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The Virasoro Algebra and Sectors with Infinite Statistical Dimension

Sebastiano Carpi*

Abstract. We show that the sectors with lowest weight $h \geq 0$, $h \neq j^2$, $j \in \frac{1}{2}\mathbb{Z}$ of the local net of von Neumann algebras on the circle generated by the Virasoro algebra with central charge $c = 1$ have infinite statistical dimension.

1 Introduction

The notion of statistical dimension of superselection sectors, introduced by Doplicher, Haag, and Roberts in [6] is one of the most important concepts emerging in the formulation of Quantum Field Theory in therms of local nets of operator algebras (see [10] for a general reference on this subject). The deep connection with Jones' theory on index for subfactors [11, 15], established by Longo [16] is a remarkable illustration of the relevance of this notion.

For an irreducible representation π of the algebra of observables $\mathcal{A}$ satisfying the DHR selection criterion the finiteness of the (statistical) dimension $d(\pi)$ is equivalent to the existence of a conjugate representation $\overline{\pi}$ corresponding to the particle-antiparticle symmetry [6], a condition which is very natural on physical grounds. In fact for local nets over a four-dimensional Minkowski space-time no example of (irreducible) sector with infinite dimension is known and the possibility that in this context the existence of such sectors can be excluded for physically reasonable algebras of observables is still open.

The situation is different in the case of conformal nets on S^1, i.e., nets associated to chiral components of 2D conformal field theories, where irreducible representations with infinite dimension seem to appear naturally. Examples have been found by Fredenhagen [7] and Rehren has given arguments indicating that for the nets generated by the Virasoro algebra with central charge $c \geq 1$ most of the irreducible representations should have infinite dimension [20]. Moreover the analysis of these representations in a model independent framework has been initiated in [1].

In this note we show (Theorem 4.4), in agreement with the arguments in [20], that the representations of the Virasoro algebra with central charge $c = 1$ and lowest weight $h \geq 0$, $h \neq j^2$, $j \in \frac{1}{2}\mathbb{Z}$ give rise to representations with infinite dimension of the corresponding conformal net $\mathcal{A}_{\mathrm{Vir}}$.

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Our strategy of proof differs from the one adopted in [7] where (partial) computation of fusion rules is used to infer infinite dimension. Part of the fusion rules for the Virasoro algebra with $c = 1$ have been recently computed by Rehren and Tuneke [22] but we shall not use their results.

Instead of the fusion structure we use a formula, which appeared in [21], giving the dimension of the restriction of a representation of a net $\mathcal{A}$ to a subsystem $\mathcal{B} \subset \mathcal{A}$ (Proposition 3.1 in this note) and well-known results on the representation theory of the Virasoro algebra [12]. As another interesting application of this formula we show, generalizing a result in [24], that for finite index subsystems of certain rational nets twisted sectors always exist (Proposition 3.3).

2 Conformal nets, their representations and subsystems

Let $\mathcal{I}$ be the set of nonempty, nondense, open intervals of unit circle S^1.

A *conformal net on* S^1 is a family $\mathcal{A} = \{\mathcal{A}(I) | I \in \mathcal{I}\}$ of von Neumann algebras, acting on a infinite-dimensional separable Hilbert space $\mathcal{H}_\mathcal{A}$, satisfying the following properties:

(i) *Isotony.*

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \text{ for } I_1 \subset I_2, \ \ I_1, I_2 \in \mathcal{I}. \tag{1}$$

(ii) *Locality.*

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \text{ for } I_1 \cap I_2 = \emptyset, \ I_1, I_2 \in \mathcal{I}. \tag{2}$$

(iii) *Conformal covariance.* There exists a strongly continuous unitary representation U of $\mathrm{PSL}(2, \mathbb{R})$ in $\mathcal{H}_\mathcal{A}$ such that

$$U(\alpha)\mathcal{A}(I)U(\alpha)^{-1} = \mathcal{A}(\alpha I) \text{ for } I \in \mathcal{I}, \ \alpha \in \mathrm{PSL}(2, \mathbb{R}), \tag{3}$$

where $\mathrm{PSL}(2, \mathbb{R})$ acts on S^1 by Moebius transformations.

(iv) *Positivity of the energy.* The *conformal Hamiltonian* L_0, which generates the restriction of U to the one-parameter group of rotations has non-negative spectrum.

(v) *Existence of the vacuum.* There exists a unique (up to a phase) U-invariant unit vector $\Omega \in \mathcal{H}_\mathcal{A}$.

(vi) *Cyclicity of the vacuum.* Ω is cyclic for the algebra $\mathcal{A}(S^1) := \bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$

Some consequences of the axioms are [8, 9]:

(vii) *Reeh-Schlieder property.* For every $I \in \mathcal{I}$, Ω is cyclic and separating for $\mathcal{A}(I)$.

(viii) *Haag duality.* For every $I \in \mathcal{I}$

$$\mathcal{A}(I)' = \mathcal{A}(I^c), \tag{4}$$

where I^c denotes the interior of $S^1 \backslash I$.

(ix) *Factoriality.* The algebras $\mathcal{A}(I)$ are type III_1 factors.

A conformal net $\mathcal{A}$ is said to be *split* if given two intervals $I_1, I_2 \in \mathcal{I}$ with the closure of I_1 contained in I_2, there exists a type I factor $\mathcal{N}(I_1, I_2)$ such that

$$\mathcal{A}(I_1) \subset \mathcal{N}(I_1, I_2) \subset \mathcal{A}(I_2). \tag{5}$$

Moreover, if for every $I, I_1, I_2 \in \mathcal{I}$ with I_1, I_2 obtained by removing a point from I we have

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I), \tag{6}$$

then $\mathcal{A}$ is said to be *strongly additive*. The split property and strong additivity do not follow from the axioms of conformal nets but they are satisfied in many interesting models.

A *representation* of a conformal net $\mathcal{A}$ is a family $\pi = \{\pi_I |\, I \in \mathcal{I}\}$ where π_I is a representation of $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$, such that

$$\pi_J|_{\mathcal{A}(I)} = \pi_I \ \text{ for } I \subset J. \tag{7}$$

Irreducibility, direct sums and unitary equivalence of representations of conformal nets can be defined in a natural way, see [8, 9]. The unitary equivalence class of an irreducible representation π on a separable Hilbert space is called a *sector* and denoted $[\pi]$. The identical representation of $\mathcal{A}$ on $\mathcal{H}_\mathcal{A}$ is called the *vacuum representation* and it is irreducible. The corresponding sector is called the *vacuum sector*.

If $\mathcal{H}_\pi$ is separable then π is automatically locally normal, namely π_I is normal for each $I \in \mathcal{I}$ and hence $\pi_I(\mathcal{A}(I))$ is a type III_1 factor. A representation π is said to be *covariant* if there is a strongly continuous unitary representation U_π on $\mathcal{H}_\pi$ of the universal covering group $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ of $\mathrm{PSL}(2,\mathbb{R})$ such that

$$\mathrm{Ad}U_\pi(\alpha)\pi_I = \pi_{\alpha I}\mathrm{Ad}U(\alpha), \tag{8}$$

where U has been lifted to $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. If a covariant representation π is irreducible then there is a unique U_π satisfying Eq. (8). Hence, in this case, the corresponding generator of rotations L_0^π is completely determined by π. Given a covariant representation π of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}_\pi$ one has the (isomorphic) inclusions $\pi_I(\mathcal{A}(I)) \subset \pi_{I^c}(\mathcal{A}(I^c))'$, $I \in \mathcal{I}$ [8]. Then the Jones (minimal) index $[\pi_{I^c}(\mathcal{A}(I^c))' : \pi_I(\mathcal{A}(I))]$ is independent of $I \in \mathcal{I}$ and the *statistical dimension* $d(\pi)$ of π is defined by

$$d(\pi) = [\pi_{I^c}(\mathcal{A}(I^c))' : \pi_I(\mathcal{A}(I))]^{\frac{1}{2}}. \tag{9}$$

The relation of the above definition with the one in [6] is given by the index-statistics theorem [9, 16]. More precisely, in analogy with [6], a definition of statistical dimension using left inverse and statistics parameter can be given in the context of conformal nets on S^1 [9] (cf. also [8]). It has been shown by Guido and Longo that for a covariant representation π with positive energy the statistical dimension is finite if and only if $[\pi_{I^c}(\mathcal{A}(I^c))' : \pi_I(\mathcal{A}(I))]^{\frac{1}{2}}$ is finite [9, Subsection

2.4] and that in this case these two numbers coincide [9, Corollary 3.7]. Actually, as a consequence of [9, Proposition 2.14] and [1, Corollary 4.4], the positivity of the energy needs not to be assumed *a priori*. In view of the above considerations it should be clear that the use of formula (9) as a definition is appropriate both in the finite and in the infinite case.

A *conformal subsystem* of a conformal net $\mathcal{A}$ is a family $\mathcal{B} = \{\mathcal{B}(I)|\ I \in \mathcal{I}\}$ of nontrivial von Neumann algebras acting on $\mathcal{H}_{\mathcal{A}}$ such that:

$$\mathcal{B}(I) \subset \mathcal{A}(I) \ \text{ for } I \in \mathcal{I}; \tag{10}$$

$$U(\alpha)\mathcal{B}(I)U(\alpha)^{-1} = \mathcal{B}(\alpha I) \ \text{ for } I, \in \mathcal{I}; \tag{11}$$

$$\mathcal{B}(I_1) \subset \mathcal{B}(I_2) \ \text{ for } I_1 \subset I_2, \ \ I_1, I_2 \in \mathcal{I}. \tag{12}$$

We shall use the notation $\mathcal{B} \subset \mathcal{A}$ for conformal subsystems. Note that $\mathcal{B}$ is not in general a conformal net since Ω is not cyclic for the algebra $\mathcal{B}(S^1) := \bigvee_{I\in\mathcal{I}} \mathcal{B}(I)$ unless $\mathcal{B} = \mathcal{A}$. However one gets a conformal net $\mathcal{B}_0$ by restriction of the algebras $\mathcal{B}(I)$, $I \in \mathcal{I}$, and of the representation U to the closure $\mathcal{H}_{\mathcal{B}}$ of $\mathcal{B}(S^1)\Omega$. Since the map

$$b \in \mathcal{B}(I) \mapsto b|_{\mathcal{H}_{\mathcal{B}}} \in \mathcal{B}_0(I)$$

is an isomorphism for every $I \in \mathcal{I}$, we shall, as usual, use the symbol $\mathcal{B}$ instead of $\mathcal{B}_0$, specifying, if necessary, when $\mathcal{B}$ acts on $\mathcal{H}_{\mathcal{A}}$ or on $\mathcal{H}_{\mathcal{B}}$.

Given a conformal subsystem $\mathcal{B} \subset \mathcal{A}$ the index of the subfactor $\mathcal{B}(I) \subset \mathcal{A}(I)$ does not depend on I and is denoted $[\mathcal{A} : \mathcal{B}]$.

3 Restricting representations

We now consider restriction of representations. Given a subsystem $\mathcal{B} \subset \mathcal{A}$ and a representation π of $\mathcal{A}$ one can define a representation π^{rest} by

$$\pi_I^{rest} = \pi_I|_{\mathcal{B}(I)} \quad I \in \mathcal{I}. \tag{13}$$

Then the following holds [21] (cf. also [23, Section 3]). We include the proof for the convenience of the reader.

Proposition 3.1. *For every conformal subsystem $\mathcal{B} \subset \mathcal{A}$ and covariant representation π of $\mathcal{A}$ on a separable Hilbert space we have*

$$d(\pi^{rest}) = [\mathcal{A} : \mathcal{B}]d(\pi). \tag{14}$$

Proof. For $I \in \mathcal{I}$ we have $d(\pi^{rest})^2 = [\pi_{I^c}(\mathcal{B}(I^c))' : \pi_I(\mathcal{B}(I))]$. Consider the inclusions

$$\pi_I(\mathcal{B}(I)) \subset \pi_I(\mathcal{A}(I)) \subset \pi_{I^c}(\mathcal{A}(I^c))' \subset \pi_{I^c}(\mathcal{B}(I^c))'.$$

Then, the multiplicativity of the index [17] implies that $d(\pi^{rest})^2$ is equal to

$$[\pi_{I^c}(\mathcal{B}(I^c))' : \pi_{I^c}(\mathcal{A}(I^c))'][\pi_{I^c}(\mathcal{A}(I^c))' : \pi_I(\mathcal{A}(I))][\pi_I(\mathcal{A}(I)) : \pi_I(\mathcal{B}(I))].$$

Since π_I is an isomorphism for every $I \in \mathcal{I}$ we have

$$[\pi_{I^c}(\mathcal{B}(I^c))' : \pi_{I^c}(\mathcal{A}(I^c))'] = [\pi_{I^c}(\mathcal{A}(I^c)) : \pi_{I^c}(\mathcal{B}(I^c))] = [\mathcal{A} : \mathcal{B}]$$

and similarly

$$[\pi_I(\mathcal{A}(I)) : \pi_I(\mathcal{B}(I))] = [\mathcal{A} : \mathcal{B}].$$

It follows that

$$d(\pi^{rest})^2 = [\mathcal{A} : \mathcal{B}]^2 d(\pi)^2.$$

□

Remark 3.2. If $N \subset M$ is an inclusion of infinite factors acting on a separable Hilbert space and ρ is a (normal, unital) endomorphism of M one can define an endomorphism ρ^{rest} of N by

$$\rho^{rest} := \gamma \circ \rho|_N, \tag{15}$$

where γ is Longo's canonical endomorphism [16]. As discussed in [19] the mapping $\rho \mapsto \rho^{rest}$ (called σ restriction in [2]) corresponds in a natural way to the restriction of representations of a net. In fact a similar argument to the one used in the proof of the previous proposition shows that

$$d(\rho^{rest}) = [M : N]d(\rho). \tag{16}$$

Here the dimension $d(\rho)$ of an endomorphism ρ of a factor M is given by the square root of the index of the subfactor $\rho(M) \subset M$.

Let $I_1, I_2 \in \mathcal{I}$ have disjoint closures, let $I_3, I_4 \in \mathcal{I}$ be the interiors of the connected components of $S^1/(I_1 \cup I_2)$ and let $\mathcal{A}$ be a conformal net on S^1. The inclusion

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset (\mathcal{A}(I_3) \vee \mathcal{A}(I_4))' \tag{17}$$

is called a 2-*interval inclusion*. A conformal net $\mathcal{A}$ is said to be *completely rational* if it is split, strongly additive and there is a 2-interval inclusion with finite index $\mu_{\mathcal{A}}$ (in this case every 2-interval inclusion has the same index [14]). It has been shown in [14] that a completely rational net has finitely many sectors which are all covariant with finite dimension. Furthermore the following holds

$$\mu_{\mathcal{A}} = \sum_i d(\pi_i)^2, \tag{18}$$

where for each sector of $\mathcal{A}$ a representation π_i has been chosen.

We now consider a conformal subsystem $\mathcal{B}$ of a completely rational net $\mathcal{A}$ such that the index $[\mathcal{A} : \mathcal{B}]$ is finite. Then $\mathcal{B}$ is completely rational [18] and the index $\mu_{\mathcal{B}}$ is given by ([14, Proposition 24.])

$$\mu_{\mathcal{B}} = [\mathcal{A} : \mathcal{B}]^2 \mu_{\mathcal{A}}. \tag{19}$$

We say that a sector of $\mathcal{B}$ is *untwisted* if it is contained in π^{rest} for some irreducible representation π of $\mathcal{A}$ on a separable Hilbert space. If it is not untwisted we say that it is *twisted*.

For every sector of $\mathcal{B}$ we choose a corresponding representation σ_i of $\mathcal{B}$. Let $\mathcal{U}$, ($\mathcal{T}$) be the set of untwisted (twisted) sectors of $\mathcal{B}$. We define

$$\mu^u{}_{\mathcal{B}} = \sum_{[\sigma_i]\in\mathcal{U}} d(\sigma_i)^2, \tag{20}$$

$$\mu^t{}_{\mathcal{B}} = \sum_{[\sigma_i]\in\mathcal{T}} d(\sigma_i)^2. \tag{21}$$

Clearly $\mu_{\mathcal{B}} = \mu^u{}_{\mathcal{B}} + \mu^t{}_{\mathcal{B}}$. In the case where $\mathcal{B} \subset \mathcal{A}$ is an orbifold inclusion, namely $\mathcal{B}$ is the fixed points net for the action of a (non-trivial) finite group G of internal symmetries of $\mathcal{A}$, it has been shown by Xu [24] that the set of twisted sectors is not empty. Actually Proposition 3.1 implies the existence of such sectors even when there is no underlying group action.

Proposition 3.3. *Let $\mathcal{B}$ be a proper conformal subsystem of a completely rational net $\mathcal{A}$, with finite index $[\mathcal{A} : \mathcal{B}]$. Then the set of twisted sectors of $\mathcal{B}$ is not empty and in fact $\mu^t{}_{\mathcal{B}} \geq 2$.*

Proof. Let $\pi_i,\ i = 0, 1, \ldots, n$ be inequivalent irreducible representations exausting all sectors of $\mathcal{A}$ and let π_0 be the vacuum representation. The set $\mathcal{U}$ of untwisted sectors of $\mathcal{B}$ can be decomposed into disjoint subsets $\mathcal{U}_i,\ i = 0, 1, \ldots, n$ in the following way: $\mathcal{U}_0$ is the set of sectors of $\mathcal{B}$ which are contained in ${\pi_0}^{rest}$ and $\mathcal{U}_k,\ k = 1, \ldots, n$ is the the set of sectors contained in ${\pi_k}^{rest}$ but not in ${\pi_i}^{rest}, i = 0, \ldots, k-1$. It follows from Proposition 3.1 and Eq. (19) that

$$\begin{aligned}\sum_i d({\pi_i}^{rest})^2 &= [\mathcal{A} : \mathcal{B}]^2 \cdot \sum_i d(\pi_i)^2 \\ &= [\mathcal{A} : \mathcal{B}]^2 \mu_{\mathcal{A}} \\ &= \mu_{\mathcal{B}}.\end{aligned}$$

Therefore

$$\begin{aligned}\mu^t{}_{\mathcal{B}} &= \mu_{\mathcal{B}} - \mu^u{}_{\mathcal{B}} \\ &= \sum_i d({\pi_i}^{rest})^2 - \sum_{[\sigma_k]\in\mathcal{U}} d(\sigma_k)^2 \\ &\geq \sum_i \Big(\sum_{[\sigma_k]\in\mathcal{U}_i} d(\sigma_k)\Big)^2 - \sum_i \Big(\sum_{[\sigma_k]\in\mathcal{U}_i} d(\sigma_k)^2\Big) \\ &\geq \Big(\sum_{[\sigma_k]\in\mathcal{U}_0} d(\sigma_k)\Big)^2 - \sum_{[\sigma_k]\in\mathcal{U}_0} d(\sigma_k)^2 \geq 2,\end{aligned}$$

where the last inequality follows from the fact that $\mathcal{U}_0$ has two or more elements when $\mathcal{B} \neq \mathcal{A}$. □

4 Virasoro algebra and infinite dimension

We begin this section with the following easy consequence of Proposition 3.1.

Proposition 4.1. *Let $\mathcal{B}$ be a conformal subsystem of a net $\mathcal{A}$ with infinite index $[\mathcal{A} : \mathcal{B}]$. Assume that there exists a covariant representation π of $\mathcal{A}$ on a separable Hilbert space whose restriction to $\mathcal{B}$ is irreducible. Then $[\pi^{rest}]$ is a covariant sector of $\mathcal{B}$ with infinite statistical dimension.*

We now come to the sectors of the conformal net $\mathcal{A}_{\rm Vir}$ generated by the Virasoro algebra with $c = 1$. We shall use the fact that $\mathcal{A}_{\rm Vir}$ can be considered has a conformal subsystem of the net $\mathcal{A}$ generated by a U(1) current $J(z)$. The net $\mathcal{A}$ is defined as follows, see [3, 5] for more details. The Hilbert space $\mathcal{H}_A$ carries a strongly continuous unitary representation U of $\mathrm{PSL}(2,\mathbb{R})$ with positive energy and a unique (up to a phase) U-invariant unit vector Ω. The U(1) current $J(z),\ z \in S^1$ is defined as operator valued distribution on $\mathcal{H}_\mathcal{A}$. Namely the operators

$$J(u) = \int \frac{dz}{2\pi i} J(z)u(z) \ \ u \in C^\infty(S^1) \tag{22}$$

have a common invariant dense domain $\mathcal{D}$ containing Ω which is also U-invariant. For each $\psi \in \mathcal{D}$ the mapping $u \mapsto J(u)\psi$ is linear and continuous from $C^\infty(S^1)$ to $\mathcal{H}_\mathcal{A}$. Moreover the vacuum Ω is cyclic for the polynomial algebra generated by the smeared currents $J(u),\ u \in C^\infty(S^1)$.

The current $J(z)$ satisfies the canonical commutation relations

$$[J(z_1), J(z_2)] = -\delta'(z_1 - z_2), \tag{23}$$

where the Dirac delta function $\delta(z_1 - z_2)$ is defined with respect to the complex measure $\frac{dz}{2\pi i}$, the hermiticity condition

$$J(z)^* = z^2 J(z), \tag{24}$$

and the covariance property

$$U(\alpha)J(u)U(\alpha)^* = J(u_\alpha),\ u \in C^\infty(S^1), \tag{25}$$

where $u_\alpha(z) := u(\alpha^{-1}z)$. For every real test function $u \in C^\infty(S^1)$ the operator $J(u)$ is essentially self-adjoint and the unitaries $W(u) := \mathrm{e}^{iJ(u)}$ satisfy the Weyl relations

$$W(u)W(v) = W(u + v)\mathrm{e}^{-\frac{A(u,v)}{2}}, \tag{26}$$

where $A(u, v) := \int \frac{dz}{2\pi i} u'(z)v(z)$. For every $I \in \mathcal{I}$ the local von Neumann algebra $\mathcal{A}(I)$ is defined by

$$\mathcal{A}(I) = \{W(u) | u \in C^\infty(S^1) \text{ real},\ \mathrm{supp}\, u \subset I\}'' \tag{27}$$

and one can show that the family $\mathcal{A}(I),\ I \in \mathcal{I}$ is a conformal net on S^1. Next we define the conformal subsystem $\mathcal{A}_{\rm Vir}$ generated by the Virasoro algebra with central charge $c = 1$. First consider the (formal) Fourier expansion of the U(1)-current

$$J(z) = \sum_n J_n z^{-n-1}, \tag{28}$$

where the Fourier modes $J_n,\ n \in \mathbb{Z}$ satisfy

$$[J_n, J_m] = n\delta_{n+m,0} \tag{29}$$

$$J_n{}^* = J_{-n}. \tag{30}$$

One can define an energy-momentum tensor $T(z)$ by the Sugawara construction

$$T(z) = \frac{1}{2} : J(z)^2 := \frac{1}{2}(J_+(z)J(z) + J(z)J_-(z)), \tag{31}$$

where,

$$J_+(z) = J(z) - J_-(z) = \sum_{n=1}^{\infty} J_{-n} z^{n-1}. \tag{32}$$

The Fourier modes in the expansion

$$T(z) = \sum_n L_n z^{-n-2} \tag{33}$$

satisfy the Virasoro algebra relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{34}$$

with central charge $c = 1$, and the hermiticity condition

$$L_n{}^* = L_{-n}. \tag{35}$$

According to our previous notations (the closure of) L_0 is the positive self-adjoint generator of the restriction of U to the one-parameter subgroup of rotations.

For $f \in C^\infty(S^1)$ the operator

$$T(f) = \int \frac{dz}{2\pi i} T(z) f(z) \tag{36}$$

is well defined on $\mathcal{D}$ and is essentially self-adjoint when $z^{-1}f(z)$ is real. The conformal subsystem $\mathcal{A}_{\mathrm{Vir}} \subset \mathcal{A}$ is then defined by

$$\mathcal{A}_{\mathrm{Vir}}(I) = \{\mathrm{e}^{iT(f)} | f \in C^\infty(S^1),\ z^{-1}f(z)\ \mathrm{real},\ \mathrm{supp}\ f \subset I\}'',\ \ I \in \mathcal{I}. \tag{37}$$

Representations of the net $\mathcal{A}$ have been studied in [3]. For every $q \in \mathbb{R}$ one can define a covariant irreducible representation (BMT-automorphism) α_q on $\mathcal{H}_q = \mathcal{H}_{\mathcal{A}}$ such that

$$\alpha_{q\,I}(W(u)) = \mathrm{e}^{q \int \frac{dz}{2\pi} z^{-1} u(z)} W(u), \tag{38}$$

for $I \in \mathcal{I}$, $u \in C^\infty(S^1)$ with support in I. Such representations have dimension $d(\alpha_q) = 1$ and correspond to (unitary) positive energy representations of the Lie algebra (29) with lowest weight q [4]. Note that α_0 is the vacuum representation of $\mathcal{A}$. Analogously to each representation of the Virasoro algebra (34) with central charge $c = 1$ and lowest weight $h \in \mathbb{R}_+$ one can associate a covariant irreducible representation π_h of $\mathcal{A}_{\mathrm{Vir}}$ which can be realized has a subrepresentation of ${\alpha_q}^{rest}$ if $h = \frac{1}{2}q^2$, see [5]. The characters of the representations α_q, $q \in \mathbb{R}$, are given by (see, e.g., [13, Section 2.2.])

$$\chi_q(t) = \mathrm{Tr}(t^{L_0^{\alpha_q}}) = t^{\frac{1}{2}q^2} p(t) \;\; t \in (0,1), \tag{39}$$

where $p(t) = \prod_{n=1}^{\infty}(1-t^n)^{-1}$. Moreover, for the representations π_h, $h \in \mathbb{R}_+$ and $t \in (0,1)$, by the results in [12] the following hold

$$\chi^h(t) := \mathrm{Tr}(t^{L_0^{\pi_h}}) = t^{j^2}(1 - t^{2|j|+1})p(t), \; h = j^2, j \in \frac{1}{2}\mathbb{Z}, \tag{40}$$

$$\chi^h(t) := \mathrm{Tr}(t^{L_0^{\pi_h}}) = t^h p(t), \; h \neq j^2, j \in \frac{1}{2}\mathbb{Z}. \tag{41}$$

Lemma 4.2. $[\mathcal{A} : \mathcal{A}_{\mathrm{Vir}}] = \infty$.

Proof. As a consequence of Proposition 3.1 we have $[\mathcal{A} : \mathcal{A}_{\mathrm{Vir}}] = d({\alpha_0}^{rest})$. Moreover it follows from the equality $\chi_0(t) = \sum_{j=0}^{\infty} \chi^{j^2}(t)$ that

$${\alpha_0}^{rest} = \oplus_{j=0}^{\infty} \pi_{j^2}$$

and this implies infinite index. □

Lemma 4.3. (cf. [13, Theorem 6.2.]) *If $h = \frac{1}{2}q^2$, $q \notin \frac{1}{\sqrt{2}}\mathbb{Z}$, then $\pi^h = {\alpha_q}^{rest}$.*

Proof. If $h = \frac{1}{2}q^2$ π^h is a subrepresentation of ${\alpha_q}^{rest}$ on a U_{α_q}-invariant subspace $\mathcal{H}_h \subset \mathcal{H}_q$. Moreover, if $q \notin \frac{1}{\sqrt{2}}\mathbb{Z}$ then $\chi^h(t) = \chi_q(t)$ and hence $\mathcal{H}_h = \mathcal{H}_q$. Accordingly we have $\pi^h = {\alpha_q}^{rest}$. □

The following theorem is a direct consequence of Proposition 4.1 and the previous two lemmata.

Theorem 4.4. *If $[\pi_h]$ belongs to the continuum sectors of $\mathcal{A}_{\mathrm{Vir}}$, i.e., $h \in \mathbb{R}_+$, $h \neq j^2$, $j \in \frac{1}{2}\mathbb{Z}$, then it has infinite statistical dimension.*

Remark 4.5. It has been shown by Rehren [20] that if $h = j^2$, $j \in \mathbb{Z}$, then $d(\pi_h) = 2|j| + 1$ and the same formula is expected to hold for every $j \in \frac{1}{2}\mathbb{Z}$.

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Sebastiano Carpi
Dipartimento di Scienze
Università "G. d'Annunzio" di Chieti-Pescara
Viale Pindaro 42
I-65127 Pescara
Italy
email: carpi@sci.unich.it

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