Ann. Henri Poincaré 4 (2003) 1001 – 1013
1424-0637/03/051001-13
DOI 10.1007/s00023-003-0153-5
Annales Henri Poincaré

The Heat Kernel Expansion for the Electromagnetic Field in a Cavity

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Abstract. We derive the first six coefficients of the heat kernel expansion for the electromagnetic field in a cavity by relating it to the expansion for the Laplace operator acting on forms. As an application we verify that the electromagnetic Casimir energy is finite.

1 Introduction

The modes of an electromagnetic field in a cavity, taken together with their unphysical, longitudinal counterparts, can be mapped onto the eigenstates of the Laplacian acting on the de Rham complex of a 3-manifold with boundary. The electric and magnetic fields are thereby associated to forms of degree $p = 1$ and $p = 2$ respectively. In this correspondence transverse modes are associated with coexact, resp. exact forms, which permits to further map longitudinal modes to forms of degree $p = 0$ and $p = 3$. We will use this observation, which is explained in detail in Section 2 below, to compute the first six coefficients of the heat kernel expansion for the electromagnetic field in a cavity. The result is used to show in a simple way that the Casimir energy in an arbitrary cavity with smooth boundaries is finite, a conclusion which has been reached previously [3]. In an appendix the derivation of the numerical coefficients of the expansion is presented.

We shall present a Hilbert space formulation of the classical Maxwell equations in a cavity $\Omega \subset \mathbb{R}^3$. In a preliminary Hilbert space $L^2(\Omega, \mathbb{R}^3)$ we define the dense subspaces

$$\begin{aligned} \mathcal{R} &= \left\{ \mathbf{V} \in L^2(\Omega, \mathbb{R}^3) \mid \operatorname{rot} \mathbf{V} \in L^2(\Omega, \mathbb{R}^3) \right\} , \\ \mathcal{R}_0 &= \{ \mathbf{V} \in \mathcal{R} \mid \langle \mathbf{U}, \operatorname{rot} \mathbf{V} \rangle = \langle \operatorname{rot} \mathbf{U}, \mathbf{V} \rangle, \ \forall \mathbf{U} \in \mathcal{R} \} \end{aligned}$$

and the (closed) operator

$$R = \operatorname{rot} \qquad \text{with domain} \quad \mathcal{D}(R) = \mathcal{R}_0 .$$

Its adjoint is then given as $R^* = \operatorname{rot}$ with $\mathcal{D}(R^*) = \mathcal{R}$. We remark that R, resp. R^*, is also the closure of rot defined on smooth vector fields $\mathbf{V}$ with boundary condition $\mathbf{V}_\parallel = 0$ on the smooth boundary $\partial\Omega$, resp. without boundary conditions. This is what is meant when we later simply say that a differential operator is defined with (or without) a certain boundary condition.

The subspace

$$\mathcal{H} = \{\mathbf{V} \in L^2(\Omega, \mathbb{R}^3) \mid \operatorname{div} \mathbf{V} = 0\} \tag{1}$$

and its orthogonal complement in $L^2(\Omega, \mathbb{R}^3)$ are preserved by R and, therefore, by R^*. We will thus view them as operators on the physical Hilbert space $\mathcal{H}$. The Maxwell equations with boundary condition $\mathbf{E}_\parallel = 0$ on the ideally conducting shell $\partial\Omega$ can now be written as

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = M \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \tag{2}$$

with

$$M = \begin{pmatrix} 0 & \mathrm{i}R^* \\ -\mathrm{i}R & 0 \end{pmatrix} = M^* \qquad \text{on } \mathcal{H} \oplus \mathcal{H} ,$$

cf. [12]. Since no boundary condition has been imposed on $\mathbf{B}$, we have $M(0, \mathbf{B}) = 0$ for all $\mathbf{B} = \nabla\psi$ with ψ harmonic, and hence

$$\dim \operatorname{Ker} M = \infty . \tag{3}$$

We shall compute the heat kernel trace

$$\mathrm{Tr}'_{\mathcal{H}\oplus\mathcal{H}}(\mathrm{e}^{-tM^2}) = \sum_k{}' \mathrm{e}^{-t\omega_k^2} ,$$

where $'$ means that the contributions of zero-modes, i.e., of eigenvalues $\omega_k = 0$ of M, have been omitted. This is necessary in view of (3), but a more physical justification, tied to the application to the Casimir effect to be discussed later, is that zero-modes are not subject to quantization.

The square of M is

$$M^2 = \begin{pmatrix} R^*R & 0 \\ 0 & RR^* \end{pmatrix} = \begin{pmatrix} -\Delta_{\mathbf{E}} & 0 \\ 0 & -\Delta_{\mathbf{B}} \end{pmatrix} , \tag{4}$$

where $\Delta_{\mathbf{E}}$, resp. $\Delta_{\mathbf{B}}$, is the Laplacian on $\mathcal{H}$ with boundary conditions

$$\mathbf{E}_\parallel = 0 , \qquad \text{resp.} \quad (\operatorname{rot} \mathbf{B})_\parallel = 0 . \tag{5}$$

The operators RR^* and R^*R have the same spectrum, including multiplicity, except for zero-modes. Incidentally, we note that eigenfunctions $(\mathbf{E}, \mathbf{B})$ corresponding to $\omega_k \neq 0$ satisfy $\mathbf{B} = -\mathrm{i}\omega_k^{-1} \operatorname{rot} \mathbf{E}$ and hence, by Stokes' theorem, the boundary condition $\mathbf{B}_\perp = 0$, which we did not impose, but which is usually also associated with ideally conducting shells. Since $\partial_t^2 + M^2 = (\mathrm{i}\partial_t - M)(-\mathrm{i}\partial_t - M)$, each pair of non-zero eigenvalues of R^*R and RR^* corresponds to a single oscillator mode for (2). We will thus discuss the heat kernel asymptotics for

$$\frac{1}{2}\,\mathrm{Tr}'_{\mathcal{H}\oplus\mathcal{H}}(\mathrm{e}^{-tM^2}) = \begin{cases} \mathrm{Tr}'_{\mathcal{H}}\,\mathrm{e}^{t\Delta_{\mathbf{E}}} & (6)\\ \mathrm{Tr}'_{\mathcal{H}}\,\mathrm{e}^{t\Delta_{\mathbf{B}}} & (7)\end{cases}$$

$$\cong \sum_{n=0}^{\infty} a_n t^{\frac{n-3}{2}}\ , \qquad (t\downarrow 0)\ . \tag{8}$$

The coefficients a_n are known, see, e.g., [5], for general operators of Laplace type. The direct application of such results is prevented by the divergence constraint in $\mathcal{H}$, see (1). In the next section we indicate how to remove it. First however we present the main result.

Let

$$L_{ab} = (\nabla_{\mathbf{e}_a}\mathbf{e}_b, \mathbf{n})\ , \qquad (a,b=1,2)\ ,$$

be the second fundamental form on the boundary $\partial\Omega$ with inward normal $\mathbf{n}$ and local orthonormal frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$. We denote by $|\Omega|$ the volume of Ω and set

$$f[\partial\Omega] = \int_{\partial\Omega} f(y)\mathrm{d}y\ ,$$

where $\mathrm{d}y$ is the (induced) Euclidean surface element on $\partial\Omega$. The corresponding Laplacian on $\partial\Omega$ is denoted by ∇^2.

Theorem 1 *Let $\Omega\subset\mathbb{R}^3$ an open, connected domain with compact closure and smooth boundary $\partial\Omega$ consisting of n components of genera $g_1, g_2, \ldots, g_n$. Then*

$$\begin{aligned}
a_0 &= 2(4\pi)^{-\frac{3}{2}}|\Omega|\ ,\\
a_1 &= 0\ ,\\
a_2 &= -\frac{4}{3}(4\pi)^{-\frac{3}{2}}(\mathrm{tr}\,L)[\partial\Omega]\ ,\\
a_3 &= \frac{1}{64}(4\pi)^{-1}\big(3(\mathrm{tr}\,L)^2 - 4\det L\big)[\partial\Omega] - \frac{1}{2}\sum_{i=1}^{n}(1+g_i) + 1\ ,\\
a_4 &= \frac{16}{315}(4\pi)^{-\frac{3}{2}}\big(2(\mathrm{tr}\,L)^3 - 9\,\mathrm{tr}\,L\cdot\det L\big)[\partial\Omega]\ ,\\
a_5 &= \frac{1}{122880}(4\pi)^{-1}\big(2295(\mathrm{tr}\,L)^4 - 12440(\mathrm{tr}\,L)^2\det L +\\
&\qquad +13424(\det L)^2 + 1200\,\mathrm{tr}\,L\cdot\nabla^2\,\mathrm{tr}\,L\big)[\partial\Omega]\ .
\end{aligned}\tag{9}$$

We will give two partially independent proofs, based on (6), resp. (7). Their agreement is related to the index theorem, as it may be seen from (4). A further, partial check of these coefficients has been made on the basis of general cylindrical domains and of the sphere, where a separation into TE and TM modes is possible.

The coefficient a_0 was computed in [13] (except for the factor 2 replaced by 3, as the divergence condition (1) was ignored), a_1, a_2 in [1]. The coefficient a_3 is closely related to a result of [3], as discussed in Section 3.

2 Proofs

We consider the space of (square integrable) forms, $\Lambda(\Omega) = \bigoplus_{p=0}^{n} \Lambda_p(\Omega)$, on the manifold Ω with boundary, together with the exterior derivative $d_{p+1} : \Lambda_p(\Omega) \to \Lambda_{p+1}(\Omega)$ defined with relative boundary condition ([11], Section 2.7.1)

$$\omega\big|_{\partial\Omega} = 0 \ ,$$

as a form $\omega|_{\partial\Omega} \in \Lambda_p(\partial\Omega)$. For later use we recall that by the de Rahm theorem for manifolds with boundary ([9] or [11], Thm. 2.7.3) we have

$$H_r^p(\Omega) \cong H_{n-p}(\Omega) \cong H_p(\Omega, \partial\Omega) \ , \tag{10}$$

where $H_r^p(\Omega) = \text{Ker } d_{p+1}/\text{Im } d_p$ is the p-th relative cohomology group, $H_p(\Omega)$ is the p-th homology group, and $H_p(\Omega, \partial\Omega)$ is the p-th relative homology group, i.e., the homology based on chains mod $\partial\Omega$.

We shall henceforth restrict to $\Omega \subset \mathbb{R}^3$ as in Theorem 1. Using either homology (10), the dimension of $H_r^p(\Omega)$ is seen to be

$$\begin{array}{cc} 0 & (p = 0) \ , \\ n - 1 & (p = 1) \ , \\ \sum_{i=1}^{n} g_i & (p = 2) \ , \\ 1 & (p = 3) \ . \end{array} \tag{11}$$

These are also the dimensions of the spaces of harmonic p-forms.

The space $\Lambda(\Omega) = \bigoplus_{p=0}^{3} \Lambda_p(\Omega)$ may be identified as

$$\Lambda(\Omega) = L^2(\Omega) \oplus L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega, \mathbb{R}^3) \oplus L^2(\Omega) \ni (\phi, \mathbf{E}, \mathbf{B}, \psi) \ ,$$

where $d : \Lambda(\Omega) \to \Lambda(\Omega)$ acts as

$$d : L^2(\Omega) \underset{\text{grad}}{\longrightarrow} L^2(\Omega, \mathbb{R}^3) \underset{\text{rot}}{\longrightarrow} L^2(\Omega, \mathbb{R}^3) \underset{\text{div}}{\longrightarrow} L^2(\Omega) \longrightarrow 0$$

with boundary conditions $\phi = 0$, $\mathbf{E}_\parallel = 0$, $\mathbf{B}_\perp = 0$ on $\partial\Omega$. Then

$$d^* : 0 \longleftarrow L^2(\Omega) \underset{-\text{div}}{\longleftarrow} L^2(\Omega, \mathbb{R}^3) \underset{\text{rot}}{\longleftarrow} L^2(\Omega, \mathbb{R}^3) \underset{-\text{grad}}{\longleftarrow} L^2(\Omega)$$

without any boundary conditions. The Laplace-Beltrami operator on forms,

$$-\Delta = \bigoplus_{p=0}^{3} (-\Delta_p) = dd^* + d^*d \ ,$$

is seen to correspond to the Euclidean Laplacian with boundary conditions

$$\begin{aligned}
&\phi = 0 && (p=0)\ ,\\
&\mathbf{E}_{\parallel} = 0\ , \quad \operatorname{div}\mathbf{E} = 0 && (p=1)\ ,\\
&\mathbf{B}_{\perp} = 0\ , \quad (\operatorname{rot}\mathbf{B})_{\parallel} = 0 && (p=2)\ ,\\
&(\operatorname{grad}\psi)_{\perp} = 0 && (p=3)\ .
\end{aligned} \tag{12}$$

Each of the four problems admits a heat kernel expansion,

$$\operatorname{Tr}_{\Lambda_p(\Omega)} \mathrm{e}^{\Delta_p t} \cong \sum_{n=0}^{\infty} a_n^{(p)} t^{\frac{n-3}{2}}\ , \tag{13}$$

whose coefficients have been computed ($n = 0, \ldots, 3$) [4] or can be computed using existing results ($n = 4, 5$) [5]. To this end we note that the boundary conditions for $p = 1, 2$ can be formulated equivalently as

$$\begin{aligned}
&\mathbf{E}_{\parallel} = 0\ , \quad \frac{\partial \mathbf{E}_{\perp}}{\partial n} - (\operatorname{tr} L)\mathbf{E}_{\perp} = 0 && (p=1)\ ,\\
&\mathbf{B}_{\perp} = 0\ , \quad \frac{\partial \mathbf{B}_{\parallel}}{\partial n} - L\mathbf{B}_{\parallel} = 0 && (p=2)\ .
\end{aligned} \tag{14}$$

First approach. We will compute (6). We observe that $-\Delta_{\mathbf{E}}$ is just the restriction of $-\Delta_1$ to its invariant subspace

$$\mathcal{H} = \left\{\mathbf{E} \in L^2(\Omega, \mathbb{R}^3) \mid \operatorname{div}\mathbf{E} = 0\right\} = \operatorname{Ker}\, d_1^*\ .$$

Hence

$$\operatorname{Tr}'_{\mathcal{H}} e^{t\Delta_{\mathbf{E}}} = \operatorname{Tr}'_{L^2(\Omega,\mathbb{R}^3)} \mathrm{e}^{t\Delta_1} - \operatorname{Tr}'_{\mathcal{H}^{\perp}} e^{t\Delta_1}\ ,$$

where the orthogonal complement of $\mathcal{H}$ in $L^2(\Omega, \mathbb{R}^3)$ is

$$\mathcal{H}^{\perp} = \overline{\operatorname{Ran}\, d_1} = \operatorname{Ran}\, d_1 = \left\{\nabla\phi \in L^2(\Omega, \mathbb{R}^3) \mid \phi = 0 \text{ on } \partial\Omega\right\}\ ,$$

(Ran d is closed by the Hodge decomposition, see, e.g., [8, 11]). By $d\Delta = \Delta d$, the operators $(-\Delta_1) \restriction_{\mathcal{H}^{\perp}}$ and $-\Delta_0$ have the same spectrum (in fact $\nabla\phi = 0$ implies $\phi = 0$ by the boundary condition). Thus, using also (11), we find

$$\begin{aligned}
\operatorname{Tr}'_{\mathcal{H}} \mathrm{e}^{t\Delta_{\mathbf{E}}} &= \operatorname{Tr}'_{L^2(\Omega,\mathbb{R}^3)} \mathrm{e}^{t\Delta_1} - \operatorname{Tr}'_{L^2(\Omega)} \mathrm{e}^{t\Delta_0}\\
&= \operatorname{Tr}_{L^2(\Omega,\mathbb{R}^3)} \mathrm{e}^{t\Delta_1} - \operatorname{Tr}_{L^2(\Omega)} \mathrm{e}^{t\Delta_0} - (n-1)\ ,
\end{aligned}$$

i.e.,

$$\begin{aligned}
a_k &= a_k^{(1)} - a_k^{(0)}\ , \qquad (k \neq 3)\ ,\\
a_3 &= a_3^{(1)} - a_3^{(0)} - n + 1\ .
\end{aligned}$$

These relations, together with the values of $a_k^{(p)}$ computed in the appendix, yield the values of the coefficients stated in the Theorem 1. In particular, we will obtain

$$a_3^{(1)} - a_3^{(0)} = \frac{1}{64}(4\pi)^{-1}\big(3(\operatorname{tr} L)^2 + 28 \det L\big)[\partial\Omega] .$$

This matches the stated value of a_3 because of

$$n = \frac{1}{2}\sum_{i=1}^{n}(1+g_i) + \frac{1}{2}\sum_{i=1}^{n}(1-g_i)$$

and of the Gauss-Bonnet theorem,

$$\frac{1}{2}\sum_{i=1}^{n}(1-g_i) = \frac{1}{2}(4\pi)^{-1}(\det L)[\partial\Omega] . \tag{15}$$

Second approach. We now compute (7). As it has been noted in the introduction, eigenmodes of $-\Delta_{\mathbf{B}}$, except for zero-modes, satisfy the boundary condition $\mathbf{B}_\perp = 0$, and are thus eigenmodes of $-\Delta_2$ belonging to its invariant subspace $\mathcal{H}$, cf. (5, 12). The converse is obvious. We conclude that

$$\operatorname{Tr}'_{\mathcal{H}} \mathrm{e}^{t\Delta_{\mathbf{B}}} = \operatorname{Tr}'_{L^2(\Omega,\mathbb{R}^3)} \mathrm{e}^{t\Delta_2} - \operatorname{Tr}'_{\mathcal{H}^\perp} \mathrm{e}^{t\Delta_2} .$$

Since

$$\mathcal{H} = \{\mathbf{B} \in L^2(\Omega,\mathbb{R}^3) \mid \operatorname{div}\mathbf{B} = 0\} = \operatorname{Ker} d_3 ,$$

we have

$$\mathcal{H}^\perp = \overline{\operatorname{Ran} d_3^*} = \operatorname{Ran} d_3^* = \big\{-\nabla\psi \in L^2(\Omega,\mathbb{R}^3) \mid \psi \in L^2(\Omega)\big\} .$$

Using $d^*\Delta = \Delta d^*$, we see that $(-\Delta_2)\restriction_{\mathcal{H}^\perp}$ and $-\Delta_3$ have the same spectrum, except for a single zero-mode (in fact, $-\nabla\psi = 0$ implies $\psi = \text{const}$). We thus find, using (11),

$$\begin{aligned}\operatorname{Tr}'_{\mathcal{H}} \mathrm{e}^{t\Delta_{\mathbf{B}}} &= \operatorname{Tr}'_{L^2(\Omega,\mathbb{R}^3)} \mathrm{e}^{t\Delta_2} - \operatorname{Tr}'_{L^2(\Omega)} \mathrm{e}^{t\Delta_3} \\ &= \operatorname{Tr}_{L^2(\Omega,\mathbb{R}^3)} \mathrm{e}^{t\Delta_2} - \operatorname{Tr}_{L^2(\Omega)} \mathrm{e}^{t\Delta_3} - \Big(\sum_{i=1}^{n} g_i \;-1\Big) ,\end{aligned}$$

i.e.,

$$\begin{aligned} a_k &= a_k^{(2)} - a_k^{(3)} , \qquad (k \neq 3) , \\ a_3 &= a_3^{(2)} - a_3^{(3)} - \sum_{i=1}^{n} g_i \; + 1 . \end{aligned}$$

From these relations and from the results of the appendix we again recover Theorem 1. In particular,

$$a_3^{(2)} - a_3^{(3)} = \frac{1}{64}(4\pi)^{-1}\big(3(\operatorname{tr} L)^2 - 36 \det L\big)[\partial\Omega]$$

leads to the claim for a_3, because of

$$\sum_{i=1}^{n} g_i = \frac{1}{2}\sum_{i=1}^{n}(1+g_i) - \frac{1}{2}\sum_{i=1}^{n}(1-g_i)$$

and of (15).

3 Application to the Casimir effect

For the purpose of this discussion we simply define the Casimir energy by the mode summation method, see, e.g., [3]. In particular, we do not address the issue [6] of whether it is the most appropriate physically. We shall however observe that the Casimir energy is finite – a conclusion obtained in [3], but questioned in [10].

Consider the cavity $\Omega \subset \mathbb{R}^3$ enclosed in a large ball Ω_0. As usual we compare the vacuum energy of the electromagnetic field in the domains $\Omega \cup (\Omega_0 \setminus \overline{\Omega})$ with that of the reference domain Ω_0. Each eigenmode of either domain contributes a zero-point energy $\omega_k/2$, resp. $\omega_k^0/2$. As a regulator for the eigenfrequencies $\omega_k = \lambda_k^{1/2}$, we choose $\mathrm{e}^{-\gamma\lambda_k}$, ($\gamma > 0$). The corresponding definition of the Casimir energy is

$$E_C = \frac{1}{2}\lim_{\Omega_0\to\infty}\lim_{\gamma\downarrow 0}\left(\sum_k \lambda_k^{\frac{1}{2}}\mathrm{e}^{-\gamma\lambda_k} - \sum_k (\lambda_k^0)^{\frac{1}{2}}\mathrm{e}^{-\gamma\lambda_k^0}\right) .$$

We shall prove that the limit $\gamma \downarrow 0$ is finite. It will also be clear that the subsequent limit $\Omega_0 \to \infty$ exists, though we shall not make the effort to prove that (see however, e.g., [8], Section 12.7 for the necessary tools). Using

$$\lambda_k^{\frac{1}{2}} = -\frac{1}{\sqrt{\pi}}\int_0^\infty \mathrm{d}t\; t^{-\frac{1}{2}}\frac{d}{dt}\mathrm{e}^{-t\lambda_k}$$

and (8) we find for the regularized sum of the eigenfrequencies

$$\sum_k \lambda_k^{\frac{1}{2}}\mathrm{e}^{-\gamma\lambda_k} \approx -\sum_{n=0}^{4}\frac{n-3}{2\sqrt{\pi}}a_n\int_0^\delta \mathrm{d}t\; t^{-\frac{1}{2}}(t+\gamma)^{\frac{n-5}{2}}$$

as $\gamma \downarrow 0$. Here $\delta > 0$ is arbitrary, but fixed, and "$\approx$" means up to terms $O(1)$. Using

$$\int_0^\delta \mathrm{d}t\; t^{-\frac{1}{2}}(t+\gamma)^{\frac{n-5}{2}} \approx \begin{cases} \frac{4}{3}\gamma^{-2} & (n=0)\,, \\ \frac{\pi}{2}\gamma^{-\frac{3}{2}} & (n=1)\,, \\ 2\gamma^{-1} & (n=2)\,, \\ \pi\gamma^{-\frac{1}{2}} & (n=3)\,, \\ -\log\gamma & (n=4)\,, \end{cases}$$

we find

$$\sum_k \lambda_k^{\frac{1}{2}} \mathrm{e}^{-\gamma\lambda_k} \approx \frac{2}{\sqrt{\pi}} a_0 \gamma^{-2} + \frac{\sqrt{\pi}}{2} a_1 \gamma^{-\frac{3}{2}} + \frac{1}{\sqrt{\pi}} a_2 \gamma^{-1} + 0 \cdot a_3 \gamma^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} a_4 \log\gamma \,.$$

Hence a finite Casimir energy requires (cf. [7]) that a_0, a_1, a_2, a_4 (but not necessarily a_3!) agree for $\Omega \cup (\Omega_0 \setminus \overline{\Omega})$ and for the reference domain Ω_0. This is indeed so for $a_0 = 2(4\pi)^{-\frac{3}{2}}|\Omega_0|$ and for $a_1 = 0$, but also for a_2, a_4 as the contributions from the two sides of $\partial\Omega$ cancel. The same conclusion is obtained if the regulator $\mathrm{e}^{-\gamma\lambda_k}$ is replaced by $\mathrm{e}^{-(\gamma\lambda_k)^{1/2}}$ (see [7], Eq. (27)):

$$\sum_k \lambda_k^{\frac{1}{2}} \mathrm{e}^{-(\gamma\lambda_k)^{1/2}} \approx \frac{24}{\sqrt{\pi}} a_0 \gamma^{-2} + 4 a_1 \gamma^{-\frac{3}{2}} + \frac{2}{\sqrt{\pi}} a_2 \gamma^{-1} + 0 \cdot a_3 \gamma^{-\frac{1}{2}} + \frac{1}{\sqrt{\pi}} a_4 \log\gamma \,.$$

Since no renormalization is necessary, the value of E_C agrees with that obtained by means of the zeta function.

In the rest of this section we compare our results with those of [2, 3]. To the extent the comparison is done we will find agreement. An important tool there is the mode generating function, Eq. (4.5) in [2],

$$\begin{aligned} \Phi(k) &\doteq \frac{1}{2} \operatorname{Tr}\left(\frac{-\Delta_{\mathbf{E}}}{-\Delta_{\mathbf{E}} - k^2} + \frac{-\Delta_{\mathbf{B}}}{-\Delta_{\mathbf{B}} - k^2} \right) \\ &\doteq \frac{k^2}{2} \operatorname{Tr}'\Big((-\Delta_{\mathbf{E}} - k^2)^{-1} + (-\Delta_{\mathbf{B}} - k^2)^{-1} \Big) \,, \qquad (k \in \mathbb{C} \setminus \mathbb{R}) \,, \end{aligned} \tag{16}$$

where "$\doteq$" means equality "within addition of some polynomial in k^2". Since the resolvents in (16) are not trace class, but their squares are, we first consider that replacement. Using $(A+\mu)^{-2} = \int_0^\infty \mathrm{d}t \; t \; \mathrm{e}^{-t(A+\mu)}$ we obtain, as $\mu \to \infty$,

$$\begin{aligned} \frac{1}{2} \operatorname{Tr}'\Big((-\Delta_{\mathbf{E}} + \mu)^{-2} + (-\Delta_{\mathbf{B}} + \mu)^{-2} \Big) &\cong \sum_{n=0}^{\infty} a_n \int_0^\infty \mathrm{d}t \cdot t^{\frac{n-3}{2}} \mathrm{e}^{-t\mu} \\ &= \sum_{n=0}^{\infty} \Gamma(\tfrac{n+1}{2}) a_n \mu^{-\frac{n+1}{2}} \end{aligned}$$

with coefficients a_n given in Theorem 1. Integrating w.r.t. μ we find

$$\frac{1}{2} \operatorname{Tr}'\Big((-\Delta_{\mathbf{E}} + \mu)^{-1} + (-\Delta_{\mathbf{B}} + \mu)^{-1} \Big) \doteq \sum_{\substack{n=0 \\ n\neq 1}}^{\infty} \Gamma(\tfrac{n-1}{2}) a_n \mu^{-\frac{n-1}{2}} \; - a_1 \log\mu$$

and hence, with $\mu^{1/2} = -\mathrm{i}k$,

$$\Phi(k) \doteq 2\sqrt{\pi} a_0 \mathrm{i} k^3 - \sqrt{\pi} a_1 k^2 \ln(-k^2) + \mathrm{i}\sqrt{\pi} a_2 k - a_3 + O(k^{-1}) \,.$$

Upon insertion of the mentioned values for $a_0, \ldots, a_3$ this agrees with Eq. (4.40) in [2], except for a_3 which is there replaced by its local part, see (9),

$$\begin{aligned} \tilde{a}_3 &= \frac{1}{64}(4\pi)^{-1}\bigl(3(\operatorname{tr} L)^2 - 4\det L\bigr)[\partial\Omega] \\ &= \frac{1}{64}\int_{\partial\Omega} \mathrm{d}\sigma \Bigl(\frac{3}{4}(\kappa_1^2+\kappa_2^2) - \kappa_1\kappa_2\Bigr) , \end{aligned}$$

where κ_1, κ_2 are the principal curvatures. Note however that this discrepancy is implicit in the definition of "$\doteq$". It is resolved in [3] by first considering $\delta\Phi(k)$, i.e., the difference of the mode generating functions corresponding to the configurations $\Omega \cup (\Omega_0 \setminus \overline{\Omega})$ and Ω_0. Thus

$$\delta\Phi(k) = -2\tilde{a}_3 + O(k^{-1}) ,$$

since the contributions to a_0, a_2 cancel, and those to $\tilde{a}_3$ double the value. Not ambiguous then is "the number of additional modes of finite frequency created by introducing the conducting surface $\partial\Omega$":

$$\mathcal{C} = \psi(0+) - \psi(\infty) ,$$

where $\psi(y) = \delta\Phi(\mathrm{i}y)$. For a connected boundary $\partial\Omega$ of genus g the value of $\psi(0+)$ has been established as $\psi(0+) = -g$ (see [3], Eq. (5.8)), resulting in

$$\mathcal{C} = 2\tilde{a}_3 - g . \tag{17}$$

This result agrees with Theorem 1: the non-local terms in (9) take the values $-\frac{1}{2}(g-1)$, $-\frac{1}{2}g$, $\frac{1}{2}$ for Ω, $\Omega_0 \setminus \overline{\Omega}$ and Ω_0 respectively. Thus,

$$\delta a_3 = 2\tilde{a}_3 - g ,$$

in agreement with (17).

A Appendix

In this appendix we compute the heat kernel coefficients in (13) for $p = 0, \ldots, 3$ and $n = 0, \ldots, 5$ on the basis of Theorems 1 and 4 in [5]. We use the same notation, together with $P = \mathbf{n} \otimes \mathbf{n}$ denoting the normal projection at the boundary. The vector bundle is $V = \Omega \times \mathbb{R}$ for $p = 0, 3$, resp. $V = T\Omega$ for $p = 1, 2$, equipped with the Euclidean connection. The decompositions of $V|_{\partial\Omega} = V_N \oplus V_D \ni (\phi^N, \phi^D)$ (with projections Π_+, resp. Π_-) and boundary conditions $\phi^N_{;n} + S\phi^N = 0$, resp.

$\phi^D = 0$, are specified as follows, cf. (14) and [5]:

$$
\begin{aligned}
&p = 0: \qquad && \begin{cases} \Pi_+ = 0\ , \\ \Pi_- = 1\ , \end{cases} \\
&p = 1: && \begin{cases} \Pi_+ = P\ , \qquad S = -L_{aa}P\ , \\ \Pi_- = 1 - P\ , \end{cases} \\
&p = 2: && \begin{cases} \Pi_+ = 1 - P\ , \qquad S = -L\ , \\ \Pi_- = P\ , \end{cases} \\
&p = 3: && \begin{cases} \Pi_+ = 1\ , \qquad S = 0\ , \\ \Pi_- = 0\ . \end{cases}
\end{aligned}
\tag{18}
$$

The result is

$$
\begin{aligned}
a_0^{(p)} &= (4\pi)^{-\frac{3}{2}} c_0^{(p)} |\Omega|\ , \\
a_1^{(p)} &= \frac{1}{4}(4\pi)^{-1} c_1^{(p)} |\partial\Omega|\ , \\
a_2^{(p)} &= \frac{1}{3}(4\pi)^{-\frac{3}{2}} c_2^{(p)} (\operatorname{tr} L)[\partial\Omega]\ , \\
a_3^{(p)} &= \frac{1}{384}(4\pi)^{-1}\big(c_{31}^{(p)}(\operatorname{tr} L)^2 + c_{32}^{(p)}(\det L)\big)[\partial\Omega]\ , \\
a_4^{(p)} &= \frac{1}{315}(4\pi)^{-\frac{3}{2}}\big(c_{41}^{(p)}(\operatorname{tr} L)^3 + c_{42}^{(p)} \operatorname{tr} L \cdot \det L\big)[\partial\Omega]\ , \\
a_5^{(p)} &= \frac{1}{245760}(4\pi)^{-1}\big(c_{51}^{(p)}(\operatorname{tr} L)^4 + c_{52}^{(p)}(\operatorname{tr} L)^2 \det L + c_{53}^{(p)}(\det L)^2 \\
&\qquad\qquad + c_{54}^{(p)} \operatorname{tr} L \cdot \nabla^2 \operatorname{tr} L\big)[\partial\Omega]
\end{aligned}
$$

with coefficients given in Table 1.

The computation of the table is based on the general result of [5], which has been applied to (18) using the following identities:

$$
\begin{aligned}
\operatorname{Tr}(P_{:a}P_{:b}) &= 2(L^2)_{ab}\ , \\
\operatorname{Tr}(P_{:a}P_{:a}P_{:b}P_{:b}) &= (L^4)_{aa} + (L^2)_{aa}(L^2)_{bb}\ , \\
\operatorname{Tr}(P_{:a}P_{:b}P_{:a}P_{:b}) &= 2(L^4)_{aa}\ , \\
\operatorname{Tr}(P_{:aa}P_{:bb}) &= 2L_{ac:a}L_{bc:b} + 4(L^4)_{aa} + 4(L^2)_{aa}(L^2)_{bb}\ , \\
\operatorname{Tr}(P_{:ab}P_{:ab}) &= 2L_{ab:c}L_{ab:c} + 6(L^4)_{aa} + 2(L^2)_{aa}(L^2)_{bb}\ .
\end{aligned}
$$

They can be derived by using $\nabla_{\mathbf{e}_a}\mathbf{n} = -L_{ab}\mathbf{e}_b$, so that

$$
P_{:a} = -L_{ac}(\mathbf{e}_c \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{e}_c)\ ,
$$

	$p=0$	$p=1$	$p=2$	$p=3$
$c_0^{(p)}$	1	3	3	1
$c_1^{(p)}$	-1	-1	1	1
$c_2^{(p)}$	1	-3	-3	1
$c_{31}^{(p)}$	3	21	33	15
$c_{32}^{(p)}$	-20	148	-220	-4
$c_{41}^{(p)}$	4	36	60	28
$c_{42}^{(p)}$	-18	-162	-186	-42
$c_{51}^{(p)}$	555	5145	8625	4035
$c_{52}^{(p)}$	-2840	-27720	-35720	-10840
$c_{53}^{(p)}$	2224	29072	29712	2864
$c_{54}^{(p)}$	120	2520	4680	2280

Table 1: These values imply Theorem 1, as explained in its proof.

and by assuming without loss that $\nabla_{\mathbf{e}_a}\mathbf{e}_b$ has no component parallel to $T_p\partial\Omega$ at the point p of evaluation, i.e., $\nabla_{\mathbf{e}_a}\mathbf{e}_b = L_{ab}\mathbf{n}$. Then

$$P_{:ab} = -L_{ac:b}(\mathbf{e}_c \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{e}_c) - 2(L^2)_{ab}P + (L_{ac}L_{bd} + L_{ad}L_{bc})\mathbf{e}_c \otimes \mathbf{e}_d \ ,$$

from which the above traces follow. In turn they allow the computation of similar traces with P replaced by $\chi = \Pi_+ - \Pi_-$, i.e., by $\chi = \pm(2P-1)$ in the cases $p = 1, 2$. In these two cases we also have

$$\begin{aligned} \operatorname{Tr} S_{:a} &= -L_{bb:a} \ , \\ \operatorname{Tr} S_{:ab} &= -L_{cc:ab} \ , \end{aligned}$$

and, moreover, for $p = 1$,

$$\begin{aligned} \operatorname{Tr}(S_{:a}S_{:a}) &= L_{bb:a}L_{cc:a} + 2L_{bb}L_{cc}(L^2)_{aa} \ , \\ \operatorname{Tr}(P_{:a}S_{:b}) &= -2(L^2)_{ab}L_{cc} \ , \\ \operatorname{Tr}(PS_{:a}S_{:a}) &= L_{bb:a}L_{cc:a} + L_{bb}L_{cc}(L^2)_{aa} \ , \end{aligned}$$

resp. for $p = 2$,

$$\mathrm{Tr}(S_{:a}S_{:a}) = L_{ab:c}L_{ab:c} + 2(L^4)_{aa} ,$$
$$\mathrm{Tr}(P_{:a}S_{:a}) = 2(L^3)_{aa} ,$$
$$\mathrm{Tr}(PS_{:a}S_{:a}) = (L^4)_{aa} .$$

Furthermore, traces of L^k, $(k \geq 2)$, were reduced to $\mathrm{tr}\, L$, $\det L$ by means of $L^2 - (\mathrm{tr}\, L)L + \det L = 0$. Finally, we used the Codazzi equation, $L_{ab:c} = L_{ac:b}$, as well as

$$L_{ab:ca} - L_{ab:ac} = L_{aa}(L^2)_{bc} - (L^2)_{aa}L_{bc} ,$$

which follows from the Gauss equation.

Acknowledgment

We thank M. Levitin and G. Scharf for discussions. The research of D. Hasler was supported in part under the EU-network contract HPRN-CT-2002-00277.

References

[1] R. Balian, C. Bloch, Distribution of eigenfrequencies for the wave equation in a finite domain. II. Electromagnetic field. Riemannian spaces, *Ann. Phys. (N.Y.)* **64**, 271 (1971); Errata, ibid. **84**, 559 (1974).

[2] R. Balian, B. Duplantier, Electromagnetic waves near perfect conductors, I. Multiple scattering expansions. Distribution of modes, *Ann. Phys. (N.Y.)* **104**, 300 (1977).

[3] R. Balian, B. Duplantier, Electromagnetic waves near perfect conductors, II. Casimir effect, *Ann. Phys. (N.Y.)* **112**, 165 (1978).

[4] N. Blažić, N. Bokan, P.B. Gilkey, Spectral geometry of the form valued Laplacian for manifolds with boundary, *Indian J. Pure Appl. Math.* **23**, 103 (1992).

[5] T.P. Branson, P.B. Gilkey, K. Kirsten, D.V. Vassilevich, Heat kernel asymptotics with mixed boundary conditions, *Nucl. Phys. B* **563**, 603 (1999).

[6] P. Candelas, Vacuum energy in the presence of dielectric and conducting surfaces, *Ann. Phys. (N.Y.)* **143**, 241 (1982).

[7] G. Cognola, L. Vanzo, S. Zerbini, Regularization dependence of vacuum energy in arbitrarily shaped cavities, *J. Math. Phys.* **33**, 222 (1992).

[8] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger operators*, Springer (1987).

[9] G.F.G. Duff, Differential forms in manifolds with boundary, *Ann. Math.* **56**, 115 (1952).

[10] D. Deutsch, P. Candelas, Boundary effects in quantum field theory, *Phys. Rev. D* **20**, 895 (1978).

[11] P.B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, CRC (1995).

[12] R. Leis, *Initial boundary value problems in mathematical physics*, Teubner/Wiley (1986).

[13] H. Weyl, Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetzte, *J. f. reine u. angew. Math.* **143**, 177 (1913).

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Communicated by Vincent Rivasseau
submitted 17/02/03, accepted 04/07/03