} (n_\beta q^2 + 2)^{-1/4}}{\frac{q^2}{m_{\beta\gamma}} + \frac{2}{m_\gamma}(p, q) + \frac{p^2}{m_{\alpha\gamma}} + 2} w(q) dq$$

acting in  $L_2^{(3)}(U_r(0))$ ,  $r = (\frac{|K|^2}{2M} + |z|)^{-\frac{1}{2}}$ . The equivalence is performed by the unitary dilation

$$\mathbf{B}_r = \text{diag}\{B_r, B_r, B_r\} : L_2^{(3)}(U_\delta(0)) \rightarrow L_2^{(3)}(U_r(0)), \quad (B_r f)(p) = (\frac{r}{\delta})^{-3/2} f(\frac{\delta}{r} p).$$

Further, we may replace

$$(n_\alpha p^2 + 2)^{-1/4}, (n_\beta q^2 + 2)^{-1/4} \quad \text{and} \quad \frac{q^2}{m_{\beta\gamma}} + \frac{2}{m_\gamma}(p, q) + \frac{p^2}{m_{\alpha\gamma}} + 2$$

by

$$(n_\alpha p^2)^{-1/4}(1 - \chi_1(p)), (n_\beta q^2)^{-1/4}(1 - \chi_1(q)) \quad \text{and} \quad \frac{q^2}{m_{\beta\gamma}} + \frac{2}{m_\gamma}(p, q) + \frac{p^2}{m_{\alpha\gamma}},$$

respectively, since the error will be a Hilbert-Schmidt operator continuous up to  $K = 0$  and  $z = 0$ . Then we get the operator  $T^{(2)}(r)$  in  $L_2^{(3)}(U_r(0) \setminus U_1(0))$  with entries

$$T_{\alpha\alpha}^{(2)}(r) = 0,$$

$$(T_{\alpha\beta}^{(2)}(r)w)(p) = (n_\alpha n_\beta)^{-\frac{1}{4}} D_{\alpha\beta} \int_{U_r(0) \setminus U_1(0)} \frac{|p|^{-1/2} |q|^{-1/2}}{\frac{q^2}{m_{\beta\gamma}} + \frac{2}{m_\gamma}(p, q) + \frac{p^2}{m_{\alpha\gamma}}} w(q) dq.$$

This operator  $T^{(2)}(r)$  is unitarily equivalent to the integral operator  $\mathbf{S}_r$  with entries (7.1). The equivalence is performed by the unitary operator  $\mathbf{M} = \text{diag}\{M, M, M\} : L_2^{(3)}(U_r(0) \setminus U_1(0)) \rightarrow L_2((0, \mathbf{r}) \times \sigma^{(3)})$ , where  $(Mf)(x, w) = e^{3x/2} f(e^x w)$ ,  $x \in (0, \mathbf{r})$ ,  $w \in \mathbf{S}^2$ .

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