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Remarks on Nonlinear Schrödinger Equations with Harmonic Potential

R. Carles

Abstract. Bose-Einstein condensation is usually modeled by nonlinear Schrödinger equations with harmonic potential. We study the Cauchy problem for these equations. We show that the local problem can be treated as in the case with no potential. For the global problem, we establish an evolution law, which is the analogue of the pseudo-conformal conservation law for the nonlinear Schrödinger equation. With this evolution law, we give wave collapse criteria, as well as an upper bound for the blow up time. Taking the physical scales into account, we finally give a lower bound for the breaking time. This study relies on two explicit operators, suited to nonlinear Schrödinger equations with harmonic potential, already known in the linear setting.

1 Introduction

This paper is devoted to existence and blow up results for the nonlinear Schrödinger equation with isotropic harmonic potential,

$$\begin{cases} i\hbar\partial_t u^\hbar + \dfrac{\hbar^2}{2}\Delta u^\hbar = \dfrac{\omega^2}{2}x^2 u^\hbar + \lambda |u^\hbar|^{2\sigma}u^\hbar, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n, \\ u^\hbar_{|t=0} = u_0^\hbar, \end{cases} \tag{1.1}$$

where $\hbar > 0$, $\lambda \in \mathbb{R}$, and $\omega, \sigma > 0$. The notation x^2 stands for $|x|^2$. Similar equations are considered for Bose-Einstein condensation (see for instance [8], [15], [16]), with $\sigma = 1$; the real λ may be positive or negative, according to the considered chemical element, and is proportional to $\hbar^2$. With the operators used in [3] and [4] (see Eq. (1.3)), we prove existence results which are analogous to the well-known results for the nonlinear Schrödinger equation with no potential (see for instance [7]). These operators simplify the proof of some results of [13], [15] and [16], and provide more general results (in particular, for the case of Bose-Einstein condensation in space dimension three). In addition, we state two evolution laws (Lemma 3.1), which can be considered as the analogue of the pseudo-conformal evolution law of the free nonlinear Schrödinger field, and allow us to prove blow up results. Precisely, if we assume that λ is negative (attractive nonlinearity) and $\sigma \geq 2/n$, then under the condition

$$\frac{1}{2}\|\hbar\nabla u_0^\hbar\|^2_{L^2} + \frac{\lambda}{\sigma+1}\|u_0^\hbar\|^{2\sigma+2}_{L^{2\sigma+2}} \leq 0,$$

the wave collapses at time $t_*^\hbar \leq \frac{\pi}{2\omega}$ (Prop. 3.2). In particular, blow up occurs for focusing cubic nonlinearities ($\lambda < 0$ and $\sigma = 1$) in space dimensions two and three, but not in space dimension one (for other reasons; see Sect. 2).

Sect. 4 is devoted to estimates from *below* of the breaking time $t_*^\hbar$, under the assumption that the initial data $u_0^\hbar$ is bounded in $\left\{u \in L^2(\mathbb{R}^n);\ xu, \nabla u \in L^2(\mathbb{R}^n)\right\}$, uniformly with respect to $\hbar \in]0,1]$ (in particular, this means that $u_0^\hbar$ is not $\hbar$-oscillatory). We prove that if λ is negative and proportional to $\hbar^2$, $\sigma = 1$ (the physical case), and $n = 2$ or 3, then the wave collapse time can be bounded from below by $\frac{\pi}{2\omega} - \Lambda\hbar^\alpha$, for some constant Λ and positive number α (Prop. 4.1). When $n = 1$, we consider the case of a quintic nonlinearity ($\sigma = 2$), which should be the right model for Bose-Einstein Condensation in low dimension (see [12]), and we prove that $t_*^\hbar \geq \frac{\pi}{2\omega} - \Lambda\hbar$, for some constant Λ. Notice that all these results are proved for fixed $\hbar$, with constants independent of $\hbar \in]0,1]$.

The following quantities are formally independent of time,

$$\begin{aligned} N^\hbar &= \|u^\hbar(t)\|_{L^2}^2, \\ E^\hbar &= \frac{1}{2}\|\hbar\nabla_x u^\hbar(t)\|_{L^2}^2 + \frac{\omega^2}{2}\|xu^\hbar(t)\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u^\hbar(t)\|_{L^{2\sigma+2}}^{2\sigma+2}. \end{aligned} \tag{1.2}$$

If $N^\hbar$ and $E^\hbar$ are defined at time $t = 0$, we prove that the solution $u^\hbar$ is defined locally in time, with the conservation of $N^\hbar$ and $E^\hbar$, provided that $\sigma < 2/(n-2)$ when $n \geq 3$. If $\lambda \geq 0$, then the solution $u^\hbar$ is defined globally in time. If $\lambda < 0$, several cases occur.

- If $\sigma < 2/n$, then the solution is defined globally in time.
- If $\sigma \geq 2/n$, then the solution is defined globally in time if $u_0^\hbar$ is sufficiently small.
- If $\sigma \geq 2/n$ and $E^\hbar \leq \frac{\omega^2}{2}\|xu_0^\hbar\|_{L^2}^2$, then the solution collapses at time $t_*^\hbar \leq \frac{\pi}{2\omega}$.

The operators on which our analysis relies are

$$J_j^\hbar(t) = \frac{\omega}{\hbar}x_j\sin(\omega t) - i\cos(\omega t)\partial_j\ ; \qquad H_j^\hbar(t) = \omega x_j\cos(\omega t) + i\hbar\sin(\omega t)\partial_j. \tag{1.3}$$

We denote $J^\hbar(t)$ (resp. $H^\hbar(t)$) the operator-valued vector with components $J_j^\hbar(t)$ (resp. $H_j^\hbar(t)$).

Lemma 1.1 *$J^\hbar$ and $H^\hbar$ satisfy the following properties.*

- *The commutation relation,*

$$\left[J^\hbar(t), i\hbar\partial_t + \frac{\hbar^2}{2}\Delta - \frac{\omega^2}{2}x^2\right] = \left[H^\hbar(t), i\hbar\partial_t + \frac{\hbar^2}{2}\Delta - \frac{\omega^2}{2}x^2\right] = 0. \tag{1.4}$$

- *Denote* $M^\hbar(t) = e^{-i\omega\frac{x^2}{2\hbar}\tan(\omega t)}$, *and* $Q^\hbar(t) = e^{i\omega\frac{x^2}{2\hbar}\cot(\omega t)}$, *then*

$$\begin{aligned} J^\hbar(t) &= -i\cos(\omega t)M^\hbar(t)\nabla_x M^\hbar(-t), \\ H^\hbar(t) &= i\hbar\sin(\omega t)Q^\hbar(t)\nabla_x Q^\hbar(-t). \end{aligned} \tag{1.5}$$

- *The modified Sobolev inequalities. For* $n \geq 2$, *and* $2 \leq r < \frac{2n}{n-2}$, *define* $\delta(r)$ *by*

$$\delta(r) \equiv n\left(\frac{1}{2} - \frac{1}{r}\right). \tag{1.6}$$

Let $2 \leq r < \frac{2n}{n-2}$ *(*$2 \leq r \leq \infty$ *if* $n = 1$*); there exists* C_r *independent of* $\hbar$ *such that, for any* $\varphi \in \Sigma$,

$$\|\varphi\|_{L^r} \leq C_r \hbar^{-\delta(r)} \|\varphi\|_{L^2}^{1-\delta(r)} \left(\|\hbar J^\hbar(t)\varphi)\|_{L^2} + \|H^\hbar(t)\varphi\|_{L^2}\right)^{\delta(r)}. \tag{1.7}$$

- *For any function* $F \in C^1(\mathbb{C}, \mathbb{C})$ *of the form* $F(z) = zG(|z|^2)$, *we have,*

$$\begin{aligned} H^\hbar(t)F(v) &= \partial_z F(v)H^\hbar(t)v - \partial_{\bar z}F(v)\overline{H^\hbar(t)v}, \ \forall t \notin \frac{\pi}{\omega}\mathbb{Z}, \\ J^\hbar(t)F(v) &= \partial_z F(v)J^\hbar(t)v - \partial_{\bar z}F(v)\overline{J^\hbar(t)v}, \ \forall t \notin \frac{\pi}{2\omega} + \frac{\pi}{\omega}\mathbb{Z}. \end{aligned} \tag{1.8}$$

Remark. Property (1.8) is a direct consequence of (1.5). Property (1.7) is a consequence of the usual Sobolev inequalities and (1.5). These operators are well-known in the linear theory (see e.g. [14] p. 108, [3]), they are the quantization of momentum and position, hence (1.4). Their action in the nonlinear setting, as stated in the above lemma, proves to be very efficient to analyze (1.1).

Notations. We work with initial data which belong to the space

$$\Sigma := \left\{u \in L^2(\mathbb{R}^n) \ ; \ xu, \nabla u \in L^2(\mathbb{R}^n)\right\}.$$

Notice that $\Sigma = D(\sqrt{-\Delta + |x|^2})$: we work in the same space as in [13].
The notation r' stands for the Hölder conjugate exponent of r.

The paper is organized as follows. In Sect. 2, we study the local Cauchy problem for (1.1), and we give sufficient conditions for the solution of (1.1) to be defined globally in time. In Sect. 3, we give a sufficient condition under which the solution blows up in finite time, and provide an upper bound for the breaking time. In Sect. 4, we give a lower bound for the breaking time, that shows that the upper bound underscored in Sect. 3 is the physical breaking time in the semi-classical limit, provided that no rapid oscillation is present in the initial data.

The results of Sections 2 and 3 were announced in [6].

2 Existence results

The solution of (1.1) with $\lambda = 0$ is given by Mehler's formula (see e.g. [9]),

$$u^\hbar(t,x) = \Big(\frac{\omega}{2i\pi\hbar \sin \omega t}\Big)^{n/2} \int_{\mathbb{R}^n} e^{\frac{i\omega}{\hbar \sin(\omega t)}\left(\frac{x^2+y^2}{2}\cos(\omega t) - x.y\right)} u_0^\hbar(y)dy =: U^\hbar(t)u_0^\hbar(x).$$

This formula defines a group $U^\hbar(t)$, unitary on L^2, for which Strichartz estimates are available, that is, mixed time-space estimates, which are exactly the same as for $U_0^\hbar(t) = e^{i\frac{t\hbar}{2}\Delta}$. Recall the main properties from which such estimates stem (see [7], or [11] for a more general argument).

- The group $U^\hbar(t)$ is unitary on L^2, $\|U^\hbar(t)\|_{L^2\to L^2} = 1$.
- For $0 < t \leq \frac{\pi}{2\omega}$, the group is dispersive, with $\|U^\hbar(t)\|_{L^1\to L^\infty} \leq C|\hbar t|^{-n/2}$.

We postpone the precise statement of Strichartz estimates to Sect. 4. Duhamel's formula associated to (1.1) reads

$$u^\hbar(t,x) = U^\hbar(t)u_0^\hbar(x) - i\lambda\hbar^{-1}\int_0^t U^\hbar(t-s)\left(|u^\hbar|^{2\sigma}u^\hbar\right)(s,x)ds.$$

Replacing $U^\hbar(t)$ by $U_0^\hbar(t)$ yields Duhamel's formula associated to

$$\begin{cases} i\hbar\partial_t u + \dfrac{\hbar^2}{2}\Delta u = \lambda|u^\hbar|^{2\sigma}u^\hbar, \\ \quad u^\hbar_{|t=0} = u_0^\hbar. \end{cases} \tag{2.1}$$

The local Cauchy problem for this equation is now well-known in many cases (see for instance [7] for a review). In particular, the local well-posedness in Σ is established thanks to the operators $\hbar\nabla_x$ and $x/\hbar + it\nabla_x$ (Galilean operator). This result is proved thanks to Strichartz inequalities, and to the following properties.

- The above two operators commute with $i\hbar\partial_t + \frac{\hbar^2}{2}\Delta$.
- They act on the nonlinearity $|u^\hbar|^{2\sigma}u^\hbar$ like derivatives.
- Gagliardo-Nirenberg inequalities.

From Lemma 1.1, the operators $H^\hbar$ and $J^\hbar$ meet all these requirements. Mimicking the classical proofs for (2.1) easily yields,

Proposition 2.1 *Let $u_0^\hbar \in \Sigma$. If $n \geq 3$, assume moreover $\sigma < 2/(n-2)$. Then there exists $T^\hbar > 0$ such that (1.1) has a unique maximal solution $u^\hbar \in C([0,T^\hbar[,\Sigma)$. $u^\hbar$ is maximal in the sense that if $T^\hbar$ is finite, then $\|u^\hbar(t)\|_\Sigma \to \infty$ as $t \uparrow T^\hbar$. Moreover $N^\hbar$ and $E^\hbar$ defined by (1.2) are constant for $t \in [0,T^\hbar[$.*

Remark. This result was proved in [13], for more general potentials. We want to underscore the fact that in the case of the harmonic potential, there is essentially nothing to prove, when using $J^\hbar$ and $H^\hbar$.

If $\lambda > 0$, the conservations of mass and energy provide *a priori* estimates on the Σ-norm of $u^\hbar(t)$, and prove global existence in Σ.

If $\lambda < 0$ and $\sigma < 2/n$, then the energy E controls the Σ-norm of $u^\hbar(t)$. Indeed, from Gagliardo-Nirenberg inequalities (1.7),

$$\|u^\hbar(t)\|_{L^{2\sigma+2}} \leq C\|u^\hbar(t)\|_{L^2}^{1-\delta(2\sigma+2)} \left(\|\hbar J^\hbar(t)u^\hbar\|_{L^2} + \|H^\hbar(t)u^\hbar\|_{L^2}\right)^{\delta(2\sigma+2)}.$$

Notice that the following identity holds point-wise,

$$|\omega x u^\hbar(t,x)|^2 + |\hbar\nabla_x u^\hbar(t,x)|^2 = |\hbar J^\hbar(t)u^\hbar(t,x)|^2 + |H^\hbar(t)u^\hbar(t,x)|^2,$$

and one can rewrite the energy as

$$E^\hbar = \frac{1}{2}\|\hbar J^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{1}{2}\|H^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u^\hbar(t)\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{2.2}$$

Therefore, using the conservation of mass $N^\hbar$ yields

$$\|\hbar J^\hbar(t)u^\hbar\|_{L^2}^2 + \|H^\hbar(t)u^\hbar\|_{L^2}^2 \leq 2E^\hbar + C(\|\hbar J^\hbar(t)u^\hbar\|_{L^2} + \|H^\hbar(t)u^\hbar\|_{L^2})^{n\sigma},$$

and if $\sigma < 2/n$, then the quantity $\|\hbar J^\hbar(t)u^\hbar\|_{L^2}^2 + \|H^\hbar(t)u^\hbar\|_{L^2}^2$ remains bounded for all times (for any fixed $\hbar$).

Similarly, global existence can be proved for small data.

Proposition 2.2 *Let $u_0^\hbar \in \Sigma$, and if $n \geq 3$, assume $\sigma < 2/(n-2)$. Then $u^\hbar$ is defined globally in time and belongs to $C([0,+\infty[,\Sigma)$ in the following cases.*

- $\lambda \geq 0$ *(repulsive nonlinearity).*
- $\lambda < 0$ *(attractive nonlinearity) and $\sigma < 2/n$.*
- $\lambda < 0$, $\sigma \geq 2/n$ *and $\|u_0^\hbar\|_\Sigma$ sufficiently small.*

Remark. In particular, in space dimension one, the solution $u^\hbar$ is always globally defined for cubic nonlinearities ($\sigma = 1$).

3 Wave collapse

Split the energy $E^\hbar$ into $E_1^\hbar + E_2^\hbar$, with

$$E_1^\hbar(t) = \frac{1}{2}\|\hbar J^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\cos^2(\omega t)\|u^\hbar(t)\|_{L^{2\sigma+2}}^{2\sigma+2},$$
$$E_2^\hbar(t) = \frac{1}{2}\|H^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\sin^2(\omega t)\|u^\hbar(t)\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

Lemma 3.1 *The quantities $E_1^\hbar$ and $E_2^\hbar$ satisfy the following evolution laws,*

$$\begin{aligned}\frac{dE_1^\hbar}{dt} &= \frac{\omega\lambda}{2\sigma+2}(n\sigma-2)\sin(2\omega t)\|u^\hbar(t)\|^{2\sigma+2}_{L^{2\sigma+2}},\\ \frac{dE_2^\hbar}{dt} &= \frac{\omega\lambda}{2\sigma+2}(2-n\sigma)\sin(2\omega t)\|u^\hbar(t)\|^{2\sigma+2}_{L^{2\sigma+2}}.\end{aligned}$$

Remark. This lemma can be regarded as the analogue of the pseudo-conformal conservation law, discovered by Ginibre and Velo ([10]) for the case with no potential ($\omega = 0$).

Sketch of the proof. Expanding $|\hbar J_j^\hbar(t)u^\hbar(t,x)|^2$ yields,

$$\begin{aligned}|\hbar J_j^\hbar(t)u^\hbar(t,x)|^2 =& \omega^2 x_j^2\sin^2(\omega t)|u^\hbar(t,x)|^2 + \hbar^2\cos^2(\omega t)|\partial_j u^\hbar(t,x)|^2\\ &+ \hbar\omega x_j\sin(2\omega t)\operatorname{Im}(\overline{u}\partial_j u).\end{aligned}$$

When differentiating the above relation with respect to time and integrating with respect to the space variable, one is led to computing the following quantities,

$$\begin{aligned}\partial_t\int|x_j u^\hbar(t,x)|^2dx =& 2\hbar\operatorname{Im}\int x_j\overline{u^\hbar}\partial_j u^\hbar,\\ \partial_t\int|\partial_j u^\hbar(t,x)|^2dx =& -2\frac{\omega^2}{\hbar}\operatorname{Im}\int x_j\overline{u^\hbar}\partial_j u^\hbar - 2\frac{\lambda}{\hbar}\operatorname{Im}\int\partial_j^2\overline{u^\hbar}|u^\hbar|^{2\sigma}u^\hbar,\\ \partial_t\operatorname{Im}\int(x_j\overline{u^\hbar}\partial_j u^\hbar) =& \frac{\hbar}{2}\int|\nabla_x u^\hbar|^2 + \frac{\omega^2}{2\hbar}\int x^2|u^\hbar|^2 + \frac{\lambda}{\hbar}\int|u^\hbar|^{2\sigma+2}\\ &-\hbar\operatorname{Re}\int x_j\partial_j\overline{u^\hbar}\Delta u^\hbar + \frac{\omega^2}{\hbar}\operatorname{Re}\int x_j\partial_j\overline{u^\hbar}x^2u^\hbar\\ &+2\frac{\lambda}{\hbar}\operatorname{Re}\int x_j\partial_j\overline{u^\hbar}|u^\hbar|^{2\sigma}u^\hbar.\end{aligned}\tag{3.1}$$

It follows,

$$\begin{aligned}\frac{d}{dt}\int|\hbar J^\hbar(t)u^\hbar(t,x)|^2dx =& \frac{\omega\sigma\lambda}{\sigma+1}\sin(2\omega t)\int|u|^{2\sigma+2}\\ &-2\lambda\hbar\cos^2(\omega t)\operatorname{Im}\int\partial_j^2\overline{u}|u|^{2\sigma}u.\end{aligned}$$

Notice that it is sensible that the right hand side is zero when $\lambda = 0$; from the commutation relation (1.4), the L^2-norm of $J^\hbar(t)u^\hbar$ is conserved when $\lambda = 0$, since $J^\hbar(t)u^\hbar$ then solves a linear Schrödinger equation.

Finally, the first part of Lemma 3.1 follows from the identity,

$$\frac{d}{dt}\|u^\hbar(t)\|^{2\sigma+2}_{L^{2\sigma+2}} = -\hbar(\sigma+1)\operatorname{Im}\int|u|^{2\sigma}\overline{u}\Delta u.$$

The second part of Lemma 3.1 follows from the relation $E_1^\hbar + E_2^\hbar = E^\hbar = cst$. The justification of these formal computations relies on a regularizing technique, which can be found for instance in [7], Lemma 6.4.3. □

As an application of this lemma, we can prove wave collapse when $E_1^\hbar(0) \leq 0$.

Proposition 3.2 *Let $u_0^\hbar \in \Sigma$ be nonzero, and if $n \geq 3$, assume $\sigma < 2/(n-2)$. Assume that the nonlinearity is attractive ($\lambda < 0$) and $\sigma \geq 2/n$. Then under the condition*

$$\frac{1}{2}\|\hbar\nabla u_0^\hbar\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u_0^\hbar\|_{L^{2\sigma+2}}^{2\sigma+2} \leq 0,$$

$u^\hbar$ collapses at time $t_*^\hbar \leq \pi/2\omega$,

$$\exists t_*^\hbar \leq \frac{\pi}{2\omega}, \quad \lim_{t\to t_*^\hbar} \|\nabla_x u^\hbar(t)\|_{L^2} = \infty, \quad \text{and} \quad \lim_{t\to t_*^\hbar} \|u^\hbar(t)\|_{L^\infty} = \infty.$$

Proof. From our assumptions, if $u^\hbar \in C([0,T];\Sigma)$ with $T \leq \pi/2\omega$,

$$E_1^\hbar(0) = E^\hbar - \frac{1}{2}\|\omega x u_0^\hbar\|_{L^2}^2 \leq 0, \text{ and } \frac{dE_1^\hbar}{dt} \leq 0, \quad \forall t \in [0,T]. \tag{3.2}$$

So long as $\nabla_x u^\hbar$ remains bounded in L^2, so does $xu^\hbar$. This follows from the conservations of mass and energy, along with Gagliardo-Nirenberg inequality. Assume $u^\hbar \in C([0,\pi/2\omega];\Sigma)$. Then letting t go to $\pi/2\omega$ yields

$$E_1\left(\frac{\pi}{2\omega}\right) \geq \frac{1}{2}\left\|\omega x u^\hbar\left(\frac{\pi}{2\omega}, x\right)\right\|_{L^2}^2,$$

which is impossible from (3.2) and the conservation of the L^2-norm of $u^\hbar$. Thus, there exists $t_*^\hbar \leq \pi/2\omega$ such that

$$\lim_{t\to t_*^\hbar} \|\nabla_x u^\hbar(t)\|_{L^2} = \infty.$$

From the conservation of energy,

$$\lim_{t\to t_*^\hbar} \|u^\hbar(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = \infty,$$

and the last part of the proposition stems from the conservation of mass. □

Remark. Notice that the blow up condition also reads

$$E^\hbar \leq \frac{\omega^2}{2}\|x u_0^\hbar\|_{L^2}^2.$$

In term of energy, this means that the blow up occurs for higher values of the Hamiltonian than in the case with no potential, where the similar condition reads $E^\hbar < 0$. This condition was found independently by Zhang [16], in the particular case $\sigma = 2/n$. In particular, our approach can treat the case of Bose-Einstein condensation in space dimension three, where the cubic nonlinearity is supercritical ($\sigma = 1 > 2/n = 2/3$).

Corollary 3.3 *Assume $\sigma \geq 2/n$, $\lambda < 0$. Let $v_0^\hbar \in \Sigma$. For $k \in \mathbb{R}$, define $u_0^\hbar = kv_0^\hbar$. Then for $|k|$ sufficiently large, $u^\hbar(t,x)$ collapses at time $t_*^\hbar \leq \pi/2\omega$, as in Prop. 3.2.*

Proof. For $|k|$ large, $E_1^\hbar(0)$ becomes negative, and one can use the results of Prop. 3.2. □

4 Lower bound for the breaking time

In this section, we specify the dependence of the coupling constant λ upon physical constants, and assume $\lambda = a\hbar^2$. We first assume that the nonlinearity is cubic, $\sigma = 1$. Physically, a is the s-wave scattering length. It is negative in the case of Bose-Einstein condensation for ^{7}Li system ([1], [2]). We prove that if the space dimension n is two or three, then the nonlinear term $a\hbar^2|u^\hbar|^2u^\hbar$ in (1.1) is negligible in the semi-classical limit $\hbar \to 0$, up to some time depending on $\hbar$. This will give us a lower bound for the breaking time $t_*^\hbar$ when $\hbar \to 0$, and prove that under the assumptions of Prop. 3.2,

$$t_*^\hbar \underset{\hbar\to 0}{\longrightarrow} \frac{\pi}{2\omega}.$$

As previously noticed, no blow up occurs for $\sigma = 1$ and $n = 1$, that is why we restrict our attention to $n = 2$ or 3. In the one-dimensional case, it has been proved in [12] that the right model for Bose-Einstein consists in replacing the cubic nonlinearity $|u^\hbar|^2u^\hbar$ by the quintic nonlinearity $|u^\hbar|^4u^\hbar$. This case is critical for global existence issues (see Prop. 2.2, Prop. 3.2), and is treated at the end of this section.

Define the function $v^\hbar$ as the solution of the linear Cauchy problem,

$$\begin{cases} i\hbar\partial_t v^\hbar + \dfrac{\hbar^2}{2}\Delta v^\hbar = \dfrac{\omega^2}{2}x^2 v^\hbar, \\ v^\hbar_{|t=0} = u_0^\hbar. \end{cases} \tag{4.1}$$

4.1 The case $n = 2$ or 3

When $n = 2$ or 3, recall that we consider now the initial value problem for $u^\hbar$,

$$\begin{cases} i\hbar\partial_t u^\hbar + \dfrac{\hbar^2}{2}\Delta u^\hbar = \dfrac{\omega^2}{2}x^2 u^\hbar + a\hbar^2|u^\hbar|^2u^\hbar, \\ u^\hbar_{|t=0} = u_0^\hbar, \end{cases} \tag{4.2}$$

where a is fixed. Our first result is independent of the sign of a.

Proposition 4.1 *Assume $n = 2$ or 3. Let $u_0^\hbar \in \Sigma$ be such that $\|u_0^\hbar\|_{L^2}$, $\|\nabla_x u_0^\hbar\|_{L^2}$ and $\|xu_0^\hbar\|_{L^2}$ are bounded,* uniformly *with $\hbar \in]0,1]$. Then there exist $C, \Lambda, \alpha > 0$ and a finite real $\underline{q}$ such that the following holds. Let $\hbar_0 > 0$ be such that $\pi/2\omega - \Lambda\hbar_0^\alpha > 0$. Then for any $\hbar \in]0,\hbar_0]$, $u^\hbar$ is defined in Σ at least up to time $\pi/2\omega - \Lambda\hbar^\alpha$, and satisfies*

$$\sup_{0\le t\le \pi/2\omega - \Lambda\hbar^\alpha} \left\| A^\hbar(t)(u^\hbar - v^\hbar)(t)\right\|_{L^2} \le C\hbar^{1/\underline{q}},$$

where $A^\hbar(t)$ can be either of the operators Id, $J^\hbar(t)$ or $H^\hbar(t)$. In particular, if $a < 0$ and $u^\hbar$ collapses at time $t_*^\hbar$, then

$$t_*^\hbar \ge \frac{\pi}{2\omega} - \Lambda\hbar^\alpha, \quad \forall \hbar \in]0,\hbar_0].$$

Remark. Notice that the assumption $\|\nabla_x u_0^\hbar\|_{L^2}$ be bounded uniformly with $\hbar$ means that $u_0^\hbar$ has no $\hbar$-dependent oscillation. This is crucial, for quadratic oscillations could lead for instance to $t_*^\hbar = \frac{\pi}{4\omega}$, see [5].

To prove Prop. 4.1, we first state precisely the Strichartz estimates we will use. Recall the classical definition (see e.g. [7]),

Definition 1 *A pair (q,r) is* admissible *if $2 \leq r < \frac{2n}{n-2}$ (resp. $2 \leq r \leq \infty$ if $n=1$, $2 \leq r < \infty$ if $n=2$) and*

$$\frac{2}{q} = \delta(r) \equiv n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Strichartz estimates provide mixed type estimates (that is, in spaces of the form $L_t^q(L_x^r)$, with (q,r) admissible) of quantities involving the unitary group

$$U_0(t) = e^{i\frac{t}{2}\Delta}.$$

A simple scaling argument yields similar estimates when U_0 is replaced with $e^{i\frac{t\hbar}{2}\Delta}$, with precise dependence upon the parameter $\hbar$. As noticed in Sect. 2, the same Strichartz estimates hold when $e^{i\frac{t\hbar}{2}\Delta}$ is replaced by $U^\hbar(t)$ (provided that only *finite* time intervals are involved).

Proposition 4.2 *Let I be a interval contained in $[0,\pi/2\omega]$. For any admissible pair (q,r), there exists C_r such that for any $f \in L^2$,*

$$\left\|U^\hbar(t)f\right\|_{L^q(I;L^r)} \leq C_r \hbar^{-1/q}\|f\|_{L^2}.$$

For any admissible pairs (q_1,r_1) and (q_2,r_2), there exists C_{r_1,r_2} such that for $F = F(t,x)$,

$$\left\|\int_{I\cap\{s\leq t\}} U^\hbar(t-s)F(s)ds\right\|_{L^{q_1}(I;L^{r_1})} \leq C_{r_1,r_2}\hbar^{-1/q_1-1/q_2}\|F\|_{L^{q_2'}(I;L^{r_2'})}. \tag{4.3}$$

The above constants are independent of $I \subset [0,\pi/2\omega]$ and $\hbar \in]0,1]$.

We now state two technical lemmas on which the proof of Prop. 4.1 relies. The first one is easy, and we leave out the proof.

Lemma 4.3 *If $n = 2$ or 3, there exists $\underline{q}$, $\underline{r}$, $\underline{s}$ and $\underline{k}$ satisfying*

$$\begin{cases} \dfrac{1}{\underline{r}'} = \dfrac{1}{\underline{r}} + \dfrac{2}{\underline{s}}, \\ \dfrac{1}{\underline{q}'} = \dfrac{1}{\underline{q}} + \dfrac{2}{\underline{k}}, \end{cases} \tag{4.4}$$

and the additional conditions:

- *The pair* $(\underline{q}, \underline{r})$ *is admissible,*
- $0 < \frac{1}{\underline{k}} < \delta(\underline{s}) < 1$.

Remark. Notice that in particular, $\underline{q}$ is finite.

Lemma 4.4 *Assume $n = 2$ or 3, and let $a^\hbar \in C(0, T; \Sigma)$ defined for some positive T, solution of*

$$\begin{cases} i\hbar\partial_t a^\hbar + \dfrac{\hbar^2}{2}\Delta a^\hbar = \dfrac{\omega^2}{2}x^2 a^\hbar + \hbar^2 F^\hbar(a^\hbar) + \hbar^2 S^\hbar, \\ \qquad\qquad a^\hbar_{|t=0} = 0. \end{cases}$$

Assume that there exists $C_0 > 0$ such that for any $T < \pi/2\omega$, and any $0 \leq t \leq T$,

$$\left\| F^\hbar(a^\hbar)(t) \right\|_{L^{\underline{r}'}} \leq \frac{C_0}{\left(\frac{\pi}{2\omega} - t\right)^{2\delta(\underline{s})}} \left\| a^\hbar(t) \right\|_{L^{\underline{r}}}.$$

Then there exist $C, \Lambda > 0$ independent of $\hbar \in [0, 1[$ such that the following holds. Let $\hbar_0 > 0$ be such that $\pi/2\omega - \Lambda\hbar_0^\alpha > 0$. Then for any $\hbar \in]0, \hbar_0]$,

$$\sup_{0 \leq t \leq \frac{\pi}{2\omega} - \Lambda\hbar^\alpha} \|a^\hbar(t)\|_{L^2} \leq C\hbar^{1-1/\underline{q}} \left\| S^\hbar \right\|_{L^{\underline{q}'}(0, \pi/2\omega - \Lambda\hbar^\alpha; L^{\underline{r}'})},$$

where $\alpha = \frac{1}{\underline{k}\delta(\underline{s}) - 1}$.

Proof of Lemma 4.4. From (4.3) with $q_1 = q_2 = \underline{q}$, for any $t < \pi/2\omega$,

$$\|a^\hbar\|_{L^{\underline{q}}(0,t;L^{\underline{r}})} \leq C\hbar^{1-2/\underline{q}} \|S^\hbar\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})} + C\hbar^{1-2/\underline{q}} \|F^\hbar(a^\hbar)\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})}. \tag{4.5}$$

From our assumptions,

$$\|F^\hbar(a^\hbar)\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})} \leq \left\| \frac{C_0}{\left(\frac{\pi}{2\omega} - s\right)^{2\delta(\underline{s})}} \|a^\hbar(s)\|_{L^{\underline{r}}_x} \right\|_{L^{\underline{q}'}(0,t)}.$$

Apply Hölder's inequality in time with (4.4),

$$\begin{aligned} \|F^\hbar(a^\hbar)\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})} &\leq C \left(\int_0^t \frac{ds}{\left(\frac{\pi}{2\omega} - s\right)^{\underline{k}\delta(\underline{s})}} \right)^{2/\underline{k}} \|a^\hbar\|_{L^{\underline{q}}(0,t;L^{\underline{r}})} \\ &\leq C \frac{1}{\left(\frac{\pi}{2\omega} - t\right)^{2\delta(\underline{s}) - 2/\underline{k}}} \|a^\hbar\|_{L^{\underline{q}}(0,t;L^{\underline{r}})}. \end{aligned}$$

Plugging this estimate into (4.5) yields, for $t \leq \pi/2\omega - \Lambda\hbar^\alpha$,

$$\|a^\hbar\|_{L^{\underline{q}}(0,t;L^{\underline{r}})} \leq C\hbar^{1-2/\underline{q}} \|S^\hbar\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})} + C\hbar^{1-2/\underline{q}} (\Lambda\hbar^\alpha)^{2/\underline{k} - 2\delta(\underline{s})} \|a^\hbar\|_{L^{\underline{q}}(0,t;L^{\underline{r}})}.$$

From (4.4), the power of $\hbar$ in the last term is canceled for $\alpha = \frac{1}{\underline{k}\delta(\underline{s})-1}$. If in addition Λ is sufficiently large, the last term of the above estimate can be absorbed by the left hand side (up to doubling the constant C for instance),

$$\|a^\hbar\|_{L^{\underline{q}}(0,t;L^{\underline{r}})} \leq C\hbar^{1-2/\underline{q}}\|S^\hbar\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})}.$$

The last three estimates also imply,

$$\|F^\hbar(a^\hbar)\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})} \leq C\|S^\hbar\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})}. \tag{4.6}$$

The lemma then follows from Prop. 4.2, (4.3), with this time $q_1 = \infty$ and $q_2 = \underline{q}$, along with (4.6). $\square$

Proof of Proposition 4.1. Denote $w^\hbar = u^\hbar - v^\hbar$ the remainder we want to assess. It solves the initial value problem,

$$\begin{cases} i\hbar\partial_t w^\hbar + \dfrac{\hbar^2}{2}\Delta w^\hbar = \dfrac{\omega^2}{2}x^2 w^\hbar + a\hbar^2|u^\hbar|^2u^\hbar, \\ w^\hbar_{|t=0} = 0. \end{cases} \tag{4.7}$$

We first want to apply Lemma 4.4 with $a^\hbar = w^\hbar$. Since $u^\hbar = v^\hbar + w^\hbar$, we can take

$$F^\hbar(w^\hbar) = a|u^\hbar|^2w^\hbar, \quad S^\hbar = a|u^\hbar|^2v^\hbar.$$

The point is now to control the $L^{\underline{s}}$-norm of $u^\hbar$. Notice that we can easily control the $L^{\underline{s}}$-norm of $v^\hbar$. Indeed, as we already emphasized, for any time t,

$$\|v^\hbar(t)\|_{L^2} = \|u_0^\hbar\|_{L^2}, \quad \|J^\hbar(t)v^\hbar\|_{L^2} = \|\nabla u_0^\hbar\|_{L^2}.$$

From Lemma 1.1, (1.5), and Gagliardo-Nirenberg inequality, we also have,

$$\begin{aligned}\|v^\hbar(t)\|_{L^{\underline{s}}} &\leq \frac{C}{|\cos(\omega t)|^{\delta(\underline{s})}}\|v^\hbar(t)\|_{L^2}^{1-\delta(\underline{s})}\|J^\hbar(t)v^\hbar\|_{L^2}^{\delta(\underline{s})} \\ &\leq \frac{C}{\left(\frac{\pi}{2\omega}-t\right)^{\delta(\underline{s})}}\|v^\hbar(t)\|_{L^2}^{1-\delta(\underline{s})}\|J^\hbar(t)v^\hbar\|_{L^2}^{\delta(\underline{s})}.\end{aligned}$$

Therefore, the assumptions of Prop. 4.1 imply that there exists $C_0 > 0$ independent of $\hbar$ such that for any $t < \pi/2\omega$,

$$\|v^\hbar(t)\|_{L^{\underline{s}}} \leq \frac{C_0}{\left(\frac{\pi}{2\omega}-t\right)^{\delta(\underline{s})}}.$$

Now $w^\hbar_{|t=0} = 0$ and we know from Prop. 2.1 that there exists $T^\hbar$ such that the Σ-norm of $w^\hbar$ is continuous on $[0, T^\hbar]$. In particular, there exists $t^\hbar > 0$ such that the following inequality,

$$\|w^\hbar(t)\|_{L^{\underline{s}}} \leq \frac{C_0}{\left(\frac{\pi}{2\omega}-t\right)^{\delta(\underline{s})}}, \tag{4.8}$$

holds for $t \in [0, t^\hbar]$. So long as (4.8) holds, we have obviously

$$\|u^\hbar(t)\|_{L^{\underline{s}}} \leq \frac{2C_0}{\left(\frac{\pi}{2\omega} - t\right)^{\delta(\underline{s})}}.$$

This estimate allows us to apply Lemma 4.4, which yields, along with (4.4), and provided that $t \leq \pi/2\omega - \Lambda\hbar^\alpha$,

$$\begin{aligned}\|w^\hbar\|_{L^\infty(0,t;L^2)} &\leq C\hbar^{1-1/\underline{q}} \||u^\hbar|^2 v^\hbar\|_{L^{\underline{q}'}(0,t;L^{\underline{r}'})}\\ &\leq C\hbar^{1-1/\underline{q}} \|u^\hbar\|^2_{L^{\underline{k}}(0,t;L^{\underline{s}})} \|v^\hbar\|_{L^{\underline{q}}(0,t;L^{\underline{r}})}\\ &\leq C\Lambda^{-2/\underline{k}}\hbar^{1/\underline{q}}.\end{aligned} \tag{4.9}$$

Now apply the operator $J^\hbar$ to (4.7). From Lemma 1.1, $J^\hbar w^\hbar$ solves the same equation as $w^\hbar$, with $|u^\hbar|^2 u^\hbar$ replaced by $J^\hbar(|u^\hbar|^2 u^\hbar)$. From (1.8),

$$|J^\hbar(t)(|u^\hbar|^2 u^\hbar)(t,x)| \leq 4|u^\hbar(t,x)|^2 |J^\hbar(t) u^\hbar(t,x)|.$$

Writing $J^\hbar u^\hbar = J^\hbar v^\hbar + J^\hbar w^\hbar$ and proceeding as above yields, so long as (4.8) holds,

$$\|J^\hbar w^\hbar\|_{L^\infty(0,t;L^2)} \leq C\Lambda^{-2/\underline{k}}\hbar^{1/\underline{q}}. \tag{4.10}$$

Combining (4.9) and (4.10), along with Gagliardo-Nirenberg inequality, yields, so long as (4.8) holds,

$$\|w^\hbar(t)\|_{L^{\underline{s}}} \leq C\frac{1}{\left(\frac{\pi}{2\omega} - t\right)^{\delta(\underline{s})}} \Lambda^{-2/\underline{k}}\hbar^{1/\underline{q}}. \tag{4.11}$$

Possibly enlarging the value of Λ, (4.11) shows that (4.8) remains valid up to time $\pi/2\omega - \Lambda\hbar^\alpha$. This proves Prop. 4.1 when $A^\hbar(t) = Id$ or $J^\hbar(t)$, from (4.9) and (4.10). The case $A^\hbar(t) = H^\hbar(t)$ is then an easy by-product. □

4.2 The case $n = 1$

We finally prove the analogue of the above results in space dimension one. When $n = 1$, one can do without Strichartz estimates, and simply use the Sobolev embedding $H^1 \subset L^\infty$,

$$\|f\|_{L^\infty} \leq C\|f\|^{1/2}_{L^2} \|\partial_x f\|^{1/2}_{L^2}.$$

The wave $u^\hbar$ now solves

$$\begin{cases} i\hbar\partial_t u^\hbar + \dfrac{\hbar^2}{2}\partial_x^2 u^\hbar = \dfrac{\omega^2}{2} x^2 u^\hbar + a\hbar^2 |u^\hbar|^4 u^\hbar, \\ u^\hbar_{|t=0} = u_0^\hbar. \end{cases} \tag{4.12}$$

We start with the analogue of Lemma 4.4.

Lemma 4.5 *Assume $n = 1$, and let $a^\hbar \in C(0,T;\Sigma)$ defined for some positive T, solution of*

$$\begin{cases} i\hbar\partial_t a^\hbar + \dfrac{\hbar^2}{2}\partial_x^2 a^\hbar = \dfrac{\omega^2}{2}x^2 a^\hbar + \hbar^2 F^\hbar(a^\hbar) + \hbar^2 S^\hbar, \\ a^\hbar_{|t=0} = 0. \end{cases} \tag{4.13}$$

Assume that there exists $C_0 > 0$ such that for any $T < \pi/2\omega$, and any $0 \le t \le T$,

$$\left\| F^\hbar(a^\hbar)(t)\right\|_{L^2} \le \frac{C_0}{\left(\frac{\pi}{2\omega} - t\right)^2} \left\| a^\hbar(t)\right\|_{L^2}.$$

Then there exists $C > 0$ independent of $\hbar \in [0,1[$ such that for any $\Lambda \ge 1$, the following holds. Let $\hbar_0 > 0$ be such that $\pi/2\omega - \Lambda\hbar_0 > 0$. Then for any $\hbar \in]0,\hbar_0]$,

$$\sup_{0 \le t \le \frac{\pi}{2\omega} - \Lambda\hbar} \|a^\hbar(t)\|_{L^2} \le C\hbar \int_0^{\pi/2\omega - \Lambda\hbar} \left\| S^\hbar(t)\right\|_{L^2} dt.$$

Proof. Multiply (4.13) by $\overline{a^\hbar}$, integrate with respect to x, and take the imaginary part of the result. This yields, from Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{d}{dt}\|a^\hbar(t)\|_{L^2} &\le 2\hbar\|F^\hbar(a^\hbar)(t)\|_{L^2} + 2\hbar\|S^\hbar(t)\|_{L^2} \\ &\le \frac{2C_0\hbar}{\left(\frac{\pi}{2\omega} - t\right)^2} \left\| a^\hbar(t)\right\|_{L^2} + 2\hbar\|S^\hbar(t)\|_{L^2}. \end{aligned}$$

The lemma then follows from the Gronwall lemma. □

We can now prove the analogue of Prop. 4.1.

Proposition 4.6 *Assume $n = 1$. Let $u_0^\hbar \in \Sigma$ be such that $\|u_0^\hbar\|_{L^2}$, $\|\partial_x u_0^\hbar\|_{L^2}$ and $\|xu_0^\hbar\|_{L^2}$ are bounded,* uniformly *with $\hbar \in]0,1]$. Then there exist $C, \Lambda > 0$ such that the following holds. Let $\hbar_0 > 0$ be such that $\pi/2\omega - \Lambda\hbar_0 > 0$. Then for any $\hbar \in]0,\hbar_0]$, $u^\hbar$ is defined in Σ at least up to time $\pi/2\omega - \Lambda\hbar$, and satisfies*

$$\sup_{0 \le t \le \pi/2\omega - \Lambda\hbar} \left\| A^\hbar(t)(u^\hbar - v^\hbar)(t)\right\|_{L^2} \le C,$$

where $A^\hbar(t)$ can be either of the operators Id, $J^\hbar(t)$ or $H^\hbar(t)$.In particular, if $a < 0$ and $u^\hbar$ collapses at time $t_*^\hbar$, then

$$t_*^\hbar \ge \frac{\pi}{2\omega} - \Lambda\hbar, \quad \forall \hbar \in]0,\hbar_0].$$

Proof. The proof follows the proof of Prop. 4.1 very closely, if we take $\underline{q} = \infty$, $(\underline{s}, \underline{k}) = (\infty, 4)$. Denote $w^\hbar = u^\hbar - v^\hbar$ the remainder we want to assess. It solves the initial value problem,

$$\begin{cases} i\hbar\partial_t w^\hbar + \dfrac{\hbar^2}{2}\partial_x^2 w^\hbar = \dfrac{\omega^2}{2}x^2 w^\hbar + a\hbar^2|u^\hbar|^4 u^\hbar, \\ w^\hbar_{|t=0} = 0. \end{cases}$$

We first want to apply the above lemma with $a^\hbar = w^\hbar$. Since $u^\hbar = v^\hbar + w^\hbar$, we can take

$$F^\hbar(w^\hbar) = a|u^\hbar|^4 w^\hbar, \quad S^\hbar = a|u^\hbar|^4 v^\hbar.$$

The point is now to control the L^∞-norm of $u^\hbar$. Notice that we can easily control the L^∞-norm of $v^\hbar$. Indeed, as we already emphasized, for any time t,

$$\|v^\hbar(t)\|_{L^2} = \|u_0^\hbar\|_{L^2}, \quad \|J^\hbar(t)v^\hbar\|_{L^2} = \|\partial_x u_0^\hbar\|_{L^2}.$$

From Lemma 1.1, (1.5), and Gagliardo-Nirenberg inequality, we also have,

$$\begin{aligned}\|v^\hbar(t)\|_{L^\infty} &\leq \frac{C}{|\cos(\omega t)|^{1/2}}\|v^\hbar(t)\|_{L^2}^{1/2}\|J^\hbar(t)v^\hbar\|_{L^2}^{1/2}\\ &\leq \frac{C}{\left(\frac{\pi}{2\omega}-t\right)^{1/2}}\|v^\hbar(t)\|_{L^2}^{1/2}\|J^\hbar(t)v^\hbar\|_{L^2}^{1/2}.\end{aligned}$$

Therefore, the assumptions of Prop. 4.6 imply that there exists $C_0 > 0$ independent of $\hbar$ such that for any $t < \pi/2\omega$,

$$\|v^\hbar(t)\|_{L^\infty} \leq \frac{C_0}{\left(\frac{\pi}{2\omega}-t\right)^{1/2}}.$$

So long as

$$\|w^\hbar(t)\|_{L^\infty} \leq \frac{C_0}{\left(\frac{\pi}{2\omega}-t\right)^{1/2}}, \tag{4.14}$$

holds, we have obviously

$$\|u^\hbar(t)\|_{L^\infty} \leq \frac{2C_0}{\left(\frac{\pi}{2\omega}-t\right)^{1/2}}.$$

This estimate allows us to apply the above lemma, which yields, provided that $t \leq \pi/2\omega - \Lambda\hbar$,

$$\begin{aligned}\|w^\hbar\|_{L^\infty(0,t;L^2)} &\leq C\hbar\||u^\hbar|^4 v^\hbar\|_{L^\infty(0,t;L^2)}\\ &\leq C\hbar\|u^\hbar\|^2_{L^4(0,t;L^\infty)}\|v^\hbar\|_{L^\infty(0,t;L^2)}\\ &\leq C\Lambda^{-1}.\end{aligned} \tag{4.15}$$

Similarly, applying the operator $J^\hbar$ to (4.7) yields, so long as (4.8) holds,

$$\|J^\hbar w^\hbar\|_{L^\infty(0,t;L^2)} \leq C\Lambda^{-1}. \tag{4.16}$$

Combining (4.15) and (4.16), along with Gagliardo-Nirenberg inequality, yields, so long as (4.14) holds,

$$\|w^\hbar(t)\|_{L^\infty} \leq C\frac{1}{\left(\frac{\pi}{2\omega}-t\right)^{1/2}}\Lambda^{-1}. \tag{4.17}$$

Taking Λ large enough, (4.17) shows that (4.14) remains valid up to time $\pi/2\omega - \Lambda\hbar$. This proves Prop. 4.6 when $A^\hbar(t) = Id$ or $J^\hbar(t)$, from (4.15) and (4.16). The case $A^\hbar(t) = H^\hbar(t)$ is then an easy by-product. $\square$

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Rémi Carles
Mathématiques Appliquées de Bordeaux
UMR 5466 CNRS
351 cours de la Libération
F-33405 Talence cedex
France
email: carles@math.u-bordeaux.fr

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