

Long Range Scattering and Modified Wave Operators for the Wave-Schrödinger System II*

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Abstract. We continue the study of scattering theory for the system consisting of a Schrödinger equation and a wave equation with a Yukawa type coupling in space dimension 3. In a previous paper, we proved the existence of modified wave operators for that system with no size restriction on the data and we determined the asymptotic behaviour in time of solutions in the range of the wave operators, under a support condition on the asymptotic state required by the different propagation properties of the wave and Schrödinger equations. Here we eliminate that condition by using an improved asymptotic form for the solutions.

1 Introduction

This paper is a sequel to a previous paper with the same title ([1], hereafter referred to as I) where we studied the theory of scattering and proved the existence of modified wave operators for the Wave-Schrödinger (WS) system

$$\begin{cases} i\partial_t u &= -(1/2)\Delta u - Au & (1.1) \\ \square A &= |u|^2 & (1.2) \end{cases}$$

where u and A are respectively a complex-valued and a real-valued function defined in space time \mathbb{R}^{3+1} . We refer to the introduction of I for general background and references and we give here only a general overview of the problem.

The main result of I was the construction of modified wave operators for the WS system, with no size restriction on the solutions. That construction basically consists in solving the Cauchy problem for the WS system with infinite initial time, namely in constructing solutions with prescribed asymptotic behaviour at infinity in time. That asymptotic behaviour is imposed in the form of suitable approximate solutions of the WS system. One then looks for exact solutions, the difference of which with the given approximate ones tends to zero at infinity in time in a suitable sense, more precisely in suitable norms. The approximate solutions are obtained as low order iterates in an iterative resolution scheme of the WS system. In I we used second order iterates. They are parametrized by data (u_+, A_+, \dot{A}_+) which play the role of (actually are in simpler cases) initial data at time zero. Those data constitute the asymptotic state for the actual solution.

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An inherent difficulty of the WS system is the difference of propagation properties of the wave equation and of the Schrödinger equation. Because of that difficulty, we had to impose in I a support condition on the Fourier transform Fu_+ of the Schrödinger asymptotic state u_+ , saying in effect that Fu_+ vanishes in a neighborhood of the unit sphere, so that u_+ generates a solution of the free Schrödinger equation which is asymptotically small in a neighborhood of the light cone. Such a support condition is unpleasant because it cannot be satisfied on a dense subspace of any reasonable space where one hopes to solve the problem, typically with u in FH^k for $k > 1/2$ (H^k is the standard L^2 based Sobolev space).

The theory of scattering and the existence of modified wave operators can also be studied for various equations and systems including the WS system by a method simpler than that of I, proposed earlier by Ozawa [5]. Contrary to that of I, that method is restricted to the case of small data and small solutions. It has been applied to various systems, in particular to the Klein-Gordon-Schrödinger (KGS) system in dimension 2, which is fairly similar to the WS system in dimension 3 from the point of view of scattering [6]. Similar propagation difficulties also appear for that system, thereby again requiring a support condition on Fu_+ in the treatment given in [6].

A progress on that problem was made recently by Shimomura [7], [8] who was able to remove the previous support condition in the construction of the modified wave operators by the Ozawa method in the case of the KGS system in dimension 2 [7] and of the WS system in dimension 3 [8]. The key of that progress consists in using an improved asymptotic form for the Schrödinger function, obtained by adding a term depending on (A_+, \dot{A}_+) which partly cancels the contribution of the asymptotic field for A in the Schrödinger equation.

Although the method used in I is more complicated than the Ozawa method (so as to accommodate arbitrarily large data and solutions), it turns out that the improved asymptotic form of u used in [8] can be transposed into the framework of the method of I, thereby allowing to remove the support condition on Fu_+ assumed in I, at least in a restricted interval of values of the parameters defining the regularity of the solutions. The purpose of the present paper is to implement that improvement, namely to rederive the main results of I without assuming the support condition used in I, by using the improved asymptotic form of the solution inspired by that of [8].

In the remaining part of this introduction, we shall briefly review the method used in I in the modified form used in the present paper. We refer to Section 2 of I for a more detailed exposition. The main result of this paper will be stated in semi-heuristic terms at the end of this introduction. The first step in that method consists in eliminating the wave equation (1.2) by solving it for A and substituting the result into the Schrödinger equation, which then becomes both non-linear and non-local in time. One then parametrizes the Schrödinger function u in terms of an amplitude w and a phase φ and one replaces the Schrödinger equation by an auxiliary system consisting of a transport equation for the amplitude and a Hamilton-Jacobi equation for the phase. One solves the Cauchy problem at infin-

ity, namely with prescribed asymptotic behaviour, for the auxiliary system, and one finally reconstructs the solution of the original WS system from that of the auxiliary system. We now proceed to the technical details. We restrict our attention to positive time, actually to $t \geq 1$.

We first eliminate the wave equation. We define

$$\omega = (-\Delta)^{1/2} \quad , \quad K(t) = \omega^{-1} \sin \omega t \quad , \quad \dot{K}(t) = \cos \omega t$$

and we replace (1.2) by

$$A = A_0 + A_1(|u|^2) \tag{1.3}$$

where

$$A_0 = \dot{K}(t)A_+ + K(t)\dot{A}_+ \quad , \tag{1.4}$$

$$A_1(|u|^2) = - \int_t^\infty dt' K(t-t')|u(t')|^2 \quad . \tag{1.5}$$

Here A_0 is a solution of the free wave equation with initial data (A_+, \dot{A}_+) at time $t = 0$. The pair (A_+, \dot{A}_+) is the asymptotic state for A .

We next perform the change of variables mentioned above on u . The unitary group

$$U(t) = \exp(i(t/2)\Delta) \tag{1.6}$$

which solves the free Schrödinger equation can be written as

$$U(t) = M(t) D(t) F M(t) \tag{1.7}$$

where $M(t)$ is the operator of multiplication by the function

$$M(t) = \exp(ix^2/2t) \quad , \tag{1.8}$$

F is the Fourier transform and $D(t)$ is the dilation operator

$$D(t) = (it)^{-n/2} D_0(t) \tag{1.9}$$

where

$$(D_0(t)f)(x) = f(x/t) \quad . \tag{1.10}$$

We parametrize u in terms of an amplitude w and of a real phase φ as

$$u(t) = M(t) D(t) \exp[-i\varphi(t)]w(t) \quad . \tag{1.11}$$

Substituting (1.11) into (1.1) yields an evolution equation for (w, φ) , namely

$$\{i\partial_t + (2t^2)^{-1}\Delta - i(2t^2)^{-1}(2\nabla\varphi \cdot \nabla + \Delta\varphi) + t^{-1}B + \partial_t\varphi - (2t^2)^{-1}|\nabla\varphi|^2\} w = 0 \tag{1.12}$$

where we have expressed A in terms of a new function B by

$$A = t^{-1} D_0 B \quad . \tag{1.13}$$

Corresponding to the decomposition (1.3) of A , we decompose

$$B = B_0 + B_1(w, w) \tag{1.14}$$

where $A_0 = t^{-1}D_0B_0$ and $A_1 = t^{-1}D_0B_1$. One computes easily

$$B_1(w_1, w_2) = \int_1^\infty d\nu \nu^{-3} \omega^{-1} \sin((\nu - 1)\omega) D_0(\nu) (\text{Re } \bar{w}_1 w_2)(\nu t) . \tag{1.15}$$

At this point, we have only one evolution equation (1.12) for two functions (w, φ) . We arbitrarily impose a second equation, namely a Hamilton-Jacobi (or eikonal) equation for the phase φ , thereby splitting the equation (1.12) into a system of two equations, the other one of which being a transport type equation for the amplitude w . For that purpose, we split B_0 and B_1 into long range and short range parts as follows. Let $\chi \in C^\infty(\mathbb{R}^3, \mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(\xi) = 1$ for $|\xi| \leq 1$, $\chi(\xi) = 0$ for $|\xi| \geq 2$ and let $0 < \beta_0, \beta < 1$. We define

$$B_0 = B_{0L} + B_{0S} \quad , \quad B_1 = B_L + B_S \tag{1.16}$$

where

$$\begin{cases} FB_{0L}(t, \xi) = \chi(\xi t^{-\beta_0}) FB_0(t, \xi) , \\ FB_L(t, \xi) = \chi(\xi t^{-\beta}) FB_1(t, \xi) . \end{cases} \tag{1.17}$$

The splitting (1.16), (1.17) differs from that made in I in two respects. First and more important is the fact that we perform that splitting both on B_0 and on B_1 , whereas in I it was done only on B_1 . Second, we use here a smooth cut-off χ instead of a sharp one. The smooth cut-off is actually needed only for B_0 . For $\beta = \beta_0$, the splitting is the same for B_0 and B_1 and can therefore be performed on B without any reference to the asymptotic state (A_+, \dot{A}_+) . The parameters β_0 and β will have to satisfy various conditions which will appear later, all of them compatible with $\beta = \beta_0 = 1/3$.

We split the equation (1.12) into the following system of two equations.

$$\begin{cases} \partial_t w = i(2t^2)^{-1} \Delta w + t^{-2} Q(\nabla \varphi, w) + it^{-1} (B_{0S} + B_S(w, w)) w \\ \partial_t \varphi = (2t^2)^{-1} |\nabla \varphi|^2 - t^{-1} B_{0L} - t^{-1} B_L(w, w) \end{cases} \tag{1.18}$$

where we have defined

$$Q(s, w) = s \cdot \nabla w + (1/2)(\nabla \cdot s) w \tag{1.19}$$

for any vector field s . The first equation of (1.18) is the transport type equation for the amplitude w , while the second one is the Hamilton-Jacobi type equation for the phase φ . Since the right-hand sides of (1.18) contain φ only through its gradient, we can obtain from (1.18) a closed system for w and $s = \nabla \varphi$ by taking the gradient of the second equation, namely

$$\begin{cases} \partial_t w = i(2t^2)^{-1} \Delta w + t^{-2} Q(s, w) + it^{-1} (B_{0S} + B_S(w, w)) w \\ \partial_t s = t^{-2} s \cdot \nabla s - t^{-1} \nabla B_{0L} - t^{-1} \nabla B_L(w, w) . \end{cases} \tag{1.20}$$

Once the system (1.20) is solved for (w, s) , one recovers φ easily by integrating the second equation of (1.18) over time. The system (1.20) will be referred to as the auxiliary system.

The construction of the modified wave operators follows the same pattern as in I. The first task is to construct solutions of the auxiliary system (1.20) with suitably prescribed asymptotic behaviour at infinity, and in particular with $w(t)$ tending to a limit $w_+ = Fu_+$ as $t \rightarrow \infty$. That asymptotic behaviour is imposed in the form of a suitably chosen pair (W, ϕ) and therefore (W, S) with $S = \nabla\phi$ with $W(t)$ tending to w_+ as $t \rightarrow \infty$. For fixed (W, S) , we make a change of variables in the system (1.18) from (w, φ) to (q, ψ) defined by

$$(q, \psi) = (w, \varphi) - (W, \phi) \tag{1.21}$$

or equivalently a change of variables in the system (1.20) from (w, s) to (q, σ) defined by

$$(q, \sigma) = (w, s) - (W, S) , \tag{1.22}$$

and instead of looking for a solution (w, s) of the system (1.20) with (w, s) behaving asymptotically as (W, S) , we look for a solution (q, σ) of the transformed system with (q, σ) (and also ψ) tending to zero as $t \rightarrow \infty$. Performing the change of variables (1.22) in the auxiliary system (1.20) yields the following modified auxiliary system for the new variables (q, σ)

$$\begin{cases} \partial_t q = i(2t^2)^{-1} \Delta q + t^{-2}(Q(s, q) + Q(\sigma, W)) + it^{-1} B_{0S} q \\ \quad + it^{-1} B_S(w, w)q + it^{-1} (2B_S(W, q) + B_S(q, q)) W - R_1(W, S) \\ \partial_t \sigma = t^{-2}(s \cdot \nabla \sigma + \sigma \cdot \nabla S) - t^{-1} \nabla (2B_L(W, q) + B_L(q, q)) - R_2(W, S) , \end{cases} \tag{1.23}$$

where the remainders $R_1(W, S)$ and $R_2(W, S)$ are defined by

$$R_1(W, S) = \partial_t W - i(2t^2)^{-1} \Delta W - t^{-2} Q(S, W) - it^{-1} (B_{0S} + B_S(W, W)) W \tag{1.24}$$

$$R_2(W, S) = \partial_t S - t^{-2} S \cdot \nabla S + t^{-1} \nabla B_{0L} + t^{-1} \nabla B_L(W, W) \tag{1.25}$$

and the dependence of the remainders on B_0 has been omitted in the notation. For technical reasons, it is useful to consider also a partly linearized version of the system (1.23), namely

$$\begin{cases} \partial_t q' = i(2t^2)^{-1} \Delta q' + t^{-2}(Q(s, q') + Q(\sigma, W)) + it^{-1} B_{0S} q' \\ \quad + it^{-1} B_S(w, w)q' + it^{-1} (2B_S(W, q) + B_S(q, q)) W - R_1(W, S) \\ \partial_t \sigma' = t^{-2}(s \cdot \nabla \sigma' + \sigma \cdot \nabla S) - t^{-1} \nabla (2B_L(W, q) + B_L(q, q)) - R_2(W, S) . \end{cases} \tag{1.26}$$

The construction of solutions (w, s) of the auxiliary system (1.20) defined for large time and with prescribed asymptotic behaviour (W, S) proceeds in two steps. The first step consists in solving the system (1.23) for (q, σ) tending to zero at infinity under suitable boundedness properties of B_0 and (W, S) and suitable time

decay properties of the remainders $R_1(W, S)$ and $R_2(W, S)$, by a minor variation of the method used in I. That method consists in first solving the linearized system (1.26) for (q', σ') with given (q, σ) , and then showing that the map $(q, \sigma) \rightarrow (q', \sigma')$ thereby defined has a fixed point, by the use of a contraction method. The second step consists in constructing (W, S) with $W(t)$ tending to w_+ as $t \rightarrow \infty$ and satisfying the required boundedness and decay properties. This is done by solving the auxiliary system (1.20) by iteration to second order as in I and then adding to W an additional term of the same form as that used in [8]. The detailed form of (W, S) thereby obtained is too complicated to be given here and will be given in Section 3 below (see (3.25)–(3.29) and (3.31)).

Once the system (1.20) is solved for (w, s) , one can proceed therefrom to the construction of a solution (u, A) of the original WS system. One first defines the phases φ and ϕ such that $s = \nabla\varphi$ and $S = \nabla\phi$ and one reconstructs (u, A) from (w, φ) by (1.11), (1.3), (1.5), thereby obtaining a solution of the WS system defined for large time and with prescribed asymptotic behaviour. The modified wave operator for the WS system is then defined as the map $\Omega : (u_+, A_+, \dot{A}_+) \rightarrow (u, A)$.

The main result of this paper is the construction of (u, A) from (u_+, A_+, \dot{A}_+) as described above, together with the asymptotic properties of (u, A) that follow from that construction. It will be stated below in full mathematical detail in Proposition 4.1. We give here a heuristic preview of that result, stripped from most technicalities. We set $\beta = \beta_0 = 1/3$ for definiteness.

Proposition 1.1. *Let $\beta_0 = \beta = 1/3$. Let (u_+, A_+, \dot{A}_+) be such that $w_+ = Fu_+ \in H^{k_+}$ for sufficiently large k_+ , that (A_+, \dot{A}_+) be sufficiently regular, and that $(FA_+, F\dot{A}_+)$ be sufficiently small near $\xi = 0$. Let (W, S) be the approximate solution of the system (1.20) defined by (3.25)–(3.29), (3.31). Then*

- (1) *There exists $T = T(u_+, A_+, \dot{A}_+)$, $1 \leq T < \infty$, such that the auxiliary system (1.20) has a unique solution (w, s) in a suitable space, defined for $t \geq T$ and such that $(w - W, s - S)$ tends to zero in suitable norms when $t \rightarrow \infty$.*
- (2) *There exists φ and ϕ such that $s = \nabla\varphi$, $S = \nabla\phi$, $\phi(1) = 0$ and such that $\varphi - \phi$ tends to zero in suitable norms when $t \rightarrow \infty$. Define (u, A) by (1.11), (1.3), (1.5). Then (u, A) solves the system (1.1), (1.2) for $t \geq T$ and (u, A) behaves asymptotically as $(MD \exp(-i\phi)W, A_0 + A_1(|DW|^2))$ in the sense that the difference tends to zero in suitable norms (for which each term separately is $O(1)$) when $t \rightarrow \infty$.*

The unspecified condition that $(FA_+, F\dot{A}_+)$ be sufficiently small near $\xi = 0$ can be shown to follow from more intuitive conditions in x -space, consisting of decay conditions at infinity in space, and, depending on the values of the parameters defining the relevant function spaces, of some moment conditions on (A_+, \dot{A}_+) .

This paper relies on a large amount of material from I. In order to bring out the structure while keeping duplication to a minimum, we give without proof a shortened logically self-sufficient sequence of those intermediate results from I that

are needed, and we provide a full exposition only for the parts that are new as compared with I. When quoting I, we shall use the notation (I.p.q) for equation (p.q) of I and Item I.p.q for Item p.q of I, such as Lemma, Proposition, etc.

The remaining part of this paper is organized as follows. In Section 2 we collect notation and some estimates of a general nature. In Section 3, we study the Cauchy problem at infinity for the auxiliary system (1.20). We recall from I the existence results of solutions under suitable boundedness properties of (W, S) and suitable decay properties of the remainders, with the appropriate modifications (Proposition 3.1). We then define (W, S) and prove that they satisfy the previous properties, concentrating on the terms in the remainders that are new as compared with I (Proposition 3.2). We then discuss the assumptions on $(FA_+, F\dot{A}_+)$ at $\xi = 0$ mentioned above. Finally in Section 4, we construct the wave operators for the WS system (1.1) (1.2) and we derive the asymptotic properties of the solution (u, A) in their range that follow from the previous estimates (Proposition 4.1).

2 Notation and preliminary estimates

In this section we introduce some notation and we collect a number of estimates which will be used throughout this paper. We denote by $\|\cdot\|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^3)$ and we define $\delta(r) = 3/2 - 3/r$. For any interval I and any Banach space X we denote by $\mathcal{C}(I, X)$ the space of strongly continuous functions from I to X and by $L^\infty(I, X)$ the space of measurable essentially bounded functions from I to X . For real numbers a and b we use the notation $a \vee b = \text{Max}(a, b)$ and $a \wedge b = \text{Min}(a, b)$. In the estimates of solutions of the relevant equations we shall use the letter C to denote constants, possibly different from an estimate to the next, depending on various parameters but not on the solutions themselves or on their initial data. We shall use the notation $C(a_1, a_2, \dots)$ for estimating functions, also possibly different from an estimate to the next, depending on suitable norms a_1, a_2, \dots of the solutions or of their initial data.

We shall use the Sobolev spaces \dot{H}_r^k and H_r^k defined for $-\infty < k < +\infty$, $1 \leq r \leq \infty$ by

$$\dot{H}_r^k = \left\{ u : \|u; \dot{H}_r^k\| \equiv \|\omega^k u\|_r < \infty \right\}$$

and

$$H_r^k = \left\{ u : \|u; H_r^k\| \equiv \|\langle \omega \rangle^k u\|_r < \infty \right\}$$

where $\omega = (-\Delta)^{1/2}$ and $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. The subscript r will be omitted if $r = 2$.

We shall look for solutions of the auxiliary system (1.20) in spaces of the type $\mathcal{C}(I, X^{k,\ell})$ where I is an interval and

$$X^{k,\ell} = H^k \oplus \omega^{-1} H^\ell$$

namely

$$X^{k,\ell} = \left\{ (w, s) : w \in H^k, \nabla s \in H^\ell \right\} \tag{2.1}$$

where it is understood that $\nabla s \in L^2$ includes the fact that $s \in L^6$, and we shall use the notation

$$\| w; H^k \| = |w|_k . \tag{2.2}$$

We shall use extensively the following Sobolev inequalities, stated here in \mathbb{R}^n , but to be used only for $n = 3$.

Lemma 2.1 *Let $1 < q, r < \infty, 1 < p \leq \infty$ and $0 \leq j < k$. If $p = \infty$, assume that $k - j > n/r$. Let σ satisfy $j/k \leq \sigma \leq 1$ and*

$$n/p - j = (1 - \sigma)n/q + \sigma(n/r - k) .$$

Then the following inequality holds

$$\| \omega^j u \|_p \leq C \| u \|_q^{1-\sigma} \| \omega^k u \|_r^\sigma . \tag{2.3}$$

The proof follows from the Hardy-Littlewood-Sobolev (HLS) inequality ([2], p. 117) (from the Young inequality if $p = \infty$), from Paley-Littlewood theory and interpolation.

We shall also use extensively the following Leibnitz and commutator estimates.

Lemma 2.2 *Let $1 < r, r_1, r_3 < \infty$ and*

$$1/r = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4 .$$

Then the following estimates hold

$$\| \omega^m(uv) \|_r \leq C (\| \omega^m u \|_{r_1} \| v \|_{r_2} + \| \omega^m v \|_{r_3} \| u \|_{r_4}) \tag{2.4}$$

for $m \geq 0$, and

$$\| [\omega^m, u]v \|_r \leq C (\| \omega^m u \|_{r_1} \| v \|_{r_2} + \| \omega^{m-1} v \|_{r_3} \| \nabla u \|_{r_4}) \tag{2.5}$$

for $m \geq 1$, where $[,]$ denotes the commutator.

The proof of those estimates is given in [3], [4] with ω replaced by $\langle \omega \rangle$ and follows therefrom by a scaling argument.

We next give some estimates of B_{0L}, B_{0S}, B_L and B_S defined by (1.16) (1.17). It follows immediately from (1.16) (1.17) that

$$\| \omega^m B_{0L} \|_2 \leq (2t^{\beta_0})^{m-p} \| \omega^p B_{0L} \|_2 \leq (2t^{\beta_0})^{m-p} \| \omega^p B_0 \|_2 \tag{2.6}$$

for $m \geq p$ and

$$\| \omega^m B_{0S} \|_2 \leq t^{\beta_0(m-p)} \| \omega^p B_{0S} \|_2 \leq t^{\beta_0(m-p)} \| \omega^p B_0 \|_2 \tag{2.7}$$

for $m \leq p$. Similar estimates hold for B_L, B_S with β_0 replaced by β . On the other hand it follows from (1.15) that

$$\| \omega^{m+1} B_1(w_1, w_2) \|_2 \leq I_m (\| \omega^m (\bar{w}_1 w_2) \|_2) \tag{2.8}$$

where I_m is defined by

$$(I_m(f))(t) = \int_1^\infty d\nu \nu^{-m-3/2} f(\nu t) . \tag{2.9}$$

We finally collect some estimates of the solutions of the free wave equation $\square A_0 = 0$ with initial data (A_+, \dot{A}_+) at time zero, given by (1.4).

Lemma 2.3 *Let $k \geq 0$. Let A_+ and \dot{A}_+ satisfy the conditions*

$$A_+, \omega^{-1} \dot{A}_+ \in H^k \quad , \quad \nabla^2 A_+, \nabla \dot{A}_+ \in H_1^k . \tag{2.10}$$

Then the following estimate holds:

$$\| \omega^m A_0 \|_r \leq b_0 t^{-1+2/r} \quad \text{for } 2 \leq r \leq \infty , \tag{2.11}$$

for $0 \leq m \leq k$ and for all $t > 0$, where b_0 depends on (A_+, \dot{A}_+) through the norms associated with (2.10).

The estimate (2.11) can be expressed in an equivalent form in terms of B_0 defined by (1.13), namely

$$\| \omega^m B_0 \|_r \leq b_0 t^{m-1/r} \quad \text{for } 2 \leq r \leq \infty . \tag{2.12}$$

Furthermore, it follows from (1.17) and (2.12) that

$$\| \omega^m B_{0L} \|_r \leq \| F^{-1} \chi \|_1 \| \omega^m B_0 \|_r \leq C b_0 t^{m-1/r} \tag{2.13}$$

where we have used the Young inequality and the fact that the L^1 -norm of $F^{-1} \chi$ is invariant under the rescaling of ξ by t^{β_0} which occurs in (1.17). From (2.12) (2.13) and (1.16) it follows that also

$$\| \omega^m B_{0S} \|_r \leq C b_0 t^{m-1/r} . \tag{2.14}$$

In the applications, the estimate (2.12) will be used mostly through its consequence (2.14).

3 Cauchy problem at infinity for the auxiliary system

In this section, we solve the Cauchy problem at infinity for the auxiliary system (1.20) in the difference form (1.23). We first solve the system (1.23) for (q, σ) tending to zero at infinity under suitable boundedness properties of (B_0, W, S) and suitable time decay properties of the remainders $R_1(W, S)$ and $R_2(W, S)$. We

then construct (W, S) with $W(t)$ tending to $w_+ = Fu_+$ as $t \rightarrow \infty$ and satisfying the required boundedness and decay properties. The method closely follows that of Sections 6 and 7 of I.

We first estimate a single solution of the linearized auxiliary system (1.26) at the level of regularity where we shall eventually solve the auxiliary system (1.20). The following lemma is basically Lemma I.6.1, restricted to the case where $1 < k < 3/2$, and sharpened in order to take into account the fact that the W used in this paper is less regular than that used in I (compare (3.1) below with (I.6.1)).

Lemma 3.1 *Let $1 < k < 3/2 < \ell$ and $\beta > 0$. Let $T \geq 1$ and $I = [T, \infty)$. Let B_0 satisfy the estimate (2.12) for $0 \leq m \leq k$. Let $(U(1/t)W, S) \in \mathcal{C}(I, X^{k+1, \ell+1}) \cap \mathcal{C}^1(I, X^{k, \ell})$ and let W satisfy*

$$\sup_{t \in I} \left\{ \|W\|_\infty \vee \|W; H^{3/2}\| \vee t^{1/2-k} \|W; \dot{H}^{k+1}\| \right\} \leq a < \infty. \tag{3.1}$$

Let $(q, \sigma), (q', \sigma') \in \mathcal{C}(I, X^{k, \ell})$ with $q \in L^\infty(I, H^k) \cap L^2(I, L^2)$ and let (q', σ') be a solution of the system (1.26) in I . Then the following estimates hold for all $t \in I$:

$$\begin{aligned} |\partial_t \|q'\|_2| \leq & C \left\{ t^{-2} a \|\nabla \sigma\|_2 + t^{-1-\beta} a^2 I_0(\|q\|_2) \right. \\ & \left. + t^{-1} a I_{-1}(\|q\|_2 \|q\|_3) \right\} + \|R_1(W, S)\|_2, \tag{3.2} \end{aligned}$$

$$\begin{aligned} |\partial_t \|\omega^k q'\|_2| \leq & C \left\{ b_0 \left(\|\omega^{k-1} q'\|_2 + t^{k-1-\delta/3} \|q'\|_r \right) \right. \\ & + t^{-2} a \left(\|\omega^k \nabla \sigma\|_2 + t^{k-1/2} \|\sigma\|_\infty \right) \\ & + t^{-2} \left(\|\nabla s\|_\infty + \|\omega^{3/2} \nabla s\|_2 \right) \|\omega^k q'\|_2 \\ & + t^{-1} a^2 \left(I_{k-1}(\|\omega^{k-1} q\|_2) + \|\omega^{1/2} q'\|_2 \right) \\ & + t^{-1} a \left(I_{k-1}(\|\omega^k q\|_2 \|q\|_3) + I_{1/2}(\|\omega^{1/2} q\|_2) \|\omega^k q'\|_2 \right) \\ & \left. + t^{-1} I_{1/2}(\|\nabla q\|_2^2) \|\omega^k q'\|_2 \right\} + \|\omega^k R_1(W, S)\|_2 \tag{3.3} \end{aligned}$$

where $s = S + \sigma$ and $0 < \delta = \delta(r) \leq k$.

$$\begin{aligned} |\partial_t \|\omega^m \nabla \sigma'\|_2| \leq & C t^{-2} \left\{ \|\nabla s\|_\infty \|\omega^m \nabla \sigma'\|_2 + \|\omega^m \nabla s\|_2 \|\nabla \sigma'\|_\infty \right. \\ & \left. + \|\omega^m \nabla \sigma\|_2 \|\nabla S\|_\infty + \|\sigma\|_\infty \|\omega^m \nabla^2 S\|_2 \right\} \\ & + C \left\{ t^{-1+\beta(m+1)} a I_0(\|q\|_2) + t^{-1+\beta(m+5/2)} I_{-3/2}(\|q\|_2^2) \right\} \\ & + \|\omega^m \nabla R_2(W, S)\|_2 \tag{3.4} \end{aligned}$$

for $0 \leq m \leq \ell$,

$$|\partial_t \|\nabla\sigma'\|_2| \leq C t^{-2} \left\{ \|\nabla s\|_\infty \|\nabla\sigma'\|_2 + \|\nabla\sigma\|_2 \left(\|\nabla S\|_\infty + \|\omega^{3/2}\nabla S\|_2 \right) \right\} \\ + C \left\{ t^{-1+\beta} a I_0 (\|q\|_2) + t^{-1+5\beta/2} I_{-3/2} (\|q\|_2^2) \right\} + \|\nabla R_2(W, S)\|_2 . \quad (3.4)_0$$

Proof. (3.2) is identical with (I.6.2) and is proved in the same way. In order to prove (3.3), we start from (cf. I.6.9))

$$\begin{aligned} |\partial_t \|\omega^k q'\|_2| &\leq t^{-1} \|\omega^k, B_{0S}\| q'\|_2 + t^{-2} (\|\omega^k, s\| \cdot \nabla q'\|_2 \\ &+ \|\nabla \cdot s\| \omega^k q'\|_2 + \|\omega^k ((\nabla \cdot s)q')\|_2 + \|\omega^k Q(\sigma, W)\|_2) \\ &t^{-1} (\|\omega^k, B_S(w, w)\| q'\|_2 + \|\omega^k (2B_S(W, q) + B_S(q, q)) W\|_2) \\ &+ \|\omega^k R_1(W, S)\|_2 \end{aligned} \quad (3.5)$$

and we estimate the various terms in the RHS successively.

The contribution of B_0 is estimated exactly as in I and yields

$$\|\omega^k, B_{0S}\| q'\|_2 \leq C b_0 \left(t \|\omega^{k-1} q'\|_2 + t^{k-\delta/3} \|q'\|_r \right) . \quad (3.6)$$

The contribution of $Q(s, q')$ is estimated by Lemmas 2.1 and 2.2 as

$$\begin{aligned} &\|\omega^k, s\| \cdot \nabla q'\|_2 + \|\nabla \cdot s\| \omega^k q'\|_2 + \|\omega^k ((\nabla \cdot s)q')\|_2 \\ &\leq C \left(\|\nabla s\|_\infty + \|\omega^{3/2}\nabla s\|_2 \right) \|\omega^k q'\|_2 \end{aligned} \quad (3.7)$$

in the same way as in I, in the case $k < 3/2$.

The contribution of $Q(\sigma, W)$ is estimated by Lemmas 2.1 and 2.2 as

$$\begin{aligned} \|\omega^k Q(\sigma, W)\|_2 &\leq C \left(\|\sigma\|_\infty \|\omega^k \nabla W\|_2 + \|\omega^k \sigma\|_6 \|\nabla W\|_3 \right. \\ &\left. + \|\omega^k \nabla \sigma\|_2 \|W\|_\infty + \|\nabla \sigma\|_r \|\omega^k W\|_{3/k} \right) \end{aligned}$$

with $\delta(r) = k$,

$$\begin{aligned} \dots &\leq C \left(\|\sigma\|_\infty \|\omega^k \nabla W\|_2 + \|\omega^k \nabla \sigma\|_2 (\|W\|_\infty + \|\omega^k W\|_{3/k}) \right) \\ &\leq C a \left(t^{k-1/2} \|\sigma\|_\infty + \|\omega^k \nabla \sigma\|_2 \right) \end{aligned} \quad (3.8)$$

by (3.1).

The contribution of B_S with $w = W + q$ yields a number of terms which we order by increasing powers of q, q' . We first expand

$$B_S(w, w) = B_S(W, W) + 2B_S(W, q) + B_S(q, q) . \quad (3.9)$$

By Lemmas 2.1 and 2.2, we estimate

$$\begin{aligned} \|\omega^k, B_S(W, W)\|q'\|_2 &\leq C (\|\nabla B_S(W, W)\|_{3/\varepsilon} + \|\omega^k B_S(W, W)\|_{3/(k-1+\varepsilon)}) \\ &\quad \times \|\omega^{k-1+\varepsilon} q'\|_2 \end{aligned}$$

for $\varepsilon > 0$. Taking $\varepsilon = 3/2 - k$ yields

$$\begin{aligned} \dots &\leq C \|\omega^{k+1} B_1(W, W)\|_2 \|\omega^{1/2} q'\|_2 \\ &\leq C I_k (\|\omega^k W\|_2 \|W\|_\infty) \|\omega^{1/2} q'\|_2 \\ &\leq C a^2 \|\omega^{1/2} q'\|_2 \end{aligned} \tag{3.10}$$

by Lemma 2.2 again and by (2.8) (3.1).

In a similar way, we estimate by Lemmas 2.1, 2.2 and by (2.8)

$$\begin{aligned} \|\omega^k, B_S(W, q)\|q'\|_2 &\leq C (\|\nabla B_S(W, q)\|_3 \|\omega^{k-1} q'\|_6 + \|\omega^k B_S(W, q)\|_{3/k} \|q'\|_r) \end{aligned}$$

with $\delta(r) = k$,

$$\begin{aligned} \dots &\leq C \|\omega^{3/2} B_1(W, q)\|_2 \|\omega^k q'\|_2 \\ &\leq C I_{1/2} (\|W\|_\infty \|\omega^{1/2} q\|_2 + \|\omega^{1/2} W\|_6 \|q\|_3) \|\omega^k q'\|_2 \\ &\leq C a I_{1/2} (\|\omega^{1/2} q\|_2) \|\omega^k q'\|_2 . \end{aligned} \tag{3.11}$$

In a similar way, we estimate

$$\begin{aligned} \|\omega^k, B_S(q, q)\|q'\|_2 &\leq C (\|\nabla B_S(q, q)\|_3 + \|\omega^k B_S(q, q)\|_{3/k}) \|\omega^k q'\|_2 \\ &\leq C \|\omega^{3/2} B_1(q, q)\|_2 \|\omega^k q'\|_2 \\ &\leq C I_{1/2} (\|\omega^{1/2} q\|_3 \|q\|_6) \|\omega^k q'\|_2 \\ &\leq C I_{1/2} (\|\nabla q\|_2^2) \|\omega^k q'\|_2 . \end{aligned} \tag{3.12}$$

We next estimate in a similar way

$$\begin{aligned} \|\omega^k (B_S(W, q)W)\|_2 &\leq C \|\omega^k B_1(W, q)\|_2 (\|W\|_\infty + \|\omega^k W\|_{3/k}) \\ &\leq C a \|\omega^k B_1(W, q)\|_2 \\ &\leq C a I_{k-1} (\|W\|_\infty + \|\omega^{k-1} W\|_{3/(k-1)}) \|\omega^{k-1} q\|_2 \\ &\leq C a^2 I_{k-1} (\|\omega^{k-1} q\|_2) . \end{aligned} \tag{3.13}$$

Finally, we estimate in a similar way

$$\begin{aligned} \|\omega^k (B_S(q, q)W)\|_2 &\leq C \|\omega^k B_1(q, q)\|_2 (\|W\|_\infty + \|\omega^k W\|_{3/k}) \\ &\leq C a I_{k-1} (\|\omega^{k-1} |q|^2\|_2) \\ &\leq C a I_{k-1} (\|\omega^k q\|_2 \|q\|_3) . \end{aligned} \tag{3.14}$$

Substituting (3.6)–(3.8) and (3.10)–(3.14) into (3.5) yields (3.3).

The estimates (3.4) and (3.4)₀ of σ' are identical with (I.6.4) and (I.6.4)₀ and have exactly the same proof. The additional term with B_{0L} in the equation for σ' is included in $R_2(W, S)$ and does not appear explicitly at this stage. \square

From there on, the treatment of the Cauchy problem at infinity for the auxiliary system follows that given in I verbatim. We need to estimate the difference of two solutions of the linearized auxiliary system (1.26), and that estimate, given by Lemma I.6.2, requires no modification because it uses regularity properties of W which are weaker than (3.1). We then solve the Cauchy problem first for the linearized auxiliary system (1.26) with finite initial time by Proposition I.6.1, then at infinity by Proposition I.6.2, and then for the auxiliary system (1.20) or (1.23) by a contraction method, by Proposition I.6.3, part (2). The only difference in the proof of Propositions I.6.2 and I.6.3 is due to the term

$$t^{-2} a t^{k-1/2} \|\sigma\|_\infty$$

in (3.3), which did not appear in Lemma I.6.1, and which is due to the fact that the assumption (3.1) is weaker than (I.6.1). That term generates an additional term

$$a Z t^{-1-\lambda-3(1-\beta)/2}$$

in the RHS of (I.6.59), with time decay strictly better than $t^{-1-\lambda}$ and therefore harmless.

We now state the first main result of this section, which corresponds to Proposition I.6.3, part (2).

Proposition 3.1 *Let $1 < k < 3/2 < \ell$. Let λ_0, λ and β satisfy the conditions*

$$\lambda > 0 \quad , \quad (1 <) \lambda + k < \lambda_0 \quad , \quad (3.15)$$

$$0 < \beta < 2/3 \quad , \quad \beta(\ell + 1) < \lambda_0 \quad . \quad (3.16)$$

Let (A_+, \dot{A}_+) satisfy the conditions (2.10). Let $(U(1/t)W, S) \in \mathcal{C}([1, \infty), X^{k+1, \ell+1}) \cap \mathcal{C}^1([1, \infty), X^{k, \ell})$ and let (W, S) satisfy the estimates

$$\text{Sup}_{t \geq 1} \left\{ \|W\|_\infty \vee \|W; H^{3/2}\| \vee t^{1/2-k} \|W; \dot{H}^{k+1}\| \right\} \leq a < \infty \quad , \quad (3.17)$$

$$\|\omega^m \nabla S\|_2 \leq b t^{1-\eta+\beta(m-3/2)} \quad (3.18)$$

for some $\eta > 0$ and for $0 \leq m \leq \ell + 1$,

$$\|R_1(W, S)\|_2 \leq c_0 t^{-1-\lambda_0} \quad , \quad (3.19)$$

$$\|\omega^k R_1(W, S)\|_2 \leq c_1 t^{-1-\lambda} \quad , \quad (3.20)$$

$$\|\omega^m \nabla R_2(W, S)\|_2 \leq c_2 t^{-1-\lambda_0+\beta(m+1)} \quad \text{for } 0 \leq m \leq \ell \quad . \quad (3.21)$$

Then there exists $T, 1 \leq T < \infty$ and positive constants Y_0, Y and Z , depending on (A_+, \dot{A}_+) through the norms in (2.10) and depending on $k, \ell, \beta, \lambda_0, \lambda, a, b$,

c_0, c_1 and c_2 such that the auxiliary system (1.20) has a unique solution $(w, s) \in \mathcal{C}(I, X^{k,\ell})$, where $I = [T, \infty)$, satisfying the estimates

$$\| w - W \|_2 \leq Y_0 t^{-\lambda_0} , \tag{3.22}$$

$$\| \omega^k(w - W) \|_2 \leq Y t^{-\lambda} , \tag{3.23}$$

$$\| \omega^m \nabla(s - S) \|_2 \leq Z t^{-\lambda_0 + \beta(m+1)} \text{ for } 0 \leq m \leq \ell , \tag{3.24}$$

for all $t \in I$.

We now turn to the construction of approximate solutions (W, S) of the system (1.20) satisfying the assumptions of Proposition 3.1 and in particular the estimates (3.17), (3.18) of (W, S) and the estimates (3.19)–(3.21) of the remainders. In I we took for (W, S) the second order approximate solution of the system (1.20) in an iterative scheme not taking into account the terms containing B_0 , thereby ending with an explicit $B_0 W$ term in the remainder $R_1(W, S)$. Here, following [8], we improve that asymptotic form by adding one more term in W , so as to partly cancel the $B_{0S} W$ term in $R_1(W, S)$. Thus we define

$$W = w_0 + w_1 + w_2 \equiv W_1 + w_2 \quad , \quad S = s_0 + s_1 \tag{3.25}$$

where w_0, s_0, w_1, s_1 are the same as in I, namely

$$w_0 = U^*(1/t)w_+ , \tag{3.26}$$

$$s_0(t) = - \int_1^t dt' t'^{-1} \nabla B_L(w_0(t'), w_0(t')) , \tag{3.27}$$

$$w_1(t) = -U^*(1/t) \int_t^\infty dt' t'^{-2} U(1/t') Q(s_0(t'), w_0(t')) , \tag{3.28}$$

$$s_1(t) = - \int_t^\infty dt' t'^{-2} s_0(t') \cdot \nabla s_0(t') + 2 \int_t^\infty dt' t'^{-1} \nabla B_L(w_0(t'), w_1(t')) . \tag{3.29}$$

In order to partly cancel $B_{0S} W$ in $R_1(W, S)$, we take $w_2 = hw_0$, thereby obtaining a linear contribution of w_2 to $R_1(W, S)$

$$\begin{aligned} (\partial_t - i(2t^2)^{-1} \Delta) w_2 &= h (\partial_t - i(2t^2)^{-1} \Delta) w_0 - it^{-2} \nabla h \cdot \nabla w_0 + (\partial_t h) w_0 \\ &\quad - i(2t^2)^{-1} (\Delta h) w_0 . \end{aligned} \tag{3.30}$$

The first term in the RHS is small, actually zero, by the choice of w_0 . We use the last term in the RHS to cancel the main contribution $B_{0S} w_0$ of $B_{0S} W$ by making the choice

$$w_2 = hw_0 \quad , \quad h = -2t \Delta^{-1} B_{0S} . \tag{3.31}$$

Note that because of the short range cut-off in B_{0S} , h is well defined, actually $h \in \mathcal{C}([1, \infty), H^{k+2})$. With that choice, the remainders become

$$R_i(W, S) = R_{i0}(W, S) + R_{i\nu}(W, S) \quad i = 1, 2 , \tag{3.32}$$

where $R_{i0}(W, s)$ are the parts not containing w_2 or B_{0L} , namely

$$R_{10}(W, S) = -t^{-2} (Q(S, w_1) + Q(s_1, w_0)) - it^{-1} (B_{0S}w_1 + B_S(W_1, W_1)W_1) \tag{3.33}$$

$$R_{20}(W, S) = -t^{-2} (s_0 \cdot \nabla s_1 + s_1 \cdot \nabla s_0 + s_1 \cdot \nabla s_1) + t^{-1} \nabla B_L(w_1, w_1) , \tag{3.34}$$

while $R_{i\nu}(W, S)$ are the parts containing w_2 or B_{0L} , namely

$$R_{1\nu}(W, S) = -t^{-2}Q(S, w_2) - it^{-1}B_{0S}w_2 + (\partial_t h)w_0 - it^{-2}\nabla h \cdot \nabla w_0 - it^{-1} (B_S(W, W)w_2 + B_S(W + W_1, w_2)W_1) , \tag{3.35}$$

$$R_{2\nu}(W, S) = t^{-1}\nabla B_{0L} + t^{-1}\nabla B_L(W + W_1, w_2) . \tag{3.36}$$

The parts R_{i0} of the remainders are the remainders occurring in I, up to the replacement of B_0 by B_{0S} and the disappearance of the term $B_{0S}w_0$, precisely the term which was responsible for the support condition in I. Up to a minor point (see below), (W_1, S) and $R_{i0}(W, S)$ have been estimated in I as follows (see Lemma I.7.1).

Lemma 3.2 *Let $0 < \beta < 1$, $k_+ \geq 3$, $w_+ \in H^{k_+}$ and $a_+ = |w_+|_{k_+}$. Then the following estimates hold for all $t \geq 1$:*

$$|w_0|_{k_+} \leq a_+ , \tag{3.37}$$

$$\| \omega^m s_0 \|_2 \leq \begin{cases} C a_+^2 \ln t & \text{for } 0 \leq m \leq k_+ \\ C a_+^2 t^{\beta(m-k_+)} & \text{for } m > k_+ , \end{cases} \tag{3.38}$$

$$|w_1|_{k_+-1} \leq C a_+^3 t^{-1}(1 + \ln t) , \tag{3.39}$$

$$\| \omega^m s_1 \|_2 \leq \begin{cases} C a_+^4 t^{-1}(1 + \ln t)^2 & \text{for } 0 \leq m \leq k_+ - 1 \\ C a_+^4 t^{-1+\beta(m+1-k_+)}(1 + \ln t) & \text{for } k_+ - 1 < m < k_+ - 1 + \beta^{-1} , \end{cases} \tag{3.40}$$

$$\| \omega^m R_{20}(W, S) \|_2 \leq \begin{cases} C(a_+) t^{-3}(1 + \ln t)^3 & \text{for } 0 \leq m \leq k_+ - 2 \\ C(a_+) t^{-3+\beta(m+2-k_+)}(1 + \ln t)^2 & \text{for } k_+ - 2 < m < k_+ - 2 + \beta^{-1} , \end{cases} \tag{3.41}$$

Let in addition $0 \leq k \leq k_+ - 1$ and let B_0 satisfy the estimate (2.12) for $0 \leq m \leq k$. Then

$$\| \omega^m R_{10}(W, S) \|_2 \leq C(a_+) \left\{ t^{-3}(1 + \ln t)^2 + t^{-1-\beta(k_+-m+1)} \right\} + C b_0 a_+^3 t^{m-5/2}(1 + \ln t) \quad \text{for } 0 \leq m \leq k . \tag{3.42}$$

Proof. The estimates (3.37)–(3.42) are those of Lemma I.7.1 except for the estimate of the term $t^{-1}B_{0S}w_1$ in $R_{10}(W, S)$ which is responsible for the last term in (3.42). We estimate that term by Lemmas 2.1 and 2.2 and by (2.12) (3.39) as

$$\begin{aligned} & \| \omega^m B_{0S} w_1 \|_2 \\ & \leq \begin{cases} C (\| \omega^m B_0 \|_2 \| w_1 \|_\infty + \| B_{0S} \|_r \| \omega^m w_1 \|_{1/m}) & \text{for } m \leq 1/2 \\ C (\| \omega^m B_0 \|_2 \| w_1 \|_\infty + \| B_{0S} \|_\infty \| \omega^m w_1 \|_2) & \text{for } m > 1/2 \end{cases} \\ & \leq C b_0 a_+^3 t^{m-3/2} (1 + \ell n t) \end{aligned} \tag{3.43}$$

for $0 \leq m \leq k$, and $1/r = 1/2 - m$ for $m \leq 1/2$, which completes the proof of (3.42). \square

We now turn to estimating $R_{i\nu}(W, S)$, $i = 1, 2$. We first reduce that question to that of estimating h and B_{0L} , assuming for the moment a boundedness property of w_2 which is part of (3.17) and which we shall prove later. We define the auxiliary space

$$Y = L^\infty([1, \infty), L^\infty \cap H^{3/2}) \tag{3.44}$$

and we remark that for $k_+ > 5/2$, it follows from (3.37), (3.39) that $W_1 = w_0 + w_1 \in Y$. We can now state the estimates of $R_{i\nu}(W, S)$.

Lemma 3.3 *Let $1 < k < 3/2$ and $k_+ \geq 3$, let $w_+ \in H^{k_+}$ and let $a_+ = |w_+|_{k_+}$. Assume that $w_2 \in Y$ and let*

$$\| W_1 \|_Y \vee \| W \|_Y \leq a < \infty . \tag{3.45}$$

Let B_0 satisfy the estimate (2.12) for $0 \leq m \leq k$. Then the following estimates hold for all $t \geq 1$:

$$\begin{aligned} \| R_{1\nu}(W, S) \|_2 & \leq C(a_+) t^{-2} \ell n t \| \nabla h \|_2 + C a_+ \left\{ b_0 t^{-1} \| h \|_2 \right. \\ & \quad \left. + \| \partial_t h \|_2 + a^2 t^{-1} (\| h \|_2 + I_0(\| h \|_2)) \right\} , \end{aligned} \tag{3.46}$$

$$\begin{aligned} \| \omega^k R_{1\nu}(W, S) \|_2 & \leq C(a_+) t^{-2} \ell n t (\| \omega^k \nabla h \|_2 + \| \omega^\delta \nabla h \|_2) \\ & + C a_+ \left\{ b_0 \left(t^{k-1-\delta/3} \| \omega^\delta h \|_2 + t^{-1} \| \omega^k h \|_2 \right) + \| \omega^k \partial_t h \|_2 + \| \omega^\delta \partial_t h \|_2 \right. \\ & \quad \left. + t^{-2} (\| \omega^{k+1} h \|_2 + \| \omega^2 h \|_2) + t^{-1} a^2 (\| \omega^k h \|_2 + I_{k-1}(\| \omega^{k-1} h \|_2)) \right\} \end{aligned} \tag{3.47}$$

for $0 < \delta < 1/2$,

$$\| \omega^m R_{2\nu}(W, S) \|_2 \leq t^{-1} \| \omega^{m+1} B_{0L} \|_2 + C a a_+ t^{-1+\beta m} I_0(\| h \|_2) \tag{3.48}$$

for all $m \geq 0$.

Proof. We first consider $\|R_{1\nu}(W, S)\|_2$. We estimate successively

$$\begin{aligned} \|Q(S, w_2)\|_2 &\leq \|S\|_\infty (\|w_0\|_\infty \|\nabla h\|_2 + \|\nabla w_0\|_3 \|h\|_6) \\ &\quad + \|\nabla S\|_3 \|w_0\|_\infty \|h\|_6 \leq C(a_+) \ell n t \|\nabla h\|_2 \end{aligned} \quad (3.49)$$

by (3.37), (3.38), (3.40) and Lemma 2.1,

$$\|B_{0S}w_2\|_2 \leq \|B_{0S}\|_\infty \|w_0\|_\infty \|h\|_2 \leq C b_0 a_+ \|h\|_2, \quad (3.50)$$

by (2.14) and (3.37),

$$\|(\partial_t h)w_0\|_2 \leq \|w_0\|_\infty \|\partial_t h\|_2 \leq C a_+ \|\partial_t h\|_2, \quad (3.51)$$

$$\|\nabla h \cdot \nabla w_0\|_2 \leq \|\nabla w_0\|_\infty \|\nabla h\|_2 \leq C a_+ \|\nabla h\|_2, \quad (3.52)$$

$$\|B_S(W, W)w_2\|_2 \leq \|B_S(W, W)\|_\infty \|w_0\|_\infty \|h\|_2 \leq C a_+ a^2 \|h\|_2, \quad (3.53)$$

by estimating $B_S(W, W)$ in a way similar to that in Lemma 3.1,

$$\begin{aligned} \|B_S(W + W_1, w_2)W_1\|_2 &\leq C \|W_1\|_3 \|\omega B_1(W + W_1, w_2)\|_2 \\ &\leq C \|W_1\|_3 \|W + W_1\|_\infty \|w_0\|_\infty I_0(\|h\|_2) \leq C a_+ a^2 I_0(\|h\|_2) \end{aligned} \quad (3.54)$$

by Lemma 2.1 and by (2.8). Collecting (3.49)–(3.54) yields (3.46).

We next consider $\omega^k R_{1\nu}(W, S)$. We estimate successively

$$\begin{aligned} \|\omega^k Q(S, w_2)\|_2 &\leq C \left\{ \|w_0\|_* \|S\|_* \|\omega^{k+1}h\|_2 \right. \\ &\quad \left. + \left(\|w_0\|_* \|\omega^{k+1}S\|_{3/(1+\delta)} + \|\omega^{k+1}w_0\|_{3/(1+\delta)} \|S\|_* \right) \|\omega^\delta \nabla h\|_2 \right\} \\ &\leq C(a_+) \ell n t (\|\omega^k \nabla h\|_2 + \|\omega^\delta \nabla h\|_2) \end{aligned} \quad (3.55)$$

by (3.37), (3.38), (3.40) and Lemmas 2.1 and 2.2, with

$$\|f\|_* = \|f\|_\infty + \|\nabla f\|_3,$$

$$\begin{aligned} \|\omega^k(B_{0S}w_2)\|_2 &\leq C \left\{ \|B_0\|_\infty \|w_0\|_\infty \|\omega^k h\|_2 + \left(\|\omega^k B_0\|_{3/\delta} \|w_0\|_\infty \right. \right. \\ &\quad \left. \left. + \|B_0\|_\infty \|\omega^k w_0\|_{3/\delta} \right) \|\omega^\delta h\|_2 \right\} \\ &\leq C b_0 a_+ \left(\|\omega^k h\|_2 + t^{k-\delta/3} \|\omega^\delta h\|_2 \right) \end{aligned} \quad (3.56)$$

by Lemmas 2.1 and 2.2 and by (2.14), (3.37),

$$\begin{aligned} \|\omega^k((\partial_t h)w_0)\|_2 &\leq C (\|w_0\|_\infty \|\omega^k \partial_t h\|_2 + \|\omega^k w_0\|_{3/\delta} \|\omega^\delta \partial_t h\|_2) \\ &\leq C a_+ (\|\omega^k \partial_t h\|_2 + \|\omega^\delta \partial_t h\|_2) \end{aligned} \quad (3.57)$$

Lemmas 2.1 and 2.2 and by (3.37),

$$\begin{aligned} \|\omega^k(\nabla h \cdot \nabla w_0)\|_2 &\leq C(\|\nabla w_0\|_\infty \|\omega^{k+1}h\|_2 + \|\omega^{k+1}w_0\|_3 \|\nabla h\|_6) \\ &\leq C a_+ (\|\omega^{k+1}h\|_2 + \|\omega^2h\|_2) \end{aligned} \tag{3.58}$$

by Lemmas 2.1 and 2.2 and by (3.37). We next estimate

$$\begin{aligned} \|\omega^k(B_S(W, W)w_2)\|_2 &\leq C(\|B_S w_0\|_\infty + \|\omega^k(B_S w_0)\|_{3/k}) \|\omega^k h\|_2 \\ &\leq C(\|B_S\|_\infty + \|\omega^k B_S\|_{3/k})(\|w_0\|_\infty + \|\omega^k w_0\|_{3/k}) \|\omega^k h\|_2 \\ &\leq C a_+ a^2 \|\omega^k h\|_2 \end{aligned} \tag{3.59}$$

where we have omitted the arguments in B_S , by Lemmas 2.1 and 2.2 and by (3.37), (3.45), and after estimating B_S in a way similar to that in Lemma 3.1. In the same way

$$\begin{aligned} \|\omega^k(B_S(W + W_1, w_2)W_1)\|_2 &\leq C(\|W_1\|_\infty + \|\omega^k W_1\|_{3/k}) \|\omega^k B_1(W + W_1, w_2)\|_2 \\ &\leq C a I_{k-1}(\|\omega^{k-1}h\|_2(\|w_0(W + W_1)\|_\infty + \|\omega^{k-1}(W + W_1)w_0\|_{3/(k-1)})) \\ &\leq C a_+ a^2 I_{k-1}(\|\omega^{k-1}h\|_2) . \end{aligned} \tag{3.60}$$

Collecting (3.55)–(3.60) yields (3.47).

We finally estimate $R_{2\nu}$. From (2.6), (2.8) we obtain

$$\begin{aligned} \|\omega^m R_{2\nu}(W, S)\|_2 &\leq t^{-1} \|\omega^{m+1} B_{0L}\|_2 \\ &+ C t^{-1+\beta m} I_0(\|W + W_1\|_\infty \|w_0\|_\infty \|h\|_2) \end{aligned}$$

which yields (3.48) by using (3.37), (3.45). □

In order to complete the estimate of the parts $R_{i\nu}(W, S)$, $i = 1, 2$, of the remainders, we now estimate h and B_{0L} . Those estimates require some restrictions on the behaviour of $(FA_+, F\dot{A}_+)$ at $\xi = 0$. Those restrictions are imposed in a dilation homogeneous way through the use of a parameter $\mu \in (-1, 1)$ in terms of quantities which have the same scaling properties as $\|A_+; \dot{H}^{-3/2-\mu}\|$ and $\|\dot{A}_+; \dot{H}^{-5/2-\mu}\|$. They will be further discussed at the end of this section.

Lemma 3.4 *Let $1 < k < 3/2$ and $-1 < \mu < 1$. Let (A_+, \dot{A}_+) satisfy the conditions*

$$A_+ \in H^{k-1} \quad , \quad \dot{A}_+ \in L^2 \quad , \tag{3.61}$$

$$xA_+ \in H^{k-1} \quad , \quad x\dot{A}_+ \in L^{3/2} \quad , \tag{3.62}$$

$$xA_+ \in \dot{H}^{-1/2-\mu} \quad , \quad A_+ , x\dot{A}_+ \in \dot{H}^{-3/2-\mu} \quad , \quad \dot{A}_+ \in \dot{H}^{-5/2-\mu} \quad . \tag{3.63}_\mu$$

Let B_{0L} and h be defined by (1.17) and (3.31).

Then the following estimates hold :

$$\| \omega^m B_{0L} \|_2 \leq C t^{m-1/2+(\beta_0-1)(m+3/2+\mu)} \left(\| A_+; \dot{H}^{-3/2-\mu} \| + \| \dot{A}_+; \dot{H}^{-5/2-\mu} \| \right) \tag{3.64}$$

for all $m \geq 0$,

$$\| \omega^m h \|_2 \leq 2t^{m-3/2} \left(1 \vee t^{(\beta_0-1)(m-1/2+\mu)} \right) \left(\| A_+; \dot{H}^\rho \| + \| \dot{A}_+; \dot{H}^{\rho-1} \| \right) \tag{3.65}$$

for all $m \leq k + 1$, where

$$\rho = (m - 2) \vee (-3/2 - \mu) = -3/2 - \mu + (m - 1/2 + \mu) \vee 0 , \tag{3.66}$$

$$\begin{aligned} \| \omega^m \partial_t h \|_2 \leq C t^{m-5/2} \left(1 \vee t^{(\beta_0-1)(m-1/2+\mu)} \right) & \left(\| xA_+; \dot{H}^{\rho+1} \| \right. \\ & \left. + \| x\dot{A}_+; \dot{H}^\rho \| + \| A_+; \dot{H}^\rho \| + \| \dot{A}_+; \dot{H}^{\rho-1} \| \right) \end{aligned} \tag{3.67}$$

for all $m \leq k$ and ρ given by (3.66),

$$\| h \|_\infty \leq C(A_+, \dot{A}_+) \tag{3.68}$$

where the constant depends on (A_+, \dot{A}_+) through the norms in (3.61), (3.63) $_\mu$.

Proof. (3.64) follows immediately from the definitions (1.4), (1.13) and (1.17) of A_0 and B_{0L} , from (2.6) and from (3.63) $_\mu$.

In order to derive the estimates of h , it is convenient to come back to the variable A_0 . The definition (3.31) of h can be rewritten as

$$h = 2t^2 \omega^{-2} D_0^{-1} A_{0S} = D_0^{-1} f \tag{3.69}$$

where

$$f = 2\omega^{-2} A_{0S} , \tag{3.70}$$

A_{0S} is defined by

$$A_{0S} = t^{-1} D_0^{-1} B_{0S} = \chi_S A_0 \equiv F^{-1} (1 - \chi(\xi t^{1-\beta_0})) F A_0 \tag{3.71}$$

and χ is defined before (1.16).

(3.65). We estimate

$$\begin{aligned} \| \omega^m h \|_2 &= t^{m-3/2} \| \omega^m f \|_2 = 2t^{m-3/2} \| \omega^{m-2} A_{0S} \|_2 \\ &\leq 2t^{m-3/2} \left(1 \vee t^{(\beta_0-1)(m-1/2+\mu)} \right) \| \omega^\rho A_0 \|_2 \end{aligned} \tag{3.72}$$

and the result follows from the assumptions (3.61) (3.63) $_\mu$.

(3.67). We use in addition the commutation relations

$$t \partial_t = D_0^{-1} P D_0 \quad , \quad P \omega^{-j} = \omega^{-j} (P + j) \quad , \quad [P, e^{i\omega t}] = 0 \tag{3.73}$$

where P is the dilation generator

$$P = t\partial_t + x \cdot \nabla .$$

In particular

$$\partial_t h = t^{-1} t \partial_t D_0^{-1} f = t^{-1} D_0^{-1} P f . \tag{3.74}$$

Using the commutation relations (3.73), we compute

$$\begin{aligned} (1/2)P f &= \omega^{-2}(P + 2)A_0 S \\ &= \omega^{-2} \cos \omega t (P + 2)\chi_S A_+ + \omega^{-3} \sin \omega t (P + 3)\chi_S \dot{A}_+ . \end{aligned}$$

Using the fact that $P + 3 = t\partial_t + \nabla \cdot x$ and the commutation relation

$$[P, \chi_S] = -\beta F^{-1} \xi t^{1-\beta_0} \cdot \nabla \chi(\xi t^{1-\beta_0}) F \equiv \tilde{\chi}$$

we obtain

$$\begin{aligned} (1/2)P f &= \omega^{-2} \cos \omega t \nabla \cdot \chi_S x A_+ + \omega^{-3} \sin \omega t \nabla \cdot \chi_S x \dot{A}_+ \\ &+ \omega^{-2} \cos \omega t (\tilde{\chi} - \chi_S) A_+ + \omega^{-3} \sin \omega t \tilde{\chi} \dot{A}_+ . \end{aligned} \tag{3.75}$$

We then estimate

$$\| \omega^m \partial_t h \|_2 = t^{m-5/2} \| \omega^m P f \|_2 \tag{3.76}$$

and we estimate the contribution of the various terms of (3.75) exactly as in the proof of (3.65), with (m, A_+, \dot{A}_+) replaced by $(m - 1, xA_+, x\dot{A}_+)$ in the first two terms, and with χ_S replaced by $\tilde{\chi} - \chi_S$ or by $\tilde{\chi}$ in the last two terms. This yields (3.67).

(3.68). By Lemma 2.1,

$$\| h \|_\infty \leq C \| \omega^{3/2-\varepsilon} h \|_2^{1/2} \| \omega^{3/2+\varepsilon} h \|_2^{1/2} \tag{3.77}$$

and (3.68) follows from (3.65) with $0 < \varepsilon \leq (k - 1/2) \wedge (\mu + 1)$. □

We now collect the results of Lemmas 3.2, 3.3 and 3.4 in order to exhibit a set of assumptions which imply those of Proposition 3.1

Proposition 3.2. *Let $1 < k < 3/2 < \ell$. Let $\mu, \lambda_0, \lambda, \beta_0, \beta$ and k_+ satisfy the conditions*

$$-1/4 < \mu \leq 1/2 \tag{3.78}$$

$$\lambda > 0 \quad , \quad (1 <)\lambda + k < \lambda_0 < 7/6 + 2\mu/3 (\leq 3/2) \tag{3.79}$$

$$0 < \beta_0 \leq \beta < 2/3 \quad , \quad \beta(\ell + 1) < \lambda_0 \tag{3.80}$$

$$\beta_0(1/2 - \mu) > \lambda_0 - 1 - \mu \tag{3.81}$$

$$\beta_0(\mu + 5/2) < 2 + \mu - \lambda_0 \tag{3.82}$$

$$k_+ \geq k + 2 \quad , \quad \beta(k_+ + 1) \geq \lambda_0 \quad , \quad \beta(\ell + 3 - k_+) < 1 . \tag{3.83}$$

Let $w_+ \in H^{k+}$ and let (A_+, \dot{A}_+) satisfy (2.10), (3.62), (3.63) $_{\mu}$. Let (W, S) be defined by (3.25)–(3.29), (3.31).

Then (W, S) satisfy the estimates (3.17), (3.18), (3.19), (3.20), (3.21) (with $0 < \eta < 1 - 3\beta/2$, in (3.18)) and all the assumptions of Proposition 3.1 are satisfied.

Proof. The contribution of the terms not containing w_2 or B_{0L} in (W, S) and in the remainders are estimated by Lemma 3.2 in the same way as in I. We concentrate on the remaining terms. The terms containing w_2 are estimated by Lemma 3.3 in terms of h , and h and B_{0L} are estimated by Lemma 3.4.

The condition (3.17) restricted to $w_2 = hw_0$ follows from the fact that it holds for h by (3.65), (3.68) and trivially for w_0 , and that it is multiplicative. Together with (3.45) for W_1 , it implies (3.45) for W .

We next consider $R_{1\nu}(W, S)$. Its L^2 norm is estimated by (3.46). By Lemma 3.4, it satisfies the estimate (3.19) provided

$$(1 - \beta_0)((1/2 - \mu) \vee 0) < 3/2 - \lambda_0 \tag{3.84}$$

which reduces to (3.81) for $\mu \leq 1/2$.

Similarly, $R_{1\nu}(W, S)$ is estimated in \dot{H}^k norm by (3.47) and satisfies the estimate (3.20) for δ sufficiently small under the condition (3.84) because the time decay of (3.47) is worse than that of (3.46) at worst by a factor $t^{k+2\delta/3}$ which is better than the allowed $t^{\lambda_0-\lambda}$ for $0 < 2\delta/3 \leq \lambda_0 - \lambda - k$.

We now turn to $R_{2\nu}(W, s)$. The contribution of B_{0L} is estimated by (3.64) and satisfies the estimate (3.21) provided

$$m + 1/2 + (\beta_0 - 1)(m + 7/2 + \mu) \leq -1 - \lambda_0 + \beta(m + 1) \tag{3.85}$$

which is implied by (3.82) for $\beta_0 \leq \beta$.

The term containing w_2 is estimated by (3.48) and satisfies the estimate (3.21) by (3.65) under the condition (3.84).

We remark here that the upper bound on λ_0 in (3.79) is the compatibility condition of (3.81), (3.82). The remaining conditions in (3.78)–(3.83) come from I. □

We now comment briefly on the various parameters that occur in Proposition 3.2 and on the conditions (3.78)–(3.83) that they have to satisfy. The parameters k and ℓ characterize the regularity of the spaces of resolution for (w, s) . As a consequence, k also characterizes the regularity of $(\hat{A}_+, \dot{\hat{A}}_+)$ as given by (2.10). The parameter μ characterizes the behaviour of $(\hat{A}_+, \dot{\hat{A}}_+) = (FA_+, F\dot{A}_+)$ at $\xi = 0$ through the condition (3.63) $_{\mu}$. The parameters λ_0 and λ are the time decay exponents of the norms of q in L^2 and in \dot{H}^k . The μ dependent upper bound on λ_0 in (3.79) ranges over $(1, 3/2]$ when μ ranges over $(-1/4, 1/2]$. Since the condition (3.84) saturates at $\lambda_0 < 3/2$ for $\mu \geq 1/2$, there is no point in considering values of $\mu > 1/2$. The parameters β_0 and β characterize the splitting of B_0 and B_1 respectively into short range and long range parts, and therefore the splitting

of the Schrödinger equation into transport and Hamilton-Jacobi equations. The parameter β should not be too large and can be taken equal to β_0 . The parameter β_0 satisfies two inequalities (3.81) and (3.82) in opposite directions, depending on λ_0 and μ , and expressing the fact that B_{0S} and B_{0L} are not too large. The upper bound on λ_0 in (3.79) is the compatibility condition of those inequalities. Whenever it is satisfied, the value $\beta_0 = 1/3$ is allowed. Actually both (3.81) and (3.82) reduce to that upper bound for $\beta_0 = 1/3$. Finally k_+ characterizes the regularity of w_+ and should be sufficiently large, depending on k, ℓ, λ_0 and β .

Remark 3.1. For $\mu = 1/2$ the short range restriction is no longer needed in the estimates of h and $\partial_t h$ in Lemma 3.4, and therefore the splitting of B_0 into short range and long range parts is no longer needed, namely B_0 can be kept entirely in the equation for w .

We finally discuss the condition $(3.63)_\mu$ of Lemma 3.4. That condition restricts the behaviour of the relevant functions for small $|\xi|$ in Fourier transformed variables. Let A be any of the functions $A_+, \dot{A}_+, xA_+, x\dot{A}_+$ and define $A_<$ by $\widehat{A}_<(\xi) = \chi(\xi)\widehat{A}(\xi)$. Then the conditions on $A_<$ contained in $(3.63)_\mu$ all take the form

$$A_< \in \dot{H}^{-3/2-\nu} \tag{3.86}$$

for $\nu = \mu, \mu \pm 1$. We first remark that in the proof of Lemma 3.4, all such conditions can be replaced by

$$|\xi|^{-\nu} \widehat{A}_< \in L^\infty \tag{3.87}$$

at the expense of inserting an additional factor $(\ell n t)^{1/2}$ in (3.65) in the case of equality, namely for $m = 1/2 - \mu$. This follows from the fact that

$$\| |\xi|^{m-3/2} \widehat{A}_{S<} \|_2 \leq \begin{cases} C \| |\xi|^{-\nu} \widehat{A}_< \|_\infty (1 \vee t^{(\beta_0-1)(m+\nu)}) & \text{for } m \neq -\nu \\ C \| |\xi|^{-\nu} \widehat{A}_< \|_\infty (\ell n t)^{1/2} & \text{for } m = -\nu \end{cases} \tag{3.88}$$

The occurrence of the factor $(\ell n t)^{1/2}$ is harmless for the applications. The condition (3.86) is weaker than (3.87) as regards the behaviour of $\widehat{A}_<$ away from zero, since it requires only $\widehat{A}_< \in L^2_{loc}$ instead of $\widehat{A}_< \in L^\infty_{loc}$. Furthermore (3.86) almost follows from (3.87), up to a change of ν into $\nu + \varepsilon$. In fact

$$\| |\xi|^{-3/2-\nu} \widehat{A}_< \|_2 \leq C \varepsilon^{-1/2} \| |\xi|^{-\nu-\varepsilon} \widehat{A}_< \|_\infty \tag{3.89}$$

for $\varepsilon > 0$. In addition, under the short range condition

$$\| |\xi|^{-3/2-\nu} \widehat{A}_{S<} \|_2 \leq C(\ell n t)^{1/2} \| |\xi|^{-\nu} \widehat{A}_< \|_\infty \tag{3.90}$$

which is the special case $m = -\nu$ of (3.88).

The restrictions on $(\widehat{A}_+, \widehat{\dot{A}}_+)$ at $\xi = 0$ expressed by (3.86) have the unpleasant feature that for $\nu \geq 0$ they cannot be ensured by imposing decay of (A_+, \dot{A}_+) at

infinity in space and that they require in addition some moment conditions. For instance even for $A \in \mathcal{S}$ one has

$$\omega^{-3/2-\nu} A_{<} = C|x|^{-3/2+\nu} * A_{<}$$

for $|\nu| < 3/2$ [9], which behaves as

$$|x|^{-3/2+\nu} \int A \, dx$$

as $|x| \rightarrow \infty$ and therefore cannot be in L^2 for $\nu \geq 0$ unless $\int A \, dx = 0$. More generally when ν increases, vanishing of the n -th moment of A is necessary as soon as $\nu \geq n$. Actually the parameter μ in $(3.63)_\mu$ has been introduced in order to minimize the number of such conditions by taking μ small.

We now give sufficient conditions on (A_+, \dot{A}_+) in terms of space decay and vanishing of suitable moments so as to ensure the low frequency part of $(3.63)_\mu$.

Lemma 3.5 *Let $-1 < \mu < 1$. Let (A_+, \dot{A}_+) satisfy (3.61), (3.62) and in addition*

$$x A_+ \in L^{3/(2+\mu)\vee 2}, \int \dot{A}_+ \, dx = 0, \langle x \rangle^{1+\mu+\varepsilon} \dot{A}_+ \in L^1, \tag{3.91}$$

$$A_+, x\dot{A}_+ \in L^{3/(3+\mu)} \quad \text{for } \mu < 0, \tag{3.92}$$

$$\int A_+ \, dx = \int x \dot{A}_+ \, dx = 0, \quad \langle x \rangle^{\mu+\varepsilon} A_+ \in L^1 \quad \text{for } \mu \geq 0. \tag{3.93}$$

Then $(3.63)_\mu$ holds.

Proof. The high frequency part of (A_+, \dot{A}_+) is controlled by (3.61), (3.62) and it is sufficient to consider $(A_{+<}, \dot{A}_{+<})$, although in some cases the high frequency parts are also controlled by (3.91), (3.92).

We first consider $x A_+$. For $-1/2 \leq \mu < 1$, we estimate

$$\| \omega^{-1/2-\mu} x A_+ \|_2 \leq C \| x A_+ \|_{3/(2+\mu)} \tag{3.94}$$

by Lemma 2.1. For $\mu \leq -1/2$, we estimate simply

$$\| \omega^{-1/2-\mu} x A_{+<} \|_2 \leq C \| x A_+ \|_2. \tag{3.95}$$

We next consider A_+ and $x\dot{A}_+$ together and we use A to denote either of them. For $-1 < \mu < 0$, we estimate

$$\| \omega^{-3/2-\mu} A \|_2 \leq C \| A \|_{3/(3+\mu)} \tag{3.96}$$

by Lemma 2.1. For $\mu \geq 0$, we estimate

$$\begin{aligned} |\xi|^{-\mu-\varepsilon} |\widehat{A}(\xi)| &= (2\pi)^{-3/2} |\xi|^{-\mu-\varepsilon} \left| \int dx (\exp(-ix\xi) - 1) A(x) \right| \\ &\leq C \| |x|^{\mu+\varepsilon} A \|_1 \end{aligned} \tag{3.97}$$

for $0 \leq \mu + \varepsilon \leq 1$. The required estimate then follows from (3.61), (3.62), (3.91) and (3.93).

We finally consider \dot{A}_+ . For $\mu < 0$, we apply the previous result with A replaced by \dot{A}_+ and μ replaced by $\mu - 1$. For $\mu \geq 0$, we estimate

$$\begin{aligned}
 |\xi|^{-1-\mu-\varepsilon} |\widehat{A}(\xi)| &= (2\pi)^{-3/2} |\xi|^{-1-\mu-\varepsilon} \int dx (\exp(-ix\xi) - 1 - ix\xi) \dot{A}_+(x) \\
 &\leq C \| |x|^{1+\mu+\varepsilon} \dot{A}_+ \|_1 \tag{3.98}
 \end{aligned}$$

for $0 \leq \mu + \varepsilon \leq 1$. The required estimate then follows from (3.61), (3.91), (3.93). \square

4 Wave operators and asymptotics for (u, A)

In this section we complete the construction of the wave operators for the system (1.1), (1.2) and we derive asymptotic properties of solutions in their range. The construction relies in an essential way on Propositions 3.1 and 3.2. So far we have worked with the system (1.20) for (w, s) and the first task is to reconstruct the phase φ . Corresponding to $S = s_0 + s_1$, we define $\phi = \varphi_0 + \varphi_1$ where

$$\varphi_0 = - \int_1^t dt' t'^{-1} B_L(w_0(t'), w_0(t')) , \tag{4.1}$$

$$\varphi_1 = - \int_t^\infty dt' (2t'^2)^{-1} |s_0(t')|^2 + 2 \int_t^\infty dt' t'^{-1} B_L(w_0(t'), w_1(t')) , \tag{4.2}$$

so that $s_0 = \nabla\varphi_0$ and $s_1 = \nabla\varphi_1$.

Let now (w, s) be the solution of the system (1.20) constructed in Proposition 3.1 and let $(q, \sigma) = (w, s) - (W, S)$. We define

$$\begin{aligned}
 \psi &= - \int_t^\infty dt' (2t'^2)^{-1} (\sigma \cdot (\sigma + 2S) + s_1 \cdot (s_1 + 2s_0)) (t') \\
 &+ \int_t^\infty dt' t'^{-1} (B_L(q, q) + 2B_L(W, q) + B_L(w_1, w_1)) (t') \tag{4.3}
 \end{aligned}$$

which is taylored to ensure that $\nabla\psi = \sigma$, given the fact that s_0, s_1 and σ are gradients. The integral is easily seen to converge in \dot{H}^1 (see I.8.4), and to satisfy

$$\| \nabla\psi \|_2 = \| \sigma \|_2 \leq C t^{-\lambda_0} . \tag{4.4}$$

Finally we define $\varphi = \phi + \psi$ so that $\nabla\varphi = s$, and (w, φ) solves the system (1.18). For more details on the reconstruction of φ from s , we refer to Section 8 of I.

We can now define the wave operators for the system (1.1), (1.2) as follows. We start from the asymptotic state (u_+, A_+, \dot{A}_+) for (u, A) . We define $w_+ = Fu_+$, we define B_0 by (1.4) (1.13), namely

$$A_0 = \dot{K}(t) A_+ + K(t) \dot{A}_+ = t^{-1} D_0 B_0 ,$$

and we define (W, S) by (3.25)–(3.29), (3.31).

We next solve the system (1.20) with infinite initial time by Propositions 3.1 and 3.2 and we reconstruct φ from s as explained above, namely $\varphi = \varphi_0 + \varphi_1 + \psi$ with φ_0, φ_1 and ψ defined by (4.1), (4.2), (4.3) with $(q, \sigma) = (w, s) - (W, S)$. We finally substitute (w, φ) thereby obtained into (1.11), (1.3) thereby obtaining a solution (u, A) of the system (1.1), (1.2). The wave operator is defined as the map $\Omega : (u_+, A_+, \dot{A}_+) \rightarrow (u, A)$.

In order to state the regularity properties of u that follow in a natural way from the previous construction, we introduce appropriate function spaces. In addition to the operators $M = M(t)$ and $D = D(t)$ defined by (1.8), (1.9), we introduce the operator

$$J = J(t) = x + it \nabla, \tag{4.5}$$

the generator of Galilei transformations. The operators M, D, J satisfy the commutation relation

$$i M D \nabla = J M D. \tag{4.6}$$

For any interval $I \subset [1, \infty)$ and any $k \geq 0$, we define the space

$$\begin{aligned} \mathcal{X}^k(I) &= \left\{ u : D^* M^* u \in \mathcal{C}(I, H^k) \right\} \\ &= \left\{ u : \langle J(t) \rangle^k u \in \mathcal{C}(I, L^2) \right\} \end{aligned} \tag{4.7}$$

where $\langle \lambda \rangle = (1 + \lambda^2)^{1/2}$ for any real number or self-adjoint operator λ and where the second equality follows from (4.6).

We now collect the information obtained for the solutions of the system (1.1), (1.2) and state the main result of this paper as follows.

Proposition 4.1. *Let $1 < k < 3/2 < \ell$. Let $\mu, \lambda_0, \lambda, \beta_0, \beta$ and k_+ satisfy the conditions (3.78)–(3.83).*

Let $u_+ \in FH^{k_+}$, let $w_+ = Fu_+$ and $a_+ = |w_+|_{k_+}$. Let (A_+, \dot{A}_+) satisfy (2.10), (3.62), (3.63) $_{\mu}$. Let (W, S) be defined by (3.25)–(3.29), (3.31). Then

- (1) *There exists $T, 1 \leq T < \infty$ such that the auxiliary system (1.20) has a unique solution $(w, s) \in \mathcal{C}([T, \infty), X^{k, \ell})$ satisfying*

$$\| w(t) - W(t) \|_2 \leq C \left(a_+, A_+, \dot{A}_+ \right) t^{-\lambda_0}, \tag{4.8}$$

$$\| \omega^k(w(t) - W(t)) \|_2 \leq C \left(a_+, A_+, \dot{A}_+ \right) t^{-\lambda}, \tag{4.9}$$

$$\| \omega^m(s(t) - S(t)) \|_2 \leq C \left(a_+, A_+, \dot{A}_+ \right) t^{-\lambda_0 + \beta m} \text{ for } 0 \leq m \leq \ell + 1, \tag{4.10}$$

for all $t \geq T$, where the constants $C(a_+, A_+, \dot{A}_+)$ depend on (A_+, \dot{A}_+) through the norms associated with (2.10), (3.62), (3.63) $_{\mu}$.

- (2) *Let $\phi = \varphi_0 + \varphi_1$ be defined by (4.1), (4.2), let $\varphi = \phi + \psi$ with ψ defined by (4.3) and $(q, \sigma) = (w, s) - (W, S)$. Let*

$$u = MD \exp(-i\varphi)w \tag{1.11} \equiv (4.11)$$

and define A by (1.3), (1.4), (1.5). Then $u \in \mathcal{X}^k([T, \infty))$, $(A, \partial_t A) \in \mathcal{C}([T, \infty), H^k \oplus H^{k-1})$, (u, A) solves the system (1.1), (1.2) and u behaves asymptotically in time as $MD \exp(-i\phi)W$ in the sense that it satisfies the following estimates:

$$\| u(t) - M(t) D(t) \exp(-i\phi(t))W(t) \|_2 \leq C(a_+, A_+, \dot{A}_+)t^{-\lambda_0} , \tag{4.12}$$

$$\| |J(t)|^k (\exp(i\phi(t, x/t))u(t) - M(t) D(t) W(t)) \|_2 \leq C(a_+, A_+, \dot{A}_+)t^{-\lambda} , \tag{4.13}$$

$$\| u(t) - M(t) D(t) \exp(-i\phi(t)) W(t) \|_r \leq C(a_+, A_+, \dot{A}_+)t^{-\lambda_0 + (\lambda_0 - \lambda)\delta(r)/k} \tag{4.14}$$

for $0 \leq \delta(r) = (3/2 - 3/r) \leq k$, for all $t \geq T$.

Define in addition

$$A_2 = A - A_0 - A_1(|DW|^2) . \tag{4.15}$$

Then A behaves asymptotically in time as $A_0 + A_1(|DW|^2)$ in the sense that A_2 satisfies the following estimates:

$$\| A_2(t) \|_2 \leq C(a_+, A_+, \dot{A}_+) t^{-\lambda_0 + 1/2} \tag{4.16}$$

$$\| \nabla A_2(t) \|_2 \leq C(a_+, A_+, \dot{A}_+) t^{-2\lambda_0 - 1/2 + (\lambda_0 - \lambda)3/2k} \tag{4.17}$$

$$\| \omega^{2k-1/2} A_2(t) \|_2 \leq C(a_+, A_+, \dot{A}_+) t^{-2\lambda - 2k+1} \tag{4.18}$$

for all $t \geq T$.

- (3) The solution (u, A) also behaves asymptotically as $(MD \exp(-i\phi)W_1, A_0 + A_1(|DW_1|^2))$ in the sense that the estimates (4.12)–(4.14) and (4.16)–(4.18) also hold with W replaced by W_1 (see (3.25)).

Sketch of proof. Part (1) is a restatement of the conclusions of Proposition 3.1 supplemented by (4.4) and follows from Propositions 3.1 and 3.2.

Part (2) follows from Part (1) and is proved in exactly the same way as Part (2) of Proposition I.8.1.

Part (3) is proved in the same way as Part (2). It follows from the fact that the only estimates of W and $q = w - W$ that are used in the proof of Part (2) are (3.45) which also holds for W_1 and (4.8)–(4.9) which also hold for w_2 . In fact, the latter estimates hold for h by Lemma 3.4, especially (3.65), under the assumptions of Proposition 3.2 and follow therefrom for w_2 in a trivial way. \square

Remark 4.1. It may seem surprising that the improved asymptotic form W for w does not give rise to better asymptotic estimates than the simpler form W_1 in the norms (4.12)–(4.14) and (4.16)–(4.18). The reason is that the additional term w_2 is small and gives rise to small contributions in terms of those norms. This does not prevent that term to give a large contribution to the time derivative $\partial_t w$ in (1.20) through the derivative term $t^{-2}\Delta w_2$. That contribution is essential to allow for the solution of the system (1.20) without assuming the support condition. The same phenomenon appears in [8].

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