Ann. Henri Poincaré 4 (2003) 847 – 896

1424-0637/03/050847-50
DOI 10.1007/s00023-003-0149-1

Annales Henri Poincaré

Surface Tension and Wulff Shape for a Lattice Model without Spin Flip Symmetry

T. Bodineau and E. Presutti

Abstract. We propose a new definition of surface tension and check it in a spin model of the Pirogov-Sinai class without symmetry. We study the model at low temperatures on the phase transitions line and prove: (i) existence of the surface tension in the thermodynamic limit, for any orientation of the surface and in all dimensions $d \geq 2$; (ii) the Wulff shape constructed with such a surface tension coincides with the equilibrium shape of the cluster which appears when fixing the total spin magnetization (Wulff problem).

1 Introduction

During the past decade, progress was made in the understanding of the phase segregation starting from microscopic models. To summarize, two approaches prevail to derive the Wulff construction for Ising type models. The first one enables to describe the phenomenon of phase coexistence in two dimensions with an extremely high accuracy, in particular it provides a sharp control of the phase boundaries w.r.t. the Hausdorff distance (see, e.g., [DKS, I1, I2, ISc, Pf, PV2]). The second strategy is much less precise and gives only $\mathbb{L}^1$ estimates; however it can be also implemented in higher dimensions (see, e.g., [ABCP, BCP, BBBP, BBP, Ce, B1, CePi1, BIV1, CePi2]).

Phase segregation occurs in a wide range of physical systems, but the two strategies mentioned above have been mainly implemented in models with symmetry among phases and in some cases, the specific microscopic structure of the interactions has been at the heart of the proofs (duality, FK representation, ferromagnetic inequalities . . .).

The goal of this paper is to extend the $\mathbb{L}^1$-approach to a class of systems without symmetry, which can be studied by the Pirogov-Sinai Theory. The $\mathbb{L}^1$-theory is at first sight not model dependent, it is based on a coarse grained description of the system and provides a general framework to relate the surface tension to $\mathbb{L}^1$-estimates. Nevertheless, its concrete implementation has been restricted to a specific class of models: Bernoulli percolation [Ce], Ising model [B1, BIV1, CePi1] and Potts model [CePi2]. These three instances have a common structure which arises in the FK representation. The coarse graining developed by Pisztora [Pi] played a key role in the derivation of the $\mathbb{L}^1$-approach for the three models above. This hinders the generalization to a broader class of models, since parts of the proof relied on Pisztora's coarse graining and thus on the FK representation. Notice also that the proofs were based on the symmetry of the model and on the ferromagnetic inequalities. In particular the analysis of the surface tension was completely model dependent.

There are some works which deal with surface tension in non-symmetric models [BKL, HKZ1, HKZ2, MMRS], but a general theory of surface tension (including the thermodynamic limit for all slopes) seems still to be missing. In this paper we propose a new definition of surface tension. The advantage is that its existence in the thermodynamic limit for arbitrary slopes of the surface and in all dimensions $d \geqslant 2$, does not rely on the symmetry of the pure phases nor on ferromagnetic inequalities, at least when the Pirogov-Sinai theory can be applied. The validity of the definition is then confirmed by the proof that the Wulff construction using this surface tension actually determines the equilibrium shape of a droplet in the system. The surface tension is characterized by two specific features, a cutoff on the interface fluctuations and the notion of perfect walls. The precise definition and heuristics are postponed to Section 3. The thermodynamic limit of the surface tension is derived by a recursive procedure. The rest of the $\mathbb{L}^1$-approach (including the coarse graining) is presented in Section 6 following the usual scheme.

In the present paper, we focus on a particular model in order to stress the main ideas in the most simple context. We actually believe that the proof holds for a general class of two phase models in the Pirogov-Sinai Theory (see the last paragraph of Subsection 2.1). The liquid/vapour phase coexistence is also the subject of current investigations and it seems possible to generalize our strategy for particles in the continuum with Kac potentials as considered by Lebowitz, Mazel, Presutti [LMP].

The main ideas in this work have been developed in collaboration with Dima Ioffe.

2 Model and main theorem

2.1 The model

We consider a lattice model on $\mathbb{Z}^d$, $d \geqslant 2$, which is made of interacting spins σ_x taking values $\{-1, 1\}$. The interaction depends on a 2^d-body potential defined so that its ground states are the configurations with all spins equal to $+1$ and all spins equal to -1. However the interaction is not invariant under spin flip and the analysis of the Gibbs measures at positive temperatures relies on the Pirogov-Sinai theory and phase coexistence occurs at non zero values of the magnetic field.

We call cell and denote it by $\mathfrak{c}$ a cube in $\mathbb{Z}^d$ of side 2 (meaning that it contains 2^d lattice sites); denoting by $\sigma_{\mathfrak{c}}$ the restriction of σ to $\mathfrak{c}$, we define the cell potential $V(\sigma_{\mathfrak{c}})$ as equal to 0 if $\sigma_{\mathfrak{c}} \equiv 1$ and $\sigma_{\mathfrak{c}} \equiv -1$, otherwise $V(\sigma_{\mathfrak{c}})$ is equal to the number of $+1$ spins present in $\sigma_{\mathfrak{c}}$. The Hamiltonian in the finite set Λ with b.c. σ_{Λ^c} is then

$$H_h^{\sigma_{\Lambda^c}}(\sigma_\Lambda) = \sum_{\mathfrak{c}\cap\Lambda\neq\emptyset} V(\sigma_{\mathfrak{c}}) - \sum_{x\in\Lambda} h\sigma_x \,.$$

If σ_{Λ^c} is the restriction to Λ^c of a configuration $\bar\sigma$ we will also write $H_h^{\bar\sigma}(\sigma_\Lambda)$.

The Gibbs measure associated to the spin system with boundary conditions $\bar\sigma$ is

$$\mu_{\beta,h,\Lambda}^{\bar\sigma}(\sigma_\Lambda) = \frac{1}{Z_{\beta,h,\Lambda}^{\bar\sigma}} \exp\left(-\beta H_h^{\bar\sigma}(\sigma_\Lambda)\right),$$

where β is the inverse of the temperature and $Z_{\beta,h,\Lambda}^{\bar\sigma}$ is the partition function. If $\bar\sigma$ is uniformly equal to 1 (resp -1), the Gibbs measure will be denoted by $\mu_{\beta,h,\Lambda}^{+}$ (resp $\mu_{\beta,h,\Lambda}^{-}$).

Classical Pirogov-Sinai theory ensures that for any β large enough, there exists a value of the magnetic field $h(\beta)$ such that a first order phase transition is located on the curve $(\beta, h(\beta))$. In particular on the phase coexistence curve, one can define (see Theorem 4.2 below) two distinct Gibbs measures $\mu_{\beta,h(\beta)}^{+}$ and $\mu_{\beta,h(\beta)}^{-}$ which are measures on the space $\{\pm1\}^{\mathbb{Z}^d}$. They are obtained by taking the thermodynamic limit of $\mu_{\beta,h(\beta),\Lambda}^{+}$ and $\mu_{\beta,h(\beta),\Lambda}^{-}$. Each of these measures represents a pure state. The averaged magnetization in each phase is denoted by

$$m_\beta^+ = \mu_{\beta,h(\beta)}^{+}(\sigma_0) \quad \text{and} \quad m_\beta^- = \mu_{\beta,h(\beta)}^{-}(\sigma_0)\,. \tag{2.1}$$

Observe that if we replace cells by bonds we recover (modulo an additive constant) the energy of the nearest neighbor Ising model. Our system is in our intentions the simplest modification of the nearest neighbor Ising model where the spin flip symmetry is broken but the ground states are kept unchanged. This choice has been to give up any attempt of generality and instead to focus on a particular model, where the main ideas are not obscured by too many technicalities. Nevertheless, we believe our analysis extends to finite range, many body Hamiltonians of the form

$$\sum_{X\subset\Lambda} V_X(\sigma_X)$$

provided they are into the Pirogov-Sinai class and under the assumptions that the potentials V_X are symmetric and translational invariant, with ground states the constant configurations. The symmetry assumption means

$$\text{for all } X, \qquad V_{\mathcal{R}X}\Big((\mathcal{R}\sigma)_{\mathcal{R}X}\Big) = V_X(\sigma_X)\,, \tag{2.2}$$

where $\mathcal{R}$ denotes the symmetry w.r.t. the origin and $(\mathcal{R}\sigma)_j = \sigma_{(\mathcal{R})^{-1}(j)}$. We will pursue the discussion on possible extensions and open questions in Subsection 3.4.

2.2 Phase coexistence

The phenomenon of phase segregation will be described in the framework of the $\mathbb{L}^1$-approach. Let us first recall the functional setting. On the macroscopic level, the system is confined in the torus $\widehat{\mathbb{T}} = [0,1]^d$ of $\mathbb{R}^d$ and a macroscopic configuration where the pure phases coexist is described by a function v taking values $\{m_\beta^-, m_\beta^+\}$. The function v should be interpreted as a signed indicator representing the local order parameter: if $v_r = m_\beta^+$ for some $r \in \widehat{\mathbb{T}}$, then the system should be locally at r in equilibrium in the $+$ phase.

To define the macroscopic interfaces, i.e., the boundary of the set $\{v = m_\beta^-\}$, a convenient functional setting is the space $\mathrm{BV}(\widehat{\mathbb{T}}, \{m_\beta^-, m_\beta^+\})$ of functions of bounded variation with values $m_\beta^\pm$ in $\widehat{\mathbb{T}}$ (see [EG] for a review). For any $v \in \mathrm{BV}(\widehat{\mathbb{T}}, \{m_\beta^-, m_\beta^+\})$, there exists a generalized notion of the boundary of the set $\{v = m_\beta^-\}$ called reduced boundary and denoted by $\partial^* v$. If $\{v = m_\beta^-\}$ is a regular set, then $\partial^* v$ coincides with the usual boundary ∂v.

The interfacial energy associated to a domain is obtained by integrating the surface tension along the boundary of the domain. The surface tension is a function $\tau_\beta : \mathbb{S}^{(d-1)} \to \mathbb{R}^+$ on the set of unit vectors $\mathbb{S}^{(d-1)}$, which in our model has the expression specified in Section 3. The Wulff functional $\mathcal{W}_\beta$ is defined in $\mathbb{L}_1(\widehat{\mathbb{T}})$ as follows

$$\mathcal{W}_\beta(v) = \begin{cases} \int_{\partial^* v} \tau_\beta(\vec{n_x})\, d\mathcal{H}_x, & \text{if} \quad v \in \mathrm{BV}(\widehat{\mathbb{T}}, \{m_\beta^-, m_\beta^+\}), \\ \infty\,, & \text{otherwise.} \end{cases} \tag{2.3}$$

To any measurable subset A of $\widehat{\mathbb{T}}$, we associate the function $\mathbb{1}_A = m_\beta^+ 1_{A^c} + m_\beta^- 1_A$ and simply write $\mathcal{W}_\beta(A) = \mathcal{W}_\beta(\mathbb{1}_A)$.

Fix an interval $[m_1, m_2]$ included in (m_β^-, m_β^+). The equilibrium crystal shapes are the solutions of the Wulff variational problem, i.e., they are the minimizers of the functional $\mathcal{W}_\beta$ under a volume constraint

$$\min \left\{ \mathcal{W}_\beta(v) \;\Big|\; v \in \mathrm{BV}(\widehat{\mathbb{T}}, \{m_\beta^-, m_\beta^+\}), \qquad m_1 \leqslant \int_{\widehat{\mathbb{T}}} v_x\, dx \leqslant m_2 \right\}. \tag{2.4}$$

Let $\mathcal{D}(m_1, m_2)$ be the set of minimizers of (2.4).

2.3 Local magnetization

The correspondence between the microscopic quantities and the functional setting described above can be obtained only after some averaging procedure, as the one we are going to describe. We first need a few extra notations. Let $\mathcal{B}^{(K)}$, $K = 2^k$, $k \in \mathbb{N}$, be the partition of $\mathbb{Z}^d$ into cubes $\mathbb{B}_K$: the seed of the partition is

$$\mathbb{B}_K(0) = \left\{ x \in \mathbb{Z}^d : 0 \leqslant x_i < K, \quad i = 1, \ldots, d \right\}$$

and the other cubes of $\mathcal{B}^{(K)}$ are obtained by translations by integer multiples of K in all coordinate directions. The sequence $\mathcal{B}^{(K)}, k \in \mathbb{N}$, is then a compatible sequence of partitions of $\mathbb{R}^d$, namely each cube $\mathbb{B}_N \in \mathcal{B}^{(N)}$ is union of cubes $\mathbb{B}_K$ in $\mathcal{B}^{(K)}$, if $K = 2^k \leqslant N = 2^n$.

Given $K = 2^k$, we denote by $\mathbb{B}_K(x)$ the box in $\mathcal{B}^{(K)}$ which contains the point $x \in \mathbb{Z}^d$. The local averaged magnetization is defined by

$$\mathcal{M}_K(x) = \frac{1}{|\mathbb{B}_K(x)|} \sum_{y \in \mathbb{B}_K(x)} \sigma_y \,. \tag{2.5}$$

By abuse of notation, $\mathcal{M}_K(\cdot)$ can be viewed also as a piecewise constant function on $\mathbb{R}^d$.

For simplicity the microscopic region Λ is chosen as $\mathbb{B}_N(0)$ and, imposing periodic b.c. it becomes the torus $\mathbb{T}_N$. We call ψ_N the map from $\mathbb{T}_N$ onto $\widehat{\mathbb{T}}$, obtained by shrinking by a factor $1/N$. We then define the local magnetization

$$\mathcal{M}_{N,K}(r) = \mathcal{M}_K\big(\psi_N^{-1}(r)\big), \qquad r \in \widehat{\mathbb{T}} \tag{2.6}$$

which is a function on $\widehat{\mathbb{T}}$ piecewise constant on the boxes $\psi_N(\mathbb{B}_K)$, $\mathbb{B}_K \subset \mathbb{T}_N$. The local order parameter $\mathcal{M}_{N,K}$ characterizes the local equilibrium. The total magnetization in $\mathbb{T}_N$ is simply denoted by $\mathcal{M}_N$.

We can now state a result on phase coexistence.

Theorem 2.1. *There exists $\beta_0 > 0$ such that for any $\beta > \beta_0$ and $[m_1, m_2] \subset (m_\beta^-, m_\beta^+)$ (with $m_1 < m_2$), the following holds: for every $\delta > 0$ there is a scale $K_0 = K_0(\beta, \delta)$ such that for any $K \geqslant K_0$*

$$\lim_{N\to\infty} \mu_{\beta,h(\beta),N}\left(\inf_{v \in \mathcal{D}(m_1,m_2)} \|\mathcal{M}_{N,K} - v\|_1 \leqslant \delta \;\middle|\; m_1 \leqslant \mathcal{M}_N \leqslant m_2 \right) = 1 \,,$$

where $\mathcal{D}(m_1, m_2)$ denotes the set of the equilibrium crystal shapes (2.4) (*where the surface tension is the one defined in Section* 3) *and $\mu_{\beta,h(\beta),N}$ is the Gibbs measure on $\mathbb{T}_N$ with periodic boundary conditions.*

3 Surface tension

For any given unit vector $\vec{n} = (n_1, \ldots, n_d)$, we are going to define the surface tension $\tau_\beta(\vec{n})$ in the direction orthogonal to $\vec{n}$. Contrary to the Ising model, the lack of symmetry between the two pure phases requires a more complex definition of surface tension which relies on two new features: a cutoff of the interface fluctuations and the introduction of perfect walls.

3.1 Interface fluctuations cutoff

We associate to any unit vector $\vec{n} = (n_1, \ldots, n_d)$ a coordinate direction $j \in \{1, \ldots, d\}$ in such a way that $n_i \leqslant n_j$ for all i while $n_i < n_j$, for any $i > j$. For notational simplicity suppose $j = d$, the other cases are treated similarly. We set

$$\Lambda_{\ell,m}(\vec{n}) = \left\{x \in \mathbb{Z}^d, \qquad \forall i < d, \quad -\ell \leqslant x_i \leqslant \ell;\ -m \leqslant (x \cdot \vec{n}) \leqslant m\right\}. \tag{3.1}$$

As $\vec{n}$ is fixed throughout this section, we will drop it from the notation.

The surface tension $\tau_\beta(\vec{n})$ will be the thermodynamic limit of ratios of partition functions defined on subsets of the slab $\Lambda_{L,\frac{11\varepsilon}{10}L}$. The limit will be taken for appropriate sequences of the parameters (L, ε), in particular we require L and $(\varepsilon/10)L$ to be in $\{2^n, n \in \mathbb{N}\}$. We will first introduce the partition function with mixed boundary conditions.

We want to impose $+$ and $-$ boundary conditions on top and bottom of our domains; it will be convenient to leave some freedom on their exact location and with this in mind we introduce the notion of barriers. A *barrier* in a slab $\Lambda_{\ell,m}$ is a connected set of cells in $\Lambda_{\ell,m}$ which connects the faces of $\Lambda_{\ell,m}$ parallel to $\vec{e}_d$ and it is such that its complement in $\Lambda_{\ell,m}$ is made of two distinct components which are not $\star$-connected. Let then $\mathcal{C}^+$ and $\mathcal{C}^-$ be two barriers in $\Lambda_{L,\frac{\varepsilon}{10}L} + \varepsilon L\vec{e}_d$ and $\Lambda_{L,\frac{\varepsilon}{10}L} - \varepsilon L\vec{e}_d$. The subset of $\Lambda_{L,\frac{11\varepsilon}{10}L}$ lying between $\mathcal{C}^+$ and $\mathcal{C}^-$ is denoted by $\Lambda(\mathcal{C}^+, \mathcal{C}^-)$. The mixed boundary conditions $\bar{\sigma}^\pm$ outside $\Lambda(\mathcal{C}^+, \mathcal{C}^-)$ are defined as follows

$$\forall x \notin \Lambda(\mathcal{C}^+, \mathcal{C}^-), \qquad \bar{\sigma}^\pm_x = \begin{cases} +1, & \text{if} \quad (x \cdot \vec{n}) \geqslant 0\,, \\ -1, & \text{if} \quad (x \cdot \vec{n}) < 0\,. \end{cases}$$

We denote by $\mathcal{S}^+$ (resp $\mathcal{S}^-$) the set of spin configurations for which there is a barrier included in $\Lambda_{L,\frac{\varepsilon}{10}L} + \frac{\varepsilon L}{2}\vec{e}_d$ (resp $\Lambda_{L,\frac{\varepsilon}{10}L} - \frac{\varepsilon L}{2}\vec{e}_d$) where all spins are equal to 1 (resp -1). Finally, we introduce the following constrained partition function on $\Lambda(\mathcal{C}^+, \mathcal{C}^-)$ with mixed boundary conditions (see Figure 1)

$$Z^{\mathcal{C}^+,\mathcal{C}^-}_{L,\varepsilon}(\mathcal{S}^+, \mathcal{S}^-) = \sum_{\sigma \in \{\pm 1\}^{\Lambda(\mathcal{C}^+,\mathcal{C}^-)}} \mathbf{1}_{\{\sigma \in \mathcal{S}^+ \cap \mathcal{S}^-\}} \exp\left(-\beta H^{\bar{\sigma}^\pm}_h(\sigma)\right). \tag{3.2}$$

The barriers $\mathcal{S}^+, \mathcal{S}^-$ act as a cutoff of the interface fluctuations: they decouple the interface from the boundary conditions outside $\Lambda(\mathcal{C}^+, \mathcal{C}^-)$. In the following, we will explain the role of this screening.

3.2 Perfect walls

A perfect wall is such that its contribution to the finite volume corrections to the pressure is infinitesimal w.r.t. the area of its surface, best examples are walls defined by periodic boundary conditions. Under suitable assumptions on the interaction it is in fact well known that with periodic boundary conditions the corrections to the pressure decay exponentially with the size of the box. Periodic boundary conditions are however not useful in our context, because we want to impose one of the two phases on some of the walls; but, as we are going to see, it is possible to define some sort of periodic conditions on single walls of the container.

We start by defining a symmetric partition of $\mathbb{Z}^d$ by the hyperplane Σ orthogonal to $\vec{n}$ and containing 0. Let us first suppose that the orientation $\vec{n}$ is such that $\Sigma \cap \mathbb{Z}^d = \{0\}$. We then set

$$\mathbb{Z}^d_+ = \left\{ x \in \mathbb{Z}^d \,\middle|\, \quad x_d \geqslant -\sum_{i=1}^{d-1} \frac{n_i}{n_d} x_i \right\} \setminus \{0\}, \qquad \mathbb{Z}^d_- = \mathbb{Z}^d \setminus (Z^d_+ \cup \{0\}) \,.$$

Then

$$\mathbb{Z}^d = \mathbb{Z}^d_+ \cup \mathbb{Z}^d_- \cup \{0\}, \qquad \mathbb{Z}^d_- = \mathcal{R}(\mathbb{Z}^d_+) \tag{3.3}$$

where $\mathcal{R}$ is the symmetry w.r.t. 0.

If there are sites in $\mathbb{Z}^d \cap \Sigma$ besides 0, we split them between $\mathbb{Z}^d_+$ and $\mathbb{Z}^d_-$ in such a way that (3.3) is preserved. Notice first that if $x \in \Sigma \cap \mathbb{Z}^d$, also $\mathcal{R}x \in \Sigma \cap \mathbb{Z}^d$. Then if $x = (x_1, \ldots, x_d) \neq 0$, we call i the first integer so that $x_i \neq 0$ (i.e., $x_1 = \cdots = x_{i-1} = 0$) and we put $x \in \mathbb{Z}^d_+$ if $x_i > 0$ and $x \in \mathbb{Z}^d_-$ otherwise. Thus

$$\begin{aligned}\mathbb{Z}^d_+ = \bigcup_{k=1}^{d} \quad \Bigg\{ \quad & \left\{ x \in \mathbb{Z}^d \,\middle|\, \forall i < k,\ x_i = 0,\ x_k > 0,\ x_d \geqslant -\sum_{j=1}^{d-1} \frac{n_j}{n_d} x_j \right\} \\ & \cup \left\{ x \in \mathbb{Z}^d \,\middle|\, \forall i < k,\ x_i = 0,\ x_k < 0,\ x_d > -\sum_{j=1}^{d-1} \frac{n_j}{n_d} x_j \right\} \Bigg\},\end{aligned} \tag{3.4}$$

A drawback of the definition is that for $\vec{n}$ oriented along one of the axis of coordinates the bottom of $\mathbb{Z}^d_+$ is not flat. This could be avoided at the price of considering a more complicated mapping than the simple symmetry w.r.t. 0.

We now proceed in defining the reflected Hamiltonian in $\mathbb{Z}^d$. The idea is to use $\mathcal{R}$ in order to glue together different regions touching the surface Σ so that if, for instance, $x \in \mathbb{Z}^d_+$ interacts across Σ with $y \in \mathbb{Z}^d_-$ then x will now interact with $\mathcal{R}(y) \in \mathbb{Z}^d_+$. As the energy is defined in terms of cells, this can be easily achieved by introducing a new set of cells $\{\mathfrak{c}\}^R$.

Cells which are entirely contained either in $\mathbb{Z}^d_+$, or in $\mathbb{Z}^d_-$ or in $B = \{-1, 0, 1\}^d$ are unchanged. Instead any cell $\mathfrak{c}$ containing sites both in $\mathbb{Z}^d_+$ and in $\mathbb{Z}^d_-$ is replaced by

$$\mathfrak{c} \to \begin{cases} \mathfrak{c}^+ = (\mathfrak{c} \setminus \mathbb{Z}^d_-) \cup \mathcal{R}(\mathfrak{c} \cap \mathbb{Z}^d_-) \\ \mathfrak{c}^- = (\mathfrak{c} \setminus \mathbb{Z}^d_+) \cup \mathcal{R}(\mathfrak{c} \cap \mathbb{Z}^d_+) \end{cases} . \tag{3.5}$$

Notice that both cells $\mathfrak{c}$ and $\mathcal{R}(\mathfrak{c})$ generate the same pair $\mathfrak{c}^{\pm}$, so that the "total number" of old and new cells is the same.

Extending the definition of $V(\sigma_{\mathfrak{c}})$ to the new set of cells, the reflected Hamiltonian is then

$$H_{h,\Lambda}^{R,\sigma_{\Lambda^c}}(\sigma) = \sum_{\substack{\mathfrak{c}\in\{\mathfrak{c}\}^R\\ \mathfrak{c}\cap\Lambda\neq\emptyset}} V(\sigma_{\mathfrak{c}}) - h\sum_{x\in\Lambda}\sigma_x\ . \tag{3.6}$$

We will always consider regions which do not contain B, so that the spins in B will act as boundary conditions: thus the structure of cells entirely contained in B is unimportant.

In preparation to the definition of the surface tension and using the notation of Subsection 3.1, we define the upper half of $\Lambda(\mathcal{C}^+,\mathcal{C}^-)$ by

$$\Lambda^+(\mathcal{C}^+) = \Lambda(\mathcal{C}^+,\mathcal{C}^-)\cap\mathbb{Z}_+^d\setminus B\,.$$

The partition function with reflection and + boundary conditions outside $\Lambda^+(\mathcal{C}^+)$

$$Z_{L,\varepsilon}^{\mathcal{C}^+,R} = \sum_{\sigma\in\{\pm1\}^{\Lambda^+(\mathcal{C}^+)}} \exp\left(-\beta H_{\Lambda^+(\mathcal{C}^+)}^{(+,R)}(\sigma)\right)\,, \tag{3.7}$$

where the Hamiltonian on the right-hand side is defined in (3.6) with Λ replaced by $\Lambda^+(\mathcal{C}^+)$. Notice that the boundary conditions outside $\Lambda^+(\mathcal{C}^+)$ are imposed also around the center of reflection on $B=\{-1,0,1\}^d$. The partition function $Z_{L,\varepsilon}^{\mathcal{C}^-,R}$ is defined similarly on the lower half, $\Lambda^-(\mathcal{C}^-)$ of $\Lambda(\mathcal{C}^+,\mathcal{C}^-)$ and with $-$ reflected boundary conditions on the top (see Figures 1 and 2).

Let Σ_L be the bottom face of $\Lambda^+(\mathcal{C}^+)$, i.e., the face with the reflected interactions (the side-length of Σ_L is L). As we will see in Lemma 4.7, away from 0 and from its boundaries, Σ_L behaves as a wall with periodic boundary conditions; indeed, the overall contribution of Σ_L to the finite volume corrections to the pressure will be proportional to L^{d-2} which is therefore a "perfect wall" in the sense specified at the beginning of this subsection.

Finally notice that one could also consider a mapping different from the symmetry w.r.t. 0 provided that it respects the topological structure of $\mathbb{Z}^d$ and that most of the points are far apart from their images. This will be made clear in Section 4.

3.3 Definition of the surface tension

We can finally introduce

Definition 3.1. *The surface tension in the direction $\vec{n}$, is defined by*

$$\tau_\beta(\vec{n}) = \liminf_{\varepsilon\to0}\ \liminf_{L\to\infty}\ \inf_{\mathcal{C}^+,\mathcal{C}^-}\ -\frac{(\vec{n}\cdot\vec{e}_d)}{\beta L^{d-1}}\log\frac{Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\mathcal{C}^+,R}\ Z_{L,\varepsilon}^{\mathcal{C}^-,R}}\ , \tag{3.8}$$

where the infimum is taken over the barriers $(\mathcal{C}^+,\mathcal{C}^-)$ in the slabs $\Lambda_{L,\frac{\varepsilon}{10}L}\pm\varepsilon L\vec{e}_d$.

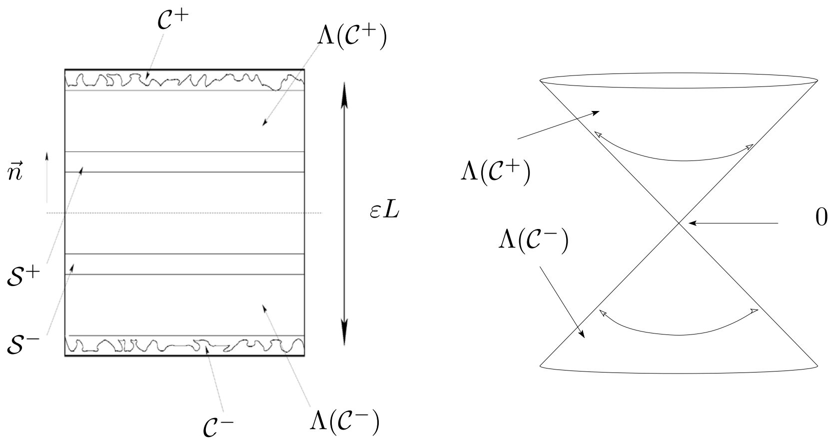

FIGURE 1. On the left, the domain $\Lambda_{L,\varepsilon L}$ is depicted with mixed boundary conditions in the direction $\vec{n}$ and with the interface cutoff. The action of the perfect walls boils down to fold $\Lambda(\mathcal{C}^+)$ and $\Lambda(\mathcal{C}^-)$ around the point 0 (see right picture and also Figure 2).

There are two important points in this definition, one is that the perfect walls should give negligible surface corrections to the pressure. Moreover, due to decay of correlations, the inf over $\mathcal{C}^+, \mathcal{C}^-$ should not matter because of cancellations among numerator and denominator: the barriers $\mathcal{S}^+$ and $\mathcal{S}^-$ screen the effect of the boundary conditions.

The main step towards the derivation of phase coexistence (Theorem 2.1) will be to prove the convergence of the thermodynamic limit for the surface tension:

Theorem 3.1. *For any β large enough (such that the model is in the Pirogov-Sinai regime, see Section* 4)*, the following holds*

$$\tau_\beta(\vec{n}) \quad = \quad \lim_{\varepsilon \to 0} \lim_{L \to \infty} \sup_{\mathcal{C}^+, \mathcal{C}^-} -\frac{(\vec{n} \cdot \vec{e}_d)}{\beta L^{d-1}} \log \frac{Z_{L,\varepsilon}^{\mathcal{C}^+, \mathcal{C}^-}(\mathcal{S}^+, \mathcal{S}^-)}{Z_{L,\varepsilon}^{\mathcal{C}^+,R} \, Z_{L,\varepsilon}^{\mathcal{C}^-,R}}, \tag{3.9}$$

where the supremum is taken over the barriers $(\mathcal{C}^+, \mathcal{C}^-)$ in the slabs $\Lambda_{L,\frac{\varepsilon}{10}L} \pm \varepsilon L \vec{e}_d$. In (3.9)*, the supremum can also be replaced by an infimum.*

The derivation of Theorem 3.1 and of the properties of the surface tension is postponed to Section 5.

3.4 Heuristics on the surface tension

We are going to discuss heuristically the representation of the surface tension and explain the choice of the perfect walls and of the cutoff. We believe that the ultimate justification for Definition 3.1 is to be the surface tension for which Theorem 2.1 is valid.

Let us start by a rough expansion of $\log Z_L^{+,-}$, which denotes the partition function on the cube $\Delta_L = \{-L, \dots, L\}^d$ with mixed boundary conditions in the direction $\vec{n}$.

$$\log Z_L^{+,-} = \frac{L^d}{2}(\mathcal{P}^+ + \mathcal{P}^-) + \frac{L^{d-1}}{(\vec{n}\cdot\vec{e}_d)}\tau_\beta(\vec{n}) + (\tau_{bd}^+ + \tau_{bd}^-)\, d\, L^{d-1} + O(L^{d-2})\,. \tag{3.10}$$

The first term is of volume order and corresponds to the pressures of the different pure phases $\mathcal{P}^+$ and $\mathcal{P}^-$ (which are equal on the curve of phase coexistence, see Lemma 4.2). The surface tension $\tau_\beta(\vec{n})$ arises at the next order, but there are as well other terms of order L^{d-1} which can be interpreted as surface energies due to the boundary conditions. The lack of symmetry of our model implies that the surface energy τ_{bd}^+ produced by the + boundary conditions differs from the surface energy τ_{bd}^- produced by the − boundary conditions.

In order to extract the surface tension factor, one has to compensate not only the bulk term, but also the surface energies τ_{bd}^+ and τ_{bd}^-. In a symmetric case (e.g., the Ising model) $\tau_{bd}^+ = \tau_{bd}^-$ therefore the partition function in Δ_L with + boundary conditions is the appropriate normalization factor. As this is no longer the case for non-symmetric models, the following alternative definition seems to be the most natural

$$\tau_\beta^\star(\vec{n}) = \lim_{L\to\infty} -\frac{(\vec{n}\cdot\vec{e}_d)}{L^{d-1}}\,\log\frac{Z_L^{+,-}}{\sqrt{Z_L^+}\,\sqrt{Z_L^-}}\,. \tag{3.11}$$

Notice that this representation of the surface tension would also require an assumption on the potential similar to (2.2) in order to produce exact cancellations between the numerator and the denominator.

The representation (3.11) of the surface tension is the most commonly used, nevertheless, to our best knowledge, the existence of its thermodynamic limit is not known in general. The surface tension can be studied for different types of models, in particular, let us mention the Ashkin-Teller model [Ve], the Blume Capel model [HK], the Potts model at the critical temperature [MMRS, LMR] and general 3D lattice models [HKZ1, HKZ2]. Depending on the dimension, the results are of different nature.

In 2 dimensions, the interface has a uni-dimensional structure and a very accurate control can be obtained by using renewal theory. In particular it should be possible to derive in a general context a complete expansion of the right-hand side of (3.11) which would include the Ornstein-Zernike corrections[1]. Such results would also provide a description of the fluctuations of the interface. We refer the reader to the paper by Hryniv and Kotecky [HK] for an implementation of these methods in the case of Blume-Capel model (see also [Al, CIV]).

[1]Private communication by D. Ioffe.

In dimension 3 or higher, if $\vec{n}$ coincides with one of the axis, the interface generated by the Dobrushin conditions is rigid and an extremely accurate description of the non-translation invariant Gibbs states can be obtained. As a byproduct of this description, (3.11) can be derived for a broad class of models (see Holicky, Kotecky, Zaradhnik [HKZ1, HKZ2]; Messager, Miracle-Solé, Ruiz, Shlosman [MMRS]). However a derivation of (3.11) in dimensions larger or equal to 3 for general slopes $\vec{n}$ seems still to be missing. In general, the ground states of tilted interfaces are degenerated, this complicates seriously the implementation of a perturbative approach of the thermodynamic limit (3.11).

The representation (3.8) of the surface tension was motivated by the Wulff construction and it has been designed primarily to prove the phase coexistence (Theorem 2.1). The first step to evaluate the surface energy of a droplet is to decompose the interface of the droplet and to estimate locally the surface tension. As the system is random, one is lead to consider partition functions with mixed boundary conditions on arbitrary domains of the type $\Lambda(\mathcal{C}^+, \mathcal{C}^-)$ and not only on regular sets like Δ_L. Locally, the occurrence of an interface means a term like the numerator of (3.8) can be factorized from the global partition function. At this point, the local surface tension factor is extracted from the global partition function by removing the numerator of (3.8) and replacing it instead by the denominator of (3.8). In (3.11), the cancellation of the terms corresponding to the boundary surface tension imposes to choose symmetric domains in the denominator. This constraint is too stringent to apply the procedure previously described to arbitrary domains. The perfect walls provide an alternative way to control the surface order corrections without using symmetry.

The second important feature of Definition 3.1 is the interface fluctuation cutoff. The Pirogov-Sinai theory describes accurately the bulk phenomena in a low temperature regime, nevertheless it cannot be applied directly to study Gibbs measures with mixed boundary conditions. The cutoff decouples the interface from the boundary conditions and therefore enables us to control the dependence between the surface tension and the domain shapes. In fact, the problem in the domain between $\mathcal{C}^+$ and $\mathcal{S}^+$ (resp $\mathcal{S}^-$ and $\mathcal{C}^-$) is set in the regime associated to the pure phase with $+$ (resp. $-$) boundary conditions where again cluster expansion applies.

The derivation of the thermodynamic limit (Theorem 3.1) relies on a recursive procedure which is reminiscent of the proof of the Wulff construction. The basic idea is to approximate the interface on large scales by using the Definition 3.1 on smaller scales. Concretely, the energy in the small regions along the interface is evaluated by pasting the a priori estimates provided by Definition 3.1. The iteration is possible thanks to the very loose structure of the definition of the surface tension. The limit w.r.t. ε has no impact on the value of the surface tension, the main motivation is technical: it is useful in the iteration procedure and afterwards in the completion of Theorem 2.1.

We are going now to compare the representations $\tau_\beta(\vec{n})$ and $\tau_\beta^\star(\vec{n})$ of the surface tension. According to Theorem 3.1 the convergence (3.9) is uniform over the domains $\Lambda(\mathcal{C}^+, \mathcal{C}^-)$ and thus it is enough to define the surface tension on regular domains of the type Δ_L. Furthermore, the perfect walls are such that

$$\lim_{L\to\infty} \frac{1}{L^{d-1}} \log \frac{\sqrt{Z_L^+}\sqrt{Z_L^-}}{Z_L^{+,R}\, Z_L^{-,R}} = 0\,.$$

It remains only to analyze the role of the cutoff of interface fluctuations. Definition 3.1 would coincide with (3.11) if the following holds

$$\lim_{L\to\infty} \frac{1}{L^{d-1}} \log \frac{Z_L^{+,-}(\mathcal{S}^+, \mathcal{S}^-)}{Z_L^{+,-}} = \lim_{L\to\infty} \frac{1}{L^{d-1}} \log \mu^{+,-}_{\beta,\Delta_L}(\mathcal{S}^+, \mathcal{S}^-) = 0\,. \tag{3.12}$$

This statement boils down to prove a very weak form of localization of the interface. In fact, a much stronger localization is expected since the fluctuation of the interface are of the order $\sqrt{L}$ in 2D and believed to be at most of the order $\sqrt{\log L}$ in 3D. For the ferromagnetic finite range Ising model and the Kac-Ising models, (3.12) holds and Definition 3.1 of the surface tension coincide with the usual one (3.11).

Since the ingredients used in the proof of Theorem 3.1 are the typical ones of cluster expansion, the extension to more general Pirogov-Sinai models, as those described at the end of Subsection 2.1, seems possible. For more general models several questions remain. In particular, Definition 3.1 does not seem appropriate to deal with periodic ground states. For multi-phase models, the solution of the variational problem is not known and thus a macroscopic description of phase coexistence is a mathematical challenge. The probabilistic point of view is slightly different since one is interested to derive the macroscopic variational problem (without solving it) from the microscopic system. In this case, the difficulties are of two distinct natures: geometric and probabilistic. For a thorough study of the geometric problems we refer the reader to Cerf, Pisztora [CePi2]. For the issues related to the coarse graining and the surface tension, we hope that our approach can provide a step towards the derivation of phase coexistence for multiphase models. Nevertheless, it should be stressed that the interesting phenomena, as boundary layers, occurring in multi-phase models cannot be capture in the $\mathbb{L}^1$-framework. A more refined analysis of the microscopic structure of the interface is necessary to describe these subtle mechanisms (see, e.g., [HK, MMRS, HKZ2]).

4 Peierls estimates, cluster expansion

In this section we will see that notion and procedures of the Pirogov-Sinai theory can be modified to apply when reflecting walls are present. In particular we will derive formulas for the finite volume corrections to the pressure which show that the contribution of the reflecting walls is negligible.

We need to generalize the context considered in the previous section because in the next ones we will have simultaneously several reflecting surfaces $\{\Sigma_i\}$ (introduced to decouple different regions of the whole domain). An example is depicted in Figure 2. These reflecting surfaces are separated in such a way that there will be no interference among them and we can consider each one separately. Let us then call Σ one of them (dropping momentarily the label i) and describe its structure. Σ is the intersection of a hyperplane $\mathcal{H}$ and a coordinate cylinder π with cubic cross section of integer side. The axis of the cylinder is the coordinate direction associated to the normal to Σ, in the sense of Subsection 3.1, and its intersection with Σ, called the center of Σ, is supposed to be in $\mathbb{Z}^d$. We then introduce the set of boundary sites associated to Σ, i.e., the sites close to the border of Σ and to the center of Σ. Defining $\mathbb{Z}^d_\pm$ as the spaces above and below $\mathcal{H}$, in the sense of Subsection 3.2, we call B' the "boundary of Σ" as the sites x of $\mathbb{Z}^d_\pm$ which are $*$ connected to $\mathbb{Z}^d_\mp$ as well as $*$ connected to π^c, if in π, and to π, if in π^c. B is defined as the union of B' with the center of Σ and the sites $*$ connected to it. We then call $\{\mathfrak{c}'\}^R$ the set of all new cells $\mathfrak{c}'$ determined by the reflection through the hyperplane $\mathcal{H}$ which are in π, are not contained in B and differ from original cells.

This refers to the generic surface Σ_i with B_i and $\{\mathfrak{c}'\}^R_i$, we are now resuming the notation with the subscript i. The union of all B_i will be called B while $\{\mathfrak{c}'\}^R$ is the union of all $\{\mathfrak{c}'\}^R_i$. We then define the set of new cells $\{\mathfrak{c}\}^R$, as the collection of $\{\mathfrak{c}'\}^R$ and of all cells which have not been modified by reflections through any of the surfaces Σ_i. Thus $\{\mathfrak{c}\}^R$ are the new cells and $\{\mathfrak{c}\}$ the old ones. The new Hamiltonian is given by the same expression (3.6) but with $\{\mathfrak{c}\}^R$ the above collection of cells. Finally, we set $\mathbb{Z}^{d,R} = \mathbb{Z}^d \setminus B$ and fix hereafter the spins in B. In the sequel Λ will denote regions in $\mathbb{Z}^{d,R}$ and the spins in B will always act as boundary conditions.

The collection $\{\mathfrak{c}\}^R$ defines a new topology, where the nearest neighbor sites of $x \in \mathbb{Z}^d$ is the union of all cells $\mathfrak{c} \in \{\mathfrak{c}\}^R$ which contain x. Without reflection, this reduces to the usual notion in $\mathbb{Z}^d$, where the n.n. sites of x are those $*$ connected to x. It is convenient to add a metric structure, defining the "ball of radius $n \in \mathbb{N}$ and center $x \in \mathbb{Z}^d$", denoted by $K(x,n)$ for the old and, respectively, by $K^R(x,n)$ for the new cells, by setting $K(x,0) = K^R(x,0) = \{x\}$ and

$$K(x,n) = \Big\{ y \in \mathbb{Z}^d : y \in \mathfrak{c}, \mathfrak{c} \cap K(x,n-1) \neq \emptyset, \mathfrak{c} \in \{\mathfrak{c}\} \Big\} \tag{4.1}$$

$$K^R(x,n) = \Big\{ y \in \mathbb{Z}^d : y \in \mathfrak{c}, \mathfrak{c} \cap K^R(x,n-1) \neq \emptyset, \mathfrak{c} \in \{\mathfrak{c}\}^R \Big\} \tag{4.2}$$

The external boundary of Λ in the old and new topology are

$$\begin{aligned} \delta(\Lambda) &= \Big\{ y \in \Lambda^c : y \in \mathfrak{c}, \mathfrak{c} \cap \Lambda \neq \emptyset, \mathfrak{c} \in \{\mathfrak{c}\} \Big\} \\ \delta^R(\Lambda) &= \Big\{ y \in \Lambda^c : y \in \mathfrak{c}, \mathfrak{c} \cap \Lambda \neq \emptyset, \mathfrak{c} \in \{\mathfrak{c}\}^R \Big\} \end{aligned} \tag{4.3}$$

where $\Lambda \subset \mathbb{Z}^{d,R}$ (we recall that B belongs to Λ^c).

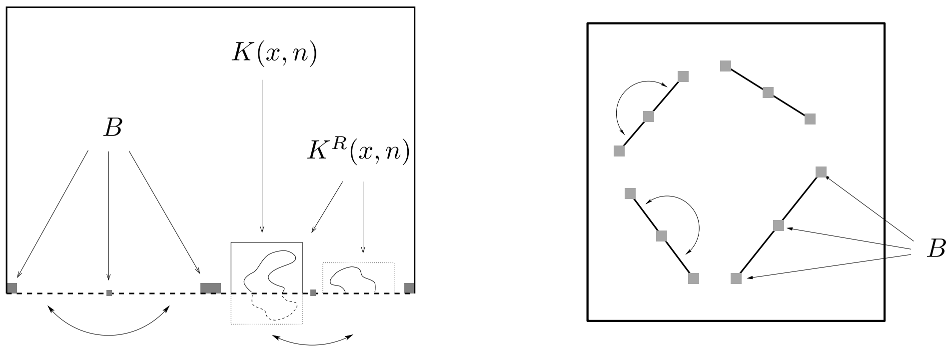

FIGURE 2. The two examples above represent the different types of reflecting surfaces which will be used in this paper. The gray rectangles stand for the location of the boundary conditions B. On the left, a domain with two reflecting surfaces on its bottom face; a reflected contour is also depicted. This type of domain will be used in the analysis of surface tension (Section 5). The domain on the right contains several reflecting surfaces where the structure of the cells is modified (see Subsections 6.3 and 6.4).

The whole analysis in this section is based on a simple geometric property of the collection $\{\mathfrak{c}\}^R$, which is a consequence of the way reflections on a single surface have been defined and the fact that the reflecting surfaces are separated from each other.

Given $x \in \mathbb{Z}^{d,R}$, call $n(x)$ the smallest integer n such that $K^R(x,n) \cap B \neq \emptyset$ and $n'(x)$ the smallest integer n such that $K^R(x,n)$ reaches two distinct reflecting surfaces Σ_i and Σ_j, i.e., contains sites on either side of Σ_i and on either side of Σ_j.

Theorem 4.1. *Suppose that for all $x \in \mathbb{Z}^{d,R}$, $n(x) < n'(x)$, then, for any $n \leqslant n(x)$, there is a bijective map $\mathcal{T}$ from $K(x,n)$ onto $K^R(x,n)$ which transforms bijectively all cells of $\{\mathfrak{c}\}$ in $K(x,n)$ onto the cells of $\{\mathfrak{c}\}^R$ in $K^R(x,n)$. Consequently, for any $\Delta \subset K^R(x,n)$ with also $\delta^R(\Delta) \subset K^R(x,n)$*

$$H_h^{R,\sigma_{\Delta^c}}(\sigma_\Delta) = H_h^{\sigma_{\mathcal{T}^{-1}(\Delta^c)}}(\sigma_{\mathcal{T}^{-1}(\Delta)}), \qquad Z_{\beta,h,\Delta}^{R,\sigma_{\Delta^c}} = Z_{\beta,h,\mathcal{T}^{-1}(\Delta)}^{\sigma_{\mathcal{T}^{-1}(\Delta^c)}} \tag{4.4}$$

Proof. Since $n < n'(x)$, it is enough to consider a reflection w.r.t. a single surface and modulo a change of variables to work in the framework of Subsection 3.2. Suppose x is in the upper part, $x \in \mathbb{Z}_+^d$, then, by induction on $k \leqslant n$ it is easy to see that $K^R(x,k) = \mathcal{T}(K(x,k))$, where $\mathcal{T}$ is equal to the identity on $K(x,n) \cap \mathbb{Z}_+^d$ and to $\mathcal{R}$ on $K(x,n) \cap \mathbb{Z}_-^d$. We next check that $\mathcal{T}$ is one-to-one. If it was not the case, there would be two distinct sites $y, z \in K(x,n)$ such that $\mathcal{T}(y) = \mathcal{T}(z)$. This would mean that $z = \mathcal{R}(y)$ and, since $K(x,n)$ is a convex set, then 0 would be in $K(x,n)$, which is excluded because $n \leqslant n(x)$. Since $\mathcal{T}$ maps the cells of $\{\mathfrak{c}\}$ in $K(x,n)$ bijectively in those $\{\mathfrak{c}\}^R$ in $K^R(x,n)$, (4.4) follows. The theorem is proved. $\square$

The previous theorem implies that away from the set B, the reflections have no impact on the energy. This will be useful to evaluate the corrections to the pressure in presence of reflected boundaries. The particular structure of the reflecting surfaces will not matter in the sequel, the analysis only relying on the following assumption:

$$\textbf{Assumption:} \text{ For all } x \in \mathbb{Z}^{d,R},\ n(x) < n'(x). \tag{4.5}$$

Having defined the setup, we can now start the analysis which begins by recalling the fundamental notion of contours, adapted to the case of reflecting surfaces.

4.1 Contours

We will refer explicitly to the case of reflections, as underlined by the superscript R; without R the expressions refer to the case without reflections for which the classical proofs apply directly and which can anyway be recovered from our analysis by replacing $\{\mathfrak{c}\}^R$ by $\{\mathfrak{c}\}$.

We define the phase indicator at x, $\eta_x^R(\sigma)$, as equal to 1 (resp. -1) if σ is identically 1 (resp. -1) on all $\mathfrak{c} \ni x$, $\mathfrak{c} \in \{\mathfrak{c}\}^R$; otherwise $\eta_x(\sigma) = 0$.

Calling R-connected two sites x and y if they both belong to a same cell in $\{\mathfrak{c}\}^R$, the spatial supports $\mathrm{sp}(\Gamma)$ of the R contours Γ of σ are the maximal R-connected components of the set $\{\eta_x^R = 0\}$. We will tacitly suppose in the sequel that they are all bounded sets. Let

$$\bar{\Gamma} = \bigcup_{x \in \mathrm{sp}(\Gamma)} K^R(x,2) \tag{4.6}$$

Then the R contours Γ of σ are the pairs $\Gamma = (\bar{\Gamma}, \sigma_{\bar{\Gamma}})$, with $\sigma_{\bar{\Gamma}}$ the restriction of σ to $\bar{\Gamma}$.

Notice that in each R connected component of $\bar{\Gamma} \setminus \mathrm{sp}(\Gamma)$, σ_x is identically equal either to 1 or to -1, while the values outside $\bar{\Gamma}$ are not determined by Γ and therefore can be arbitrary. Let

$$D := \bar{\Gamma} \setminus \mathrm{sp}(\Gamma) \tag{4.7}$$

and call D_0 and $D_i^{\pm}$ the maximal R connected components of D. D_0 is the one which is R connected to the unbounded component of $\bar{\Gamma}^c$, D_i^+ (resp. D_i^-) are the components where σ_x (as specified by Γ) is equal to 1 (resp. -1). We also call $\mathrm{int}_i^{\pm}(\Gamma)$ the component of $\bar{\Gamma}^c$ which is R connected to $D_i^{\pm}$. Finally Γ is a $\pm$ contour, if $\sigma = \pm 1$ on D_0.

The R contours in a bounded domain $\Lambda \subset \mathbb{Z}^{d,R}$ with $+$ $[-]$ boundary conditions are defined as the contours of the configuration $(\sigma_\Lambda, \mathbf{1}_{\Lambda^c})$ [resp. and of $(\sigma_\Lambda, -\mathbf{1}_{\Lambda^c})$].

The weight $w^{R,+}(\Gamma)$ of a $+$ R contour is

$$w^{R,+}(\Gamma) = \frac{e^{-\beta H^R_h(\sigma_{\bar\Gamma})}}{e^{\beta h|\bar\Gamma|}} \prod_{i=1}^{n^-} \frac{Z^{R,-}_{\beta,h,\mathrm{int}_i^-(\Gamma)}}{Z^{R,+}_{\beta,h,\mathrm{int}_i^-(\Gamma)}} \ . \tag{4.8}$$

The superscript R recalls that all quantities are defined using the collection of cells $\{\mathfrak{c}\}^R$. The term $e^{\beta h|\bar\Gamma|}$ in the denominator is the Gibbs factor of the configuration $\mathbf{1}_{\bar\Gamma}$ identically equal to 1 in $\bar\Gamma$, $e^{-\beta H_h(\mathbf{1}_{\bar\Gamma})} = e^{\beta h|\bar\Gamma|}$.

The weight $w^{R,-}(\Gamma)$ of a $-$ R contour is defined symmetrically with the role of $+$ and $-$ interchanged. With these definitions, we have the identity

$$Z^{R,\pm}_{\beta,h,\Lambda} = e^{\pm\beta h|\Lambda|} \sum_{\{\Gamma_i\}^\pm_\Lambda} \prod_{\{\Gamma_i\}^\pm_\Lambda} w^{R,\pm}(\Gamma_i) \ , \tag{4.9}$$

where $\{\Gamma_i\}^+_\Lambda$ $[\{\Gamma_i\}^-_\Lambda]$ is a compatible collection of $+$ $[-]$ R contours in Λ. Two contours are compatible iff their spatial supports are not R-connected.

For the case without reflections we can apply directly the classical Pirogov-Sinai theory:

Theorem 4.2. *There is $c > 0$ and, for any β large enough, $h(\beta) \in (0, ce^{-\beta/2})$ so that the thermodynamic limits of $\mu^\pm_{\beta,h(\beta),\Lambda}$ define distinct DLR measures. Moreover, for any contour Γ, the weight without reflection satisfy*

$$0 < w^\pm(\Gamma) \leqslant e^{-\beta N_\Gamma/2} \ , \tag{4.10}$$

where N_Γ is the number of distinct cells which cover $sp(\Gamma)$.

In the following the bound (4.10) will be referred as a Peierls estimate since it leads

$$\mu^\pm_{\beta,h(\beta),\Lambda}(\Gamma) \leqslant e^{-\beta N_\Gamma/2} \tag{4.11}$$

The bound (4.10) is actually the crucial point of the theorem, the small weight of the contours is in fact responsible for the memory of the boundary conditions to survive the thermodynamic limit, thus yielding the phase transition. Moreover, as we will see, if β is large, (and the weight small), by cluster expansion, it is possible to exponentiate the right-hand side of [the analogue without reflections] of (4.9) and thus to compute the finite volume corrections to the pressure. This is on the other hand also the key point in the proof of (4.10), which at first sight makes all the above to look circular. The main goal in this section is to prove the bound (4.10) in case of reflections.

4.2 Restricted ensembles

Following Zahradnik, we construct a much simpler, fictitious model which, as a miracle, in the end, turns out to coincide with the real one. In the whole sequel β is large enough and $h = h(\beta)$, see Theorem 4.2, will often drop from the notation.

Inspired by (4.9), we set for any bounded region $\Lambda \subset \mathbb{Z}^{d,R}$,

$$\Xi^{R,\pm}_{\beta,\Lambda} = e^{\pm\beta h|\Lambda|} \sum_{\{\Gamma_i\}^{\pm}_{\Lambda}} \prod_{\{\Gamma_i\}^{\pm}_{\Lambda}} \hat{w}^{R,\pm}(\Gamma_i) \tag{4.12}$$

$$\hat{w}^{R,+}(\Gamma) = \min\Big\{ e^{-\beta N^R_\Gamma/2}; \frac{e^{-\beta H^R_h(\sigma_{\bar{\Gamma}})}}{e^{\beta h|\bar{\Gamma}|}} \prod_{i=1}^{n^-} \frac{\Xi^{R,-}_{\beta,\mathrm{int}^-_i(\Gamma)}}{\Xi^{R,+}_{\beta,\mathrm{int}^-_i(\Gamma)}} \Big\}, \qquad \Gamma \text{ a } + R \text{ contour} \tag{4.13}$$

$$\hat{w}^{R,-}(\Gamma) = \min\Big\{ e^{-\beta N^R_\Gamma/2}; \frac{e^{-\beta H^R_h(\sigma_{\bar{\Gamma}})}}{e^{-\beta h|\bar{\Gamma}|}} \prod_{i=1}^{n^+} \frac{\Xi^{R,+}_{\beta,\mathrm{int}^+_i(\Gamma)}}{\Xi^{R,-}_{\beta,\mathrm{int}^+_i(\Gamma)}} \Big\}, \qquad \Gamma \text{ a } - R \text{ contour} \tag{4.14}$$

where N^R_Γ is the number of R cells in $\mathrm{sp}(\Gamma)$. In this way the weights automatically satisfy the crucial bound (4.10), but first let us check that (4.12)–(4.14) do really define the "partition functions" $\Xi^{R,\pm}_{\beta,\Lambda}$ and the "weights" $\hat{w}^{R,\pm}(\Gamma)$. Indeed, the triple (4.12)–(4.14) should be regarded as an equation in the unknowns $\Xi^{R,\pm}_{\beta,\Lambda}$ and $\hat{w}^{R,\pm}(\Gamma)$. Existence and uniqueness are proved by induction on $|\Lambda|$. If $|\Lambda| = 1$, any contour in Λ has no interior, hence (4.13)–(4.14) specify its weight and consequently (4.12) determines $\Xi^{R,\pm}_{\beta,\Lambda}$ for such a Λ. If on the other hand we know $\Xi^{R,\pm}_{\beta,\Lambda}$ for all $\Lambda \subset \mathbb{Z}^{d,R}$, $|\Lambda| \leqslant n$, we can use (4.13)–(4.14) to determine the weights of all Γ if all their interior parts have volume $\leqslant n$; since regions Λ with $|\Lambda| = n+1$ cannot contain contours whose interior part has volume $> n$, we can use (4.12) to determine $\Xi^{R,\pm}_{\beta,\Lambda}$ for such a Λ, and the induction step is proved.

For β large enough the weights $\hat{w}^{R,\pm}(\Gamma)$ become so small, that the general theory of cluster expansion can be applied, see for instance [KP], so that

$$\log \Xi^{R,\pm}_{\beta,\Lambda} = \pm\beta h|\Lambda| + \sum_{\pi \in \Pi^{R,\pm}_\Lambda} \hat{\omega}^{R,\pm}(\pi) , \tag{4.15}$$

where $\Pi^{R,+}_\Lambda$ $[\Pi^{R,-}_\Lambda]$ is the collection of all $+$ $[-]$ polymers π contained in Λ and $\hat{\omega}^{R,\pm}(\cdot)$ their weights, all such notions being defined next. Analogous expressions are valid in the absence of reflections.

A $+$ R polymer $\pi = [\Gamma_i^{\varepsilon_i}]$ in $\Pi^{R,+}_\Lambda$ (the definition of $-$ polymers is similar and omitted), is an unordered, finite collection of distinct $+$ R contours Γ_i taken with positive integer multiplicity ε_i, and such that, setting

$$X(\pi) = \bigcup_i \mathrm{sp}(\Gamma_i), \qquad \pi = [\Gamma_i^{\varepsilon_i}] \tag{4.16}$$

$X(\pi)$ is a R connected subset of Λ.

The weights $\hat{\omega}^{R,\pm}(\pi)$, $\pi = [\Gamma_i^{\varepsilon_i}]$, are given in terms of the weights of contours, $\hat{w}^{R,\pm}(\Gamma)$:

$$\hat{\omega}^{R,\pm}(\pi) = r(\pi) \prod_i \hat{w}^{R,\pm}(\Gamma_i)^{\varepsilon_i} , \tag{4.17}$$

where

$$r(\pi) = \prod_i (\varepsilon_i!)^{-1} \sum_{\mathcal{G}' \subset \mathcal{G}(\pi)} (-1)^{|\mathcal{G}'|} ,$$

with $\mathcal{G}(\pi)$ the (abstract) graph of π, which consists of vertices, labelled by the $\sum_i \varepsilon_i$ contours in π, and of edges, which join any two vertices labelled by contours with intersecting supports. By definition $\mathcal{G}(\pi)$ is connected and the sum in (4.18) is over all the connected subgraphs $\mathcal{G}'$ of $\mathcal{G}(\pi)$ which contain all the $\sum_i \varepsilon_i$ vertices; $|\mathcal{G}'|$ denotes the number of edges in $\mathcal{G}'$.

The number of connections of each site is not increased by the reflection procedure. Thus, for β large enough, [KP], the series on the right-hand side of (4.15) is absolutely convergent and, given any finite sequence $\Gamma_1, \dots, \Gamma_n$ of contours,

$$\sum_{\pi \in \Pi^{R,\pm}, \pi \ni \Gamma_i, i=1,\dots,n} |\hat{\omega}^{R,\pm}(\pi)| \leqslant \prod_{i=1}^{n} e^{-N_{\Gamma_i}(\beta/2 - 2^d \alpha)} , \tag{4.18}$$

where $\Pi^{R,\pm}$ denotes the collection of all $+$ $[-]$ polymers in the whole space $\mathbb{Z}^{d,R}$ and $\alpha > 0$ is large enough, in particular we will also require that

$$\sum_{D \ni 0} 2^{2|D|} e^{-\alpha |D|} < 1 , \tag{4.19}$$

where the sum is over all R connected sets D in $\mathbb{Z}^{d,R}$ which contain the origin (supposing $0 \in \mathbb{Z}^{d,R}$). D represents the spatial support of a contour and $2^{|D|}$ bounds the number of contours with same spatial support D. The extra 2 in $2^{2|D|}$ is for convenience. The factor 2^d in the last term of (4.18) enters via the relation $(2^d)N_D \geq |D|$, N_D the number of cells needed to cover D.

Since by Theorem 4.2, the weights $w^{\pm}(\Gamma)$ satisfy the same bounds as the $\hat{w}^{R,\pm}(\Gamma)$, we have, analogously to (4.15),

$$\log Z^{\pm}_{\beta,\Lambda} = \pm \beta h |\Lambda| + \sum_{\pi \in \Pi^{R,\pm}_{\Lambda}} \omega^{\pm(\pi)} , \tag{4.20}$$

with $\omega^{\pm}(\pi)$ defined by (4.17) having $w^{\pm}(\Gamma)$ in the place of $\hat{w}^{R,\pm}(\Gamma)$. As in (4.18),

$$\sum_{\pi \in \Pi, \pi \ni \Gamma_i, i=1,\dots,n} |\omega^{\pm}(\pi)| \leqslant \prod_{i=1}^{n} e^{-N_{\Gamma_i}(\beta/2 - 2^d \alpha)} . \tag{4.21}$$

We will often use the following corollary of (4.19)–(4.21):

Lemma 4.1. *For any β large enough and any $x \in \mathbb{Z}^{d,R}$*

$$\sum_{X(\pi) \ni x} |\hat{\omega}^{R,\pm}(\pi)| \leqslant e^{-\beta/2 + 2^{d+1}\alpha} \tag{4.22}$$

and, for any x and n,

$$\sum_{X(\pi)\ni x, X(\pi)\cap K^R(x,n)^c\neq\emptyset} |\hat{\omega}^{R,\pm}(\pi)| \leqslant e^{-(\beta/2-2^{d+1}\alpha)n} \tag{4.23}$$

Both (4.22) *and* (4.23) *remain valid in the case without reflections.*

Proof. By (4.18),

$$\begin{aligned}\sum_{X(\pi)\ni 0} |\hat{\omega}^{R,\pm}(\pi)| \leqslant &\sum_{\Gamma:\mathrm{sp}(\Gamma)\ni 0} e^{-N_\Gamma(\beta/2-2^d\alpha)} \\ &\leqslant e^{-(\beta/2-2^{d+1}\alpha)} \sum_{\Gamma:\mathrm{sp}(\Gamma)\ni 0} e^{-\alpha|\mathrm{sp}(\Gamma)|} ,\end{aligned}$$

where we used that $|\mathrm{sp}(\Gamma)| = 2^d N_\Gamma$. Applying (4.19), we obtain (4.22).

To prove (4.23), we denote by $\{\Gamma_1,\ldots,\Gamma_k\}$ any sequence of contours such that $\mathrm{sp}(\Gamma_1)\ni x$, $\mathrm{sp}(\Gamma_k)\cap K^c\neq\emptyset$, $K\equiv K^R(x,n)$, and $\mathrm{sp}(\Gamma_i)\sim\mathrm{sp}(\Gamma_{i+1})$, $i=1,\ldots,k-1$, (where $A\sim B$ shorthands that A is R connected to B). Then the left-hand side of (4.23) is bounded by

$$\begin{aligned}\sum_{k,\{\Gamma_1,\ldots,\Gamma_k\}} \sum_{\pi:\Gamma_i\in\pi, i=1,\ldots,k} |\hat{\omega}^{R,\pm}(\pi)| &\leqslant \sum_{k,\{\Gamma_1,\ldots,\Gamma_k\}} \prod_{i=1}^k e^{-N_{\Gamma_i}(\beta/2-2^d\alpha)} \\ &\leqslant e^{-(\beta/2-2^{d+1}\alpha)n} \sum_k \sum_{D_1\ni x, D_j\sim D_{j+1}, j=1,\ldots,k-1} \prod_{i=1}^k 2^{|D_i|} e^{-\alpha|D_i|}\end{aligned}$$

which proves (4.23) because, as we are going to see, the sum over k, that we denote by $S(x)$, is bounded by 1.

Calling $S_N(x)$ the sum with $k\leq N$, since $S(x)$ is the limit as $N\to\infty$ of $S_N(x)$, it suffices to prove that for all y and N, $S_N(y)\leqslant 1$. The proof is by induction on N. $S_1(y)<1$ by (4.19). Suppose $S_{N-1}(x)\leqslant 1$ for all x, then

$$S_N(x) \leqslant \sum_{D_1\ni x} 2^{|D_1|} e^{-\alpha|D_1|} \prod_{y\in D_1} \Big(1+S_{N-1}(y)\Big) \leqslant \sum_{D_1\ni x} 2^{|D_1|} 2^{|D_1|} e^{-\alpha|D_1|}$$

the second factor $2^{|D_1|}$ coming from the induction hypothesis. Then, by (4.19), $S_N(x)\leqslant 1$ for any x and (4.23) is proved. The lemma is proved. □

By the analogue of (4.22) we conclude convergence of the series on the right-hand side of

$$\mathcal{P}_\pm := \pm h + \frac{1}{\beta} \sum_{\pi\in\Pi(\pm), X(\pi)\ni 0} \frac{\omega^\pm(\pi)}{|X(\pi)|} \tag{4.24}$$

To study the weights of the polymers obtained by reflection we will use the following three lemmas, where Λ is tacitly supposed to be a bounded region in $\mathbb{Z}^d$.

They refer to the case without reflections and since the magnetic field h is equal to $h(\beta)$ they are part of the classical Pirogov-Sinai Theory. For convenience, we give an explicit proof, consequence of Theorem 4.2.

Lemma 4.2. *For β large enough,*

$$\mathcal{P}_+ = \mathcal{P}_- = \mathcal{P} \tag{4.25}$$

where $\mathcal{P}$ is the thermodynamic pressure at inverse temperature β and magnetic field $h = h(\beta)$. Moreover,

$$\log Z^\pm_{\beta,\Lambda} = \pm\beta h|\Lambda| + \sum_{x\in\Lambda}\sum_{x\in X(\pi)\subset\Lambda} \frac{\omega^\pm(\pi)}{|X(\pi)|} \tag{4.26}$$

$$= \beta|\Lambda|\mathcal{P} - \sum_{X(\pi)\cap\Lambda^c\neq\emptyset} \frac{|X(\pi)\cap\Lambda|}{|X(\pi)|}\,\omega^\pm(\pi) \tag{4.27}$$

Proof. (4.26) is just a rewriting of (4.20); (4.25) follows from (4.26) by taking the thermodynamic limit and using Lemma 4.1. (4.27) is also a rewriting of (4.26). The lemma is proved. □

Lemma 4.3. *For β large enough, and calling $\delta(\Lambda)$ the union of all sites in Λ^c which are $*$-connected to Λ,*

$$\Big|\log Z^\pm_{\beta,\Lambda} - \beta|\Lambda|\mathcal{P}\Big| \leqslant e^{-\beta/2+2^{d+1}\alpha}|\delta(\Lambda)| \tag{4.28}$$

Proof. By (4.27)

$$\Big|\log Z^\pm_{\beta,\Lambda} - \beta|\Lambda|\mathcal{P}\Big| \leqslant \sum_{x\in\delta(\Lambda)}\sum_{X(\pi)\ni x}|\omega^\pm(\pi)|$$

which, by (4.22), yields (4.28). The lemma is proved. □

The final lemma proves that the bound (4.10) was too conservative.

Lemma 4.4. *There is a constant c so that, for β large enough,*

$$\Big|\log Z^+_{\beta,\Lambda} - \log Z^-_{\beta,\Lambda}\Big| \leqslant 2e^{-\beta/2+2^{d+1}\alpha}|\delta(\Lambda)| \tag{4.29}$$

$$w^\pm(\Gamma) \le \exp\Big\{-\beta N_\Gamma\Big(1 - ce^{-\beta/2}\Big)\Big\} \tag{4.30}$$

Proof. (4.29) follows directly from (4.28). By the analogue of (4.8) without reflections,

$$w^\pm(\Gamma) \le \exp\Big\{-\beta N_\Gamma + 2\beta|h||\bar\Gamma| + 2e^{-\beta/2+2^{d+1}\alpha}\sum_i|\delta(\mathrm{int}_i^-(\Gamma))|\Big\}$$

Notice also that the constraint on $h(\beta)$ can be easily recovered. By equating the two right-hand side of (4.24) and then using (4.22) in the version without reflections, we get

$$\beta|h| \leqslant e^{-\beta/2+2^{d+1}\alpha} \tag{4.31}$$

Moreover, if $x \in \delta(\mathrm{int}_i^-(\Gamma))$ there is a cell $\mathfrak{c}$ such that $\mathfrak{c} \cap K(x,2) \neq \emptyset$, $\mathfrak{c} \cap \mathrm{sp}(\Gamma) \neq \emptyset$, so that to each $x \in \cup_i \delta(\mathrm{int}_i^-(\Gamma))$ we can associate a cell contributing to N_Γ, in such a way that the same cell is counted at most $|K(0,3)|$ times. Thus

$$w^{\pm}(\Gamma) \leq \exp\Big\{ -\beta N_\Gamma + 2e^{-\beta/2+2^{d+1}\alpha}|\bar{\Gamma}| + 2e^{-\beta/2+2^{d+1}\alpha}|K(0,3)|N_\Gamma \Big\} \tag{4.32}$$

The inequality

$$|\bar{\Gamma}| \leq |\mathrm{sp}(\Gamma)||K(2,0)| \leq N_\Gamma 2^d |K(2,0)|$$

concludes the proof of the lemma. □

We turn now back to the main goal of the section, namely to prove that the bound (4.10) holds also for the weights with reflections. The proof is obtained in two steps.

Theorem 4.3. *For any β large enough the following holds. Let $x \in \mathbb{Z}^{d,R}$ and $n \leqslant n(x)$; then if $\Lambda \cup \partial^R(\Lambda) \subset K^R(x,n)$, $\Xi^{R,\pm}_{\beta,\Lambda} = Z^{R,\pm}_{\beta,\Lambda}$ and if Γ is a $\pm$, R contour with $\bar{\Gamma} \subset K^R(x,n)$, then $\hat{w}^{R,\pm}(\Gamma) = w^{R,\pm}(\Gamma) < e^{-\beta N_\Gamma/2}$.*

Proof. Under the assumption that $n \leqslant n(x)$, Theorem 4.1 applies and therefore the proof will follow from the previous results on the weights without reflection and from the one-to-one correspondence between $K(x,n)$ and $K^R(x,n)$. In particular (4.4) implies that for domains strictly contained in $K^R(x,n)$

$$Z^{\pm}_{\beta,\mathcal{T}^{-1}(\Lambda)} = Z^{R,\pm}_{\beta,\Lambda} \tag{4.33}$$

In the case $|\Lambda| = 1$, any contour in Λ has no interior and (4.12)–(4.13) allow to compute $\hat{w}^{R,\pm}(\Gamma)$, getting, as in the proof of Lemma 4.4,

$$\hat{w}^{R,\pm}(\Gamma) \leqslant \exp\Big\{ -\beta N_\Gamma + 2\beta|h||\bar{\Gamma}|\Big\}$$

hence, for β large enough, $\hat{w}^{R,\pm}(\Gamma) = w^{\pm}(\Gamma) < e^{-\beta N_\Gamma/2}$. Suppose by induction that for any $|\Lambda| \leqslant k$ (Λ as in the text of the theorem), $\Xi^{R,\pm}_{\beta,\Lambda} = Z^{\pm}_{\beta,\mathcal{T}^{-1}(\Lambda)} = Z^{R,\pm}_{\beta,\Lambda}$. Then if Γ is as in the text of the theorem and moreover all its interior parts have volume $\leqslant k$, then the second term on the right-hand side of (4.12)–(4.13) is equal to $w^{\pm}(\mathcal{T}^{-1}\Gamma)$, with the obvious meaning of the notation, which by Lemma 4.4 is, for β large enough, $< e^{-\beta N_\Gamma/2}$. Then the second term on the right-hand side of (4.12)–(4.13) is smaller than the first one, hence $\hat{w}^{R,\pm}(\Gamma) = w^{R,\pm}(\Gamma)$. Since all contours inside Λ have interior parts with volume $\leqslant k$, (4.10) shows that $\Xi^{R,\pm}_{\beta,\Lambda} = Z^{\pm,R}_{\beta,\Lambda} = Z^{\pm}_{\beta,\mathcal{T}^{-1}(\Lambda)}$, thus proving the induction step. The theorem is proved. □

Before extending the result to general Λ, we state and prove the following lemma.

Lemma 4.5. *For β large enough,*

$$\log \Xi^{R,\pm}_{\beta,\Lambda} = \beta|\Lambda|\mathcal{P} + \sum_{x\in\Lambda} \Big\{ \sum_{X(\pi)\cap K^R(x;n(x))^c\neq\emptyset; x\in X(\pi)\subset\Lambda} \frac{\hat\omega^{R,\pm}(\pi)}{|X(\pi)|} - \sum_{X(\pi)\cap K(0;n(x))^c\neq\emptyset; 0\in X(\pi)} \frac{\omega^{\pm}(\pi)}{|X(\pi)|}\Big\}. \quad (4.34)$$

Proof. We write

$$\log \Xi^{R,\pm}_{\beta,\Lambda} = \pm\beta h|\Lambda| + \sum_{x\in\Lambda} \Big\{ \sum_{x\in X(\pi)\subset K^R(x;n(x))} \frac{\omega^{R,\pm}(\pi)}{|X(\pi)|} + \sum_{X(\pi)\cap K^R(x;n(x))^c\neq\emptyset; x\in X(\pi)\subset\Lambda} \frac{\omega^{R,\pm}(\pi)}{|X(\pi)|}\Big\}$$

$$\beta|\Lambda|\mathcal{P} = \pm\beta h|\Lambda| + \sum_{i\in\Lambda} \Big\{ \sum_{X(\pi)\subset K(x;n(x)); x\in X(\pi)} \frac{\omega^{\pm}(\pi)}{|X(\pi)|} + \sum_{X(\pi)\cap K(x;n(x))^c\neq\emptyset; x\in X(\pi)} \frac{\omega^{\pm}(\pi)}{|X(\pi)|}\Big\}$$

Then $\log \Xi^{R,\pm}_{\beta,\Lambda} - \beta|\Lambda|\mathcal{P}$ is equal to the difference of the right-hand side of the last two equations. The first terms in the sum over x cancel with each other, see the proof of Theorem 4.1, and (4.34) follows after recalling that the weights without reflections are translational invariant. The theorem is proved. □

Theorem 4.4. *For any β large enough, for any bounded $\Lambda \subset \mathbb{Z}^{d,R}$, $\Xi^{R,\pm}_{\beta,\Lambda} = Z^{R,\pm}_{\beta,\Lambda}$ and for any bounded, $\pm$, R contour Γ, $\hat w^{R,\pm}(\Gamma) = w^{R,\pm}(\Gamma) < e^{-\beta N_\Gamma/2}$.*

Proof. By (4.34) and (4.22), denoting by $n(x,y)$ the maximal integer such that $y \notin K^R(x;n(x,y))$,

$$\begin{aligned}\Big|\log \Xi^{R,\pm}_{\beta,\Lambda} - \beta|\Lambda|\mathcal{P}\Big| &\leqslant 2\sum_{x\in\Lambda} e^{-[\beta/2-2^{d+1}\alpha]\, n(x)} \\ &\leqslant 2\sum_{x\in\Lambda}\sum_{y\in\delta^R(\Lambda)} e^{-[\beta/2-2^{d+1}\alpha]\, n(x,y)} \\ &\leqslant 2\sum_{y\in\delta^R(\Lambda)}\sum_{n\geq 1} e^{-(\beta/2-2^{d+1}\alpha)n}(2n+1)^d\end{aligned}$$

which yields

$$\Big|\log \Xi^{R,\pm}_{\beta,\Lambda} - \beta|\Lambda|\mathcal{P}\Big| \leqslant c_\alpha e^{-\beta/2+2^{d+1}\alpha}\big|\delta^R(\Lambda)\big| \ , \tag{4.35}$$

with

$$c_\alpha = 2e^\alpha \sum_{n\geq 1} \ e^{-\alpha n}(2n+1)^d \ . \tag{4.36}$$

An inductive argument as in the proof of Theorem 4.3 completes then the proof of the theorem. □

Before ending this section, we collect some estimates used in the next sections.

Lemma 4.6. *Given any positive integer n,*

$$\Big|\log Z^{\pm}_{\beta,\Lambda} - \{\pm\beta h|\Lambda| + \sum_{x\in\Lambda} \ \sum_{x\in X(\pi)\subset K(x;n)\cap\Lambda} \frac{\omega^{\pm}(\pi)}{|X(\pi)|}\}\Big| \leqslant |\Lambda| e^{-(\beta/2-2^d\alpha)n} \tag{4.37}$$

$$\Big|\log Z^{R,\pm}_{\beta,\Lambda} - \{\pm\beta h(\beta)|\Lambda| + \sum_{x\in\Lambda} \ \sum_{x\in X(\pi)\subset K^R(x,n)\cap\Lambda} \frac{\omega^{R,\pm}(\pi)}{|X(\pi)|}\}\Big| \leqslant |\Lambda| e^{-(\beta/2-2^d\alpha)n} \tag{4.38}$$

Proof. (4.37) and (4.38) follow from (4.23) and its analogue without reflections. □

This lemma will enable us to estimate the corrections to the pressure. Let us also examine two other consequences which will be crucial in the rest of the paper.

The first consequence justifies the notion of perfect walls introduced in Subsection 3.2. We consider the slab $\Lambda_{L,\varepsilon}$ and the reflection w.r.t. to the hyperplane $\Sigma = \{x \in \mathbb{R}^d, \quad (\vec{n}\cdot x) = 0\}$ which splits $\Lambda_{L,\varepsilon}$ into two non-interacting domains.

Lemma 4.7. *There exists $c > 0$ such that*

$$\Big|\log Z^+_{L,\varepsilon} - \log Z^{R,+}_{L,\varepsilon}\Big| \leqslant c e^{-(\beta/2-2^d\alpha)} L^{d-2} \, , \tag{4.39}$$

where $Z^+_{L,\varepsilon}$ denotes the partition function on $\Lambda_{L,\varepsilon L}$ with $+$ boundary conditions and $Z^{R,+}_{L,\varepsilon}$ is the partition function obtained by reflection (see Subsection 3.2). The same statement holds with $-$ boundary conditions.

Proof. Let $B = \{-1,0,1\}^d$. For any x in $\Lambda_{L,\varepsilon}$, we set

$$\bar{n}(x) = \min\{n, \quad K(x,n)\cap B \neq \emptyset, K(x,n)\cap \Lambda^c_{L,\varepsilon} \neq \emptyset\}$$

Then

$$\Big|\log Z^+_{L,\varepsilon} - \log Z^{+,R}_{L,\varepsilon}\Big| \leqslant \sum_{x\in\Lambda_{L,\varepsilon L}} \Big(\sum_{X(\pi)\cap K^R(x;\bar{n}(x))^c\neq\emptyset; x\in X(\pi)\subset\Lambda_{L,\varepsilon L}} \frac{\omega^{R,+}(\pi)}{|X(\pi)|} + \sum_{X(\pi)\cap K(x;\bar{n}(x))^c\neq\emptyset; x\in X(\pi)} \frac{\omega^{+}(\pi)}{|X(\pi)|}\Big) \ .$$

The contribution of the polymers with $X(\pi)$ in $K(x,n)$ and the reflected ones in $X(\pi)$ in $K^R(x,n)$ with $n \leqslant \bar{n}(x)$ cancel with each other by Theorem 4.1.

Since the weights of the polymers are exponentially small (see Lemma 4.1), the result follows. □

The second consequence will be used in Section 6. Let $\mathbb{T}_N$ be the torus $\{-N,\dots,N\}^d$ and we consider a collection of reflections inside $\mathbb{T}_N$ for which the assumption (4.5) is satisfied. Let B denote the boundary conditions imposed by the reflections, i.e., the centers and the boundaries of each reflecting surfaces. We have

$$\left| \log Z^R_{\beta,N} - \log Z_{\beta,N} \right| \leqslant c_\alpha e^{-(\beta/2-2^d\alpha)} |B| \,, \tag{4.40}$$

where $Z^R_{\beta,N}$ (resp. $Z_{\beta,N}$) denotes the partition functions in $\mathbb{T}_N$ with periodic boundary conditions and with (resp. without) reflection.

5 Properties of the surface tension

In the following, β is fixed large enough such that the results of Section 4 are satisfied and h refers to $h(\beta)$. We first derive the existence of the thermodynamic limit for the surface tension and then its convexity and positivity.

5.1 Proof of Theorem 3.1

The proof can be split into three steps. First, we are going to prove that the choice of the barriers $(\mathcal{C}^+,\mathcal{C}^-)$ has almost no contribution on the ratio of the partition functions. Then, an inductive procedure enables us to improve (3.8) and to derive the convergence (3.9).

Step 1:

The first step is to prove that

$$\tau_\beta(\vec{n}) \quad = \quad \liminf_{\varepsilon\to 0}\ \liminf_{L\to\infty}\ \sup_{\mathcal{C}^+,\mathcal{C}^-} -\frac{(\vec{n}\cdot\vec{e}_d)}{\beta L^{d-1}} \log \frac{Z^{\mathcal{C}^+,\mathcal{C}^-}_{L,\varepsilon}(\mathcal{S}^+,\mathcal{S}^-)}{Z^{\mathcal{C}^+,R}_{L,\varepsilon}\, Z^{\mathcal{C}^-,R}_{L,\varepsilon}} \,. \tag{5.1}$$

This boils down to check that there are constants (C_1,C_2) such that for any (L,ε) and for any $(\mathcal{C}^+,\mathcal{C}^-)$ and $(\tilde{\mathcal{C}}^+,\tilde{\mathcal{C}}^-)$

$$\left| \log \frac{Z^{\mathcal{C}^+,\mathcal{C}^-}_{L,\varepsilon}(\mathcal{S}^+,\mathcal{S}^-)}{Z^{\mathcal{C}^+,R}_{L,\varepsilon}\, Z^{\mathcal{C}^-,R}_{L,\varepsilon}} - \log \frac{Z^{\tilde{\mathcal{C}}^+,\tilde{\mathcal{C}}^-}_{L,\varepsilon}(\mathcal{S}^+,\mathcal{S}^-)}{Z^{\tilde{\mathcal{C}}^+,R}_{L,\varepsilon}\, Z^{\tilde{\mathcal{C}}^-,R}_{L,\varepsilon}} \right| \leqslant C_1 L^d \exp(-C_2 \varepsilon L) \,. \tag{5.2}$$

The events $\mathcal{S}^+,\mathcal{S}^-$ decouple the interface from the boundary effects thus (5.2) can be derived by using only estimates in a pure phase.

It is enough to consider $\tilde{\mathcal{C}}^- = \mathcal{C}^-$. In this case, (5.2) becomes

$$\left| \log \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)} - \log \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,R}}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,R}} \right| \leqslant C_1 L^d \exp(-C_2 \varepsilon L)\,. \tag{5.3}$$

For any spin configuration in $\mathcal{S}^+$, let us denote by s^+ the support of the $+$ barrier in $\Lambda_{L,\frac{\varepsilon}{10}L} + \frac{\varepsilon L}{2}\vec{e}_d$ which is the closest to the hyperplane $\Sigma = \{x;\ (x\cdot\vec{n}) = 0\}$. This particular choice of s^+ will be stressed by the notation $s^+ \rightsquigarrow \mathcal{S}^+$. The constrained partition function can be decomposed as follows

$$Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-) = \sum_{s^+} e^{\beta h|s^+|} Z_{L,\varepsilon}^{\mathcal{C}^+,s^+}\, Z_{L,\varepsilon}^{s^+,\mathcal{C}^-}(s^+ \rightsquigarrow \mathcal{S}^+,\mathcal{S}^-)\,,$$

with the first partition function free of constraints so that cluster expansion applies and the second partition function which takes into account the constraint that there is no $+$ barrier in $\Lambda_{L,\frac{\varepsilon}{10}L} + \frac{\varepsilon L}{2}\vec{e}_d$ below s^+.

We first write

$$\frac{Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)} = \sum_{s^+} \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,s^+}}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,s^+}}\, e^{\beta h|s^+|}\, \frac{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,s^+}\, Z_{L,\varepsilon}^{s^+,\mathcal{C}^-}(s^+ \rightsquigarrow \mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)}\,. \tag{5.4}$$

Let $N = \varepsilon L/10$, (suppose, for notational simplicity, N an integer), then

$$\exp\{-4L^d e^{-(\beta/2-2\alpha)N}\} \leqslant \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,s^+}}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,s^+}}\, \frac{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,R}}{Z_{L,\varepsilon}^{\mathcal{C}^+,R}} \leqslant \exp\{4L^d e^{-(\beta/2-2\alpha)N}\} \tag{5.5}$$

follows from crossed cancellations among the terms in the numerator and denominator. We are going to apply the expansion of the partition function derived in Lemma 4.6 with $n = N$. The factor 4 is because there are 4 partition functions involved. With reference to (4.37) and (4.38), the contribution of x such that the scalar product $(x\cdot\vec{n}) \geqslant 8\varepsilon L/10$ coming from $Z_{L,\varepsilon}^{\mathcal{C}^+,s^+}$ and $Z_{L,\varepsilon}^{\mathcal{C}^+,R}$ cancel with each other, as well as those from $Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,s^+}$ and $Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,R}$. Symmetrically, the contribution of x such that $(x\cdot\vec{n}) < 8\varepsilon L/10$ arising from $Z_{L,\varepsilon}^{\mathcal{C}^+,s^+}$ and $Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,s^+}$ cancel with each other, as well as those from $Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,R}$ and $Z_{L,\varepsilon}^{\mathcal{C}^+,R}$.

Finally, by applying (5.5), we get from (5.4):

$$\begin{aligned}
&\frac{Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)} \\
&\qquad\leqslant \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,R}}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,R}} e^{4L^d e^{-(\beta/2-2\alpha)N}} \sum_{s^+} \frac{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,s^+}\, e^{\beta h|s^+|}\, Z_{L,\varepsilon}^{s^+,\mathcal{C}^-}(s^+ \rightsquigarrow \mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)} \\
&\qquad\leqslant \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,R}}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,R}}\, e^{4L^d e^{-(\beta/2-2\alpha)N}}
\end{aligned}$$

In the same way we get

$$\frac{Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)} \quad \geqslant \quad \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,R}}{Z_{L,\varepsilon}^{\tilde{\mathcal{C}}^+,R}}\, e^{-4L^d e^{-(\beta/2-2\alpha)N}} \,.$$

Recalling that $N = \varepsilon L/10$, we have thus completed the proof of (5.2).

<u>*Step 2:*</u>

The goal is to derive a lower bound for

$$\phi(L,\varepsilon,\mathcal{C}^+,\mathcal{C}^-) = \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S})}{Z_{L,\varepsilon}^{\mathcal{C}^+,R}\, Z_{L,\varepsilon}^{\mathcal{C}^-,R}}$$

in terms of $\tau_\beta(\vec{n})$. For simplicity $\mathcal{S} = (\mathcal{S}^+,\mathcal{S}^-)$.

The previous step (see (5.1)) implies that there exists a sequence $(\varepsilon_k, L_k)_{k\,\geqslant\,0}$ such that

$$\left| \frac{(\vec{n}\cdot\vec{e}_d)}{\beta L_k^{d-1}} \inf_{\mathcal{C}^+,\mathcal{C}^-} \; \log\phi(L_k,\varepsilon_k,\mathcal{C}^+,\mathcal{C}^-) + \tau_\beta(\vec{n}) \right| \leqslant \frac{1}{k}\,. \tag{5.6}$$

We fix (ε_k, L_k) and consider a pair (ε, L) such that $\varepsilon_k L_k \ll \varepsilon L$ and $L_k \ll L$. In order to derive a lower bound on $\phi(L,\varepsilon,\mathcal{C}^+,\mathcal{C}^-)$, we are going to localize the interface in the slab $\Lambda_{L,\varepsilon_k L_k}$.

We set $\Lambda_k^0 = \Lambda_{L_k,\varepsilon_k L_k}$, the upper-script 0 is to distinguish it from its translates (which will be introduced below). We call $\mathcal{C}_k^0 = (\mathcal{C}_k^{+,0},\mathcal{C}_k^{-,0})$ and $\mathcal{S}_k^0 = (\mathcal{S}_k^{+,0},\mathcal{S}_k^{-,0})$ the set of all spin configurations which have $\pm$ barriers as required from the definition of the surface tension. The "maximal barriers" are denoted by $c_k^{\pm,0}$, meaning that $c_k^{\pm,0}$ is the first barrier coming from the top [resp. the bottom] of Λ_k^0. We also write $c_k^{\pm,0} \rightsquigarrow \mathcal{C}_k^{\pm,0}$ for the event where $c_k^{\pm,0}$ are the maximal barriers in $\mathcal{C}_k^{\pm,0}$. We finally call $\mathcal{U}_k^{\pm,0}$ the union of all sites outside Λ_k^0 and at distance 1 from its faces parallel to $\vec{n}$; The $\pm$ labels distinguish those where the b.c. in the definition of the surface tension are set equal to ± 1.

Let (Λ_k^i) be those translates of Λ_k^0 which are contained in $\Lambda_{L,\varepsilon L}$, where

$$\forall i = (i_1,\ldots,i_d);$$

$$\Lambda_k^i = \Lambda_k^0 + \left((L_k+2)i_1,\ldots,(L_k+2)i_{d-1}, -\sum_{j=1}^{d-1}(L_k+2)\frac{n_j}{n_d}\, i_j + \xi_i \right),$$

with $\xi_i \in [0,1)$ chosen such that $\Lambda_k^i \subset \mathbb{Z}^d$. The same translation which carries Λ_k^0 onto Λ_k^i is used to define $\mathcal{C}_k^i = (\mathcal{C}_k^{+,i},\mathcal{C}_k^{-,i})$, $\mathcal{S}_k^i = (\mathcal{S}_k^{+,i},\mathcal{S}_k^{-,i})$, $c_k^{\pm,i} \rightsquigarrow \mathcal{C}_k^{\pm,i}$, $\mathcal{U}_k^{\pm,i}$ as translates of the corresponding quantities with $i = 0$. Notice that the distance between two distinct Λ_k^i and Λ_k^j is always larger than the range of the interaction and indeed two distinct $\mathcal{U}_k^{\pm,i}$ have at most their external surfaces in common. We denote by $\mathcal{U}_k^+$ the union of all $\mathcal{U}_k^{+,i}$ with the addition of the regions $\Lambda_k^i \cap \Lambda_{L,\varepsilon L} \cap \{(x \cdot \vec{n}) \geqslant 0\}$, when i ranges over all values such that Λ_k^i is not contained in $\Lambda_{L,\varepsilon L}$. $\mathcal{U}_k^-$ is defined analogously and $\mathcal{U}_k = \mathcal{U}_k^+ \cup \mathcal{U}_k^-$.

The volume of $\mathcal{U}_k$ is bounded (for L so large that $(L_k+2)^2 < L$) by

$$|\mathcal{U}_k| \leqslant \{(L_k+2)^{d-2}2\}\varepsilon_k L_k \frac{L^{d-1}}{(L_k+2)^{d-1}} + L^{d-2}(L_k+2)\varepsilon_k L_k \leqslant 4\varepsilon_k L^{d-1} \,. \quad (5.7)$$

The first term bounds the contribution of all i where $\Lambda_k^i \subset \Lambda_{L,\varepsilon L}$, the second term the remaining ones; the final estimate uses that $(L_k+2)^2 < L$.

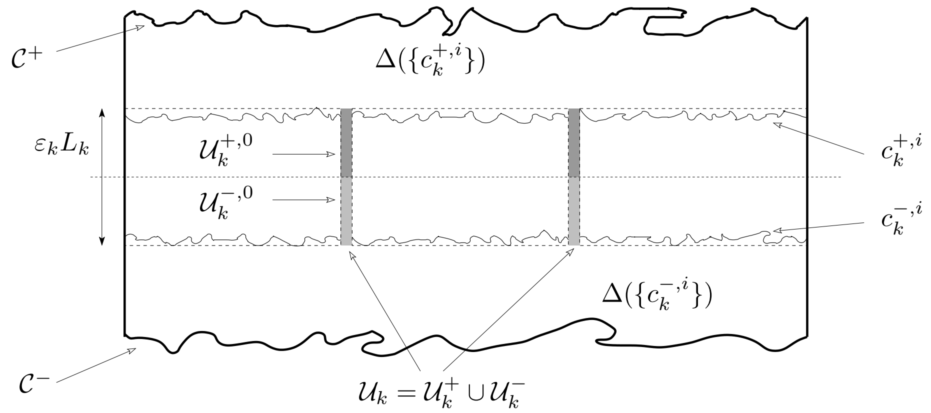

FIGURE 3. Decomposition at the scale L_k of the domain $\Lambda(\mathcal{C}^+,\mathcal{C}^-)$ by means of the subsets $(\Lambda_k^i)_i$ (depicted by dashed boxes).

Let $\mathcal{Q}_k$ be the intersection of the events $\mathcal{C}^i = (\mathcal{C}_k^{+,i},\mathcal{C}_k^{-,i})$, $\mathcal{S}_k^i = (\mathcal{S}_k^{+,i},\mathcal{S}_k^{-,i})$ over all i such that $\Lambda_k^i \subset \Lambda_{L,\varepsilon L}$. Call $c_k^{\pm,i}$ the maximal barriers realizing the event $\mathcal{C}_k^{\pm,i}$ (maximal in the sense described previously). In order to decouple the events in the different regions (Λ_k^i), we fix the spin configurations in $\mathcal{U}_k$ as equal to $\mathbf{1}^{\pm}_{\mathcal{U}_k}$, where the latter is the configuration where the spins are equal to ± 1 on $\mathcal{U}_k^{\pm}$, we call $\mathcal{Q}'_k$ such a further constraint. On $\mathcal{Q}_k$ we set $\Lambda(c_k^{+,i},c_k^{-,i})$ as the region in Λ_k^i which goes from the maximal top barrier $c_k^{+,i}$ down to the maximal bottom barrier $c_k^{-,i}$ (both included), and set

$$\begin{aligned}
\Delta(\{c_k^{\pm,i}\}) = \Lambda(\mathcal{C}^+,\mathcal{C}^-) \quad &\setminus \quad \left(\bigcup_i \Lambda(c_k^{+,i},c_k^{-,i}) \bigcup \mathcal{U}_k\right), \\
\Delta^+(\{c_k^{+,i}\}) \quad &= \quad \Delta(\{c_k^{\pm,i}\}) \cap \{x;\ (x\cdot\vec{n}) \geqslant 0\}, \\
\Delta^-(\{c_k^{-,i}\}) \quad &= \quad \Delta(\{c_k^{\pm,i}\}) \cap \{x;\ (x\cdot\vec{n}) < 0\}\,.
\end{aligned}$$

Imposing the constraint $\mathcal{Q}_k, \mathcal{Q}'_k$, and decomposing the partition function with respect to $(\mathcal{C}_k^{+,i}, \mathcal{C}_k^{-,i})$, we get

$$
\begin{aligned}
&Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}) \geqslant Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}\big(\mathcal{S} \bigcap \mathcal{Q}_k \bigcap \mathcal{Q}'_k\big) \qquad (5.8)\\
&= \sum_{(c_k^{+,i}, c_k^{-,i})} e^{-\beta H_h(\mathbf{1}^{\pm}_{\mathcal{U}_k})} Z^{\mathcal{C}^+}_{\Delta^+(\{c_k^{+,i}\})}\big(\mathcal{S}^+, c_k^{+,i} \rightsquigarrow \mathcal{C}_k^{+,i}\big) Z^{\mathcal{C}^-}_{\Delta^-(\{c_k^{-,i}\})}\big(\mathcal{S}^-, c_k^{-,i} \rightsquigarrow \mathcal{C}_k^{-,i}\big)\\
&\times \prod_i \left\{ e^{\beta h(|c^{+,i}| - |c^{-,i}|)} Z_{L_k,\varepsilon_k}^{c_k^{+,i}, c_k^{-,i}}(\mathcal{S}_k^i) \right\}.
\end{aligned}
$$

By introducing the partition functions in each Λ_k^i with reflected boundary conditions at the scale L_k, we will recover an approximation of the surface tension. For each factor $Z_{L_k,\varepsilon_k}^{c_k^{+,i}, c_k^{-,i}}(\mathcal{S}_k)$ in the last product, we write (see (5.6))

$$
Z_{L_k,\varepsilon_k}^{c_k^{+,i}, c_k^{-,i}}(\mathcal{S}_k) \geqslant Z_{L_k,\varepsilon_k}^{c_k^{+,i}, R(k)} Z_{L_k,\varepsilon_k}^{c_k^{-,i}, R(k)} \exp\left(- \frac{\beta L_k^{d-1}}{(\vec{n} \cdot \vec{e}_d)} \big(\tau_\beta(\vec{n}) + 1/k\big) \right),
$$

we are using the notation of Subsection 3.2 with $R(k)$ instead of R to underline that the partition functions $Z_{L_k,\varepsilon_k}^{c_k^{\pm,i}, R(k)}$ take into account the multiple reflections at the scale L_k (see Figure 2). By taking the product over all i, we get

$$
\prod_i Z_{L_k,\varepsilon_k}^{c_k^{+,i}, c_k^{-,i}}(\mathcal{S}_k) \geqslant \exp\left(- \frac{\beta L^{d-1}}{(\vec{n} \cdot \vec{e}_d)} \big(\tau_\beta(\vec{n}) + 1/k\big) \right) \prod_i Z_{L_k,\varepsilon_k}^{c_k^{+,i}, R(k)} Z_{L_k,\varepsilon_k}^{c_k^{-,i}, R(k)}.
$$

We are going to plug the previous inequality in (5.8) in order to reconstruct two partition functions on the domains

$$
\Delta^{\pm} = \bigcup_i \left\{ \Delta^{\pm}(\{c_k^{\pm,i}\}_k) \cup \Lambda^{\pm}(c^{\pm,i}, R(k)) \cup c_k^{\pm,i} \right\}.
$$

Notice that the sets $\Delta^{\pm}$ are slightly different from $\Lambda^{\pm}(\mathcal{C}^{\pm})$ since they are built according to the rules of the reflection at the scale L_k. We finally obtain

$$
\begin{aligned}
Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}\big(\mathcal{S} \bigcap \mathcal{Q}_k \bigcap \mathcal{Q}'_k\big) \quad \geqslant \quad & Z_{\Delta^+}^{\mathcal{C}^+, R(k)}\big(\mathcal{S}^+, \mathcal{C}_k^{+,i}\big) Z_{\Delta^-}^{\mathcal{C}^-, R(k)}\big(\mathcal{S}^-, \mathcal{C}_k^{-,i}\big) e^{-\beta H_h(\mathbf{1}^{\pm}_{\mathcal{U}_k})}\\
& \exp\left(- \frac{\beta L^{d-1}}{(\vec{n} \cdot \vec{e}_d)} \big(\tau_\beta(\vec{n}) + 1/k\big) \right),
\end{aligned}
$$

where $Z_{\Delta^+}^{\mathcal{C}^+, R(k)}\big(\mathcal{S}^+, \mathcal{C}_k^{+,i}\big)$ denotes the partition function on Δ^+ with a perfect wall made of multiple reflections on the scale L_k and taking into account the occurrence of the barriers $\mathcal{S}^+$ and $\{\mathcal{C}_k^{+,i}\}_i$.

By (5.7)

$$
|H_h(\mathbf{1}^{\pm}_{\mathcal{U}_k})| \leqslant c|\mathcal{U}_k| \leqslant c4\varepsilon_k L^{d-1}
$$

so that, it only remains to check that

$$\lim_{k\to\infty}\lim_{L\to\infty}\inf_{\mathcal{C}^+,\mathcal{C}^-}\frac{1}{L^{d-1}}\log\frac{Z_{\Delta^+}^{\mathcal{C}^+,R(k)}\big(\mathcal{S}^+,\mathcal{C}_k^{+,i}\big)Z_{\Delta^-}^{\mathcal{C}^-,R(k)}\big(\mathcal{S}^-,\mathcal{C}_k^{-,i}\big)}{Z_{L,\varepsilon}^{\mathcal{C}^+,R}\,Z_{L,\varepsilon}^{\mathcal{C}^-,R}}\geqslant 0\,, \tag{5.9}$$

because, if we suppose that the previous inequality holds, then

$$\liminf_{L\to\infty}\frac{(\vec{n}\cdot\vec{e}_d)}{\beta L^{d-1}}\inf_{\mathcal{C}^+,\mathcal{C}^-}\log\phi(L,\varepsilon,\mathcal{C}^+,\mathcal{C}^-)\geqslant\,-\tau_\beta(\vec{n})\,,$$

which completes the Theorem 3.1.

Step 3:

The final step is devoted to the derivation of (5.9). This amounts to prove that the corrections to the pressure for the different types of reflected boundary conditions are negligible.

First, we check that the constrained partition function $Z_{\Delta^+}^{\mathcal{C}^+,R(k)}\big(\mathcal{S}^+,\mathcal{C}_k^{+,i}\big)$ is asymptotically equivalent to the non-constrained partition function $Z_{\Delta^+}^{\mathcal{C}^+,R(k)}$. Let $\mu_{\Delta^+}^{+,R(k)}$ be the corresponding Gibbs measure. Then the following holds

$$\mu_{\Delta^+}^{+,R(k)}\big(\mathcal{S}^+,\mathcal{C}_k^{+,i}\big)\geqslant\big(1-L^{d-1}\exp(-c\varepsilon L)\big)\big(1-L_k^{d-1}\exp(-c\varepsilon_k L_k)\big)^{\mathcal{N}_k}\,. \tag{5.10}$$

This can be derived as follows. The occurrence of a barrier with blocks uniformly labelled by 1 in the slab $\Lambda_{l,m}$ implies that there is no connected set of blocks labelled by -1 joining the two faces of $\Lambda_{l,m}$ orthogonal to $\vec{n}$. Under $\mu_{\Delta^+}^{+,R(k)}$, a Peierls estimate similar to (4.11) (see Theorem 4.4). A Peierls type argument implies then that a connected set of $-$ blocks with length at least m has a probability smaller than $\exp(-\frac{\beta}{2}m)$. Applying recursively the Peierls argument, we derive (5.10).

By hypothesis on the sequence (ε_k, L_k), for k large enough (5.10) implies

$$\mu_{\Delta^+}^{+,R(k)}\big(\mathcal{S}^+,\mathcal{C}_k^{+,i}\big)\geqslant 2^{-1-L^{d-1}\exp(-c\varepsilon_k L_k)}\,.$$

Therefore

$$\lim_{k\to\infty}\lim_{L\to\infty}\frac{1}{L^{d-1}}\inf_{\mathcal{C}^+}\log\frac{Z_{\Delta^+}^{\mathcal{C}^+,R(k)}\big(\mathcal{S}^+,(\mathcal{C}_k^{+,i})\big)}{Z_{L,\varepsilon}^{\mathcal{C}^+,R(k)}}\geqslant 0\,. \tag{5.11}$$

This reduces the proof of (5.9) to

$$\lim_{k\to\infty}\lim_{L\to\infty}\frac{1}{L^{d-1}}\inf_{\mathcal{C}^+,\mathcal{C}^-}\log\frac{Z_{\Delta^+}^{\mathcal{C}^+,R(k)}\,Z_{\Delta^-}^{\mathcal{C}^-,R(k)}}{Z_{L,\varepsilon}^{\mathcal{C}^+,R}\,Z_{L,\varepsilon}^{\mathcal{C}^-,R}}=0\,. \tag{5.12}$$

Again this estimate will follow from cross cancellations between the 4 partition functions.

Following the strategy of step 1, the bulk contribution and the correction to the pressure from the boundary terms $\mathcal{C}^+, \mathcal{C}^-$ can be estimated by Lemma 4.6; they are of the order

$$L^d \exp\left\{2L^d e^{-(\beta/2-2\alpha)\varepsilon L/10}\right\} .$$

Thus it is enough to check that the contribution of the perfect walls involved in each partition function will be negligible w.r.t. to the surface order. We first consider the partition functions $Z_{L,\varepsilon}^{\mathcal{C}^+,R}, Z_{L,\varepsilon}^{\mathcal{C}^-,R}$. By an analogous argument of the one used to derive (4.39), we see that the corrections to the pressure induced by the perfect wall Σ are of the order L^{d-2}.

We consider now the partition functions with multiple reflections. The perfect wall associated to the box Λ_k^i is denoted Σ_i and to each reflection corresponds a particular set B_i of boundary conditions. The set B_i comprises the sites around the center of reflection in Σ_i as well as the sites outside Λ_k^i which are connected to Σ_i. The union of the B_i is denoted by B (see Figure 2).

In order to use the estimate of Section 4, we should first check that the assumption (4.5) holds for the multiple reflections at the scale L_k. Suppose that for some $x, n'(x) \leqslant n(x)$. Following the proof of Theorem 4.1 there exists a bijective map $\mathcal{T}$ such that $K^R(x,n) = \mathcal{T}(K(x,n))$ for any $n < n'(x)$. Thus $K^R(x, n'(x)-1)$ contains only sites in $K(x, n'(x)-1)$ or in the reflection of $K(x, n'(x)-1)$ w.r.t. one perfect wall. By construction $K^R(x, n'(x))$ is obtained by adding all the cells connected to $K^R(x, n'(x)-1)$, so that it is impossible that $K^R(x, n'(x))$ contains sites in two distinct perfect walls Σ_i and Σ_j without intersecting the boundaries of Σ_i and Σ_j which are included in B. This shows that $n(x) < n'(x)$ and that assumption (4.5) is satisfied.

In each partition function $Z_{\Delta^+}^{\mathcal{C}^+,R(k)}$ or $Z_{\Delta^-}^{\mathcal{C}^-,R(k)}$ there are $\left(\frac{L}{L_k}\right)^{d-1}$ reflections at the scale L_k. Each reflection leads to corrections of the order L_k^{d-2} and overall we get an effect of the order $\frac{L^{d-1}}{L_k}$. As k diverges this leads to vanishingly small contributions w.r.t. the surface order L^{d-1}.

Combining the previous estimates, we conclude (5.12).

5.2 Properties

We are going to establish some basic properties of the surface tension

Proposition 5.1. *For any β large enough such that the model is in the Pirogov Sinai regime*

$$\inf_{\vec{n} \in \mathbb{S}^{d-1}} \tau_\beta(\vec{n}) > 0.$$

The positivity of the surface tension defined in (3.11) was already derived in [BKL] (nevertheless the existence of the thermodynamic limit was an assumption in [BKL]).

The homogeneous extension on $\mathbb{R}^d$ of the surface tension is defined by

$$\forall x \in \mathbb{R}^d, \qquad \tau_\beta(x) = \|x\|_2\, \tau_\beta\left(\frac{x}{\|x\|_2}\right), \qquad \tau_\beta(0) = 0\,.$$

Proposition 5.2. *The surface tension τ_β is convex on $\mathbb{R}^d$.*

As a consequence [Am], the functional $\mathcal{W}_\beta$ is lower semi-continuous.

The definition (3.8) of the surface tension in the direction $\vec{n}$ relies on the arbitrary choice of the orientation of the slab along one of the axis (see Section 3). Nevertheless, since τ_β is convex, it is also continuous and therefore the value of the surface tension is independent of the arbitrary choices in the definition.

Proof of Proposition 5.1.
According to Theorem 3.1, it is enough to prove that there is $c_\beta > 0$ such that uniformly over $\vec{n}$ the following holds

$$\forall L > 0, \forall \varepsilon > \frac{1}{L}, \qquad \inf_{\mathcal{C}^+,\mathcal{C}^-} \log \frac{Z_{L,\varepsilon}^{\mathcal{C}^+,\mathcal{C}^-}(\mathcal{S}^+,\mathcal{S}^-)}{Z_{L,\varepsilon}^{\mathcal{C}^+,R}\, Z_{L,\varepsilon}^{\mathcal{C}^-,R}} \leqslant -\, c_\beta L^{d-1}\,. \tag{5.13}$$

At this stage the constraint $(\mathcal{S}^+,\mathcal{S}^-)$ plays no role and can be dropped. Furthermore, it is enough to select the most simple barriers $\mathcal{C}^+,\mathcal{C}^-$ and to derive

$$\forall L > 0, \forall \varepsilon > \frac{1}{L}, \qquad \log \frac{Z_{L,\varepsilon}^{+,-}}{Z_{L,\varepsilon}^{+,R}\, Z_{L,\varepsilon}^{-,R}} \leqslant -\, c_\beta L^{d-1}\,, \tag{5.14}$$

where $Z_{L,\varepsilon}^{+,-}$ denotes the partition function with mixed boundary conditions in the domain $\Lambda_{L,\varepsilon L}$. For simplicity we suppose that $n_d = (\vec{n}\cdot\vec{e}_d) \geqslant 1/\sqrt{d}$.

As explained after the heuristic expansion (3.10), the precise derivation of the surface tension requires to compensate precisely the boundary surface tensions τ_{bd}^+ and τ_{bd}^- appearing in the numerator and the denominator. For (5.13), only a crude bound on τ_{bd}^+ and τ_{bd}^- is necessary. More precisely, by (4.38), there is $C_1 > 0$ such that

$$\left| \log Z_{L,\varepsilon}^{+,R} + \log Z_{L,\varepsilon}^{-,R} - \beta\mathcal{P}|\Lambda_{L,\varepsilon L}| \right| \leqslant \frac{C_1}{n_d} L^{d-1} e^{-\beta/2}\,. \tag{5.15}$$

Due to the mixed $\pm$ b.c. the spin configurations which contribute to $Z_{L,\varepsilon}^{+,-}$ have necessarily an "open" contour Γ whose spatial support, $\mathrm{sp}(\Gamma)$, $*$-disconnects the top and bottom faces of $\Lambda_{L,\varepsilon L}$. The complement of $\bar{\Gamma}$, see Subsection 4.1 for definitions, is made by a finite number of regions, say $\Delta_1,\dots,\Delta_n$, with their boundaries, $\delta\Delta_i$ (i.e., all cells in Δ_i^c, $*$-connected to Δ_i) where the spins have a constant sign, denoted by ξ_i. Then

$$Z_{L,\varepsilon}^{+,-} = \sum_\Gamma e^{-\beta H_h(\sigma_{\bar{\Gamma}})} \prod_{i=1}^n Z_{\Delta_i}^{\xi_i}\,.$$

By (4.28), we get

$$Z_{L,\varepsilon}^{+,-} \leqslant e^{\beta|\Lambda_{L,\varepsilon L}|\mathcal{P}} \sum_{\Gamma} e^{-\beta H_h(\sigma_{\bar\Gamma})+\beta|\bar\Gamma|\mathcal{P}} \prod_{i=1}^{n} e^{e^{-\beta/2+2\alpha} N_{\delta\Delta_i}} .$$

In the last product we use the inequality

$$\sum_{i=1}^{n} N_{\delta\Delta_i} \leqslant 3^d N_\Gamma$$

(as each cell in $\delta\Delta_i$ is $*$-connected to a cell of $\mathrm{sp}(\Gamma)$ and the correspondence is at most 3^d to 1). Moreover, by the definition of contours and using the fact that h belongs to $(0, e^{-\beta/2+2^{d+1}\alpha})$ (see (4.31))

$$-\beta H_h(\sigma_{\bar\Gamma}) \leqslant \beta|h||\bar\Gamma| - \beta N_\Gamma \leqslant -\beta[1-e^{-\beta/2+2^{d+1}\alpha}6^d]N_\Gamma .$$

The previous estimate implies that for β large enough,

$$\begin{aligned} Z_{L,\varepsilon}^{+,-} &\leqslant e^{\beta|\Lambda_{L,\varepsilon L}|\mathcal{P}} \sum_{\Gamma} \\ &\exp\Big\{-\Big(\beta[1-e^{-\beta/2+2^{d+1}\alpha}6^d] - 6^d 2 e^{-\beta/2+2^{d+1}\alpha} - 3^d e^{-\beta/2+2^{d+1}\alpha}\Big)N_\Gamma\Big\} \\ &\leqslant e^{\beta|\Lambda_{L,\varepsilon L}|\mathcal{P}} \sum_{D\ni x^*, N_D \geqslant 2^{-d}L^{d-1}} e^{-\beta/2 N_D} 2^{3^d|D|} , \end{aligned}$$

where the sum is over all connected sets D of cells (D standing for $\mathrm{sp}(\Gamma)$) which contain x^* a point of $\Lambda_{L,\varepsilon L}$ $*$-connected to the surface which separates the $+$ and $-$ boundary conditions; $2^{3^d|D|}$ counts the number of contours with given spatial support. This leads to

$$Z_{L,\varepsilon}^{+,-} \leqslant e^{\beta|\Lambda_{L,\varepsilon L}|\mathcal{P}} e^{-(\beta/2-\alpha)2^{-d}L^{d-1}} . \tag{5.16}$$

Inequalities (5.15) and (5.16) imply

$$\frac{Z_{L,\varepsilon}^{+,-}}{Z_{L,\varepsilon}^{+,R}\, Z_{L,\varepsilon}^{-,R}} \leqslant \exp\Big\{-L^{d-1}\big(2^{-d}(\frac{\beta}{2}-\alpha) - \frac{C_1}{n_d}e^{-\beta/2}\big)\Big\} .$$

Since $n_d \geqslant 1/\sqrt{d}$, for β large enough (5.14) holds.

Proof of Proposition 5.2.
The convexity is equivalent to the pyramidal inequality (see, e.g., [MMR]). To any collection of unit vectors $(\vec n_1, \dots, \vec n_{d+1})$, one associates a pyramid $\Delta(\vec n_1, \dots, \vec n_{d+1})$ with faces $(\mathcal{F}_i)_i$ orthogonal to $(\vec n_i)_i$. Let $|\mathcal{F}_i|$ be the area of $\mathcal{F}_i$. Then the pyramidal inequality means that

$$|\mathcal{F}_1|\,\tau_\beta(\vec n_1) \leqslant \sum_{i=2}^{d+1} |\mathcal{F}_i|\,\tau_\beta(\vec n_i) . \tag{5.17}$$

The derivation of the pyramidal inequality follows closely the approximation scheme explained in the second step of the proof of Theorem 3.1. For a given (L, ε), instead of approximating the surface tension in the slab $\Lambda_{L,\varepsilon L}(\vec{n}_1)$ by localizing the interface in the smaller slabs $\Lambda_{L_k,\varepsilon_k L_k}(\vec{n}_1)$, the interface is constrained to follow a more complicated periodic pattern.

More precisely, the hyperplan orthogonal to $\vec{n_1}$ and going through 0, is paved by unit $(d-1)$-dimensional cubes denoted by $(C^{(\ell)})_\ell$. For any ℓ, let $\mathcal{F}_1^{(\ell)}$ be a translate of $\mathcal{F}_1$ rescaled appropriately to fit in the cube $C^{(\ell)}$. The corresponding pyramid is denoted by $\Delta^{(\ell)}$. In this way, a periodic structure is created

$$\mathcal{Q} = \bigcup_\ell \left(C^{(\ell)} \cup \Delta^{(\ell)}\right) \setminus \mathcal{F}_1^{(\ell)} .$$

The interface will be forced to cross $\Lambda_{L,\varepsilon L}(\vec{n}_1)$ by following the periodic pattern $N\mathcal{Q}$, where $N = \varepsilon^2 L$. This is done by decomposing each flat region of $N\mathcal{Q}$ orthogonal to $\vec{n}_i$ into slabs $\Lambda_{L_k,\varepsilon_k L_k}(\vec{n}_i)$, with $L_k \ll N$. The interface is allowed to fluctuate inside each slab, thus an approximation of the surface tension in each direction $\vec{n}_i$ can be recovered. Since the portion of the interface outside the slabs is small w.r.t. the surface order, its contribution is negligible and we obtain

$$\frac{L^{d-1}}{n_d}\tau_\beta(\vec{n}_1) \leqslant \sum_\ell \left\{ \frac{|C^{(\ell)} \setminus \mathcal{F}_1^{(\ell)}|}{n_d}\tau_\beta(\vec{n}_1) + \sum_{i=2}^{d+1} \frac{|\mathcal{F}_i^{(\ell)}|}{n_d}\,\tau_\beta(\vec{n}_i) \right\} . \tag{5.18}$$

Thus inequality (5.18) follows.

6 Wulff construction

In this section, β is fixed large enough such that the results of Section 4 on the phase transition regime hold. The Gibbs measure with magnetic field $h(\beta)$ and periodic boundary conditions on $\mathbb{T}_N$ is denoted by $\mu_{\beta,N}$.

6.1 Coarse graining

A key step in the analysis of the equilibrium crystal shapes is to extract a precise information from the $\mathbb{L}^1$-estimates by means of a coarse graining. For this purpose, we adapt in our context a coarse graining which was introduced in [B2].

The typical spin configurations are defined at the mesoscopic scale $K = 2^k$. Let $\partial\mathbb{B}_K = \mathbb{B}_{K+K^\alpha} \setminus \mathbb{B}_K$ be the enlarged external boundary of the box $\mathbb{B}_K$, where α is in $(0,1)$. The parameter $\zeta > 0$ will control the accuracy of the coarse graining.

Let x be in $\mathbb{T}_N$ and denote by $\mathbb{B}_K(x)$ the corresponding $\mathcal{B}^{(K)}$-measurable box. For any $\varepsilon = \pm 1$, the box $\mathbb{B}_K(x)$ is ε-*good* if the spin configuration inside the enlarged box $\mathbb{B}_{K+K^\alpha}(x)$ is typical, i.e.,

(P1) The box $\mathbb{B}_K(x)$ is surrounded by at least a connected surface of cells in $\partial\mathbb{B}_K(x)$ with η-labels uniformly equal to ε.

(P2) The average magnetization inside $\mathbb{B}_K(x)$ is close to the equilibrium value m_β^ε of the corresponding pure phase

$$\left|\mathcal{M}_K(x) - m_\beta^\varepsilon\right| \leqslant \zeta \quad \text{and} \quad \mathcal{M}_K(x) = \frac{1}{(2K+1)^d} \sum_{i \in \mathbb{B}_K(x)} \sigma_i \,.$$

See Figure 4.

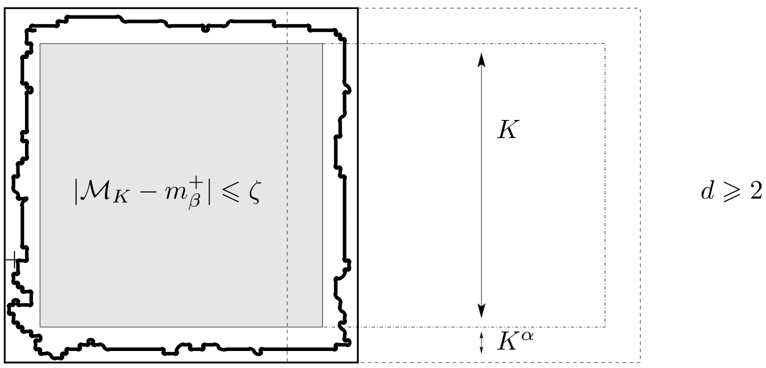

FIGURE 4. Coarse grained configuration with overlapping + good blocks.

On the mesoscopic level, each $\mathcal{B}^{(K)}$-measurable box $\mathbb{B}_K(x)$ is labelled by a mesoscopic phase label

$$\forall x \in \mathbb{T}_N, \qquad u_K^\zeta(x) = \begin{cases} m_\beta^\varepsilon, & \text{if } \mathbb{B}_K(x) \text{ is } \varepsilon\text{-}good\,, \\ 0, & \text{otherwise.} \end{cases}$$

For large mesoscopic boxes, the typical spin configurations occur with overwhelming probability.

Theorem 6.1. *Then for any $\zeta > 0$, the following holds uniformly over N*

$$\forall \{x_1, \ldots, x_\ell\}, \quad \mu_{\beta,N}\left(u_K^\zeta(x_1) = 0, \ldots, u_K^\zeta(x_\ell) = 0\right) \leqslant \left(\rho_K^\zeta\right)^\ell, \tag{6.1}$$

where the parameter ρ_K^ζ vanishes as K goes to infinity.

Despite the fact that the mesoscopic phase labels are not independent, the theorem above ensures that the occurrence of the bad-blocks is dominated by a Bernoulli measure. For the sake of completeness, the proof of Theorem 6.1 is recalled in the Appendix.

As in (2.6), the macroscopic counterpart of the phase labels is defined by

$$u^{\zeta}_{N,K}(x) = u^{\zeta}_{K}\big(\psi_N^{-1}(x)\big), \qquad x \in \widehat{\mathbb{T}}\,.$$

The images of $\mathcal{B}^{(K)}$ boxes by ψ_N are denoted by $\widehat{\mathbb{B}}_{N,K}(x)$.

Any discrepancy in the $\mathbb{L}^1$-norm between the coarse graining and the local order parameter can be neglected with superexponential probability. By construction, for any $x \in \widehat{\mathbb{T}}$ either $|\mathcal{M}_{N,K}(x) - u^{\zeta}_{N,K}(x)|$ is smaller than ζ or the block $\widehat{\mathbb{B}}_{N,K}(x)$ has label $u^{\zeta}_{N,K}(x) = 0$. Using the domination by Bernoulli percolation, the following holds. Given any $\delta > 0$, one can choose the accuracy ζ of the coarse graining and a scale $K_0(\delta, \beta)$ such that for any mesoscopic $K \geqslant K_0$

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N} \Big(\|\mathcal{M}_{N,K} - u^{\zeta}_{N,K}\|_1 > \delta \Big) \;=\; -\infty\,. \tag{6.2}$$

This estimate will enable us to rephrase statements on the local parameter in terms of the phase labels $u^{\zeta}_{N,K}$ which are much easier to handle.

6.2 Equilibrium crystal shapes

The concentration in $\mathbb{L}^1$ of $\mathcal{M}_{N,K}$ to the solutions of the variational problem requires the derivation of precise logarithmic asymptotic in terms of the surface tension.

Proposition 6.1. *Let v be in $\mathrm{BV}(\widehat{\mathbb{T}}, \{m_\beta^-, m_\beta^+\})$, then one can choose $\delta_0 = \delta_0(v)$, such that uniformly in $\delta < \delta_0$*

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N}\big(\|\mathcal{M}_{N,K} - v\|_1 \leqslant \delta\big) \geqslant \, -\mathcal{W}_\beta(v) - o(\delta)\,,$$

where the function $o(\cdot)$ depends only on β and v and vanishes as δ goes to 0.

Proposition 6.2. *For all v in $\mathrm{BV}(\widehat{\mathbb{T}}, \{m_\beta^-, m_\beta^+\})$ such that $\mathcal{W}_\beta(v)$ is finite, one can choose $\delta_0 = \delta_0(v)$, such that uniformly in $\delta < \delta_0$*

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N}\big(\|\mathcal{M}_{N,K} - v\|_1 \leqslant \delta\big) \leqslant \, -\mathcal{W}_\beta(v) + o(\delta)\,,$$

where the function $o(\cdot)$ depends only on β and v and vanishes as δ goes to 0.

6.3 Upper bound

The proof of Proposition 6.2 follows the general scheme of the $\mathbb{L}^1$ Theory. First the boundary $\partial^* v$ is approximated; this enables us to reduce the proof to local computations in small regions. Then in each region the interface is localized on the mesoscopic level by using the minimal section argument. In the last step, the representation of the surface tension (see Definition 3.1) enables us to conclude.

Step 1: Approximation procedure.

We approximate $\partial^* v$ with a finite number of parallelepipeds.

Theorem 6.2. *For any δ positive, there exists s positive such that there are ℓ disjoint parallelepipeds $\widehat{R}^1, \dots, \widehat{R}^\ell$ included in $\widehat{\mathbb{T}}$ with basis $\widehat{B}^1, \dots, \widehat{B}^\ell$ of size length s and height δs. The basis $\widehat{B}^i$ divides $\widehat{R}^i$ in 2 parallelepipeds $\widehat{R}^{i,+}$ and $\widehat{R}^{i,-}$ and the normal to $\widehat{B}^i$ is denoted by $\vec{n}_i$. Furthermore, the parallelepipeds satisfy the following properties*

$$\int_{\widehat{R}^i} |\mathcal{X}_{\widehat{R}^i}(x) - v(x)|\, dx \leqslant \delta\, vol(\widehat{R}^i) \quad \text{and}$$

$$\Big| \sum_{i=1}^{\ell} \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x^{(d-1)} - \mathcal{W}_\beta(v) \Big| \leqslant \delta,$$

where $\mathcal{X}_{\widehat{R}^i} = m_\beta^+ 1_{\widehat{R}^{i,+}} + m_\beta^- 1_{\widehat{R}^{i,-}}$ and the volume of $\widehat{R}^i$ is $vol(\widehat{R}^i) = \delta s^d$.

The proof follows from standard arguments of geometric measure theory (see for example [Ce, B1]). Theorem 6.2 enables us to decompose the boundary into regular sets (see Figure 5) so that it will be enough to consider events of the type

$$\left\{ \mathcal{M}_{N,K} \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i)) \right\},$$

where $\mathcal{V}(\widehat{R}^i, \varepsilon)$ is the ε-neighborhood of $\mathcal{X}_{\widehat{R}^i}$

$$\mathcal{V}(\widehat{R}^i, \varepsilon) = \left\{ v' \in \mathbb{L}^1(\widehat{\mathbb{T}}) \;\middle|\; \int_{\widehat{R}^i} |v'(x) - \mathcal{X}_{\widehat{R}^i}(x)|\, dx \leqslant \varepsilon \right\}.$$

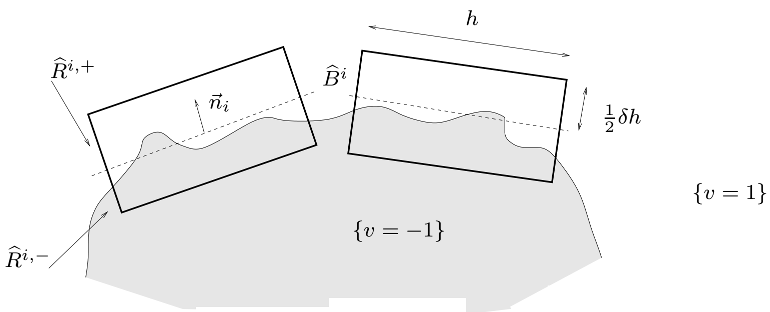

FIGURE 5. Approximation by parallelepipeds.

According to (6.2), the local averaged magnetization can be replaced by the mesoscopic phase labels. Therefore Proposition 6.2 is equivalent to the following statement: for any δ positive, there exists $K_0 = K_0(\delta, h)$, $\zeta_0 = \zeta_0(\delta, h)$ such that uniformly in $K \geqslant K_0$, $\zeta \leqslant \zeta_0$

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N} \left(u_{N,K}^{\zeta} \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i)) \right) \leqslant -\mathcal{W}_\beta(v) + C(\beta, v)\delta. \quad (6.3)$$

The previous inequality localizes the $\mathbb{L}^1$-estimates into regular macroscopic domains R_N^i which are the counterparts of the domains $\Lambda_{sN,\delta sN}(\vec{n}_i)$ introduced in Section 3. To use the definition of the surface tension, one has first to establish the existence of 4 barriers in R_N^i which will play the roles of $\mathcal{C}^+, \mathcal{C}^-$ and $\mathcal{S}^+, \mathcal{S}^-$. The derivation of this boils down to transfer the macroscopic $\mathbb{L}^1$-bounds into a microscopic statement on the localization of an interface inside each R_N^i. This is a key step in the $\mathbb{L}^1$-approach and the coarse graining will play a major role.

Step 2: Minimal section argument.

The microscopic images of $\widehat{R}^{i,\pm}$ in $\mathbb{T}_N$ are denoted $R_N^{i,\pm}$ and we set $R_N^i = R_N^{i,+} \cup R_N^{i,-}$. For simplicity, we will only prove the existence of a + barrier $\mathcal{C}^{i,+}$ lying in the upper part of R_N^i and refer to [B2] for a complete derivation. We consider $\partial^{\mathrm{top}} R_N^i$ the face of R_N^i orthogonal to the vector $\vec{n}_i$ and contiguous to $R_N^{i,+}$. Let $R_N^{i,\mathrm{top}}$ be the set of sites in $R_N^{i,+}$ at distance smaller than $\frac{\delta s}{10} N$ of $\partial^{\mathrm{top}} R_N^i$. At a given mesoscopic scale K, we associate to any spin configuration the set of *bad* boxes which are the boxes $\mathbb{B}_K$ intersecting $R_N^{i,\mathrm{top}}$ with u_K^ζ labels equal to 0 or -1. For any integer j, we set $B_N^{i,j} = B_N^i + j\, c_d K\, \vec{n}_i$ and define

$$B_N^{i,j} = \left\{ y \in R_N^{i,\mathrm{top}} \mid \exists x \in B_N^{i,j}, \qquad \|y - x\| \leqslant 10 \right\}.$$

The sections $\mathcal{B}_j^i$ of the parallelepiped R_N^i are defined as the smallest connected set of $\mathcal{B}^{(K)}$-measurable boxes $\mathbb{B}_K$ intersecting $B_N^{i,j}$. The parameter c_d is chosen such that the $\mathcal{B}_j^i$ are disjoint surfaces of boxes. For j positive, let $n_i^+(j)$ be the number of *bad* boxes in $\mathcal{B}_j^i$ and define

$$n_i^+ = \min \left\{ n_i^+(j) : \qquad \frac{9\delta s}{10 c_d} \frac{N}{K} < j < \frac{\delta s}{c_d} \frac{N}{K} \right\}.$$

Call j^+ the smallest location where the minimum is achieved and define the minimal section in $R_N^{i,\mathrm{top}}$as $\mathcal{B}_{j^+}^i$ (see Figure 6).

For any spin configuration such that $u_{N,K}^\zeta$ belongs to $\bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))$, the number of *bad* boxes in a minimal section is bounded by

$$n_i^+ \leqslant \delta \mathrm{vol}(\widehat{R}^i) \frac{10 c_d}{\delta s} \left(\frac{N}{K} \right)^{d-1} \leqslant 10 c_d \delta s^{d-1} \left(\frac{N}{K} \right)^{d-1}.$$

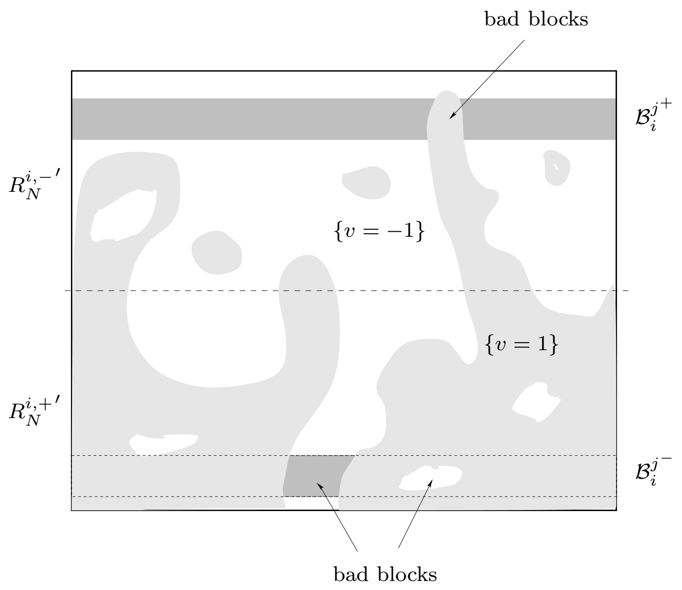

FIGURE 6. Minimal sections.

As $\sum_{i=1}^{\ell} |\widehat{B}^i| = \ell s^{d-1}$ can be controlled in terms of the perimeter of $\partial^* v$, the total number of *bad* boxes is bounded by

$$\sum_{i=1}^{\ell} n_i^+ \leqslant \delta\, C(v) \left(\frac{N}{K}\right)^{d-1} . \tag{6.4}$$

From the very construction of the coarse graining, the + spin surfaces associated to overlapping boxes with $u_{N,K}^{\zeta}$ labels equal to 1 are connected. As each minimal section contains mainly + good blocks, there exist almost a + barrier in each minimal section. By modifying the spin configurations σ on the *bad* boxes, we will complete these + barriers.

More precisely, we associate to any configuration σ the configuration $\bar{\sigma}$ with spins equal to + on the boundary of each *bad* box in the minimal section $\mathcal{B}_{j+}^i$ and equal to σ otherwise. The cost of this surgical procedure can be estimated as follows.

$$\mu_{\beta,N}\left(u_{N,K}^{\zeta} \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))\right) \leqslant \sum_{(i_1,\dots,i_k)} \sum_{(j_1,\dots,j_k)} \sum_{(n_1^+,\dots,n_k^+)} \mu_{\beta,N}\big(\{n_1^+,\dots,n_k^+\}\big) . \tag{6.5}$$

The right-hand side takes into account the fact that in the domains $R^{i_1},\dots,R^{i_k}$, the minimal sections are at heights $j_1,\dots,j_k \in [\frac{9\delta s N}{10K}, \frac{\delta s N}{K}]$ and contain $n_1^+,\dots,n_k^+$

bad boxes such that (6.4) holds. Once the location of the *bad* boxes is fixed, the number of spin flips to modify σ into $\bar\sigma$ is at most $C(v)\left(\frac{N}{K}\right)^{d-1} K^{d-1}$. By construction $\bar\sigma$ belongs to the set $\mathcal{A}_1$ of spin configurations which contain a + barrier in the upper part of each domain R_N^i

$$\mu_{\beta,N}\big(\{n_1^+,\dots,n_k^+\}\big) \leqslant \prod_{\alpha=1}^{k} \begin{pmatrix} (sN/K)^{d-1} \\ n_\alpha^+ \end{pmatrix} \exp\big(\delta\, C_2(v,\beta)N^{d-1}\big)\, \mu_{\beta,N}\big(\mathcal{A}_1\big)\,,$$

where $(sN/K)^{d-1}$ refers to the total number of blocks in each minimal section. Summing over all the configurations and using (6.4) again, we obtain

$$\sum_{(n_1^+,\dots,n_k^+)} \mu_{\beta,N}\big(\{n_1^+,\dots,n_k^+\}\big) \leqslant \exp\big(o(\delta)\, C_3(v,\beta)N^{d-1}\big)\, \mu_{\beta,N}\big(\mathcal{A}_1\big)\,. \tag{6.6}$$

Finally replacing (6.6) in (6.5), we get

$$\mu_{\beta,N}\left(u^{\zeta}_{N,K} \in \bigcap_{i=1}^{\ell} \mathcal{V}(\widehat{R}^i, \delta \mathrm{vol}(\widehat{R}^i))\right) \leqslant 2^{\ell}\left(\frac{N}{K}\right)^{\ell} \exp\big(o(\delta)\, C_3(v,\beta)N^{d-1}\big)\, \mu_{\beta,N}\big(\mathcal{A}_1\big)\,. \tag{6.7}$$

Repeating the same argument, we can consider instead of $\mathcal{A}_1$ an event $\mathcal{A}$ which contains at least 4 barriers in each R_N^i. For any spin configuration in $\mathcal{A}$, we define the set of sites $\mathcal{C}^{i,+}$ as the support of the + barrier in $R_N^{i,+}$ which is the closest to $\partial^{\mathrm{top}} R_N^i$. In the same way, $\mathcal{C}^{i,-}$ is the location of the $-$ barrier in the lower part of R_N^i which is the closest to $(R_N^i)^c$. By analogy with the notation of Section 3, the set of spin configurations which contain a + and a $-$ barrier in the domain $\Lambda(\mathcal{C}^{i,+},\mathcal{C}^{i,-})$ is denoted by $\mathcal{S}^i = (\mathcal{S}^{i,+},\mathcal{S}^{i,-})$.

Step 3: Surface tension estimates.

As a consequence of the previous step, for any spin configuration in $\mathcal{A}$, there exists a microscopic interface localized in each cube R_N^i. Thus we are now in a good shape to check that

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N}(\mathcal{A}) \leqslant -\sum_{i=1}^{\ell} \int_{\widehat{B}_i} \tau_\beta(\vec n_i)\, d\mathcal{H}_x + C(\beta,v,\delta)\,, \tag{6.8}$$

where $C(\beta,v,\delta)$ vanishes as δ tends to 0. Combining the previous inequality with (6.7), we deduce (6.3). We now proceed in deriving (6.8).

We first pin the interfaces on the sides of each R_N^i by imposing that the boxes on the boundary of each $R_N^{i,+}$ (resp $R_N^{i,-}$) parallel to $\vec n_i$ have η labels equal to 1 (resp -1). Since the height of R_N^i is δs, this procedure requires to modify at most

$\delta s^{d-1} N^{d-1}$ spins. Therefore this has no further impact on the evaluation of the statistical weights of the configurations because the cost of flipping these spins is bounded by $\exp(\delta C(v)\, N^{d-1})$.

In this way, the domain $\mathbb{T}_N$ is partitioned into the domains $\Lambda(\mathcal{C}^{i,+}, \mathcal{C}^{i,-})$ and a remainder which will be denoted by Δ.

$$\mu_{\beta,N}(\mathcal{A}) = \frac{1}{Z_{\beta,N}} \sum_{(\mathcal{C}^{i,+}, \mathcal{C}^{i,-})} Z_\Delta^\omega \prod_{i=1}^{\ell} Z_{N,\delta N}^{\mathcal{C}^{i,+}, \mathcal{C}^{i,-}} (\mathcal{S}^i)\,,$$

where the boundary conditions ω are imposed by the values of the spins outside $\cup_i \Lambda(\mathcal{C}^{i,+}, \mathcal{C}^{i,-})$.

Introducing by force the partition functions with the perfect walls we get

$$\mu_{\beta,N}(\mathcal{A}) = \frac{1}{Z_{\beta,N}} \sum_{(\mathcal{C}^{i,+}, \mathcal{C}^{i,-})} Z_\Delta^\omega \prod_{i=1}^{\ell} Z_{N,\delta N}^{\mathcal{C}^{i,+},R} Z_{N,\delta N}^{\mathcal{C}^{i,-},R} \prod_{i=1}^{\ell} \frac{Z_{N,\delta N}^{\mathcal{C}^{i,+}, \mathcal{C}^{i,-}} (\mathcal{S}^i)}{Z_{N,\delta N}^{\mathcal{C}^{i,+},R} Z_{N,\delta N}^{\mathcal{C}^{i,-},R}}\,. \tag{6.9}$$

By Definition 3.1 of the surface tension, the last term in the right-hand side is bounded by

$$\prod_{i=1}^{\ell} \frac{Z_{N,\delta N}^{\mathcal{C}^{i,+}, \mathcal{C}^{i,-}} (\mathcal{S}^i)}{Z_{N,\delta N}^{\mathcal{C}^{i,+},R} Z_{N,\delta N}^{\mathcal{C}^{i,-},R}} \leqslant \exp\left(-N^{d-1} \left[\sum_{i=1}^{\ell} \int_{\widehat{B}_i} \tau_\beta(\vec{n}_i) d\mathcal{H}_x + |\widehat{B}_i| c(\beta, N, \delta) \right] \right), \tag{6.10}$$

where the remainder $c(\beta, N, \delta)$ satisfies

$$\limsup_{\delta \to 0} \limsup_{N \to 0}\ c(\beta, N, \delta) = 0\,.$$

In order to complete the derivation of (6.8), it remains to check that

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log \left(\frac{1}{Z_{\beta,N}} \sum_{(\mathcal{C}^{i,+}, \mathcal{C}^{i,-})} Z_\Delta^\omega \prod_{i=1}^{\ell} Z_{N,\delta N}^{\mathcal{C}^{i,+},R} Z_{N,\delta N}^{\mathcal{C}^{i,-},R} \right) = \lim_{N\to\infty} \frac{1}{N^{d-1}} \log \frac{Z_{\beta,N}^R}{Z_{\beta,N}} = 0\,,$$

where $Z_{\beta,N}^R$ denotes the partition function in $\mathbb{T}_N$ where the interactions have been reflected in the middle of each R_N^i. The previous statement follows readily from (4.40) where the contribution of the reflected boundary conditions to the pressure are proven to be of order N^{d-2}. Nevertheless in order to apply (4.40), we have first to check that the assumption (4.5) holds for the particular topology imposed by the reflections. If assumption (4.5) fails, it is easy to see that one can decompose each parallelepiped R_N^i into smaller parallelepipeds $\{R_N^{i,k}\}_k$ of side-length $h' \ll h$

for which Theorem 6.2 still holds (see the proof in [B1]). If h' is smaller than the mutual distance between the parallelepipeds $\{R_N^i\}_i$, a set $K^R(x,n)$ cannot intersect two regions $R_N^{j,k}$ and $R_N^{j',k'}$ with $j \neq j'$ without touching the boundary conditions B. Following the argument detailled in the third step of Subsection 5.1, we can then exclude multiple reflections between cubes $\{R_N^{i,k}\}_k$. Thus assumption (4.5) is also valid in this setup.

6.4 Lower bound

In order to derive Proposition 6.1, it is enough to consider the typical spin configurations which contain a microscopic contour in a neighborhood of the boundary of $\partial^* v$. At this stage, Theorem 3.1 becomes necessary.

Step 1: Approximation procedure.

We first start by approximating the boundary $\partial^* v$ by a regular surface $\partial\widehat{V}$. A polyhedral set has a boundary included in the union of a finite number of hyperplanes. The surface $\partial^* v$ can be approximated as follows (see Figure 7)

Theorem 6.3. *For any δ positive, there exists a polyhedral set $\widehat{V}$ such that*

$$\|\mathbb{1}_{\widehat{V}} - v\|_1 \leqslant \delta \qquad \text{and} \qquad \big|\mathcal{W}_\beta(\widehat{V}) - \mathcal{W}_\beta(v)\big| \leqslant \delta.$$

For any s small enough there are ℓ disjoint parallelepipeds $\widehat{R}^1,\dots,\widehat{R}^\ell$ with basis $\widehat{B}^1,\dots,\widehat{B}^\ell$ included in $\partial\widehat{V}$ of side-length s and height δs. Furthermore, the sets $\widehat{B}^1,\dots,\widehat{B}^\ell$ cover $\partial\widehat{V}$ up to a set of measure less than δ denoted by $\widehat{U}^\delta = \partial\widehat{V} \setminus \bigcup_{i=1}^\ell \widehat{B}^i$ and they satisfy

$$\Big|\sum_{i=1}^{\ell} \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x^{(d-1)} - \mathcal{W}_\beta(v)\Big| \leqslant \delta,$$

where the normal to $\widehat{B}^i$ is denoted by $\vec{n}_i$.

The proof is a direct application of Reshtnyak's Theorem and can be found in the paper of Alberti, Bellettini [AlBe].

Using Theorem 6.3, we can reduce the proof of Proposition 6.1 to the computation of the probability of $\{\|\mathcal{M}_{N,K} - \mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\}$. According to (6.2) the estimates can be restated in terms of the mesoscopic phase labels. It will be enough to show that: for any $\delta > 0$, there exists $\zeta = \zeta(\delta)$ and $K_0(\delta)$ such that for all $K \geqslant K_0$

$$\liminf_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N} \left(\|u_{N,K}^\zeta - \mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\right) \geqslant -\mathcal{W}_\beta(\widehat{V}) - o(\delta), \tag{6.11}$$

where the function $o(\delta)$ vanishes as δ goes to 0.

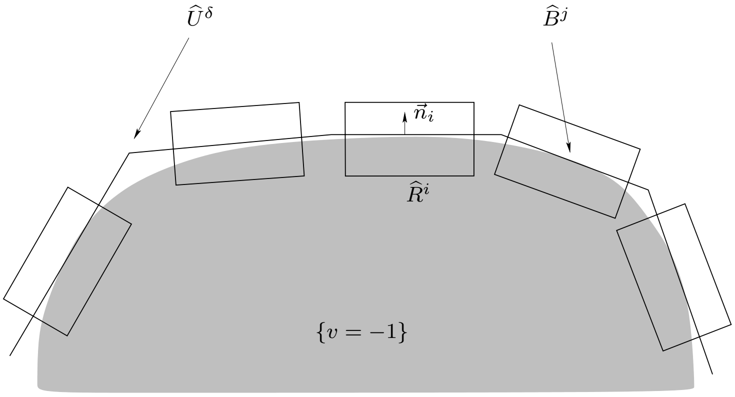

FIGURE 7. Polyhedral approximation.

Step 2: Localization of the interface.

The images of $\widehat{V}$, $\widehat{R}^i$ and $\widehat{U}^\delta$ in $\mathbb{T}_N$ will be denoted by V_N, R_N^i and U_N^δ. We split R_N^i into $R_N^{i,-}$ and $R_N^{i,+}$ which are the microscopic counterparts of $\widehat{V}\cap\widehat{R}^i$ and $\widehat{R}^i\setminus\widehat{V}$.

We will enforce the occurrence of a microscopic interface along the boundary $\partial\widehat{V}$. As in the derivation of the upper bound, the domains R_N^i are the counterparts of $\Lambda_{N,\delta N}(\vec{n}_i)$. Let $\mathcal{A}^{i,+}$ be the event that there are two + barriers in $R_N^{i,+}$ and $\mathcal{A}^{i,-}$ the analogous event with two − barriers in $R_N^{i,-}$. The ± barrier in $R_N^{i,\pm}$ which is the closest from $(R_N^i)^c$ is denoted by $\mathcal{C}^{i,\pm}$. We set $\mathcal{A}=\bigcap_{i=1}^{\ell}\mathcal{A}^{i,+}\cap\mathcal{A}^{i,-}$. Let us also define $\mathcal{D}^{i,+}$ (resp $\mathcal{D}^{i,-}$) the set of spin configurations such that the η-labels are equal to 1 (resp −1) on the sides of $R_N^{i,+}$ (resp $R_N^{i,-}$) parallel to $\vec{n}_i$. In order to construct a closed contour of spins surrounding V_N, we define $\mathcal{D}$ as the set of configurations in $\mathcal{D}^{i,+}$ and $\mathcal{D}^{i,-}$ such that the blocks on one side of U_N^δ have η-labels − and + in the other side.

Any spin configuration in $\mathcal{A}\cap\mathcal{D}$ contains a microscopic interface which decouples V_N from its complement. One has

$$\mu_{\beta,N}\Big(\|u_{N,K}^{\zeta}-\mathbb{1}_{\widehat{V}}\|_1\leqslant\delta\Big)\geqslant\mu_{\beta,N}\Big(\Big\{\|u_{N,K}^{\zeta}-\mathbb{1}_{\widehat{V}}\|_1\leqslant\delta\Big\}\cap\mathcal{A}\cap\mathcal{D}\Big).\qquad(6.12)$$

The spin configurations inside V_N (resp V_N^c) are surrounded by − (resp +) boundary conditions, so that they are in equilibrium in the − (resp +) pure phase. Bulk estimate imply that one can choose s small enough, $\zeta'=\zeta'(\delta)$ and $K_0'=K_0'(\delta)$ such that

$$\lim_{N\to\infty}\ \mu_{\beta,N}\Bigg(\int_{\widehat{V}^c}|u_{N,K}^{\zeta'}(x)-m_\beta^+|\,dx\geqslant\frac{\delta}{2}\ \text{or}\ \int_{\widehat{V}}|u_{N,K}^{\zeta'}(x)-m_\beta^-|\,dx\geqslant\frac{\delta}{2}\ \Bigg|\ \mathcal{A}\cap\mathcal{D}\Bigg)=0\,.$$

(This limit can be obtained by using a proof similar to the one of Theorem 6.4.)

So that (6.12) can be rewritten for N large enough as

$$\mu_{\beta,N}\left(\|u_{N,K}^{\zeta'} - \mathbb{1}_{\widehat{V}}\|_1 \leqslant \delta\right) \geqslant \frac{1}{2}\mu_{\beta,N}\left(\mathcal{A}\cap\mathcal{D}\right). \tag{6.13}$$

Step 3: Surface tension.

Let Λ be the union of the sets $\Lambda_i = \Lambda(\mathcal{C}^{i,+},\mathcal{C}^{i,-})$. The configurations in the event $\mathcal{A}\cap\mathcal{D}$ contain two closed surfaces with $+$ and $-$ blocks which partition the domain $\mathbb{T}_N$ into 3 regions.

$$\mathbb{T}_N = \Lambda \cup \Delta^+ \cup \Delta^-,$$

where $\Delta^\pm$ represents the location of the $\pm$ pure phases and Λ is concentrated along the interface. We proceed now to evaluate the right-hand side of (6.13)

$$\mu_{\beta,N}\left(\mathcal{A}\cap\mathcal{D}\right) \geqslant \frac{1}{Z_N}\sum_{\mathcal{C}^{i,+},\mathcal{C}^{i,-}} Z^+_{\Delta^+} Z^-_{\Delta^-}\prod_i Z_{\Lambda_i}^{\mathcal{C}^{i,+},\mathcal{C}^{i,-}}(\mathcal{S}_i),$$

where we used analogous notation to Section 3 for the partition function with mixed boundary conditions. Introducing the partition functions with reflected boundary conditions we get

$$\mu_{\beta,N}\left(\mathcal{A}\cap\mathcal{D}\right) \geqslant \frac{1}{Z_N}\sum_{\mathcal{C}^{i,+},\mathcal{C}^{i,-}} Z^+_{\Delta^+} Z^-_{\Delta^-} Z_{\Lambda_i}^{\mathcal{C}^{i,+},R} Z_{\Lambda_i}^{\mathcal{C}^{i,-},R} \prod_i \frac{Z_{\Lambda_i}^{\mathcal{C}^{i,+},\mathcal{C}^{i,-}}(\mathcal{S}_i)}{Z_{\Lambda_i^+}^{\mathcal{C}^{i,+},R} Z_{\Lambda_i^-}^{\mathcal{C}^{i,-},R}}, \tag{6.14}$$

where $\Lambda_i^\pm$ refers to the sets $\Lambda^\pm(\mathcal{C}^{i,\pm})$ which were introduced in Subsection 3.2. The last term in the right-hand side is an approximation of the surface tension in each domain Λ_i, therefore Theorem 3.1 implies

$$\begin{aligned}\inf_{\mathcal{C}^{i,+},\mathcal{C}^{i,-}} \frac{1}{N^{d-1}}\sum_i \log \frac{Z_{\Lambda_i}^{\mathcal{C}^{i,+},\mathcal{C}^{i,-}}(\mathcal{S}_i)}{Z_{\Lambda_i^+}^{\mathcal{C}^{i,+},R} Z_{\Lambda_i^-}^{\mathcal{C}^{i,-},R}} \\ \geqslant -\sum_i \int_{\widehat{B}^i} \tau_\beta(\vec{n}_i)\, d\mathcal{H}_x^{(d-1)} - P(v)c(\delta,N),\end{aligned} \tag{6.15}$$

where $\lim_{\delta\to 0}\lim_{N\to\infty} c(\delta,N) = 0$ and $P(v)$ is the perimeter of v.

It remains to check that

$$\lim_{N\to\infty}\frac{1}{N^{d-1}}\log\left(\frac{1}{Z_N}\sum_{\mathcal{C}^{i,+},\mathcal{C}^{i,-}} Z^+_{\Delta^+} Z^-_{\Delta^-} Z_{\Lambda_i^+}^{\mathcal{C}^{i,+},R} Z_{\Lambda_i^-}^{\mathcal{C}^{i,-},R}\right) = 0. \tag{6.16}$$

Combining inequalities (6.15) and (6.16) we see that

$$\liminf_{N\to\infty}\frac{1}{N^{d-1}}\log\mu_{\beta,N}\left(\mathcal{A}\cap\mathcal{D}\right) \geqslant -\sum_{i=1}^{\ell}\int_{\widehat{B}^i}\tau_\beta(\vec{n}_i)\, d\mathcal{H}_x^{(d-1)} - o(\delta).$$

Using Theorem 6.3 and letting δ vanish, we conclude the proof of Proposition 6.1.

We turn now to the derivation of (6.16). Since the reflected boundary conditions decouple the system, the numerator should be understood as the product of two partition functions associated to the sets $\bar{\Delta}^+ = \Delta^+ \cup_i \Lambda^+(\mathcal{C}^{i,+})$ and $\bar{\Delta}^- = \Delta^- \cup_i \Lambda^-(\mathcal{C}^{i,-})$, where $\Lambda(\mathcal{C}^{i,\pm})$ denotes the part of Λ_i. It is important to note that contrary to $\Delta^\pm$, the sets $\bar{\Delta}^\pm$ are independent of the choice of the surfaces $\mathcal{C}^{i,\pm}$. In particular, following the notation of Section 3,

$$\sum_{\mathcal{C}^{i,+}} Z^+_{\Delta^+}\, Z^{\mathcal{C}^{i,+},R}_{\Lambda_i^+} = Z^R_{\bar{\Delta}^+}(\mathcal{C}^{i,+}),$$

where the right-hand side denotes the partition function on $\bar{\Delta}^+$ under the constraint that in each $\mathfrak{R}^{i,+}_N$ there is a + barrier. Applying the same strategy as for the derivation of (5.11), we can check that

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log \frac{Z^R_{\bar{\Delta}^+}(\mathcal{C}^{i,+})}{Z^R_{\bar{\Delta}^+}} = 0\,.$$

This implies that (6.16) is equivalent to

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log \frac{Z^R_{\bar{\Delta}^+}\, Z^R_{\bar{\Delta}^-}}{Z_N} = 0\,. \tag{6.17}$$

The partition functions in the numerator take also into account the constraints imposed by the set $\mathcal{D}$ on the spins along the set U^δ_N and on the sides of R^i_N parallel to $\vec{n}_i$. These constraints can be released up to a small cost w.r.t. the surface order. This comes from the fact that the event $\mathcal{D}$ is supported by at most $c(d,\delta)N^{d-1}$ edges where $c(d,\delta)$ vanishes as δ goes to 0. Therefore the probability of $\mathcal{D}$ is negligible with respect to a surface order and we get

$$\left| \log \frac{Z^R_{\bar{\Delta}^+}\, Z^R_{\bar{\Delta}^-}}{Z^R_N} \right| \leqslant c(d,\delta)N^{d-1}\,, \tag{6.18}$$

where Z^R_N is the unconstrained partition function on $\mathbb{T}_N$ for which the interactions in the middle of each R^i_N have been modified and replaced by perfect walls. Again by the same considerations as in the last argument of the proof of the upper bound (see Subsection 6.3), one check that one can find a polyhedral approximation for which assumption (4.5) is satisfied. The corrections to the pressure induced by the reflection are negligible w.r.t. the surface order (see (4.40)) so that

$$\lim_{N\to\infty} \frac{1}{N^{d-1}} \log \frac{Z^R_N}{Z_N} = 0\,.$$

This, combined with (6.18) implies the validity of (6.16).

6.5 Exponential tightness

The purpose of this subsection is to prove that phase coexistence cannot occur by creation of many small droplets. Rephrased in a mathematical way, this means that with an overwhelming probability, the configurations will concentrate close to the compact set

$$\mathcal{K}_a = \big\{ v \in \mathrm{BV}(\widehat{\mathbb{T}}, \{m_\beta^-, m_\beta^+\}) \mid \quad P(\{v = m_\beta^-\}) \leqslant a \big\}, \tag{6.19}$$

where P denotes the perimeter and a will be chosen large enough.

Proposition 6.3. *There exists a constant $C(\beta) > 0$ such that for all δ positive one can find $K_0(\delta)$ such that for $K \geqslant K_0$*

$$\forall a > 0, \qquad \limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N}\left(\mathcal{M}_{N,K} \notin \mathcal{V}(\mathcal{K}_a, \delta)\right) \leqslant -C(\beta)\, a,$$

where $\mathcal{V}(\mathcal{K}_a, \delta)$ is the δ-neighborhood of $\mathcal{K}_a$ in $\mathbb{L}^1(\widehat{\mathbb{T}})$.

The estimate (6.2) allows us to shift our attention from the local averaged magnetization to the mesoscopic phase labels. In particular Proposition 6.3 follows from

Theorem 6.4. *Fix $\zeta > 0$. For every $a > 0$ and $\delta > 0$ there exists a finite scale $K_0(\delta)$, such that for all $K \geqslant K_0$*

$$\limsup_{N\to\infty} \frac{1}{N^{d-1}} \log \mu_{\beta,N}\Big(u^{\zeta}_{N,K} \notin \mathcal{V}(\mathcal{K}_a, 2\delta)\Big) \leqslant -c(\beta, K)a\,, \tag{6.20}$$

where $c(\beta, K)$ is a positive constant.

The core of the proof relies on the control of the phase of small contours by means of an entropy/energy argument. The argument is standard and depends only on the structure of the coarse graining. We refer the reader to [BIV1] (Theorem 2.2.1), where Proposition 6.4 was derived in a complete generality. Finally, notice that similar arguments can easily be adapted to multi-phase models (see Remark 3.4 in [BIV2]).

Theorem 2.1 can be obtained by combining Propositions 6.3, 6.1, 6.2. Since $\mathcal{K}_a$ is compact with respect to the $\mathbb{L}^1$ topology (see [EG]), the exponential tightness property 6.3 enables us to focus only on a finite number of configurations close to $\mathcal{K}_a$. The precise asymptotic of these configurations is then estimated by Propositions 6.1, 6.2 (see [B1] for details).

A Proof of Theorem 6.1

The magnetic field is equal to $h(\beta)$ and omitted from the notation throughout the proof. The proof follows the argument developed in [B2].

Step 1. Let us start with a single box. If $\mathbb{B}_K(x)$ is not a *good* box then either there is a contour of length at least K^α crossing the enlarged boundary or conditionally on the event that the box $\mathbb{B}_K(x)$ is surrounded by a surface of η-block spins of sign ε_x, the magnetization $\mathcal{M}_K(x)$ is atypical. These two occurrences can be estimated separately. Applying the Peierls estimate (4.11), we get

$$\mu_{\beta,N}\big(\text{there is a contour crossing } \partial\mathbb{B}_K(x)\big) \leqslant K^{d-1}\exp(-c_\beta K^\alpha)\,. \tag{A.1}$$

Conditionally on the occurrence of a connected surface $\mathcal{S}$ of η-block spins of sign ε_x surrounding the box $\mathbb{B}_K(x)$, the configurations inside $\mathbb{B}_K(x)$ are decoupled from the exterior. We first use Tchebyshev inequality

$$\mu_{\beta,N}\left(\{|\mathcal{M}_K(x) - m_\beta^{\varepsilon_x}| \geqslant \zeta\} \;\big|\; \mathcal{S}\right) \leqslant \frac{1}{\zeta^2 K^{2d}}\mu^{\varepsilon_x}_{\beta,\mathrm{int}(\mathcal{S})}\left(\Big(\sum_{i\in\mathbb{B}_K(x)}\sigma_i - m_\beta^{\varepsilon_x}\Big)^2\right),$$

where $\mathrm{int}(\mathcal{S})$ is the region surrounded by $\mathcal{S}$. As $\mathcal{S}$ has been chosen as the closest surface to $(\mathbb{B}_{K+K^\alpha})^c$, the magnetization inside the box $\mathbb{B}_K(x)$ is measurable after the conditioning. Classical Pirogov-Sinai theory ensures also that under the assumptions of Theorem 4.2, the correlations decay exponentially in the ε_x-pure phase, so that we obtain

$$\mu^{\varepsilon_x}_{\beta,h,\mathrm{int}(\mathcal{S})}\left(\{|\mathcal{M}_K(x) - m_\beta^{\varepsilon_x}| \geqslant \zeta\}\right) \le \frac{1}{\zeta^2 K^d}\chi\,, \tag{A.2}$$

where the susceptibility $\chi = \sum_{i\in\mathbb{Z}^d}\mu^+_\beta(\sigma_0;\sigma_i)$ is finite.

Step 2. In order to evaluate the probability of the event

$$\left\{u_K^\zeta(x_1) = 0, \dots, u_K^\zeta(x_\ell) = 0\right\}$$

the partition $\mathcal{B}^{(K)}$ is sub-divised into c_d sub-partitions $(\mathcal{B}_i^{(K)})_{i\,\leqslant\, c_d}$ such that two cubes of size $K+K^\alpha$ centered on two sites of $\mathcal{B}_i^{(K)}$ are disjoint. By applying Hölder inequality, the estimate (6.1) is reduced to cubes which are not nearest neighbors.

$$\mu_{\beta,N}\left(u_K^\zeta(x_1) = 0, \dots, u_K^\zeta(x_\ell) = 0\right) \leqslant \prod_{i=1}^{c_d}\mu_{\beta,N}\left(\forall x_j \in \mathcal{D}_i^{(K)}, \quad u_K^\zeta(x_j) = 0\right)^{\frac{1}{c_d}}.$$

Step 3. The event $\left\{u_K^\zeta(x_1) = 0, \dots, u_K^\zeta(x_\ell) = 0\right\}$ can be decomposed into 2 terms: on ℓ' boxes the density is atypical, whereas there are contours crossing the $\ell-\ell'$ enlarged boundaries of the remaining boxes.

For a given collection of j boxes, we define

$$\begin{aligned}\mathcal{A}_j &= \{\text{The } j \text{ boxes are surrounded by } \pm \text{ surfaces, but their averaged}\\ &\qquad\quad \text{magnetizations are non-typical}\}\\ \mathcal{B}_j &= \{\text{There are contours crossing the } j \text{ enlarged boundaries of the boxes}\}\,.\end{aligned}$$

The probabilities of both events can be evaluated as follows. As the j boxes are disjoint and the surfaces of blocks decouple the configurations inside each box

$$\mu_{\beta,N}(\mathcal{A}_j) \leqslant \big(\mu_{\beta,N}(\mathcal{A}_1)\big)^j \leqslant \big(\alpha_K\big)^j \,,$$

where the constant $\alpha_K = \frac{\chi}{\zeta^2 K^d}$ was introduced in (A.2).

$$\mu_{\beta,N}(\mathcal{B}_j) = \sum_{i=1}^{j} \mu_{\beta,N}\big(\{\exists\ i \text{ contours crossing the } j \text{ enlarged boundaries}\}\big)\,.$$

We choose i blocks as starting points of these contours. Then we have to evaluate

$$\sum_{|\Gamma_1|+\cdots+|\Gamma_i| \,\geqslant\, jK^\alpha} \mu_{\beta,N}(\Gamma_1,\dots,\Gamma_i)\,,$$

where the contours $(\Gamma_1,\dots,\Gamma_i)$ have also to cross each boundaries of the j cubes.

Let n_r be the number of boundaries crossed by the contour r

$$\sum_{|\Gamma_1|+\cdots+|\Gamma_i| \,\geqslant\, jK^\alpha} \mu_{\beta,N}(\Gamma_1,\dots,\Gamma_i) \leqslant \sum_{n_1+\cdots+n_i=j} \ \sum_{(\Gamma_r,n_r)} \mu_{\beta,N}(\Gamma_1,\dots,\Gamma_i)\,.$$

If a contour crosses n_r boundaries then it has a length at least $n_r K^\alpha + (n_r-1)K$ because the distance between the boxes is at least K. Thus

$$\begin{aligned}
&\sum_{|\Gamma_1|+\cdots+|\Gamma_i| \,\geqslant\, jK^\alpha} \mu_{\beta,N}(\Gamma_1,\dots,\Gamma_i)\\
&\qquad\leqslant \sum_{n_1+\cdots+n_i=j} \prod_{r=1}^{i} \exp(-c_\beta n_r K^\alpha - c_\beta(n_r-1)K)\\
&\qquad\leqslant \exp(-c_\beta jK^\alpha)\left(\sum_{n=1}^{\infty} \exp(-c_\beta(n-1)K)\right)^i\\
&\qquad\leqslant C^i \exp(-c_\beta jK^\alpha)\,.
\end{aligned}$$

Finally

$$\begin{aligned}
\mu_{\beta,N}(\mathcal{B}_j) &\leqslant \sum_{i=1}^{j} \binom{j}{i} K^{(d-1)i} C^i \exp(-c_\beta jK^\alpha)\\
&\leqslant \exp(-c_\beta\, jK^\alpha)(1+CK^{d-1})^j = (\alpha'_K)^j\,,
\end{aligned}$$

where the constant α'_K vanishes as K goes to infinity.

Combining both estimates, we obtain

$$\mu_{\beta,N}\left(u_K^\zeta(x_1)=0,\ldots,u_K^\zeta(x_\ell)=0\right) \leqslant \sum_{\ell'=1}^{\ell}\binom{\ell}{\ell'}\mu_{\beta,N}(\mathcal{A}_{\ell'})^{1/2}\mu_{\beta,N}(\mathcal{B}_{\ell-\ell'})^{1/2} \leqslant \left(\alpha_K+\alpha'_K\right)^{\ell}.$$

This completes the proof.

Acknowledgment

We are indebted to R. Kotecky, S. Shlosman and Y. Velenik for many helpful discussions. The first part of this work was done at the IHP where both authors participated to the semester on "hydrodynamic limits". T.B. acknowledges kind hospitality at Roma Tor Vergata where this paper has been completed.

References

[AlBe] G. Alberti, G. Bellettini, Asymptotic behavior of a non local anisotropic model for phase transition, *J. Math. Ann.* **310**, No. 3, 527–560, (1998).

[Am] L. Ambrosio, On the lower semicontinuity of quasiconvex integrals in SBV(Ω, R^k), *Nonlinear Anal.* **23**, No. 3, 405–425, (1994).

[ABCP] G. Alberti, G. Bellettini, M. Cassandro, E. Presutti, Surface tension in Ising system with Kac potentials, *J. Stat. Phys.* **82**, 743–796 (1996).

[Al] K. Alexander, Cube-root boundary fluctuations for droplets in random cluster models, *Comm. Math. Phys.* **224**, No. 3, 733–781 (2001).

[BCP] G. Bellettini, M. Cassandro, E. Presutti, Constrained minima of non local free energy functionals, *J. Stat. Phys.* **84**, 1337–1349 (1996).

[BBBP] O. Benois, T. Bodineau, P. Butta, E. Presutti, On the validity of van der Waals theory of surface tension, *Mark. Proc. and Rel. Fields* **3**, 175–198 (1997).

[BBP] O. Benois, T. Bodineau, E. Presutti, Large deviations in the van der Waals limit, *Stoch. Proc. and Appl.* **75**, 89–104 (1998).

[B1] T. Bodineau, The Wulff construction in three and more dimensions, *Comm. Math. Phys.* **207**, 197–229 (1999).

[B2] T. Bodineau, *Phase coexistence for the Kac Ising models*, 75–111, Progr. Probab., 51, Birkhäuser Boston, Boston, MA, (2002).

[BIV1] T. Bodineau, D. Ioffe, Y. Velenik, Rigorous probabilistic analysis of equilibrium crystal shapes, *J. Math. Phys.* **41**, No.3, 1033–1098 (2000).

[BIV2] T. Bodineau, D. Ioffe, Y. Velenik, Winterbottom construction for finite range ferromagnetic models: a $\mathbb{L}^1$-approach, *J. Stat. Phys.* **105**, No 1/2, 93–131 (2001).

[BKL] J. Bricmont, K. Kuroda, J. Lebowitz, Surface tension and phase coexistence for general lattice systems, *J. Stat. Phys.* **33**, 59–75 (1983).

[Ce] R. Cerf, Large deviations for three-dimensional supercritical percolation, *Astérisque* **267** (2000).

[CePi1] R. Cerf, A. Pisztora, On the Wulff crystal in the Ising model, *Ann. Probab.* **28**, No. 3, 947–1017 (2000).

[CePi2] R. Cerf, A. Pisztora, Phase coexistence in Ising, Potts and percolation models, *Ann. Inst. H. Poincaré Probab. Statist.* **37**, No. 6, 643–724 (2001).

[CIV] M. Campanino, D. Ioffe, Y. Velenik, Ornstein-Zernike Theory for the finite range Ising models above T_c, *Probab. Theory Related Fields* **125**, No. 3, 305–349 (2003).

[DKS] R.L. Dobrushin, R. Kotecký, S. Shlosman, *Wulff construction: a global shape from local interaction*, AMS translations series, vol 104, Providence R.I. (1992).

[EG] L. Evans, R. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, London (1992).

[F] I. Fonseca, The Wulff theorem revisited, *Proc. Royal London Soc.* Sect. A, **432**, 125–145 (1991).

[FM] I. Fonseca, S. Mueller, A uniqueness proof of the Wulff Theorem, *Proc. Roy. Soc. Edinburgh*; Sect A, **119**, 125–136 (1991).

[HK] O. Hryniv, R. Kotecky, Surface tension and the Ornstein-Zernike behaviour for the 2D Blume-Capel model, *Jour. Stat. Phys.*, **106**, No. 3–4, 431–476 (2002).

[HKZ1] P. Holicky, R. Kotecky, M. Zahradnik, Rigid interfaces for lattice models at low temperatures., *Jour. Stat. Phys.* **50**, No. 3–4, 755–812, (1988).

[HKZ2] P. Holicky, R. Kotecky, M. Zahradnik, Phase Diagram of Horizontally Invariant Gibbs States for Lattice Models, *Ann. Henri Poincaré Phys. Theo.*, **3**, no. 2, 203–267 (2002).

[KP] R. Kotecky and D. Preiss, Cluster Expansion for Abstract Polymer Models, *Commun. Math. Phys.*, **103**, 491–498 (1986).

[I1] D. Ioffe, Large deviations for the 2D Ising model: a lower bound without cluster expansions, *J. Stat. Phys.* **74**, 411–432 (1994).

[I2] D. Ioffe, Exact deviation bounds up to T_c for the Ising model in two dimensions, *Prob. Th. Rel. Fields* **102**, 313–330 (1995).

[ISc] D. Ioffe, R. Schonmann, Dobrushin-Kotecký-Shlosman theory up to the critical temperature, *Comm. Math. Phys.* **199**, 117–167 (1998).

[LMR] L. Laanait, A. Messager, J. Ruiz, Phases coexistence and surface tensions for the Potts model., *Comm. Math. Phys.* **105**, No. 4, 527–545 (1986).

[LMP] J. Lebowitz, Mazel, Presutti, Liquid-vapor phase transitions for systems with finite-range interactions., *J. Stat. Phys.* **94**, No.5-6, 955–1025 (1999).

[MMR] A. Messager, S. Miracle-Solé, J. Ruiz, Surface tension, step free energy and facets in the equilibrium crystal, *J. Stat. Phys.* **67**, No. 3–4, 449–470 (1992).

[MMRS] A. Messager, S. Miracle-Solé, J. Ruiz, S. Shlosman, Interfaces in the Potts model. II. Antonov's rule and rigidity of the order disorder interface, *Comm. Math. Phys.* **140**, No.2, 275–290 (1991).

[Pf] C.E. Pfister, Large deviations and phase separation in the two-dimensional Ising model, *Helv. Phys. Acta* **64**, 953–1054 (1991).

[PV2] C.E. Pfister, Y. Velenik, Large deviations and continuum limit in the 2D Ising model, *Prob. Th. Rel. Fields* **109**, 435–506 (1997).

[Pi] A. Pisztora, Surface order large deviations of Ising, Potts and percolation models, *Prob. Th. Rel. Fields* **104**, 427–466 (1996).

[Ve] Y. Velenik, *Phase separation as a large deviations problem: a microscopic derivation of surface thermodynamics for some 2D spin systems*, Thèse **1712** EPF-L, (1997).

T. Bodineau†
Université Paris 7
and Laboratoire de Probabilités et Modèles Aléatoires C.N.R.S. UMR 7599
U.F.R. Mathématiques, Case 7012
2, Place Jussieu
F-75251 Paris, France
email: bodineau@gauss.math.jussieu.fr

E. Presutti
Errico Presutti
Dipartimento di Matematica
Università di Roma Tor Vergata
I-00133 Roma, Italy
email: presutti@mat.uniroma2.it

Communicated by Vincent Rivasseau
submitted 24/01/03, accepted 12/04/03

†Research partially supported by MURST and NATO Grant PST.CLG.976552