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# $L^p$ Norms of Eigenfunctions in the Completely Integrable Case

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**Abstract.** The eigenfunctions $e^{i\langle\lambda,x\rangle}$ of the Laplacian on a flat torus have uniformly bounded $L^p$ norms. In this article, we prove that for every other quantum integrable Laplacian, the $L^p$ norms of the joint eigenfunctions blow up at least at the rate $\|\varphi_k\|_{L^p} \geq C(\epsilon)\lambda_k^{\frac{p-2}{4p}-\epsilon}$ when $p > 2$. This gives a quantitative refinement of our recent result [TZ1] that some sequence of eigenfunctions must blow up in $L^p$ unless $(M, g)$ is flat. The better result in this paper is based on mass estimates of eigenfunctions near singular leaves of the Liouville foliation.

## 0 Introduction

This paper, a companion to [TZ1], is concerned with the growth rate of the $L^p$-norms of $L^2$-normalized $\Delta$-eigenfunctions

$$\Delta\varphi_j = \lambda_j^2\varphi_j, \quad \langle\varphi_j, \varphi_k\rangle = \delta_{jk}$$

on compact Riemannian manifolds $(M, g)$ with completely integrable geodesic flow $G^t$ on $S^*M$. The motivating problem is to relate sizes of eigenfunctions to dynamical properties of its geodesic flow $G^t$ on $S^*M$. In general this is an intractable problem, but much can be understood by studying it in the special case of integrable systems. To be precise, we assume that $\Delta$ is *quantum completely integrable* or QCI in the sense that there exist $n = \dim M$ first-order pseudo-differential operators $P_1, \ldots, P_n$ such that

$$P_1 = \sqrt{\Delta}, \ \ [P_i, P_j] = 0 \tag{1}$$

and whose symbols $(p_1, \ldots, p_n)$ satisfy the independence condition $dp_1 \wedge dp_2 \wedge \cdots \wedge dp_n \neq 0$ on a dense open set $\Omega \subset T^*M - 0$. Since $\{p_i, p_j\} = 0$, their joint Hamiltonian flow $\Phi_t(x,\xi) := \exp(t_1 X_{p_1}) \circ \cdots \circ \exp(t_1 X_{p_n})(x,\xi)$ (where $X_p$ denotes the Hamiltonian vector field of $p$) defines a Hamiltonian $\mathbb{R}^n$ action with $\mathbb{R}^+$-homogeneous moment map

$$\mathcal{P} = (p_1, \ldots, p_n) : T^*M - 0 \to \mathbb{R}^n. \tag{2}$$

Throughout, we will assume that the orbits of this action are non-degenerate in the sense of Eliasson (see [El] and Definitions 2 and 3).

The main result of [TZ1] was that the $L^\infty$-norms of the $L^2$-normalized joint eigenfunctions $\{\varphi_\lambda\}$ of $(P_1, \ldots, P_n)$ are unbounded unless $(M, g)$ is flat. The rough idea was to prove that orbits of the $\mathbb{R}^n$ action which had singular projection to $M$ (under the natural projection $\pi : T^*M \to M$) caused sup-norm blow-up in associated sequences of eigenfunctions. In this paper, we make use of singular orbits rather than just singular projections of possibly regular orbits. The nice feature of singular orbits is that the associated modes blow up at high rates. To use this approach, we first prove that singular orbits must occur unless $(M, g)$ is a flat torus, and we relate blow-up rates of $L^p$-norms of sequences $||\varphi_\lambda||_{L^p}$ of eigenfunctions to the dimensions of the singular orbits. For the definition of singular orbits and Eliasson non-degeneracy, see Definition (1).

**Theorem 1** *Suppose that* $(M, g)$ *is a compact Riemannian manifold whose Laplacian* $\Delta$ *is quantum completely integrable as in* (1) *and suppose that the Hamiltonian* $\mathbb{R}^n$ *action defined by* (2) *satisfies Eliasson's non-degeneracy condition (Definition* 2*). Then, unless* $(M, g)$ *is a flat torus, this action must have a singular orbit of dimension* $< n$*. If the minimal dimension of the singular orbits is* $\ell$*, then for every* $\epsilon > 0$*, there exists a sequence of eigenfunctions satisfying:*

$$\begin{cases} \|\varphi_k\|_{L^\infty} \geq C(\epsilon)\lambda_k^{\frac{n-\ell}{4}-\epsilon}. \\ \\ \|\varphi_k\|_{L^p} \geq C(\epsilon)\lambda_k^{\frac{(n-\ell)(p-2)}{4p}-\epsilon}, \;\; 2 < p < \infty. \end{cases}$$

The proof does not determine the minimal dimension $\ell$. By taking products of lower-dimensional manifolds, it is easy to construct examples with any value of $\ell = 1, \ldots, n-1$. But, as will be discussed in §1, it is quite plausible that one-dimensional orbits 'often' occur for Hamiltonian $\mathbb{R}^n$ actions on cotangent bundles. Such leaves correspond to closed geodesics which are invariant under the $\mathbb{R}^n$ action. Simple examples are given by the 'equatorial geodesics' on convex surfaces of revolution, i.e., geodesics invariant the $S^1$ action. In such cases, the eigenfunction blow-up estimate becomes the optimal rate $\|\varphi_k\|_{L^\infty} \geq C(\epsilon)\lambda_k^{\frac{n-1}{4}-\epsilon}$. In the case of the sphere or other convex surfaces of revolution, the blow-up rate is achieved by 'highest weight' eigenfunctions (corresponding to joint spectral points on the boundary of the image of the moment map $\mathcal{P}$).

Regarding the sharpness of the result, based on the case of the quantum Euler top [T1], it seems reasonable to conjecture that $\|\varphi_{\lambda_j}\|_\infty \geq C\lambda_j^{\frac{n-\ell}{8}}$ in the elliptic case and $\|\varphi_{\lambda_j}\|_\infty \geq C\lambda_j^{\frac{n-\ell}{4}}(\log \lambda_j)^{-\alpha}$ for some $\alpha > 0$ in the hyperbolic case.

The method of proof of Theorem 1 is partly based on a study of trace formulae, as in [TZ1], and partly on a study of Birkhoff normal forms of quantum integrable systems and their modes and quasi-modes near singular torus orbits. We derive $L^p$ estimates from studying matrix elements $\langle A\varphi_\mu, \varphi_\mu\rangle$ of pseudodifferential operators relative to the joint eigenfunctions $\varphi_\mu$. In the case of multiplications $A = 1_\Omega$ by (smoothed out) characteristic functions of domains $\Omega \subset M$, such matrix elements measure the $L^2$- *mass* $\int_\Omega |\varphi_\mu(x)|^2 dV$ of the eigenfunction in $\Omega$. To obtain $L^p$ norm information from $L^2$-mass estimates, we study *small scale* eigenfunction mass, that is, the mass in shrinking tubes. The main estimate is given in Lemma 8.

To our knowledge, these and the results of [TZ1, TZ2] are the first lower bounds on eigenfunctions in the completely integrable case. They were contained in the original posted version (arXiv (math-ph/0002038)) of our article [TZ1]. In revising that paper for publication, we decided to separate the results into the qualitative one of [TZ1] and the present quantitative ones.

A recent paper of Donnelly [D] has obtained a similar maximal blow up rate $\lambda_k^{\frac{n-1}{4}}$ for $L^\infty$ norms of eigenfunctions on compact Riemannian manifolds $(M, g)$ with warped product metrics or isometric $S^1$ actions under certain non-degeneracy assumptions. In the $S^1$-action case, the assumption is analogous to the existence of a singular Eliasson non-degenerate one-dimensional orbit. Donnelly's setting has two commuting operators of a very special type (i.e., $S^1$ acts on the base), whereas our QCI setting has $n$ commuting operators of a more general type. Comparison of the results suggests a common generalization to partially integrable Laplacians $P_1 = \sqrt{\Delta}, P_2, \ldots, P_m, m \leq n, [P_i, P_j] = 0$. If the associated $\mathbb{R}^m$ action has a one-dimensional orbit satisfying an analogue of Eliasson non-degeneracy, then it is reasonable to expect the existence of a sequence of joint eigenfunctions which blow up at the maximal rate.

In another paper [TZ2], we give a proof of the eigenfunction blow-up result of [TZ1] which in some ways is closer in spirit to the approach of this paper than that of [TZ1]. The method is to relate norms of modes to norms of "quasi-modes", i.e., approximate eigenfunctions associated to Bohr-Sommerfeld leaves of the Liouville foliation. That paper also gives a number of detailed examples of quantum completely integrable systems such as Liouville tori, surfaces of revolution, ellipsoids, tops and so on.

We close by stating a conjecture on a much larger class of Riemannian manifolds which is suggested by the recent results:

**Conjecture 2** *Any compact $(M, g)$ with a stable elliptic orbit has a sequence of eigenfunctions whose $L^\infty$ norms blow up at the rate $\lambda_k^{\frac{n-1}{4}}$.*

Such orbits occur in the KAM setting of perturbations of integrable systems. There surely exist quasi-modes with this property, and the difficulty is to show that this implies the existence of modes with this blow-up rate.

## 1 Geometry of completely integrable systems

This section is devoted to the geometric aspects of our problem. We begin with some preliminary background on completely integrable systems. In particular, we describe Eliasson's normal form theorem for completely integrable Hamiltonians near non-degenerate singular orbits. We then prove the existence of singular orbits of Hamiltonian $\mathbb{R}^n$ actions commuting with geodesic flows on manifolds other than flat tori.

A completely integrable system is defined by an Abelian subalgebra

$$\mathbf{p} = \mathbb{R}\{p_1, \dots, p_n\} \subset (C^\infty(T^*M - 0), \{,\}). \tag{3}$$

Here, $\{,\}$ is the standard Poisson bracket. We assemble the generators into the moment map

$$\mathcal{P} = (p_1, \dots, p_n) : T^*M \to B \subset \mathbb{R}^n. \tag{4}$$

The Hamiltonians $p_j$ generate the $\mathbb{R}^n$-action

$$\Phi_t = \exp t_1 \Xi_{p_1} \circ \exp t_2 \Xi_{p_2} \cdots \circ \exp t_n \Xi_{p_n}.$$

We denote $\Phi_t$-orbits by $\mathbb{R}^n \cdot (x, \xi)$. By the Liouville-Arnold theorem [AM], the orbits of the joint flow $\Phi_t$ are diffeomorphic to $\mathbb{R}^k \times T^m$ for some $(k, m), k + m \leq n$. We now consider level sets $\mathcal{P}^{-1}(b)$ of the moment map and their decompositions into orbits.

First, we suppose that $b$ is a regular value. Since $\mathcal{P}$ is proper, a regular level has the form

$$\mathcal{P}^{-1}(b) = \Lambda^{(1)}(b) \cup \cdots \cup \Lambda^{(m_{cl})}(b), \quad (b \in B_{reg}) \tag{5}$$

where each $\Lambda^{(l)}(b) \simeq T^n$ is an $n$-dimensional Lagrangian torus. Here, $m_{cl}(b) = \#\mathcal{P}^{-1}(b)$ is the number of orbits on the level set $\mathcal{P}^{-1}(b)$. In sufficiently small neighbourhoods $\Omega^{(l)}(b)$ of each component torus, $\Lambda^{(l)}(b)$, the Liouville-Arnold theorem also gives the existence of local action-angle variables $(I_1^{(l)}, \dots, I_n^{(l)}, \theta_1^{(l)}, \dots, \theta_n^{(l)})$ in terms of which the joint flow of $\Xi_{p_1}, \dots, \Xi_{p_n}$ is linearized [AM].

Now let us consider singular levels of the moment map and singular orbits of the $\mathbb{R}^n$-action. We use the following notations:

**Definition 1**

- *A point $(x, \xi)$ is called a singular point of $\mathcal{P}$ if $dp_1 \wedge \cdots \wedge dp_n(x, \xi) = 0$.*
- *A level set $\mathcal{P}^{-1}(c)$ of the moment map is called a singular level if it contains a singular point $(x, \xi) \in \mathcal{P}^{-1}(c)$. (We then say $c$ is a singular value and write $c \in B_{sing}$.)*
- *A connected component of $\mathcal{P}^{-1}(c)$ is a singular component if it contains a singular point.*
- *An orbit $\mathbb{R}^n \cdot (x, \xi)$ of $\Phi_t$ is singular if it is non-Lagrangian, i.e., has dimension $< n$;*

Suppose that $c \in B_{sing}$. We first decompose the singular level

$$\mathcal{P}^{-1}(c) = \cup_{j=1}^r \Gamma_{sing}^{(j)}(c) \tag{6}$$

into connected components $\Gamma_{sing}^{(j)}(b)$ and then decompose

$$\Gamma_{sing}^{(j)}(c) = \cup_{k=1}^p \mathbb{R}^n \cdot (x_k, \xi_k) \tag{7}$$

each component into orbits. Both decompositions can take a variety of forms. The regular components $\Gamma^{(j)}_{sing}(b)$ must be Lagrangian tori. Under a non-degeneracy assumption (see Definition 1.1 ), the singular component consists of finitely many orbits. The orbit $\mathbb{R}^n \cdot (x, \xi)$ of a singular point is necessarily singular, hence it has the form $\mathbb{R}^k \times T^m$ for some $(k, m)$ with $k + m < n$. Regular points may of course also occur on a singular component; their orbits are Lagrangian and can take any one of the forms $\mathbb{R}^k \times T^m$ for some $(k, m)$ with $k + m = n$.

Now let $v \in \mathcal{P}^{-1}(c)$ and assume the orbit $\mathbb{R}^n \cdot (v) := \{\exp t_1 \Xi_{p_1} \circ \cdots \circ \exp t_k \Xi_{p_k}(v);\ t = (t_1, \dots, t_k) \in \mathbb{R}^n\}$ is compact and of rank $k$ in the sense that

$$\operatorname{rank}(dp_1, \dots, dp_n)|_v = \operatorname{rank}(dp_1, \dots, dp_k) = k < n. \tag{8}$$

Following [El] (see p. 9), we observe that the Hessians $d_v^2 p_j$ determine an Abelian subalgebra

$$d_v^2 \mathbf{p} \subset S^2(K/L, \omega_v)^* \tag{9}$$

of quadratic forms on the reduced symplectic subspace $K/L$, where we put

$$K = \bigcap_{i=1}^n \ker dp_i(v), \quad L = \operatorname{span}\{\Xi_{p_1}(v), \dots, \Xi_{p_n}(v)\}.$$

**Definition 2** *[El] The orbit $\mathbb{R}^n \cdot v$ is said to be non-degenerate of rank $k$ if $d_v^2 \mathbf{p}$ is a Cartan subalgebra of $S^2(K/L, \omega_v)^*$.*

A Cartan subalgebra is a maximal Abelian subalgebra generated by semi-simple elements. The above definition is (superficially) more general than the one in [El] (p. 6), since Eliasson assumes through most of [El] that the subalgebra is elliptic (in a sense we describe below). However, most of Eliasson's ideas apply to generic integrable systems where the Cartan subalgebra is of mixed type, with real or complex hyperbolic generators as well as elliptic ones, as discussed in the last section of [El] and in [El2]. Also, our assumption that (9) is a CSA is somewhat stronger than in [El].

The definition can be rephrased in terms of reduced Hamiltonian systems, as follows. First, there is a singular Liouville-Arnold theorem (cf. [N]) which produces action variables conjugate to the angle variables on the singular orbit. As in (8), we choose indices so that $dp_1, \dots, dp_k$ are linearly independent everywhere on $\mathbb{R}^n \cdot (v_0)$. The singular Liouville-Arnold theorem [AM] states that there exists local canonical transformation

$$\psi = \psi(I, \theta, x, y) : \mathbb{R}^{2n} \to T^*M - 0,$$

where

$$I = (I_1, \dots, I_k), \theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k,\ x = (x_1, \dots, x_{n-k}),$$
$$y = (y_1, \dots, y_{n-k}) \in \mathbb{R}^{n-k} m$$

defined in an invariant neighbourhood of $\mathbb{R}^n \cdot (v)$ such that

$$p_i \circ \psi = I_i \;\; (i = 1, \ldots, k), \tag{10}$$

and such that the symplectic form $\omega$ on $T^*M$ takes the form

$$\psi^* \omega = \sum_{j=1}^{k} dI_j \wedge d\theta_j + \sum_{j=1}^{n-k} dx_j \wedge dy_j. \tag{11}$$

As for the remaining Hamiltonians $p_j$, there exist constants $c_{ij}$ with $i = k+1, \ldots, n$ and $j = 1, \ldots, k$, such that at each point of the orbit, $\mathbb{R}^n \cdot (v)$,

$$dp_i = \sum_{j=1}^{k} c_{ij} dp_j. \tag{12}$$

Since $dp_1, \ldots, dp_k$ are linearly independent in a sufficiently neighbourhood $U$ of $v \in \mathcal{P}^{-1}(c)$, the action of the flows corresponding to the Hamilton vector fields, $\Xi_{p_1}, \ldots, \Xi_{p_k}$ generates a symplectic $\mathbb{R}^k$ action on $\mathcal{P}^{-1}(c_0) \cap U$. We reduce $U$ with respect to the partial moment map $\mathcal{I} := (I_1, \ldots, I_k)(= (p_1, \ldots, p_k))$, i.e., we take $\{\mathcal{I} = 0\}$ and divide by the Hamiltonian flow. This produces a $2(n-k)$-dimensional symplectic manifold,

$$\Sigma_k := \mathcal{P}^{-1}(c_0) \cap U / \mathbb{R}^k, \tag{13}$$

with the induced symplectic form, $\sigma$. We will denote the canonical projection map by:

$$\pi_k : \mathcal{P}^{-1}(c_0) \cap U \longrightarrow \Sigma_k.$$

Since $\{p_i, p_j\} = 0$ for all $i, j = 1, \ldots, n$, it follows that $p_{k+1}, \ldots, p_n$ induce $C^\infty$ functions on $\Sigma_k$, which we will, with some abuse of notation, continue to write as $p_{k+1}, \ldots, p_n$. From (12), it follows that

$$dp_i(\pi_k(v)) = 0; \;\; i = k+1, \ldots, n.$$

Here, we denote the single point $\pi_k(\mathbb{R}^n \cdot (v))$ by $\pi_k(v)$. We thus obtain an Abelian subalgebra $\mathbf{p}_{red} = \mathbb{R}\{p_{k+1}, \ldots, p_n\}$ of $(C^\infty(\Sigma_k), \{,\})$ equipped with the Poisson bracket defined by $\sigma$, consisting of functions with a critical point at $\pi_k(v)$. Equivalent to Definition 2 is:

**Definition 3** *The orbit $\mathbb{R}^n \cdot v$ is non-degenerate of rank $k$ if $d_v^2 \mathbf{p}_{red}$ is a Cartan subalgebra of the Lie subalgebra of quadratic forms in $(C^\infty(\Sigma_k), \{,\})$.*

## 1.1 Normal forms of integrable Hamiltonians near non-degenerate singular orbits

Eliasson's normal form theorem for completely integrable systems near a compact non-degenerate singular orbit $\Lambda \subset \mathcal{P}^{-1}(c)$ of rank $k$ expresses the Hamiltonians $p_j$

in terms of the linear action variables $I_k$ of (10) and of additional action variables in the symplectic transversal (or reduced space). Before stating the normal form theorem, we recall the definitions of the action variables.

Let $\mathcal{Q}(2m)$ denote the Lie algebra of quadratic forms on $\mathbb{R}^{2m}$ equipped with its standard Poisson bracket. It contains the following special elements (action variables):

(i) Real hyperbolic: $I_i^h = x_i \xi_i$;

(ii) Elliptic: $I_i^e = x_i^2 + \xi_i^2$;

(iii) Complex hyperbolic: $I_i^{ch} = x_i \xi_{i+1} - x_{i+1}\xi_i + \sqrt{-1}(x_i\xi_i + x_{i+1}\xi_{i+1})$.

Let us call the reduced (or transversal) Hamiltonian system around the equilibrium point (or singular orbit) non-degenerate elliptic, if it is non-degenerate in the sense of Definitions 2 and 3 and if the generators of $d_v^2 \mathbf{p}_{red}$ are elliptic as in (ii). Elliason's elliptic normal form theorem states that in this non-degenerate elliptic case, there exists a local symplectic diffeomorphism

$$\kappa : V \to U, \quad \kappa(\mathbf{T}^k \times 0) = \mathbb{R}^n \cdot v$$

from a neighbourhood $V$ of $\mathbf{T}^k \times 0$ in $T^*(\mathbf{T}^k \times \mathbb{R}^{n-k})$ to a neighborhood $U$ of the orbit and a locally defined $C^\infty$ function $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ such that

$$p_i \circ \kappa^{-1} - c_i = f(I_1^e, \dots, I_{n-k}^e, I_1, \dots, I_k). \tag{14}$$

There is a corresponding normal form theorem in the hyperbolic case or in the case of mixed elliptic-hyperbolic systems. The statement and proof are alluded to in [El] and discussed in detail in [El2]. We let $2m = 2(n-k) = \dim K/L$ as above. By our assumption, the sub-algebra $d_v^2\mathbf{p}$ is a Cartan subalgebra of $\mathcal{Q}(2m)$. By simultaneously diagonalizing the quadratic forms, we can find a basis of $d_v^2\mathbf{p}$ consisting of generators of the above types. The normal form theorem on the reduced (or transversal) space now states that there exists a locally-defined canonical mapping, $\kappa : U \to U_0$, from a small neighbourhood, $U$, of $\pi_k(v) \in \Sigma_k$ to a neighbourhood, $U_0$, of $0 \in \mathbb{R}^{2m}$, with the property that:

$$\forall i, j \ \{p_i \circ \kappa^{-1}, I_j^e\} = \{p_i \circ \kappa^{-1}, I_j^h\} = \{p_i \circ \kappa^{-1}, I_j^{ch}\} = 0. \tag{15}$$

Here, $p_j$ are actually the functions induced by $p_{k+1}, \dots, p_n$ on $\Sigma_k$. By making a second-order Taylor expansion about $I^e = I^{ch} = I^h = 0$, it follows from (15) that $\forall i = 1, \dots, n$, there locally exist $M_{ij} \in C^\infty(U_0)$ with $\{I_i^e, M_{ij}\} = \{I_i^h, M_{ij}\} = \{I_i^{ch}, M_{ij}\} = 0$ such that:

$$p_i \circ \kappa^{-1} - c_i = \sum_{j=1}^{H} M_{ij} \cdot I_j^h + \sum_{j=H+1}^{H+L+1} M_{ij} \cdot I_j^{ch} + \sum_{j=H+L+1}^{n} M_{ij} \cdot I_j^e. \tag{16}$$

Non-degeneracy is easily seen to be equivalent to:

$$(M_{ij})(0) \in Gl(n; \mathbb{R}).$$

The corresponding result for the original integrable system near a singular compact orbit (Theorem C of [El]) is a parameter-dependent version of the reduced normal form theorem, generalizing (14). With the same assumptions, there exists a neighbourhood, $\Omega$, of the orbit, $\mathbb{R}^n \cdot (v)$, and a canonical map $\kappa : \Omega \to T^*(\mathbb{T}^k) \times D$ with the property that, for all $i = 1, \ldots, n$,

$$p_i \circ \kappa^{-1} - c_i = \sum_{j=1}^{H} M_{ij} \cdot I_j^h + \sum_{j=H+1}^{H+L+1} M_{ij} \cdot I_j^{ch} + \sum_{j=H+L+1}^{n-k} M_{ij} \cdot I_j^e + \sum_{j=n-k+1}^{n} M_{ij} \cdot I_{n+1-j}. \tag{17}$$

Here $I^h := (I_1^h, \ldots, I_H^h)$, $I^{ch} := (I_{H+1}^{ch}, \ldots, I_{H+L+1}^{ch})$ and $I^e := (I_{H+L+2}^e, \ldots, I_{n-k}^e)$ denote the elements defined above and $I := (I_1, \ldots, I_k)$ are momentum coordinates of $T^*(\mathbb{T}^k)$. The $M_{ij}$ Poisson-commute with all the action functions.

As mentioned above, the proofs of (16) and (17) are similar to the elliptic case in [El]; for discussion of how the results can be extended to mixed elliptic-hyperbolic systems we refer to [El, El2, CP, VN, VN2].

## 1.2 Existence of singular orbits

In this section, we prove the first part of Theorem 1:

**Lemma 3** *Let $(M, g)$ be a compact Riemannian manifold whose geodesic flow commutes with a Hamiltonian $\mathbb{R}^n$ action. Then, unless $(M, g)$ is a flat torus, the action must have a singular orbit of dimension $< n$.*

*Proof.* The hypothesis that the Liouville foliation is non-singular has two immediate geometric consequences:

(i) By Mane's theorem [M] , $(M, g)$ is a manifold without conjugate points;

(ii) The (homogeneous) moment map $\mathcal{P} : T^*M - 0 \to \mathbb{R}^n - 0$ is a torus fibration by $T^n$.

Statement (ii) follows from the Liouville-Arnold theorem. On a non-singular leaf, we must have $dp_1 \wedge \cdots \wedge dp_n \neq 0$. Since this holds everywhere, $\mathcal{P}$ is a submersion; and since it is proper, it is a fibration. The fiber must be $T^n$, again by the Liouville-Arnold theorem. Since $\mathcal{P}$ is homogeneous, the image $\mathcal{P}(T^*M - 0) = \mathbb{R}_+ \cdot \mathcal{P}(S^*M)$. Since $\mathcal{P}$ is a submersion, the image is a smooth compact submanifold of $S^{n-1}$ hence must be all of $S^{n-1}$.

By (i) it follows that $M = \tilde{M}/\Gamma$ where $(\tilde{M}, \tilde{g})$ is the universal Riemannian cover of $(M, g)$ (diffeomorphic to $\mathbb{R}^n$), and where $\Gamma \sim \pi_1(M)$ is the group of covering transformations.

We claim that (ii) implies $\pi_1(M) = \mathbb{Z}^n$.

Indeed, $T^*M - 0$ is a double fibration (with indicated fibers):

$$\begin{array}{ccccc} & & T^*M - 0 & & \\ & \pi \swarrow (\mathbb{R}^n - 0) & & (T^n) \searrow \mathcal{P} & \\ M & & & & \mathbb{R}^n - 0 \end{array} .$$

By the homotopy sequence of a fibration $\pi : E \to B$,

$$\cdots \pi_q(F) \to \pi_q(E) \to \pi_q(B) \to \pi_{q-1}(F) \cdots \to \pi_0(E) \to \pi_0(B) \to 0$$

and using that $\pi_2(T^n) = \mathbf{1}$ and that $\pi_2(M) = \mathbf{1}$ by (i), we obtain

$$\mathbf{1} \to \pi_1(S^{n-1}) \to \pi_1(S^*M) \to \pi_1(M) \to \pi_0(S^{n-1}) \to \pi_0(S^*M) \to \pi_0(M) \to \mathbf{1}$$

$$\mathbf{1} \to \pi_2(S^*M) \to \pi_2(S^{n-1}) \to \pi_1(T^n) \to \pi_1(S^*M) \to \pi_1(S^{n-1}) \to$$

$$\to \pi_0(T^n) \to \pi_0(S^*M) \to \pi_0(S^{n-1}) \to \mathbf{1} \tag{18}$$

Since $\pi_2(N) = \pi_2(\tilde{N})$ (where $\tilde{N}$ is the universal cover), and since $\tilde{S^*M} = \tilde{S^{n-1}} \times \mathbb{R}^n$, we have $\pi_2(S^*M) = \pi_2(S^{n-1})$. From its definition, we see that the homomorphism $\pi_2(S^*M) \to \pi_2(S^{n-1})$ is an isomorphism, hence the second sequence simplifies to

$$\mathbf{1} \to \pi_1(T^n) \to \pi_1(S^*M) \to \pi_1(S^{n-1}) \to \pi_0(T^n) \to \pi_0(S^*M) \to \pi_0(S^{n-1} \to \mathbf{1} \tag{19}$$

Let us first assume that $n \geq 3$. Then $\pi_1(S^*M) = \pi_1(M)$ and $\pi_1(S^{n-1}) = \mathbf{1}$, so we get

$$\mathbf{1} \to \pi_1(T^n) \to \pi_1(M) \to \mathbf{1},$$

i.e., $\pi_1(T^n) \cong \pi_1(M)$.

We now consider dimension 2. By (i), the genus $g \geq 1$. The unit tangent bundles of surfaces of genus $g \geq 2$ do not fiber over a circle, so we can disqualify them and conclude $g = 1$. (In fact, it follows by a classic result of Kozlov [K] (see also [TAI]) that the only surfaces that can possibly have a completely integrable geodesic flow (even with singularities) are $M = S^2, T^2$.)

It follows then that $M$ is diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^n$, i.e., $M$ is a torus. Since $g$ has no conjugate points, the proof is concluded by Burago-Ivanov's theorem that metrics on tori without conjugate points are flat [BI]. □

### 1.2.1 Problem on homogeneous moment maps

As mentioned above, the existence proof of singular orbits does not give any information on the dimension of such orbits. Do there exist one-dimensional orbits in 'generic' cases?

In the case of Hamiltonian torus $T^n$ actions, one-dimensional orbits are detected by the image $\mathcal{C}$ of $T^*M - 0$ under the moment map $(I_1, \ldots, I_n) : T^*M - 0 \to \mathbb{R}^n$. By assumption, the $I_j$ generate $2\pi$-periodic Hamiltonian flows. In the case of the flat torus, $\mathcal{C} = R^N - 0$, but in other cases $\mathcal{C}$ is a convex polyhedral cone with boundary. The boundary faces correspond to singular orbits and, in particular, an edge (a one-dimensional boundary face) corresponds to a periodic orbit.

The moment polyhedra of toric moment maps have been studied by Lerman and Lerman-Shirokova [LS, L2]. Lerman has proved that if no subtorus acts freely on $T^*M - 0$, then $\mathcal{C}$ always has an edge [L3]. The problem we would like to pose is to formulate and prove an analogous result for homogeneous $\mathbb{R}^n$ actions on cotangent bundles. Can one detect singular torus orbits of dimension one from $\mathcal{C}$ and must $\mathcal{C}$ have an edge? By taking products with flat tori one can see that not all torus actions have a one-dimensional orbit, but perhaps 'generic' examples do.

To our knowledge, the cone $\mathcal{C}$ has not been studied to date for general homogeneous $\mathbb{R}^n$ actions. Clearly it depends on a choice of generators of the $\mathbb{R}^n$ action. In the toric case, the special generators gave rise to a polyhedron $\mathcal{C}$ from which one could read off the existence of singular orbits. Is there a good replacement in the case of general $\mathbb{R}^n$ actions, one which does not already presuppose a knowledge of the singular orbits?

## 2 Quantum integrable systems and Birkhoff normal forms

Our purpose in this section is to construct a microlocal Birkhoff normal form for a QCI (quantum completely integrable) system near a singular orbit. We first set up some notation for the joint spectrum of quantum integrable systems. Throughout, we follow the notation and terminology of [TZ1] and refer there for background on QCI systems.

### 2.1 Quantum integrable systems

In this article, the only Hamiltonians we consider are Laplacians $\Delta$ on compact Riemannian manifolds $(M, g)$, although many of the methods and results extend to Schrödinger operators as in [TZ1]. We therefore assume that $P_1 = \sqrt{\Delta}$, and that the other commuting operators $P_2, \ldots, P_n$ are classical pseudodifferential operators of order one.

Although our operators are homogeneous pseudodifferential operators, it is convenient to use the notation and methods of semiclassical microlocal analysis. We therefore rescale our operators to obtain

$$Q_j := \hbar P_j \in Op(S^{0,1}_{cl}).$$

In forming microlocal models, we will need to introduce more general kinds of semiclassical quantum integrable systems, so we pause to define the class.

**Definition 4** *We say that the operators $Q_j \in Op_\hbar(S^{m,k}_{cl})$; $j = 1, \ldots, n$, generate a semiclassical quantum completely integrable system if*

$$[Q_i, Q_j] = 0;\ \forall 1 \leq i, j \leq n,$$

*and the respective semiclassical principal symbols $q_1, \ldots, q_n$ generate a classical integrable system with $dq_1 \wedge dq_2 \wedge \cdots \wedge dq_n \neq 0$ on a dense open set $\Omega \subset T^*M - 0$.*

Here, we use the notation of Dimassi-Sjöstrand [DSj] for operator classes: Given an open $U \subset \mathbb{R}^n$, we say that $a(x, \xi; \hbar) \in C^\infty(U \times \mathbb{R}^n)$ is in the symbol class $S^{m,k}(U \times \mathbb{R}^n)$, provided

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; \hbar)| \leq C_{\alpha\beta} \hbar^{-m} (1 + |\xi|)^{k - |\beta|}.$$

We say that $a \in S^{m,k}_{cl}(U \times \mathbb{R}^n)$ provided there exists an asymptotic expansion:

$$a(x, \xi; \hbar) \sim \hbar^{-m} \sum_{j=0}^{\infty} a_j(x, \xi) \hbar^j,$$

valid for $|\xi| \geq \frac{1}{C} > 0$ with $a_j(x, \xi) \in S^{0, k-j}(U \times \mathbb{R}^n)$ on this set. The associated $\hbar$ Kohn-Nirenberg quantization is given by

$$Op_\hbar(a)(x, y) = (2\pi\hbar)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi/\hbar} \, a(x, \xi; \hbar) \, d\xi.$$

As is well known, the definition can be globalized to $M$ using a partition of unity. We denote this class by $Op_\hbar(S^{m,k})(T^*M)$. The symbol of the composition is given by the usual formula: Given $a \in S^{m_1, k_1}$ and $b \in S^{m_2, k_2}$, the composition $Op_\hbar(a) \circ Op_\hbar(b) = Op_\hbar(c) + \mathcal{O}(\hbar^\infty)$ in $L^2(M)$ where locally,

$$c(x, \xi; \hbar) \sim \hbar^{-(m_1 + m_2)} \sum_{|\alpha|=0}^{\infty} \frac{(-i\hbar)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha a) \cdot (\partial_x^\alpha b).$$

For further details, we refer to [DSj].

For a general QCI system, we denote by $\{\varphi_\mu\}$ an orthonormal basis of joint eigenfunctions,

$$Q_j \varphi_\mu = \mu_j(\hbar) \varphi_\mu, \quad \langle \varphi_\mu, \varphi_{\mu'} \rangle = \delta_{\mu, \mu'}, \tag{20}$$

and the joint spectrum by

$$\Sigma(\hbar) := \{\mu(\hbar) := (\mu_1(\hbar), \ldots, \mu_n(\hbar))\}. \tag{21}$$

### 2.2 Model cases

Quantum Birkhoff normal forms are microlocal expressions of a given QCI system in terms of certain model system. Model quantum completely integrable systems are direct sums of the quadratic Hamiltonians:

- $\hat{I}^h := \hbar(D_y y + y D_y)$ (hyperbolic Hamiltonian),
- $\hat{I}^e := \hbar^2 D_y^2 + y^2$, (elliptic Hamiltonian),
- $\hat{I}^{ch} := \hbar\,[(y_1 D_{y_1} + y_2 D_{y_2}) + \sqrt{-1}(y_1 D_2 - y_2 D_{y_1})]$ (complex hyperbolic Hamiltonian),
- $\hat{I} := \hbar\, D_\theta$, (regular Hamiltonian).

The corresponding model eigenfunctions are:

- $u_h(y;\lambda,\hbar)$ $= |\log\hbar|^{-1/2}[c_+(\hbar)Y(y)|y|^{-1/2+i\lambda(\hbar)/\hbar} + c_-(\hbar)Y(-y)|y|^{-1/2+i\lambda(\hbar)/\hbar}]$; $|c_-(\hbar)|^2 + |c_+(\hbar)|^2 = 1; \lambda(\hbar) \in \mathbb{R}$.
- $u_e(y;n,\hbar) = \hbar^{-1/4}\exp(-y^2/\hbar)\ \Phi_n(\hbar^{-1/2}y);\ n \in \mathbb{N}$.
- $u_{ch}(r,\theta;t_1,t_2,\hbar) = |\log\hbar|^{-1/2} r^{(-1+it_1(\hbar))/\hbar}\, e^{it_2(\hbar)\theta/\hbar};\ t_1(\hbar), t_2(\hbar) \in \mathbb{R}$.
- $u_{reg}(\theta;m,\hbar) = e^{im\theta};\ m \in \mathbb{Z}$.

Here, $Y(x)$ denotes the Heaviside function, $\Phi_n(y)$ the $n$th Hermite polynomial and $(r,\theta)$ polar variables in the $(y_1, y_2)$ complex hyperbolic plane.

The important part of a model eigenfunctions is its microlocalization to a neighborhood of $x = \xi = 0$, so we put:

$$\psi(x;\hbar) := Op_\hbar(\,\chi(x)\chi(y)\chi(\xi)\,)\cdot u(y;\hbar),$$

where $\epsilon > 0$ and $\chi \in C_0^\infty([-\epsilon,\epsilon])$. In the hyperbolic, complex hyperbolic, elliptic and regular cases, we write $\psi_h(y;\hbar), \psi_{ch}(y;\hbar), \psi_e(y;\hbar)$ and $\psi_{reg}(y;\hbar)$ respectively. A straightforward computation [T2] shows that when $t_1(\hbar), t_2(\hbar), n\hbar, m\hbar = \mathcal{O}(\hbar)$ the model quasimodes are $L^2$-normalized; that is

$$\|Op_\hbar(\chi(x)\chi(y)\chi(\xi)\,)\,u(y;\hbar)\|_{L^2} \sim 1 \tag{22}$$

as $\hbar \to 0$. Note that, although the model eigenfunctions above are not in general smooth functions, the microlocalizations are $C^\infty$ and supported near the origin.

### 2.3 Ladders of eigenfunctions

Semiclassical limits are taken along ladders in the joint spectrum. For fixed $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, we define a ladder of joint eigenvalues of the original homogeneous problem $P_1 = \sqrt{\Delta}, P_2, \ldots, P_n$ by:

$$\{(\lambda_{1k}, \ldots, \lambda_{nk}) \in Spec(P_1, \ldots, P_n);\ \forall j = 1, \ldots, n,\ \lim_{k\to\infty} \frac{\lambda_{jk}}{|\lambda_k|} = b_j\}, \tag{23}$$

where $|\lambda_k| := \sqrt{\lambda_{1k}^2 + \cdots + \lambda_{nk}^2}$.

In the corresponding $\hbar$-scaled system, the ladder will be denoted by:

$$\Sigma_b(\hbar) := \{\mu(\hbar) := (\mu_1(\hbar), \dots, \mu_n(\hbar)) \in \mathrm{Spec}(Q_1, \dots, Q_n);\ |\mu_j(\hbar) - b_j| \\ \leq C\hbar,\ j = 1, \dots, n\}. \quad (24)$$

We define the joint eigenspace corresponding to $\Sigma_b(\hbar)$ as follows: For $b \in \mathcal{P}(T^*M - 0)$, define

$$V_b(\hbar) := \{\varphi_\mu; \|\varphi_\mu\|_{L^2} = 1 \,\mathrm{with}\, \mu(\hbar) \in \Sigma_b(\hbar)\}. \quad (25)$$

## 2.4 Microlocal solution space

Following [CP] Definition 2, we call a family of distributions $u_\hbar; \hbar \in (0, \hbar_0]$ *admissible* if there exist constants $N_j; j = 1, 2, 3$ such that for any $\varphi \in S(\mathbb{R})$,

$$\left| \int \varphi(x) u_\hbar(x) dx \right| \leq \hbar^{-N_1} ||(1 + x^2)^{N_2} \varphi||_{C^{N_3}}.$$

We now define the microlocal (quasi-)eigenvalue problem in $\Omega$. We denote $\hbar$-microlocal equivalence on an open set $\Omega$ by $=_\Omega$ [CP]. We say that $\nu_k(\hbar)$ is a quasi-classical eigenvalue if there exists a non-trivial, admissible solution of the eigenvalue problem

$$Q_j \psi_\nu =_\Omega \nu_j(\hbar) \psi_\nu; j = 1, \dots, n. \quad (26)$$

The set of quasi-classical eigenvalues around $c$ is thus:

$$Q\Sigma_c(\hbar) := \{\nu(\hbar) : (26) \text{ holds, with } \ |\nu(\hbar) - c| \leq C\hbar,\ j = 1, \dots, n\}. \quad (27)$$

We define the corresponding microlocal solution space as the span of

$$QV_c(\hbar) := \{\psi_\nu; \|\psi_\nu\|_{L^2} = 1,\ (26) \text{ holds with}\, \mu(\hbar) \in Q\Sigma_b(\hbar)\}. \quad (28)$$

The solution space (28) can be characterized uniquely (up to a $\mathbb{C}(\hbar)$-multiple) in terms of the model quasimodes $\psi_e$, $\psi_h$, $\psi_{ch}$ and $\psi_{reg}$. In the following, we use the abbreviation $(u_e(y; n, \hbar) \cdot u_h(y; \lambda_k(\hbar), \hbar) \cdot u_{ch}(y; t_{1,k}(\hbar), t_{2,k}(\hbar), \hbar) \prod_{j=1}^k e^{im_j\theta_j}\ )$ for the expression

$$\Pi_{j=1}^H \psi_{hj}(y_j; \hbar) \otimes \Pi_{j=H+1}^{H+L} \psi_{chj}(y_j; \hbar) \otimes \Pi_{j=H+L+1}^{H+L+E} \psi_{ej}(y_j; \hbar) \otimes \\ \Pi_{j=L+H+E+1}^{n} \psi_{rj}(y_j; \hbar)$$

in which the $(m, n, \lambda, t_1, t_2)$ parameters are put in.

**Proposition 4** *For any $\psi_{\nu(\hbar)} \in QV_c(\hbar)$, there exist $t_1(\hbar), \lambda(\hbar) \in \mathbb{R}$ and $t_2(\hbar), n(\hbar)$, $m(\hbar) \in \mathbb{Z}$ and an $\hbar$-dependent constant $c(\hbar)$ such that*

$$\psi_\nu =_\Omega c(\hbar)\, F\ \ (u_e(y; n, \hbar) \cdot u_h(y; \lambda(\hbar), \hbar) \cdot u_{ch}(y; t_1(\hbar), t_2(\hbar), \hbar) \prod_{j=1}^k e^{im_j\theta_j}\ ),$$

*where $F_\hbar$ is the microlocally unitary $\hbar$-Fourier integral operator in Lemma* (5).

*Proof.* The microlocal solutions of the model eigenfunction equations are unique up to $\mathbb{C}(\hbar)$-multiples. This was proved for the strictly hyperbolic case in [CP] and for the complex hyperbolic case in [VN2]. The elliptic case follows from the local normal form. An application of Lemma (5) then gives the result. □

### 2.5 Singular Birkhoff normal form

In this section we introduce Birkhoff normal forms for a quantum completely integrable system near a singular orbit. The main result is the following quantum analogue (see also [VN]) of the classical Eliasson normal form in (17). In the following lemma, $\epsilon^{e,h,ch,reg}(\hbar)$ are each classical symbols of order 1 in $\hbar \in (0, \hbar_0]$; that is, there exist $\epsilon_j^{e,h,ch,reg} \in \mathbb{C}; j = 1, 2, \dots$ such that $\epsilon^{e,h,ch,reg}(\hbar) \sim \sum_{j=1}^{\infty} \epsilon_j^{e,h,ch,reg} \hbar^j$ as $\hbar \to 0^+$.

**Lemma 5** *Let $c \in \mathbb{R}^n$ be a singular value of the moment map $\mathcal{P}$ and $\mathbb{R}^n \cdot v$ be a rank-$k$ Eliasson non-degenerate orbit of the joint flow. Then, there exists a microlocally elliptic $\hbar$-Fourier integral operator, $F_\hbar$, and a microlocally invertible $n \times n$ matrix of $\hbar$-pseudodifferential operators, $\mathcal{M}_{ij}$, with $[F_\hbar^{-1} \mathcal{M}_{ij} F_\hbar, Q_k] =_\Omega 0; k = 1, \dots, n$ and satisfying:*

$$F_\hbar^{-1} (Q_1 - c_1, \dots, Q_n - c_n) F_\hbar =_\Omega \mathcal{M} \cdot ( \hat{I}^h - \epsilon^h(\hbar), \hat{I}^{ch} - \epsilon^{ch}(\hbar), \hat{I}^e - \epsilon^e(\hbar), \hbar D_\theta - \epsilon^{reg}(\hbar) ) + \mathcal{O}(\hbar^\infty). \quad (29)$$

*Here,* $\hat{I}_j^h = \hbar(D_{y_j} y_j + y_j D_{y_j})$, $\hat{I}_j^e = \hbar^2 D_{y_j}^2 + y_j^2$, $\hat{I}^{ch} = \hbar[(y_j D_{y_j} + y_{j+1} D_{y_{j+1}}) + \sqrt{-1}(y_j D_{j+1} - y_{j+1} D_{y_j})]$ *and* $\hat{I}_j = \hbar D_{\theta_j}$.

*Proof.* The proof is essentially the same as in ([VN] Theorem 3.6). The only complication here is that since $\mathbb{R}^n \cdot (v)$ is a $k < n$-dimensional torus and not a point, $\hbar D_{\theta_1}, \dots, \hbar D_{\theta_k}$ must be added to the space of model operators. The proof can be reduced to that in [VN] by making Fourier series decompositions in the $(\theta_1, \dots, \theta_k)$ variables (see, for instance [T2] Theorem 3). □

## 3 Blow-up of eigenfunctions attached to singular orbits of the Lagrangian fibration

The purpose of this section is to prove Theorem 1. We break up the proof into a sequence of three Lemmas, each concerned with estimates of matrix elements relative to the eigenfunctions. They culminate in an estimate in Lemma 8 of the *small scale* $L^2$ mass of certain eigenfunctions $\varphi_\mu \in V_c(\hbar)$ near any singular orbit $\Lambda \subset \Gamma_{sing}(c)$. The end result is that for each $\hbar \in (0, \hbar_0]$, there exist $\varphi_\mu \in V_c(\hbar)$ with

$$( Op_\hbar(\chi_1^\delta(x;\hbar)) \varphi_\mu, \varphi_\mu ) \gg |\log \hbar|^{-m} \quad (30)$$

for some $m > 0$ and where $\chi_1^\delta(x;\hbar) := \chi_1(\hbar^{-\delta} x)$ with $\chi_1(x)$ a cut-off function supported near $\pi(\Lambda)$.

In the proof, we will need to use additional pseudo-differential operators belonging to a more refined semi-classical calculus, containing cut-offs such as $\chi_1(\hbar^{-\delta}x)$, which involve length scales $\sim \hbar^{\delta}$ with $0 < \delta < 1/2$.

### 3.1 Eigenfunction mass near non-degenerate, singular orbits

In this section, we give a lower bound for the microlocal mass on 'large' length scales of joint eigenfunctions with joint eigenvalues $\mu(\hbar) \in \Sigma_c(\hbar)$ near any compact, singular orbit $\Lambda \subset \Sigma_c(\hbar)$ satisfying Eliasson's non-degeneracy condition. Note that each connected component $\Gamma_{sing}(c)$ of a singular level always contains a $k < n$-dimensional compact orbit $\Lambda$ (see Proposition 1.3 of [TZ1]). Our lower bound is analogous to that of Lemma 2.8 of [TZ1] in the regular case, which proves that each component torus of a regular level carries an amount of eigenfunction mass bounded below by a constant independent of $\hbar$. In the singular case, the same kind of proof gives a weaker result in which the mass can decrease but at most logarithmically. It would be interesting to know if in fact there exists a uniform lower bound for the mass as in the regular case, but we do not need such a strong result for our application and do not pursue the matter here.

In the following lemma, we write $f(\hbar) \gg g(\hbar)$ if there exists a constant $C_0 > 0$ such that for $\hbar$ sufficiently small, $|f(\hbar)| \geq \frac{1}{C_0}|g(\hbar)|$.

**Lemma 6** *Let $\Lambda := \mathbb{R}^n \cdot (v) \subset \Gamma_{sing}(c)$ be an Eliasson non-degenerate orbit, and let $\chi_\Lambda(x,\xi) \in C_0^\infty(\Omega;[0,1])$ be a cut-off function supported in an invariant neighbourhood $\Omega$ of $\Lambda$ and identically equal to one on a smaller neighbourhood $\tilde{\Omega}$ of $\Lambda$. Then for each $\hbar \in (0,\hbar_0]$ there exist $\varphi_\mu \in V_c(\hbar)$ and a constant $m \geq 0$ such that,*

$$(Op_\hbar(\chi_\Lambda)\varphi_\mu, \varphi_\mu) \gg |\log \hbar|^{-m}.$$

*Proof.* Let $\Lambda_j; j = 1,\ldots,K$ denote all the compact $\mathbb{R}^n$-orbits contained in the singular component, $\Gamma_{sing}(c)$ and $\chi_j(x,\xi) \in C_0^\infty; j = 1,\ldots,K$ cut-off functions with $\chi_j = 1$ near $\Lambda_j$. We choose the indices so that $\Lambda_1 = \Lambda$ and put $\chi := \chi_1$. Let $F \in S(R)$ with $F \geq 0, F(0) = 1$ and $\check{F} \in C_0^\infty(\mathbb{R})$ with sufficiently small support near $0 \in \mathbb{R}$. We define $\hat{f}(t_1,\ldots,t_n) := \prod_{j=1}^n \hat{F}(t_j)$. We have

$$\sum_{\mu\in\Sigma(\hbar)} \langle Op_\hbar(\chi)\varphi_\mu, \varphi_\mu\rangle\, f(\hbar^{-1}(\mu(\hbar)-c)) = \int_{\mathbb{R}^n} \check{f}(t) Tr Op_\hbar(\chi) e^{i\sum_{j=1}^n t_j[Q_j(\hbar)-c_j]/\hbar} dt. \quad (31)$$

We calculate the trace using the microlocal conjugation of $e^{i\sum_{j=1}^n t_j[Q_j(\hbar)-c_j]}$ to its normal form given in Lemma 5. Then, by a well-known parametrix construction ([BPU] Lemma 2.1) for $Op_\hbar(\chi\circ\kappa)\cdot e^{i\sum_{i=1}^n t_i\mathcal{M}_i^j \hat{I}_j/\hbar}$ for $|t|$ small, we have that

$\int_{\mathbb{R}^n} \check{f}(t) Tr Op_\hbar(\chi) e^{i\sum_{j=1}^n t_j [Q_j(\hbar)-c_j]/\hbar}\, dt$ is locally a sum of integrals of the form

$$T := (2\pi\hbar)^{-n} \int\int\int e^{-i\langle M(x,\xi)\cdot(I_e,I_h,I_{ch},I),t\rangle/\hbar}\, \check{f}(t)\, \chi\circ\kappa(x,\xi)\, a(t,x,x,\xi)\, dtdxd\xi + \mathcal{O}(\hbar), \quad (32)$$

where $a \in C_0^\infty$ with $a(0,x,x,\xi) = 1$. Here, $I_h(x,\xi) := (I_h(x_1,\xi_1),\ldots,I_h(x_H,\xi_H))$, $I_{ch}(x,\xi) = (I_{ch}(x_{H+1},\xi_{H+1}),\ldots,I_h(x_{H+L+1},\xi_{H+L+1}))$, $I_e(x,\xi) = (I_e(x_{L+H+2}, \xi_{L+H+2}),\ldots,I_e(x_{n-k},\xi_{n-k}))$ and $I(x,\xi) = (I_{n-k+1}),\ldots,I_n)$. Next, make the change of variables $(t_1,\ldots,t_n) \mapsto^t M\cdot(t_1,\ldots,t_n)$ in (32) and get that

$$T := (2\pi\hbar)^{-n} \int\int\int e^{-i\langle (I_e,I_h,I_{ch},I),s\rangle/\hbar}\, b(s,x,x,\xi)\, dsdxd\xi + \mathcal{O}(\hbar), \quad (33)$$

where,

$$b(s,x,x,\xi) = \check{f}({}^tM^{-1}(x,\xi)\cdot s)\, \chi\circ\kappa(x,\xi)\, a({}^tM^{-1}(x,\xi)s,x,x,\xi)\, |\det M(x,\xi)|^{-1}.$$

Since $M(0)\in GL_n(\mathbb{R})$, by choosing supp $\chi$ sufficiently small it follows that $b\in C_0^\infty$. Then, by carrying out the iterated $(s_i,x_i,\xi_i)$-integrals in (33), we are reduced to computing the asymptotics of the integrals:

$$T_{reg} := (2\pi\hbar)^{-1} \int_{-\infty}^{\infty}\int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\xi s/\hbar}\hat{F}(s)c(s,x,\xi)dsdxd\xi \sim c_{reg},$$

$$T_e := (2\pi\hbar)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{is(x^2+\xi^2)/\hbar}c(s,x,\xi)\hat{F}(s)dsdxd\xi \sim c_e$$

$$T_h := (2\pi\hbar)^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{isx\xi/\hbar}c(s,x,\xi)\hat{F}(s)dsdxd\xi \sim c_h|\log\hbar|,$$

$$T_{ch} := (2\pi\hbar)^{-2} \int_0^{2\pi}\int_0^{2\pi}\int_0^{\infty}\int_0^{\infty} e^{ir\rho[s_1\cos\theta - s_2\sin\alpha]/\hbar}c(s,r,\rho,\theta,\alpha)$$

$$\times\hat{F}(s_1)\hat{F}(s_2)ds_1ds_2rdr\rho d\rho d\theta d\alpha \sim c_{ch}|\log\hbar|.$$

where, $c\in C_0^\infty$ with $c(0)\neq 0$ and $c_{reg}, c_e, c_h, c_{ch}$ denote non-zero constants. The estimate for $T_{reg}$ follows by stationary phase in the $(s,\xi)$-variables, whereas for $T_e, T_h, T_{ch}$ the asymptotics follow from Proposition 3.4 and Theorem 3.5 in [BPU].

It follows that for some non-zero constant $C_0$

$$\sum_{\mu\in\Sigma(\hbar)} f(\hbar^{-1}(\mu(\hbar)-c))\langle Op_\hbar(\chi_\Lambda)\varphi_\mu,\varphi_\mu\rangle \sim C_0|\log\hbar|^{m_1}, \quad (35)$$

where, $m_1$ denote the total number of complex and real hyperbolic summands in $\Lambda_1$. The same estimate holds for all singular tori.

Put

$$m = \max_{j=1,\ldots,K}\{m_j\} - m_1. \quad (36)$$

We now argue by contradiction that the estimate in the Lemma is correct. If not, then for all $\varphi_\mu$'s the matrix element $(Op_\hbar(\chi_\Lambda)\varphi_\mu, \varphi_\mu) = o(|\log \hbar|^{-m})$, and therefore

$$\sum_{\mu \in \Sigma(\hbar)} f(\hbar^{-1}(\mu_j(\hbar) - c))\langle Op_\hbar(\chi_\Lambda)\varphi_\mu, \varphi_\mu\rangle = o(|\log \hbar|^{-m}) \sum_{\mu \in \Sigma(\hbar)} f(\hbar^{-1}(\mu(\hbar) - c)). \tag{37}$$

Since the $\Lambda_j; j = 1, \dots, K$ are $\omega$-limit sets for the joint flow on $\Gamma_{sing}(c)$, it follows by the semiclassical Egorov theorem and the Gårding inequality ([T2] Proposition 1) that for any joint eigenfunction $\varphi_\mu$ with $|\mu(\hbar) - c| = o(1)$,

$$(\sum_{i=1}^{K}(Op_\hbar(\chi_i)\varphi_\mu, \varphi_\mu)) >> 1. \tag{38}$$

Thus,

$$\begin{aligned} \textstyle\sum_{\mu \in \Sigma(\hbar)} f(\hbar^{-1}(\mu(\hbar) - c)) &\ll \textstyle\sum_{\mu \in \Sigma(\hbar)} \left(\sum_{i=1}^{K}(Op_\hbar(\chi_i)\varphi_\mu, \varphi_\mu)\right) f(\hbar^{-1}(\mu(\hbar) - c)) \\ &\ll \textstyle\sum_{i=1}^{K} |\log \hbar|^{m_i}. \end{aligned} \tag{39}$$

In the last line, we used (35) in each term.

Combining (35) and (39), we get the contradiction

$$|\log \hbar|^{m_1} \leq o(|\log \hbar|^{-m}) \sum_{i=1}^{K} |\log \hbar|^{m_i} = o(|\log \hbar|^{m_1}), \tag{40}$$

by the choice of $m$ in (36). □

### 3.2 Localization on singular orbits

Let $\Lambda$ be any Eliasson non-degenerate compact orbit. We claim that the joint eigenfunctions $\varphi_\mu \in V(\hbar)$ satisfying the estimate in Lemma 6 must blow up along $\pi(\Lambda)$. The first way of quantifying this blowup involves computing the asymptotics for the expected values $(Op_\hbar(q) \cdot Op_\hbar(\chi_\Lambda)\varphi_\mu, Op_\hbar(\chi_\Lambda)\varphi_\mu)$ where $q \in C_0^\infty(T^*M)$.

**Lemma 7** *Let $\varphi_\mu \in V(\hbar)$ satisfy the bound in Lemma 6. Then:*

$$(Op_\hbar(q) \cdot Op_\hbar(\chi_\Lambda)\varphi_\mu, Op_\hbar(\chi_\Lambda)\varphi_\mu) = |c(\hbar)|^2 \left( \int_{\mathbb{R}^n \cdot (v)} q \, d\mu + \mathcal{O}(|\log \hbar|^{-1/2}) \right), \tag{41}$$

*again with $|c(\hbar)| \gg |\log \hbar|^{-m}$ for some $m \geq 0$.*

*Proof.* Since $\varphi_\mu$ solves the equation (26) exactly (and a fortiori microlocally on $\Omega$), we may express it by Proposition (4) in the form:

$$\varphi_\mu =_\Omega c(\hbar)\; F\; u_\mu, \tag{42}$$

for some constant $c(\hbar)$. Here,

$$u_\mu = (u_e(y; n, \hbar) \cdot u_h(y; \lambda(\hbar), \hbar) \cdot u_{ch}(y; t_1(\hbar), t_2(\hbar), \hbar) \prod_{j=1}^{k} e^{im_j\theta_j}\ ) \tag{43}$$

Here, by applying the operators on both sides of the QBNF in Lemma (5) to the model eigendistributions $u_\mu$ and using the uniqueness result in Lemma (4), it follows that for some $n \times n$ matrix $M$ with $M(0) \in GL_n$,

$$M(m\hbar, n\hbar, \lambda(\hbar), t_1(\hbar), t_2(\hbar)) \cdot (m\hbar, n\hbar, \lambda(\hbar), t_1(\hbar), t_2(\hbar)) = \mu(\hbar).$$

By the inverse function theorem, the $(m\hbar, n\hbar, \lambda(\hbar), t_1(\hbar), t_2(\hbar))$ are uniquely determined (modulo $\mathcal{O}(\hbar^\infty)$ ) by the joint eigenvalues $\mu(\hbar)$ and moreover, when $\mu(\hbar) \in \Sigma(\hbar)$ it follows that $m\hbar, n\hbar, \lambda(\hbar), t_1(\hbar), t_2(\hbar) = \mathcal{O}(\hbar)$. By Lemma (6), by (22) and by (42), it follows that for $\hbar \in (0, \hbar_0]$,

$$|\log \hbar|^{-m} \ll (Op_\hbar(\chi_\Lambda)\varphi_\mu, \varphi_\mu) = |c(\hbar)|^2 (F^* Op_\hbar(\chi_\Lambda) F u_\mu, u_\mu) \leq \ \ |c(\hbar)|^2. \tag{44}$$

Granted this lower bound on $|c(\hbar)|$, the Lemma reduces to estimating matrix elements of model eigenfunctions. We now evaluate the matrix elements case by case. The most interesting case is where the orbit $\Lambda$ is strictly real or complex hyperbolic. We use (42) to conjugate to the model setting. The function $q$ goes to $q \circ \chi$ where $\chi$ is the canonical transformation underlying $F$. The model $\mathbb{R}^n$- action locally reduces to a compact torus $T^k$-action, so we can average the function $q \circ \chi$ over the action to obtain a smooth invariant function. We then Taylor expand this averaged function in the directions $(y, \eta)$ transverse to the action. We obtain:

$$\begin{aligned}&(Op_\hbar(q) \circ Op_\hbar(\chi_\Lambda)\varphi_\mu, Op_\hbar(\chi_\Lambda)\varphi_\mu)\\ &\quad = |c(\hbar)|^2 \left( \int_{\mathbb{R}^n \cdot (v)} q \, d\mu + (Op_\hbar(r_h)u_\mu, u_\mu) + (Op_\hbar(r_{ch})u_\mu, u_\mu) + \mathcal{O}(\hbar) \right).\end{aligned} \tag{45}$$

where, $r_h, r_{ch} \in C_0^\infty(\Omega)$ are the Taylor remainders with $r_h, r_{ch} = \mathcal{O}(|y| + |\eta|)$. A direct computation for the model distributions, $u_\mu$ (see [T2] Lemma 5 and Proposition 3) shows that:

$$(Op_\hbar(r_h)u_\mu, u_\mu) = \mathcal{O}(|\log \hbar|^{-1/2}), \ \ (Op_\hbar(r_{ch})u_\mu, u_\mu) = \mathcal{O}(|\log \hbar|^{-1/2}). \tag{46}$$

The remaining cases are where elliptic (i.e., Hermite factors). Each such factor satisfies

$$(Op_\hbar(r_e)u_\mu, u_\mu) = \mathcal{O}(\hbar) \tag{47}$$

so is better than what is claimed. □

### 3.3 Mass concentration on small length scales

Let $\Lambda := \mathbb{R}^n \cdot v$ be a compact, $k$-dimensional singular orbit of the Hamiltonian $\mathbb{R}^n$-action generated by $(p_1, \dots, p_n)$. In this section, we study mass concentration of modes in shrinking tubes of radius $\sim \hbar^\delta$ for $0 < \delta < 1/2$ around $\pi(\Lambda)$ in $M$, where $\pi : T^*M \longrightarrow M$ denotes the canonical projection map. Such small scale concentration of mass estimates quickly lead to sup-norm estimates.

For the sake of simplicity we will assume in this section that

$$\pi : \Lambda \to M$$

is an embedding. This seems to be a reasonable assumption since $\dim \Lambda = k < \dim M$, and it is satisfied in most (if not all) the examples we know. As we explain in §3.5, the proof extends with only minor modifications to the general case.

We denote by $T_\epsilon(\pi(\Lambda))$ the set of points of distance $< \epsilon$ from $\pi(\Lambda)$. For $0 < \delta < 1/2$, we introduce a cut-off $\chi_1^\delta(x;\hbar) \in C_0^\infty(M)$ with $0 \leq \chi_1^\delta \leq 1$, satisfying

(i) $\operatorname{supp} \chi_1^\delta \subset T_{\hbar^\delta}(\pi(\Lambda))$

(ii) $\chi_1^\delta = 1$ on $T_{3/4\hbar^\delta}(\pi(\Lambda))$.

(iii) $|\partial_x^\alpha \chi_1^\delta(x;\hbar)| \leq C_\alpha h^{-\delta|\alpha|}$.

Under the assumption that $\Lambda$ is an embedded submanifold of $M$, the functions

$$\chi_1^\delta(x;\hbar) = \zeta_1(\hbar^{-2\delta} d^2(x, \pi(\Lambda))) \tag{48}$$

are smooth on $T_\epsilon(\pi(\Lambda))$ and satisfy the conditions. Here, $d(.,.)$ is the Riemannian distance function. Also, $\zeta_1 \in C_0^\infty(\mathbb{R})$ with $0 \leq \zeta_1 \leq 1, \zeta_1(x) = 1$ for $|x| \leq 3/4$ and $\operatorname{supp} \zeta_1 \subset (-1, 1)$.

**Lemma 8** *Let $\varphi_\mu \in V_c(\hbar)$ satisfy the bounds in Lemma 6. Then for any $0 \leq \delta < 1/2$, $(Op_\hbar(\chi_1^\delta)\varphi_\mu, \varphi_\mu) \gg |\log \hbar|^{-m}$.*

#### 3.3.1 Outline of proof

The proof uses the somewhat technical properties of *small scale pseudodifferential operators*. We first sketch the proof without these technicalities.

Let $\chi_2^\delta(x, \xi; \hbar) \in C_0^\infty(T^*M; [0,1])$ be a second cut-off supported in a radius $\hbar^\delta$ tube, $\Omega(\hbar)$, around $\Lambda$ with

$$\Omega(\hbar) \subset \operatorname{supp} \chi_1^\delta \tag{49}$$

and such that

$$\chi_1^\delta = 1 \text{ on } \operatorname{supp} \chi_2^\delta. \tag{50}$$

Then, clearly

$$\chi_1^\delta(x, \xi) \geq \chi_2^\delta(x, \xi), \tag{51}$$

for any $(x,\xi) \in T^*M$. Modulo small errors (see (57)), inequality (51) implies the corresponding operator bound for the matrix elements:

$$( Op_\hbar(\chi_1^\delta)\varphi_\mu, \varphi_\mu ) \gg ( Op_\hbar(\chi_2^\delta)\varphi_\mu, \varphi_\mu ). \tag{52}$$

Now, take $\varphi_\mu \in V_c(\hbar)$ satisfying the bounds in Lemma 6. We basically use (42) and (44) to estimate the matrix elements on the RHS of (52) from below in terms of the masses of the model distributions $u(y,\theta;\hbar) = \prod_{j=1}^k e^{im_j\theta_j} u_e(y) \cdot u_h(y) \cdot u_{ch}(y)$. But to obtain the fine estimate in the lemma, we need to compute these masses on *shrinking* neighbourhoods of diameter $\hbar^\delta$ centered around a singular $\ell < n$-dimensional orbit, $\Lambda$. Therefore we need to introduce appropriate classes of small scale pseudodifferential operators. By estimating matrix elements of such operators, we show that in shrinking neighbourhoods of diameter $\hbar^\delta$, the model distributions have finite mass bounded from below by a positive constant independent of $\hbar \in (0,\hbar_0]$ provided we choose $0 \leq \delta < 1/2$, giving the estimate stated in the Lemma.

### 3.3.2 Small scale semiclassical pseudo-differential calculus

The more refined symbols are defined as follows: Given an open set $U \in \mathbb{R}^n$ and $0 \leq \delta < \frac{1}{2}$, we say that $a(x,\xi;\hbar) \in S^m_\delta(U \times \mathbb{R}^n)$ if

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;\hbar)| \leq C_{\alpha\beta}\hbar^{-\delta(|\alpha|+|\beta|)}. \tag{53}$$

Model symbols include cut-offs of the form $\chi(h^{-\delta}x, h^{-\delta}\xi)$ with $\chi \in C_0^\infty(\mathbb{R}^{2n})$. There is a pseudo-differential calculus $Op_\hbar S^m_\delta(U \times \mathbb{R}^n)$ associated with such symbols with the usual symbolic composition formula and Calderon-Vaillancourt $L^2$-boundedness theorem [Sj]. Composition with operators in our original class $Op_\hbar S^{m,0}(U \times \mathbb{R}^n)$ preserves $Op_\hbar S^m_\delta(U \times \mathbb{R}^n)$.

We can now give the proof.

*Proof.* We need to define shrinking cut-offs around $\Lambda \subset T^*M$, and therefore introduce a Riemannian distance function on $T^*M$. A natural choice is to use the Riemannian metric induced by the Riemannian connection of $(M,g)$ on $T^*M \times T^*M$: the connection induces a splitting $T(T^*M) = H \oplus V$ into horizontal and vertical sub-bundles. We define a metric by requiring that $H \perp V$; on $H$ we lift the metric $g$ under $\pi$; for $V$ we use the Euclidean metrics; $g$ induces on the vertical spaces.

We then let $\tilde{d}(.,.)$ be the associated distance function between points of $T^*(M)$. For $\epsilon > 0$ sufficiently small we define

$$A_\epsilon := \{(x,\xi) \in T^*(M); \tilde{d}((x,\xi),\Lambda) \leq \epsilon\}. \tag{54}$$

We then choose $\chi_2^\delta(x,\xi;\hbar) \in C_0^\infty(T^*M$ with $0 \leq \chi_2^\delta \leq 1$ so that:

$$\operatorname{supp}\chi_2 \subset A_{\frac{3}{4}\hbar^\delta}, \;\; \chi_2 = 1 \;\text{ on }\; A_{\frac{1}{2}\hbar^\delta}, \;\text{ and so that }\; \chi_2^\delta(x,\xi;\hbar) \in S^0_\delta(T^*M). \tag{55}$$

For example, we can define

$$\chi_2^\delta(x,\xi;\hbar) := \zeta_2(\hbar^{-2\delta}\tilde{d}^2((x,\xi);\Lambda)), \tag{56}$$

where $\zeta_2 \in C_0^\infty(\mathbb{R}), \leq \zeta_2 \leq 1$ with $\zeta_2(x) = 1$ for $|x| \leq 1/2$ and supp $\zeta_2 \subset (-3/4, 3/4)$. We choose the cut-off $\chi_1^\delta$ as defined in (48). Clearly, $\chi_j^\delta \in S_\delta^0(T^*M)$; $j = 1, 2$, and

$$\chi_1^\delta(x,\xi) \geq \chi_2^\delta(x,\xi), \quad \forall (x,\xi) \in T^*M.$$

By the Gårding inequality, there exists a constant $C_1 > 0$ such that:

$$(Op_\hbar(\chi_1^\delta)\varphi_\mu, \varphi_{\mu)}) \geq (Op_\hbar(\chi_2^\delta)\varphi_\mu, \varphi_\mu) - C_1\hbar^{1-2\delta}. \tag{57}$$

We now conjugate the right side to the model by the $\hbar$-Fourier integral operator $F$ of Lemma (5). Since $F$ is a microlocally elliptic $\hbar$-Fourier integral operator associated to a canonical transformation $\kappa$, it follows by Egorov's theorem

$$(Op_\hbar(\chi_2^\delta)\varphi_\mu, \varphi_\mu) = |c(\hbar)|^2 (Op_\hbar(\chi_2^\delta \circ \kappa)u_\mu, u_\mu) - C_3\hbar^{1-2\delta} \tag{58}$$

where $c(\hbar)u_\mu(y,\theta;\hbar)$ is the microlocal normal form (43) for the eigenfunction $\varphi_\mu$. Since $\varphi_\mu \in V_c(\hbar)$ satisfies the bounds in Lemma (6) it follows that $|c(\hbar)|^2 \gg |\log\hbar|^{-m}$ and from (58) we are left with estimating the matrix elements $(Op_\hbar(\chi_2^\delta \circ \kappa)u_\mu, u_\mu)$ from below. To simplify the calculation, define the product-type cut-off function

$$\chi_2'^\delta \circ \kappa(y,\eta,I;\hbar) = \prod_{j=1}^{L+N} \chi(\hbar^{-\delta}y_j)\,\chi(\hbar^{-\delta}\eta_j) \cdot \prod_{j=L+N+1}^{L+M+N+1} \chi(\hbar^{-\delta}\rho_j)\chi(\hbar^{-\delta}\alpha_j) \cdot \prod_{j=n-l+1}^{n} \chi(\hbar^{-\delta}I_{n+1-j}). \tag{59}$$

Here $(r_j, \alpha_j)$ denote radial variables in the $j$th complex hyperbolic summand and $\chi(x) \in C_0^\infty(\mathbb{R})$ is a cut-off equal to one near zero. Since $y = \eta = I = 0$ on $(\kappa^{-1})^*\Lambda$, it follows that for $\chi(x)$ with sufficiently small support,

$$\chi_2^\delta \circ \kappa(y,\eta,I;\hbar) \geq \chi_2'^\delta \circ \kappa(y,\eta,I;\hbar). \tag{60}$$

Thus it suffices to estimate $\chi_2'^\delta \circ \kappa(y,\eta,I;\hbar)$.from below. To simplify the notation a little, we will write $\chi^\delta(x;\hbar) := \chi(\hbar^{-\delta}x)$ below. Now, $(Op_\hbar(\chi_2'^\delta \circ \kappa)u, u)$ consists of products of four types of terms. The first three are:

$$M_e = \int_{-\infty}^{\infty} \chi^\delta(\eta;\hbar)\,|\widehat{\chi^\delta u_e}(\eta;\hbar)|^2 d\eta,$$

$$M_h = \int_{-\infty}^{\infty} \chi^\delta(\eta;\hbar)\,|\widehat{\chi^\delta u_h}(\eta;\hbar)|^2 d\eta,$$

and finally,

$$M_{ch} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi^{\delta}(\eta_1, \eta_2; \hbar) \, |\widehat{\chi^{\delta} u_{ch}}(\eta_1, \eta_2; \hbar)|^2 \, d\eta_1 d\eta_2.$$

To estimate $M_e$, we note that, since $\varphi_\mu \in V(\hbar)$, and

$$\mathcal{F}(e^{-|y|^2/\hbar} \Phi_n(y\hbar^{-1/2}))(\eta) = e^{-|\eta|^2/\hbar} \Phi_n(\eta\hbar^{-1/2}),$$

it follows that,

$$M_e = \int_{-\infty}^{\infty} e^{-2|\eta|^2/\hbar} |\Phi_n(\hbar^{-1/2}\eta)|^2 d\eta + \mathcal{O}(\hbar^{\infty})$$

and so for $\hbar \in (0, \hbar_0]$, $M_e(\hbar) \sim 1$.

To estimate $M_h$, we recall that $\lambda(\hbar) = \mathcal{O}(\hbar)$ and that by [CP] Section 4.3 it suffices to estimate:

$$\frac{1}{\log \hbar} \left( \int_0^{\infty} \chi(\hbar\xi/\hbar^{\delta}) \left| \int_0^{\infty} e^{-ix} x^{-1/2+i\lambda/\hbar} \chi(x/\hbar^{\delta}\xi) dx \right|^2 \frac{d\xi}{\xi} \right). \tag{61}$$

The integral in (61) equals:

$$(\log \hbar)^{-1} \int_0^{\hbar^{\delta-1}} \frac{d\xi}{\xi} \left| \int_0^{\hbar^{\delta}\xi} e^{-ix} x^{-1/2+i\lambda/\hbar} dx \right|^2 + \mathcal{O}(|\log \hbar|^{-1}). \tag{62}$$

To estimate this last integral, assume first that $\xi \in [0, \hbar^{-\delta}]$. Then,

$$\int_0^{\hbar^{\delta}\xi} e^{-ix} x^{-1/2+i\lambda/\hbar} dx = \mathcal{O}(|\hbar^{\delta}\xi|^{1/2})$$

and so, (61) equals

$$|\log \hbar|^{-1} \int_{\hbar^{-\delta}}^{\hbar^{\delta-1}} \frac{d\xi}{\xi} \left| \int_0^{\hbar^{\delta}\xi} e^{-ix} x^{-1/2+i\lambda/\hbar} dx \right|^2 + \mathcal{O}(|\log \hbar|^{-1}). \tag{63}$$

From (63), it follows that:

$$M_h = C(\delta) + \mathcal{O}(|\log \hbar|^{-1}) \tag{64}$$

where $C(\delta) > 0$ when $0 < \delta < \frac{1}{2}$.

Finally, we are left with the integral $M_{ch}$ corresponding to a loxodromic subspace. Since $|\mathcal{J}_k(\rho)| \leq 1$ for all $k \in \mathbb{Z}$ and $\rho \in \mathbb{R}$, and $k(\hbar) = \mathcal{O}(M), t(\hbar) = \mathcal{O}(\hbar)$, it follows that:

$$M_{ch} = |\log \hbar|^{-1} \int_{\hbar^{-\delta}}^{\hbar^{\delta-1}} \left| \int_0^{\hbar^{\delta}\alpha} \mathcal{J}_k(\rho) \rho^{it/\hbar} d\rho \right|^2 \frac{d\alpha}{\alpha} + \mathcal{O}(|\log \hbar|^{-1}). \tag{65}$$

Here, $\mathcal{J}_k(\rho)$ denotes the $k$th integral Bessel function of the first kind [AS]. For $\alpha \geq \hbar^{-\delta}$,

$$\left| \int_0^{\hbar^\delta \alpha} \mathcal{J}_k(\rho) \rho^{it/\hbar} d\rho \right| = \left| \frac{2^{it/\hbar} \Gamma \frac{k+1+it/\hbar}{2})}{\Gamma(\frac{k+1-it/\hbar}{2})} \right| + \mathcal{O}(|\hbar^\delta \alpha|^{-1/2}) = 1 + \mathcal{O}(|\hbar^\delta \alpha|^{-1/2}), \quad (66)$$

and so,

$$M_{ch} = |\log \hbar|^{-1} \int_{\hbar^{-\delta}}^{\hbar^{\delta-1}} \frac{d\alpha}{\alpha} + \mathcal{O}(|\log \hbar|^{-1}) = 1 - 2\delta + \mathcal{O}(|\log \hbar|^{-1}). \quad (67)$$

Consequently, given $\delta = 1/2 - \epsilon$ it again follows that $M_{ch} = C(\epsilon) > 0$ uniformly for $\hbar \in (0, \hbar_0(\epsilon)]$.

The final step involves estimating $(Op_\hbar(\chi^\delta(I))e^{im\theta}, e^{im\theta})$. An integration by parts in the $I_1, \ldots, I_\ell$ variables shows that:

$$(Op_\hbar(\chi^\delta(I))e^{im\theta}, e^{im\theta}) = 1 + \mathcal{O}(\hbar^{1-\delta}). \quad (68)$$

As a consequence of the estimates above for $M_h, M_{ch}, M_e$ and the estimates in (57),(58) and (60), it follows that for *any* $\epsilon > 0$ and $\delta = 1/2 - \epsilon$, there exists a constant $C(\epsilon) > 0$ such that for all $\varphi_\mu \in V_c(\hbar)$ satisfying the bounds in Lemma 6,

$$(Op_\hbar(\chi_1^\delta)\varphi_\mu, \varphi_\mu) \geq C(\epsilon) |\log \hbar|^{-m}. \quad (69)$$

### 3.4 Completion of the proof of Theorem 1

Since

$$\begin{aligned} \int_M |\varphi_\mu(x)|^2 \chi_1^\delta(x;\hbar)\, dvol(x) &\leq \sup_{x \in T_{\hbar^{2\delta}}(\pi(\Lambda))} |\varphi_\mu(x)|^2 \int_M \chi_1^\delta(x;\hbar)\, dvol(x) \\ &\leq \|\varphi_\mu\|_{L^\infty}^2 \cdot \int_M \chi_1^\delta(x;\hbar)\, dvol(x) \end{aligned} \quad (70)$$

it follows from Lemma 8 or (69) that

$$\|\varphi_\mu\|_{L^\infty}^2 \cdot \left( \int_M \chi_1^\delta(x;\hbar)\, dvol(x) \right) \geq C(\epsilon) |\log \hbar|^{-m}, \quad (71)$$

uniformly for $\hbar \in (0, \hbar_0(\epsilon)]$. Since

$$\int_M \chi_1^\delta(x;\hbar)\, dvol(x) = \mathcal{O}(\hbar^{\delta(n-\ell)}), \quad (72)$$

the lower bound coming from (71) is:

$$\|\varphi_\mu\|_{L^\infty}^2 \geq C(\epsilon) \hbar^{-\frac{1}{2}(n-\ell)+\epsilon} |\log \hbar|^{-m}.$$

Since we take $\hbar^{-1} \in \{\lambda_j; \lambda_j \in Spec - \sqrt{\Delta}\}$, this gives:

$$\|\varphi_{\lambda_j}\|_{L^\infty} \geq C(\epsilon) \lambda_j^{\frac{n-\ell}{4} - \epsilon}.$$

The lower $L^p$ bounds when $2 < p < \infty$ follow by applying the Hölder inequality in the estimate (69). □

We doubt that much is lost in the second inequality in (70), since we expect $\varphi_\mu(x)$ to take its supremum at or near $\pi(\Lambda)$ and to have a roughly constant size on this set.

This completes the proof of Theorem 1 when $\pi(\Lambda)$ is embedded. We now briefly explain how to modify the proof in the general case.

### 3.5 General $\pi(\Lambda)$

The only problem is that the function $d^2(x, \pi(\Lambda))$ is not generally smooth if $\pi(\Lambda)$ has singularities (such as self-intersections). We therefore need to use different cut-offs from $\zeta_1(h^{-2\delta} d^2(x, \pi(\Lambda)))$. It turns out to be sufficient just to localize our argument as follows.

For any $C^\infty$ map such as $\pi : \Lambda \to M$, there exists a relatively open dense set $U$ on which $\pi : \Lambda \to \pi(\Lambda)$ attains its maximum rank. By the implicit function theorem, there further exists a relatively open set $V \subset U$ on which $\pi : V \to \pi(V)$ is one to one. For small $\epsilon$, the image of the normal bundle of radius $\epsilon$ along the relative interior of $\pi(V)$ exponentiates to a product tubular neighborhood of the form $\pi(V) \times D_\epsilon^{n-k}$ where $D_\epsilon^r$ is the $r$-dimensional ball of radius $\epsilon$. We use the corresponding Fermi normal product coordinates $(y, v)$. We pick a function $F_V(y)$, compactly supported in the interior of $\pi(V)$ and equal to one on a somewhat smaller open subset $V' \subset \pi(V)$ and with the same $\delta$ as above we define the cut-off

$$\chi_1^\delta(x; \hbar) = F_V(y)\zeta_1(\hbar^{-2\delta}|v|^2)\,. \tag{73}$$

Due to $F_V$ there is no singularity at the boundary of $F(V)$. The cut-off is smooth and satisfies

(i) supp $\chi_1^\delta \subset T_{\hbar^\delta}(\pi(V))$

(ii) $\chi_1^\delta = 1$ on a product neighborhood of the form $V' \times D^{n-r}_{3/4h^\delta}$;

(iii) $|\partial_x^\alpha \chi_1^\delta(x; \hbar)| \le C_\alpha h^{-\delta|\alpha|}$.

Correspondingly, we define

$$\chi_2^\delta(x, \xi; \hbar) := F_V(\theta)\zeta_2(\hbar^{-2\delta}\tilde{d}^2((x, \xi); V)), \quad \theta \in V,\ \pi(\theta) = y. \tag{74}$$

Here, we are using the singular action-angle coordinates $(I, \theta)$ near $\Lambda$.

The argument is then the same as in the embedded case, except that we need to multiply by the additional cut-off factor $F_V(\theta)$ in the angle coordinate. If $\pi : V \to \pi(V)$ has rank $k$, this modification therefore does not change the remaining calculations in any essential way, since the eigenfunctions corresponding to the singular action variables, i.e., the exponentials $e^{i\langle k, \theta\rangle}$, have modulus one. So the $d\theta$ integral simply becomes the integral of $F_V$ over $\Lambda$ rather than the integral of 1, and this merely changes the lower bounds in (68)–(69) by an $\hbar$-independent factor.

If on the other hand, $\pi : V \to \pi(V)$ has rank $< k$, then the estimates actually improve because the tube volume in (72) decays to a higher power in $\hbar$. We leave the details of this to the reader.

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