Ann. Henri Poincaré 3 (2002) 673 – 691

1424-0637/02/040673-19

Annales Henri Poincaré

Uniform Estimates of the Resolvent of the Laplace-Beltrami Operator on Infinite Volume Riemannian Manifolds. II

F. Cardoso* and G. Vodev

Abstract. We prove uniform weighted high frequency estimates for the resolvent of the Laplace-Beltrami operator on connected infinite volume Riemannian manifolds under some natural assumptions on the metric on the ends of the manifold. This extends previous results by Burq [3] and Vodev [8].

1 Introduction and statement of results

The purpose of this paper is to extend the results in [8] to more general Riemannian manifolds (which may have cusps). Let (M, g) be an n-dimensional unbounded, connected Riemannian manifold with a Riemannian metric g of class $C^\infty(\overline{M})$ and a compact C^∞-smooth boundary ∂M (which may be empty), of the form $M = X_0 \cup X_1 \cup X_2$, where X_0 is a compact, connected Riemannian manifold with a metric $g_{|X_0}$ of class $C^\infty(\overline{X}_0)$ with a compact boundary $\partial X_0 = \partial M \cup \partial X_1 \cup \partial X_2$, $\partial M \cap \partial X_1 = \emptyset$, $\partial M \cap \partial X_2 = \emptyset$, $\partial X_1 \cap \partial X_2 = \emptyset$, $X_k = [r_k, +\infty) \times S_k$, $r_k \gg 1$, with metric $g_{|X_k} := dr^2 + \sigma_k(r)$, $k = 1, 2$. Here $(S_k, \sigma_k(r))$, $k = 1, 2$, are $n - 1$ dimensional compact Riemannian manifolds without boundary equipped with families of Riemannian metrics $\sigma_k(r)$ depending smoothly on r which can be written in any local coordinates $\theta \in S_k$ in the form

$$\sigma_k(r) = \sum_{i,j} g^k_{ij}(r, \theta) d\theta_i d\theta_j, \quad g^k_{ij} \in C^\infty(X_k).$$

Denote $X_{k,r} = [r, +\infty) \times S_k$. Clearly, $\partial X_{k,r}$ can be identified with the Riemannian manifold $(S_k, \sigma_k(r))$ with the Laplace-Beltrami operator $\Delta_{\partial X_{k,r}}$ written as follows

$$\Delta_{\partial X_{k,r}} = -p_k^{-1} \sum_{i,j} \partial_{\theta_i}(p_k g_k^{ij} \partial_{\theta_j}),$$

where (g_k^{ij}) is the inverse matrix to (g^k_{ij}) and $p_k = (\det(g^k_{ij}))^{1/2} = (\det(g_k^{ij}))^{-1/2}$. Let Δ_g denote the Laplace-Beltrami operator on (M, g). We have

$$\Delta_{X_k} := \Delta_g|_{X_k} = -p_k^{-1}\partial_r(p_k \partial_r) + \Delta_{\partial X_{k,r}} = -\partial_r^2 - \frac{p_k'}{p_k}\partial_r + \Delta_{\partial X_{k,r}}.$$

*Partially supported by CNPq (Brazil)

Throughout this paper given a function $p(r,\theta)$, p', p'' and etc. will denote the first, the second and etc. derivative with respect to r. It is easy to check the identity

$$p_k^{1/2}\Delta_{X_k}p_k^{-1/2} = -\partial_r^2 + \Lambda_{k,r} + q_k(r,\theta), \tag{1.1}$$

where

$$\Lambda_{k,r} = -\sum_{i,j}\partial_{\theta_i}(g_k^{ij}\partial_{\theta_j}),$$

and q_k is an effective potential given by

$$q_k(r,\theta) = (2p_k)^{-2}\left(\frac{\partial p_k}{\partial r}\right)^2 + (2p_k)^{-2}\sum_{i,j}\frac{\partial p_k}{\partial\theta_i}\frac{\partial p_k}{\partial\theta_j}g_k^{ij} + 2^{-1}p_k\Delta_{X_k}(p_k^{-1}).$$

We make the following assumptions:

$$|q_k(r,\theta)| \le C, \quad \frac{\partial q_1}{\partial r}(r,\theta) \le Cr^{-1-\delta_0}, \quad -\frac{\partial q_2}{\partial r}(r,\theta) \le Cr^{-1}, \tag{1.2}$$

with constants $C,\delta_0 > 0$. Denote by h_k the principal symbol of $\Delta_{\partial X_{k,r}}$, that is,

$$h_k(r,\theta,\xi) = \sum_{i,j} g_k^{ij}(r,\theta)\xi_i\xi_j, \quad (\theta,\xi)\in T^*S_k.$$

Clearly, $-\partial h_k/\partial r$ can be interpreted as being the second fundamental form of the surface $\partial X_{k,r}$. We suppose that

$$(-1)^k\frac{\partial h_k}{\partial r}(r,\theta,\xi) \ge \frac{C}{r}h_k(r,\theta,\xi), \quad \forall(\theta,\xi)\in T^*S_k, \tag{1.3}$$

with a constant $C > 0$. In particular, this means that $\partial X_{1,r}$ (resp. $\partial X_{2,r}$) is strictly convex (resp. strictly concave) viewed from $X_{1,r}$ (resp. $X_{2,r}$). This implies that the commutators $(-1)^k[\partial_r,\Lambda_{k,r}]$, $k = 1,2$, are strictly positive.

Denote by G the selfadjoint realization of Δ_g on the Hilbert space $H = L^2(M, d\mathrm{Vol}_g)$ with Dirichlet or Neumann boundary conditions on ∂M. Given $s_1,s_2\in\mathbf{R}$, choose a real-valued positive function $\chi_{s_1,s_2}\in C^\infty(\overline{M})$, $\chi_{s_1,s_2} = 1$ on $M\setminus(X_{1,r_1+1}\cup X_{2,r_2+1})$, $\chi_{s_1,s_2} = r^{-s_k}$ on X_{k,r_k+2}. Also, given $a > r_1$ choose a real-valued positive function $\eta_a\in C^\infty(\overline{M})$, $\eta_a = 0$ on $M\setminus X_{1,a}$, $\eta_a = 1$ on $X_{1,a+1}$. Our main result is the following

Theorem 1.1 *Under the assumptions (1.2) and (1.3), for every $s_1 > 1/2$, $s_2 > 1$, there exist positive constants $C_0, C > 0$, $a > r_1$ so that for $z\in\mathbf{R}$, $z\ge C_0$, the limit*

$$R^+_{s_1,s_2}(z) := \lim_{\varepsilon\to 0^+}\chi_{s_1,s_2}(G - z + i\varepsilon)^{-1}\chi_{s_1,s_2} : H\to H$$

exists and satisfies the bounds

$$\|R^+_{s_1,s_2}(z)\|_{\mathcal{L}(H)} \le e^{Cz^{1/2}}, \tag{1.4}$$

$$\|\eta_a R^+_{s_1,s_2}(z)\eta_a\|_{\mathcal{L}(H)} \le Cz^{-1/2}. \tag{1.5}$$

Suppose that there exist metrics $\widetilde{\sigma}_k(r)$ depending smoothly on $r \in (-\infty, +\infty)$ such that $\widetilde{\sigma}_k(r) = \sigma_k(r)$ for $r \geq r_k$ and the resolvents (defined for $\operatorname{Im} z < 0$, $\operatorname{Re} z > 0$)

$$R_{X_k^0}(z) := (\Delta_{X_k^0} - z)^{-1} : L^2_{\mathrm{comp}}(X_k^0, d\mathrm{Vol}_{g_{X_k^0}}) \to H^2_{\mathrm{loc}}(X_k^0, d\mathrm{Vol}_{g_{X_k^0}}),$$

where $X_k^0 = (-\infty, +\infty) \times S_k$ with metric $g_{X_k^0} = dr^2 + \widetilde{\sigma}_k(r)$, $\Delta_{X_k^0}$ denoting the selfadjoint realization of the Laplace-Beltrami operator on X_k^0 on the Hilbert space $L^2(X_k^0, d\mathrm{Vol}_{g_{X_k^0}})$, extend analytically to $\operatorname{Im} z \leq e^{-\gamma_1 |z|^{1/2}}$, $\operatorname{Re} z \geq C_1$, $\gamma_1, C_1 > 0$, and satisfy in this region the bounds (with $\alpha = 0, 1$):

$$\|\partial_z^\alpha \chi R_{X_k^0}(z)\chi\|_{\mathcal{L}(L^2(X_k^0, d\mathrm{Vol}_{g_{X_k^0}}))} \leq C_2 e^{\gamma_2 |z|^{1/2}}, \quad \forall \chi \in C_0^\infty(X_k^0), \tag{1.6}$$

with some constants $C_2, \gamma_2 > 0$. As a consequence of Theorem 1.1 we get the following

Corollary 1.2 *Under the assumptions (1.2), (1.3) and (1.6), the resolvent (defined for* $\operatorname{Im} z < 0$, $\operatorname{Re} z > 0$*)*

$$R_M(z) := (G - z)^{-1} : L^2_{\mathrm{comp}}(M, d\mathrm{Vol}_g) \to H^2_{\mathrm{loc}}(M, d\mathrm{Vol}_g),$$

extends analytically to $\operatorname{Im} z \leq e^{-\gamma |z|^{1/2}}$, $\operatorname{Re} z \geq C_0$, *and satisfies in this region the bound*

$$\|\chi R_M(z)\chi\|_{\mathcal{L}(H)} \leq C e^{\gamma |z|^{1/2}}, \tag{1.7}$$

$\forall \chi \in C^\infty(\overline{M})$ *of compact support, with some constants* $C_0, C, \gamma > 0$.

Remark. It is easy to see that the above results hold for more general connected Riemannian manifolds of the form

$$M = X_0 \cup X_1^1 \cup \cdots \cup X_1^J \cup X_2^1 \cup \cdots \cup X_2^I, \quad I \geq 0, J \geq 1,$$

with X_1^j like X_1, X_2^i like X_2, and X_0 being a compact Riemannian manifold with boundary $\partial X_0 = \partial M \cup \partial X_1^1 \cup \cdots \cup \partial X_1^J \cup \partial X_2^1 \cup \cdots \cup \partial X_2^I$, $\partial M \cap \partial X_1^j = \emptyset$, $\partial M \cap \partial X_2^i = \emptyset$, $\partial X_1^j \cap \partial X_2^i = \emptyset$, $\partial X_1^{j_1} \cap \partial X_1^{j_2} = \emptyset$, $j_1 \neq j_2$, $\partial X_2^{i_1} \cap \partial X_2^{i_2} = \emptyset$, $i_1 \neq i_2$.

This corollary can be derived from the bounds (1.4) and (1.6) in precisely the same way as in the proof of Theorem 1.2 of [8] and this is why we omit the proof.

Another consequence of the above theorem is that we get uniform high frequency resolvent estimates for long-range perturbations of the Euclidean metric. Let $\mathcal{O} \subset \mathbf{R}^n$, $n \geq 2$, be a bounded domain with a C^∞-smooth boundary Γ and a connected complement $\Omega = \mathbf{R}^n \setminus \mathcal{O}$. Let g be a Riemannian metric in Ω of the form

$$g = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j, \quad g_{ij}(x) \in C^\infty(\overline{\Omega}).$$

We make the following assumption:

$$|\partial_x^\alpha (g_{ij}(x) - \delta_{ij})| \leq C_\alpha \langle x \rangle^{-\delta_0 - |\alpha|}, \tag{1.8}$$

for every multi-index α, with constants $C_\alpha, \delta_0 > 0$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$ and δ_{ij} denotes the Kronecker symbol. Denote by Δ_g the corresponding Laplace-Beltrami operator, i.e.

$$\Delta_g = -f^{-1/2} \sum_{i,j=1}^{n} \partial_{x_i}(f^{1/2} g^{ij} \partial_{x_j}),$$

where (g^{ij}) is the inverse matrix to (g_{ij}) and $f = \det(g_{ij})$. Denote by G the self-adjoint realization of Δ_g on the Hilbert space $H = L^2(\Omega; d\mathrm{Vol}_g)$, $d\mathrm{Vol}_g := f^{1/2} dx$, with Dirichlet or Neumann boundary conditions on Γ. It is not hard to see (e.g. see the appendix of [3] for the proof of an analytic version) that under the assumption (1.8), there exists a global smooth change of variables, $(r, \theta) = (r(x), \theta(x))$, for $|x| \gg 1$, where $r \in [r_0, +\infty)$, $r_0 \gg 1$, $\theta \in S = \{y \in \mathbf{R}^n : |y| = 1\}$, which transforms the metric g in the form

$$dr^2 + r^2 \sum_{i,j} h_{ij}(r, \theta) d\theta_i d\theta_j, \tag{1.9}$$

where $h_{ij} \in C^\infty$ satisfy the inequalities

$$|\partial_r^\alpha \partial_\theta^\beta (h_{ij}(r, \theta) - h_{ij}^0(\theta))| \leq C_{\alpha,\beta} r^{-\delta_0 - \alpha} \tag{1.10}$$

for all multi-indexes α and β. Here $\sum_{i,j} h_{ij}^0(\theta) d\theta_i d\theta_j$ is the metric on S induced by the Euclidean one. The coordinates (r, θ) are just the normal geodesics coordinates which are well defined outside a sufficiently large compact since the metric g is close to the Euclidean one. In other words, the Riemannian manifold (Ω, g) is isometric to a connected Riemannian manifold (M, g) of the form $M = Y_0 \cup Y$, where Y_0 is a compact connected Riemannian manifold with boundary $\partial Y_0 = \partial M \cup \partial Y$, $\partial M \cap \partial Y = \emptyset$, and $Y = [r_0, +\infty) \times S$, $r_0 \gg 1$, with metric given by (1.9) and satisfying (1.10). Therefore, Y is a particular case of the manifold X_1 above, and we get the following consequence of Theorem 1.1.

Corollary 1.3 *Under the assumption (1.8), for every $s > 1/2$ there exist constants $C_0, C > 0$ and $a \gg 1$ so that for $z \in \mathbf{R}, z \geq C_0$, the limit*

$$R_s^+(z) := \lim_{\varepsilon \to 0^+} \langle x \rangle^{-s} (G - z + i\varepsilon)^{-1} \langle x \rangle^{-s} : H \to H$$

exists and satisfies the bounds

$$\|R_s^+(z)\|_{\mathcal{L}(H)} \leq e^{Cz^{1/2}}, \tag{1.11}$$

$$\|\chi_a R_s^+(z)\chi_a\|_{\mathcal{L}(H)} \leq Cz^{-1/2}, \tag{1.12}$$

where χ_a denotes the characteristic function of $|x| \geq a$.

Remark. It is easy to see from the proof that it suffices to have (1.10) for $\alpha+|\beta| \leq 3$.

When $g_{ij} = \delta_{ij}$ outside some compact the bound (1.11) follows from the results of Burq [2], where he proved a similar bound for the cutoff resolvent. This was improved in [7] for metrics satisfying $g_{ij} - \delta_{ij} = O(e^{-|x|^{2+\epsilon_0}})$, $\epsilon_0 > 0$. Burq [3] has recently extended his result to long-range metric perturbations assuming that g_{ij} admit an analytic extension from $\{x \in \mathbf{R}^n : |x| \geq \rho_0\}$, $\rho_0 \gg 1$, to $\{z \in \mathbf{C}^n : |\mathrm{Re}\, z| \geq \rho_0, |\mathrm{Im}\, z| \leq \gamma_0|\mathrm{Re}\, z|\}$, $\gamma_0 > 0$. In particular, this implies that if (1.8) holds with $\alpha = 0$, it holds for any α. He used the complex scaling method to show that there are no resonances in an exponentially small neighbourhood of the real axis. In particular, it follows from [3] that one has an analogue of (1.11) for the cutoff resolvent, which combined with the result of Bruneau-Petkov [1] imply the bound (1.11) itself in that case. Burq [3] has also proved an analogue of (1.12) with χ_a replaced by the characteristic function of $a < |x| < b$ with $b > a \gg 1$.

Note that the class of manifolds, (M, g), we study includes hyperbolic ones with negative curvature, κ, satisfying $C^{-1} \leq -\kappa \leq C$ on M for some constant $C > 0$. In fact, the methods we develop in the present paper apply to infinite volume Riemannian manifolds with infinity consisting of a finite number of two type of ends - elliptic ends (like X_1 above) whose number is ≥ 1 and cusps (like X_2 above) whose number is ≥ 0. An elliptic end satisfying (1.2) and (1.3) with $k = 1$ is of infinite volume. The condition (1.2) on the effective potential together with (1.3) guarantee that the (Dirichlet) self-adjoint realization of Δ_{X_1} on $L^2(X_1, d\mathrm{Vol}_g)$ has no discrete spectrum (except for possibly a finite number of eigenvalues). Moreover, if we consider the generalized geodesic flow in X_1, as (1.3) implies that ∂X_1 is strictly convex, every geodesic coming from the infinity of X_1 is allowed to hit the boundary either transversally or at a diffractive point, so it escapes back to infinity. This suggests that the operator Δ_{X_1} should have properties typical for the so called nontrapping operators. This in turn suggests that the resolvent of the global operator Δ_g cut off on the both sides by a cutoff function supported in X_1 should satisfy the same high frequency estimates as does the resolvent of Δ_{X_1}. We show that this is exactly what happens - see the bound (1.5) which without cutoffs is known to hold for nontrapping perturbations. The key point of our proof is the estimate (2.22) proved in Section 2. It seems that the assumptions (1.2) and (1.3) with $k = 1$ are the weakest ones under which (2.22) holds true.

The situation on a cusp X_2 is exactly opposite and this is why in (1.5) we cannot take the function η with support on X_2. In fact, the conditions (1.2) and (1.3) with $k = 2$ do not imply that the volume of X_2 must be finite, but we will keep the notion *cusp* in this case as well. Of course, there are finite volume hyperbolic cusps, X_2, (with negative curvature) satisfying (1.2) and (1.3) with $k = 2$. An interesting example of two dimensional hyperbolic manifolds our results apply to is $X_k = [a_k, +\infty)_r \times (\mathbf{R} \setminus \ell_k \mathbf{Z})_t$, $a_k, \ell_k > 0$, $k = 1, 2$, with metrics $g|_{X_1} = dr^2 + \cosh^2 r dt^2$, $g|_{X_2} = dr^2 + e^{-2r} dt^2$. Note that for such manifolds the bound (1.4) as well as Corollary 1.2 have been already proved in [8], but the bound (1.5) seems to be new. We expect that Theorem 1.1 (or at least (1.4)) holds for more general infinite volume hyperbolic manifolds with a more complex structure at infinity, as for example manifolds with non-maximal cusps.

The bound (1.4) is proved in [8] for manifolds which have a similar structure at infinity as the manifold M above, but under the restriction that the metric on the ends X_k, $k = 1, 2$, is of the form $dr^2 + p_k(r)^{-2}\sigma_k$, where σ_k does not depend on r, and $p_k(r)$ are smooth positive functions satisfying conditions analogous to (1.2) and (1.3) above. The fact that we have a separation of variables was used in an essential way in the methods developed in [8]. In the situation we treat in the present paper we do not have such a separation of variables, which requires a different approach. It is based on an idea of Burq [3] which consists of using Carleman estimates outside a sufficiently large compact with a real-valued phase function, $\varphi(r)$, with $\varphi'(r) > 0$, depending on the spectral parameter (in our case $\lambda \gg 1$) such that $\varphi' = O(\lambda^{-1} r^{-1})$ outside another compact (in which region the estimates are no longer of Carleman type). We apply this on the elliptic (infinite volume) end X_1 - see Proposition 2.3 which is essentially due to Burq (see Propositions 6.2 and 7.2 of [3]), but here we give a different proof in a little bit more general situation. Moreover, our construction of the phase function φ is simpler than that one in [3]. Then the problem is to paste together this estimate with estimates on the compact part of the manifold essentially due to Lebeau-Robbiano [4], [5] (see Proposition 4.1 and also Theorem A.2 of [7]), with weighted estimates at the infinity of X_1 (see Proposition 2.4) as well as with weighted Carleman estimates on X_2 (see Proposition 3.1). This is carried out in Section 4.

Acknowledgements. A part of this work was carried out while the second author was visiting Universidade Federal de Pernambuco, Recife, Brazil, in March-April 2001, and he would like to thank this institution for the hospitality and the nice working conditions. The first author was also partially supported by the agreement Brazil-France in Mathematics - Proc. 69.0014/01-5.

2 Uniform a priori estimates on X_1

We begin this section by constructing a real-valued phase function, φ, with properties described in Lemma 2.1 below. A similar phase function was first constructed by Burq [3]. Here we simplify this construction (as well as some of his arguments) adapting it to our approach.

Let $\lambda \gg 1$ be a big parameter, let $0 < \delta \ll 1$ be independent of λ, and let $\gamma_0 > 1$ be independent of λ and δ. In what follows, C will denote a positive constant independent of λ, while C' will denote a positive constant independent of λ and δ. Define the continuous function $\widetilde{\varphi}_1(r)$ so that $\widetilde{\varphi}_1(r) = (Ar^{-\delta} - 1)^{1/2}$ for $r_1 \le r < a_1 = A^{1/\delta}$, $\widetilde{\varphi}_1(r) = 0$ for $r \ge a_1$, where $A = (r_1+2)^\delta(\gamma_0+1)^2/4+1$. Choose a real-valued function $\phi \in C_0^\infty((-1,1))$ such that $\phi \ge 0$, $\int \phi = 1$ and $\phi'^2 \le C_0\phi$ with some constant $C_0 > 0$, and set $\phi_\epsilon(r) = \epsilon^{-1}\phi(r/\epsilon)$, $0 < \epsilon \ll 1$. Let $\zeta \in C_0^\infty(\mathbf{R})$ be a real-valued function, $\zeta \ge 0$, equal to 1 in a small neighbourhood of a_1 and to zero outside another small neighbourhood of a_1. Then the function

$$\varphi_1 = (1-\zeta)\widetilde{\varphi}_1 + \phi_\epsilon \star (\zeta\widetilde{\varphi}_1)$$

belongs to $C^\infty([r_1,\infty))$ and vanishes for $r \ge a_1 + 1$. Moreover, since $\varphi_1'\varphi_1 \to \widetilde{\varphi}_1'\widetilde{\varphi}_1 = -2^{-1}\delta A r^{-1-\delta}$ if $r < a_1$ and to zero if $r > a_1$ as $\epsilon \to 0$, taking $\epsilon > 0$ small enough we can arrange

$$-\varphi_1'(r)\varphi_1(r) \le C'\delta r^{-1}, \quad \forall r \ge r_1. \tag{2.1}$$

Also, the choice of ϕ guarantees the bound

$$\varphi_1'(r)^2 \le C\varphi_1(r), \quad \forall r \ge r_1. \tag{2.2}$$

Define a real-valued function $\varphi \in C^\infty([r_1,+\infty))$ such that $\varphi(r_1) = -1$ and

$$\varphi'(r) = \varphi_1(r) + \lambda^{-1/2}r^{-1}\varphi_2(r)(1+\lambda^{1/2}\varphi_3(r))^{-1},$$

where $\varphi_j \in C^\infty([r_1,+\infty))$, $j = 2,3$, are real-valued functions independent of λ, $0 \le \varphi_j(r) \le 1$, $\varphi_j'(r) \ge 0$, $\forall r$, chosen so that $\varphi_2 = 0$ for $r \le a_1'$, $\varphi_2 = 1$ for $r \ge a_1''$, $r_1 + 2 < a_1' < a_1'' \in \operatorname{supp}(1-\zeta)$, $\varphi_3 = 0$ for $r \le a_2'$, $\varphi_3 = 1$ for $r \ge a_2''$, $a_1 + 1 < a_2' < a_2''$. We also require that

$$r\varphi_3'(r) \le \frac{1}{4}, \quad \forall r. \tag{2.3}$$

Moreover, near a_1' we choose φ_2 in the form $\varphi_2(r) = \exp((a_1'-r)^{-1})$ if $r > a_1'$, which guarantees the inequality

$$\varphi_2'(r)^2 \le C\varphi_2(r), \quad \forall r \ge r_1. \tag{2.4}$$

It is easy also to see that we have the inequalities

$$|\varphi_j'(r)| + |\varphi_j''(r)| + |\varphi_j'''(r)| \le Cr^{-1}\varphi_2(r),\ j = 1,3,$$
$$|\varphi_2'(r)| + |\varphi_2''(r)| + |\varphi_2'''(r)| \le C\varphi_1(r). \tag{2.5}$$

Note that the choice of the constant A guarantees that $\varphi(r_1+2) \ge \gamma_0$.

Lemma 2.1 *The following inequalities hold for $\lambda \ge \lambda_0(\delta) \gg 1$ and $\forall r \ge r_1$:*

$$C\lambda^{-1}r^{-1} \le \varphi'(r) \le Cr^{-1}, \tag{2.6}$$

$$-\varphi'(r)\varphi''(r) \le C'\delta r^{-1}, \tag{2.7}$$

$$|\varphi''(r)| \le C\lambda^{1/2}r^{-1}\varphi'(r), \quad \varphi''(r)^2 \le C\lambda^{1/2}r^{-1}\varphi'(r), \tag{2.8}$$

$$|\varphi'''(r)| \le C\lambda r^{-1}\varphi'(r), \quad |\varphi'''(r)| \le C\lambda^{1/2} r^{-1}, \tag{2.9}$$

$$|\varphi^{(4)}(r)| \le C\lambda^{3/2} r^{-1}\varphi'(r), \tag{2.10}$$

$$2\lambda\varphi'(r)^2 + \varphi''(r) \ge C' r^{-1}\varphi'(r). \tag{2.11}$$

Proof. We have

$$C\lambda^{-1}r^{-1} \le \lambda^{-1}r^{-1}(r\varphi_1(r)+\varphi_2(r)) \le \varphi'(r) \le r^{-1}(r\varphi_1(r)+\lambda^{-1/2}\varphi_2(r)) \le Cr^{-1},$$

which proves (2.6). To prove (2.7) observe that

$$\varphi''(r) = \varphi_1'(r) - \lambda^{-1/2}r^{-2}\varphi_2(r)(1+\lambda^{1/2}\varphi_3(r))^{-1}$$
$$+\lambda^{-1/2}r^{-1}\varphi_2'(r)(1+\lambda^{1/2}\varphi_3(r))^{-1} - r^{-1}\varphi_2(r)\varphi_3'(r)(1+\lambda^{1/2}\varphi_3(r))^{-2},$$

and hence, in view of (2.1),

$$-\varphi'\varphi'' = -\varphi_1\varphi_1' + \lambda^{-1/2}r^{-2}\varphi_1\varphi_2(1+\lambda^{1/2}\varphi_3)^{-1}$$
$$-\lambda^{-1/2}r^{-1}(\varphi_1'\varphi_2+\varphi_1\varphi_2')(1+\lambda^{1/2}\varphi_3)^{-1}$$
$$+\lambda^{-1}r^{-2}\varphi_2^2(1+\lambda^{1/2}\varphi_3)^{-2} - \lambda^{-1}r^{-2}\varphi_2\varphi_2'(1+\lambda^{1/2}\varphi_3)^{-2}$$
$$+\lambda^{-1/2}r^{-2}\varphi_2^2\varphi_3'(1+\lambda^{1/2}\varphi_3)^{-2} \le C'\delta r^{-1} + C\lambda^{-1/2}r^{-1} \le 2C'\delta r^{-1}.$$

Moreover, in view of (2.5) we have

$$|\varphi''| \le Cr^{-2}\varphi_2(1+\lambda^{1/2}\varphi_3)^{-1} \le C\lambda^{1/2}r^{-1}\varphi'.$$

On the other hand,

$$\varphi''^2 \le 4\varphi_1'^2 + Cr^{-2}(\varphi_2+\varphi_2'^2)(1+\lambda^{1/2}\varphi_3)^{-1},$$

and hence (2.8) follows in view of (2.2) and (2.4). Furthermore, we have

$$\varphi''' = \varphi_1'' + 2\lambda^{-1/2}r^{-3}\varphi_2(1+\lambda^{1/2}\varphi_3)^{-1} - 2\lambda^{-1/2}r^{-2}\varphi_2'(1+\lambda^{1/2}\varphi_3)^{-1}$$
$$+2r^{-2}\varphi_3'(1+\lambda^{1/2}\varphi_3)^{-2} + \lambda^{-1/2}r^{-1}\varphi_2''(1+\lambda^{1/2}\varphi_3)^{-1}$$
$$-r^{-1}\varphi_3''(1+\lambda^{1/2}\varphi_3)^{-2} + 2\lambda^{1/2}r^{-2}\varphi_3'^2(1+\lambda^{1/2}\varphi_3)^{-3},$$

and hence $|\varphi'''| \le C\lambda^{1/2}r^{-1}$. On the other hand, in view of (2.5) we have

$$|\varphi'''| \le Cr^{-2}\varphi_2 + \lambda^{-1/2}\varphi_1 + C\lambda^{1/2}r^{-2}\varphi_2(1+\lambda^{1/2}\varphi_3)^{-1} \le C\lambda r^{-1}\varphi',$$

which proves (2.9). In the same way,

$$|\varphi^{(4)}| \le |\varphi_1'''| + C\lambda r^{-2}\varphi_2(1+\lambda^{1/2}\varphi_3)^{-1} + C\lambda^{-1/2}(|\varphi_2'|+|\varphi_2''|+|\varphi_2'''|) \le C\lambda^{3/2}r^{-1}\varphi'.$$

To prove (2.11) observe that

$$2\lambda\varphi'^2 + \varphi'' \ge 2\lambda\varphi_1^2 + 2r^{-2}\varphi_2^2(1+\lambda^{1/2}\varphi_3)^{-2} + \varphi_1'$$
$$-\lambda^{-1/2}r^{-2}\varphi_2(1+\lambda^{1/2}\varphi_3)^{-1} - r^{-1}\varphi_2\varphi_3'(1+\lambda^{1/2}\varphi_3)^{-2}.$$

For $r \geq a_1 + 1$ we have $\varphi_1 = \varphi_1' = 0$ and hence, in view of (2.3),

$$2\lambda\varphi'^2 + \varphi'' \geq (1 - 2r\varphi_3' - \lambda^{-1/2})r^{-2}(1 + \lambda^{1/2}\varphi_3)^{-2} \geq C'r^{-1}\varphi'.$$

For $r < a_1 + 1$ we have $\varphi_3 = 0$ and hence

$$2\lambda\varphi'^2 + \varphi'' \geq 2\lambda\varphi_1^2 + \varphi_1' + r^{-2}\varphi_2^2.$$

Since $\varphi_1'(a_1 + 1) = 0$, there exists $a_0 < a_1 + 1$ such that $|\varphi_1'| \leq (2r)^{-2}\varphi_2^2$ for $a_0 \leq r \leq a_1 + 1$. Hence, for $a_0 \leq r \leq a_1 + 1$,

$$2\lambda\varphi'^2 + \varphi'' \geq \lambda\varphi'^2 \geq C\lambda^{1/2}r^{-1}\varphi'. \tag{2.12}$$

For $r_1 \leq r \leq a_0$, we have $|\varphi_1'| \leq C\varphi_1^2$, which again implies (2.12). □

Throughout this section $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will denote the norm and the scalar product on $L^2(S_1)$, while the Sobolev space $H^1(X_1, d\mathrm{Vol}_g)$ will be equipped with the semiclassical norm given by

$$\|u\|^2_{H^1(X_1,d\mathrm{Vol}_g)}$$
$$= \|u\|^2_{L^2(X_1,d\mathrm{Vol}_g)} + \|\mathcal{D}_r u\|^2_{L^2(X_1,d\mathrm{Vol}_g)} + \int_{r_1}^{\infty} \sum_{i,j} \langle p_1 g_1^{ij}\mathcal{D}_{\theta_i}u(r,\cdot), \mathcal{D}_{\theta_j}u(r,\cdot)\rangle dr,$$

where $\mathcal{D}_r = (i\lambda)^{-1}\partial_r$, $\mathcal{D}_{\theta_j} = (i\lambda)^{-1}\partial_{\theta_j}$. Denote by $L^2(X_1)$ and $H^1(X_1)$ the spaces equipped with the norms

$$\|u\|^2_{L^2(X_1)} = \int_{r_1}^{\infty} \|u(r,\cdot)\|^2 dr,$$
$$\|u\|^2_{H^1(X_1)} = \int_{r_1}^{\infty} \left(\|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2 + \sum_{i,j} \langle g_1^{ij}\mathcal{D}_{\theta_i}u(r,\cdot), \mathcal{D}_{\theta_j}u(r,\cdot)\rangle \right) dr.$$

It is easy to see that

$$\|u\|_{L^2(X_1,d\mathrm{Vol}_g)} = \|p_1^{1/2}u\|_{L^2(X_1)}, \quad \|u\|_{H^1(X_1,d\mathrm{Vol}_g)} \simeq \|p_1^{1/2}u\|_{H^1(X_1)}.$$

Finally, given an $a \geq r_1$ and functions $u(r,\theta)$, $v(r,\theta)$, we denote

$$\|u\|_{L^2(\partial X_{1,a})} := \|u(a,\cdot)\|, \quad \langle u, v\rangle_{L^2(\partial X_{1,a})} := \langle u(a,\cdot), v(a,\cdot)\rangle,$$

$$\|u\|^2_{H^1(\partial X_{1,a})} := \|u(a,\cdot)\|^2 + \sum_{i,j} \langle g_1^{ij}\mathcal{D}_{\theta_i}u(a,\cdot), \mathcal{D}_{\theta_j}u(a,\cdot)\rangle.$$

It is clear from the definition of the function φ above that there exists an $a \geq r_1$ such that $\varphi'(r) = \lambda^{-1}r^{-1}$ for $r \geq a$. The main result in this section is the following

Theorem 2.2 *Let $u \in H^2(X_1, d\mathrm{Vol}_g)$, $u = 0$, $\partial_r u = 0$ on ∂X_1, be such that $r^s(\Delta_{X_1} - \lambda^2 + i\varepsilon)u \in L^2(X_1, d\mathrm{Vol}_g)$ for $\lambda > 0$, $0 < \varepsilon \le 1$ and $0 < s - 1/2 \ll 1$. Then, for a suitable choice of the parameter $\delta > 0$, there exist constants $C_1, C_2, \lambda_0 > 0$ (independent of λ and ε) so that for $\lambda \ge \lambda_0$ we have*

$$\|e^{\lambda(\varphi(r)-\varphi(a))}u\|^2_{H^1(X_1\setminus X_{1,a},d\mathrm{Vol}_g)} + \|r^{-s}u\|^2_{H^1(X_{1,a},d\mathrm{Vol}_g)}$$
$$\le C_1\lambda^{-2}\|e^{\lambda(\varphi(r)-\varphi(a))}(\Delta_{X_1} - \lambda^2 + i\varepsilon)u\|^2_{L^2(X_1\setminus X_{1,a},d\mathrm{Vol}_g)}$$
$$+C_1\lambda^{-2}\|r^s(\Delta_{X_1} - \lambda^2 + i\varepsilon)u\|^2_{L^2(X_{1,a},d\mathrm{Vol}_g)} - C_2\lambda^{-1}\mathrm{Im}\langle\partial_r u, u\rangle_{L^2(\partial X_{1,a})}. \quad (2.13)$$

Proof. Denote

$$P = p_1^{1/2}(\lambda^{-2}\Delta_{X_1} - 1 + i\varepsilon)p_1^{-1/2} = \mathcal{D}_r^2 + L_r - 1 + V + i\varepsilon,$$

where $0 < \varepsilon = O(\lambda^{-2})$, $L_r = \lambda^{-2}\Lambda_{1,r}$, $V = \lambda^{-2}q_1$, and

$$P_\varphi = e^{\lambda\varphi}Pe^{-\lambda\varphi} = P - \varphi'(r)^2 + \lambda^{-1}\varphi''(r) + 2i\varphi'(r)\mathcal{D}_r.$$

We will first prove the following

Proposition 2.3 *Let $u \in H^2(X_1 \setminus X_{1,a})$, $u = 0$, $\partial_r u = 0$ on $\partial X_1 \cup \partial X_{1,a}$. Then, there exist constants $C, \lambda_0 > 0$ (independent of λ and ε) so that for $\lambda \ge \lambda_0$ we have*

$$\|(\varphi'/r)^{1/2}u\|_{H^1(X_1\setminus X_{1,a})} \le C\lambda^{1/2}\|P_\varphi u\|_{L^2(X_1\setminus X_{1,a})}. \quad (2.14)$$

Proof. Let $\psi(r) \in C^\infty([r_1, a])$ be a real-valued function. Integrating by parts one can easily get the identity

$$\mathrm{Re}\,\langle\psi P_\varphi u, u\rangle_{L^2(X_1\setminus X_{1,a})} = \langle\psi\mathcal{D}_r u, \mathcal{D}_r u\rangle_{L^2(X_1\setminus X_{1,a})} + \langle\psi L_r u, u\rangle_{L^2(X_1\setminus X_{1,a})}$$
$$-\langle(\psi + \psi\varphi'^2 - \lambda^{-2}q_1 + \lambda^{-1}\varphi'\psi' + 2^{-1}\lambda^{-2}\psi'')u, u\rangle_{L^2(X_1\setminus X_{1,a})}. \quad (2.15)$$

Set

$$F(r) = -\langle(L_r - 1 + W)u(r,\cdot), u(r,\cdot)\rangle + \|\mathcal{D}_r u(r,\cdot)\|^2,$$

where $W = \lambda^{-2}q_1 - \varphi'^2 + \lambda^{-1}\varphi''$. We have

$$F'(r) =$$
$$= -2\mathrm{Re}\,\langle L_r u(r,\cdot), u'(r,\cdot)\rangle - 2\mathrm{Re}\,\langle\mathcal{D}_r^2 u(r,\cdot), u'(r,\cdot)\rangle + 2\mathrm{Re}\,\langle(1 - W)u(r,\cdot), u'(r,\cdot)\rangle$$
$$- \langle[\partial_r, L_r]u(r,\cdot), u(r,\cdot)\rangle - \langle W'u(r,\cdot), u(r,\cdot)\rangle$$
$$= -2\mathrm{Re}\,\langle P_\varphi u(r,\cdot), u'(r,\cdot)\rangle + 4\lambda\varphi'\|\mathcal{D}_r u(r,\cdot)\|^2 - 2\varepsilon\mathrm{Im}\,\langle u(r,\cdot), u'(r,\cdot)\rangle$$
$$- \langle[\partial_r, L_r]u(r,\cdot), u(r,\cdot)\rangle - \langle W'u(r,\cdot), u(r,\cdot)\rangle.$$

Multiplying this identity by φ' and integrating with respect to r lead to

$$\int_{r_1}^a \varphi'F'dr = -2\mathrm{Re}\int_{r_1}^a \langle\varphi'P_\varphi u, u'\rangle dr + 4\lambda\int_{r_1}^a \|\varphi'\mathcal{D}_r u\|^2 dr \quad (2.16)$$
$$- 2\varepsilon\mathrm{Im}\int_{r_1}^a \langle\varphi'u, u'\rangle dr - \int_{r_1}^a \langle\varphi'[\partial_r, L_r]u, u\rangle dr - \int_{r_1}^a \langle\varphi'W'u, u\rangle dr.$$

On the other hand, we have

$$\int_{r_1}^{a} \varphi' F' dr = -\int_{r_1}^{a} \varphi'' F dr$$

$$= \mathrm{Re} \int_{r_1}^{a} \langle \varphi'' L_r u, u\rangle dr - \int_{r_1}^{a} \langle \varphi'' \mathcal{D}_r u, \mathcal{D}_r u\rangle dr - \int_{r_1}^{a} \langle \varphi''(1-W)u, u\rangle dr$$

$$= \mathrm{Re} \int_{r_1}^{a} \langle \varphi'' P_\varphi u, u\rangle dr - 2\int_{r_1}^{a} \langle \varphi'' \mathcal{D}_r u, \mathcal{D}_r u\rangle dr$$

$$+ \int_{r_1}^{a} \langle (\lambda^{-1}\varphi''^2 + \lambda^{-1}\varphi'\varphi''' + 2^{-1}\lambda^{-2}\varphi^{(4)})u, u\rangle dr, \tag{2.17}$$

where we have used (2.15) with $\psi = \varphi''$. Combining (2.16) and (2.17) we get the identity

$$2\int_{r_1}^{a} \langle (2\lambda\varphi'^2 + \varphi'')\mathcal{D}_r u, \mathcal{D}_r u\rangle dr - \int_{r_1}^{a} \langle \varphi' [\partial_r, L_r] u, u\rangle dr$$

$$= 2\mathrm{Re} \int_{r_1}^{a} \langle \varphi' P_\varphi u, u'\rangle dr + \mathrm{Re} \int_{r_1}^{a} \langle \varphi'' P_\varphi u, u\rangle dr + 2\varepsilon \mathrm{Im} \int_{r_1}^{a} \langle \varphi' u, u'\rangle dr$$

$$+ \int_{r_1}^{a} \langle (-2\varphi'^2\varphi'' + \lambda^{-1}\varphi''^2 + 2\lambda^{-1}\varphi'\varphi''' + 2^{-1}\lambda^{-2}\varphi^{(4)} + \lambda^{-2}\varphi' q_1')u, u\rangle dr. \tag{2.18}$$

It is easy to see that (1.3) implies

$$-[\partial_r, L_r] \geq \frac{C}{r} L_r, \quad C > 0, \tag{2.19}$$

and hence in view of (1.2) and Lemma 2.1 we conclude from (2.18)

$$\int_{r_1}^{a} \|(\varphi'/r)^{1/2}\mathcal{D}_r u\|^2 dr + \int_{r_1}^{a} \|(\varphi'/r)^{1/2} L_r^{1/2} u\|^2 dr$$

$$\leq O(\lambda) \int_{r_1}^{a} \|P_\varphi u\|^2 dr + C\delta \int_{r_1}^{a} \|(\varphi'/r)^{1/2} u\|^2 dr, \tag{2.20}$$

for $\lambda \geq \lambda_0(a, \delta) \gg 1$, where $C > 0$ does not depend on λ, δ and a. On the other hand, by (2.15) used with $\psi = r^{-1}\varphi'$ we have

$$\int_{r_1}^{a} \Big\langle \big(r^{-1}\varphi'(1 + \varphi'^2 + \lambda^{-1}\varphi'' - \lambda^{-1}r^{-1}\varphi' + \lambda^{-2}r^{-2} - \lambda^{-2}q_1)$$

$$-\lambda^{-2}r^{-2}\varphi'' + 2^{-1}\lambda^{-2}r^{-1}\varphi'''\big)\, u, u\Big\rangle\, dr$$

$$= \int_{r_1}^{a} \langle r^{-1}\varphi' \mathcal{D}_r u, \mathcal{D}_r u\rangle dr + \int_{r_1}^{a} \langle r^{-1}\varphi' L_r u, u\rangle dr - \mathrm{Re} \int_{r_1}^{a} \langle P_\varphi u, r^{-1}\varphi' u\rangle dr,$$

and hence, in view of Lemma 2.1 and (1.2), we get

$$\frac{1}{4}\int_{r_1}^{a}\|(\varphi'/r)^{1/2}u\|^2dr \tag{2.21}$$
$$\le \int_{r_1}^{a}\|(\varphi'/r)^{1/2}\mathcal{D}_r u\|^2dr + \int_{r_1}^{a}\|(\varphi'/r)^{1/2}L_r^{1/2}u\|^2dr + \int_{r_1}^{a}\|P_\varphi u\|^2dr.$$

Now (2.14) follows from (2.20) and (2.21), provided $\delta > 0$ is taken small enough. □

Proposition 2.4 *Let $u \in H^2(X_{1,a})$ be such that $r^sPu \in L^2(X_{1,a})$ for $1/2 < s \le (1+\delta_0)/2$. Then, $\forall 0 < \gamma \ll 1$ there exist constants $C_1, C_2, \lambda_0 > 0$ (which may depend on γ but are independent of λ and ε) so that for $\lambda \ge \lambda_0$ we have*

$$\|r^{-s}u\|^2_{H^1(X_{1,a+1})}$$
$$\le C_1\lambda^2\|r^sPu\|^2_{L^2(X_{1,a})} - C_2\lambda^{-1}\mathrm{Im}\,\langle\partial_r u, u\rangle_{L^2(\partial X_{1,a})} + \gamma\|u\|^2_{H^1(X_{1,a}\setminus X_{1,a+1})}. \tag{2.22}$$

Proof. Choose a real-valued function $\phi \in C^\infty(\mathbf{R})$, $0 \le \phi \le 1$, such that $\phi(r) = 0$ for $r \le a + 1/2$, $\phi(r) = 1$ for $r \ge a + 2/3$ and $\phi'(r) \ge 0$, $\forall r$. Integrating by parts we get

$$\langle r^{-2s}(L_r - 1 + V)\phi u, \phi u\rangle_{L^2(X_{1,a})} + \|r^{-s}\mathcal{D}_r(\phi u)\|^2_{L^2(X_{1,a})}$$
$$= \mathrm{Re}\,\langle r^{-2s}P(\phi u), \phi u\rangle_{L^2(X_{1,a})} + 2s\lambda^{-2}\mathrm{Re}\,\langle r^{-2s-1}(\phi u)', \phi u\rangle_{L^2(X_{1,a})},$$

and hence

$$\left|\langle r^{-2s}(L_r - 1 + V)\phi u, \phi u\rangle_{L^2(X_{1,a})} + \|r^{-s}\mathcal{D}_r(\phi u)\|^2_{L^2(X_{1,a})}\right|$$
$$\le O(\lambda)\|P(\phi u)\|^2_{L^2(X_{1,a})} + O(\lambda^{-1})\left(\|r^{-s}\phi u\|^2_{L^2(X_{1,a})} + \|r^{-s}\mathcal{D}_r(\phi u)\|^2_{L^2(X_{1,a})}\right). \tag{2.23}$$

We also have

$$\varepsilon\|u\|^2_{L^2(X_{1,a})} = \mathrm{Im}\,\langle Pu, u\rangle_{L^2(X_{1,a})} - \lambda^{-2}\mathrm{Im}\,\langle u', u\rangle_{L^2(\partial X_{1,a})}$$
$$\le \gamma^{-1}\lambda\|r^sPu\|^2_{L^2(X_{1,a})} + \gamma\lambda^{-1}\|r^{-s}u\|^2_{L^2(X_{1,a})} - \lambda^{-2}\mathrm{Im}\,\langle u', u\rangle_{L^2(\partial X_{1,a})},$$

$\forall\gamma > 0$, and

$$\|\mathcal{D}_r(\phi u)\|^2_{L^2(X_{1,a})} \le 2\|\phi u\|^2_{L^2(X_{1,a})} + \|P(\phi u)\|^2_{L^2(X_{1,a})}$$
$$\le 2\|u\|^2_{L^2(X_{1,a})} + \|Pu\|^2_{L^2(X_{1,a})} + O(\lambda^{-2})\|\phi_1 u\|^2_{H^1(X_{1,a})},$$

where $\phi_1 \in C_0^\infty([a, a+1])$, $\phi_1 = 1$ on $[a + 1/3, a + 3/4]$. Hence,

$$\varepsilon\lambda\left(\|\phi u\|^2_{L^2(X_{1,a})} + \|\mathcal{D}_r(\phi u)\|^2_{L^2(X_{1,a})}\right) \tag{2.24}$$
$$\le O_\gamma(\lambda^2)\|r^sPu\|^2_{L^2(X_{1,a})} + \gamma\|r^{-s}u\|^2_{H^1(X_{1,a})} - 3\lambda^{-1}\mathrm{Im}\,\langle u', u\rangle_{L^2(\partial X_{1,a})},$$

$\forall\gamma > 0$. Set

$$E(r) = -\langle (L_r - 1 + V)\phi u(r,\cdot), \phi u(r,\cdot)\rangle + \|\mathcal{D}_r(\phi u)(r,\cdot)\|^2.$$

We have

$$E'(r) = -\langle [\partial_r, L_r]\phi u(r,\cdot), \phi u(r,\cdot)\rangle - \langle V'\phi u(r,\cdot), \phi u(r,\cdot)\rangle$$
$$-2\varepsilon \mathrm{Im}\,\langle \phi u(r,\cdot), (\phi u)'(r,\cdot)\rangle - 2\lambda \mathrm{Im}\,\langle P(\phi u)(r,\cdot), \mathcal{D}_r(\phi u)(r,\cdot)\rangle$$
$$= -\langle [\partial_r, L_r]\phi u(r,\cdot), \phi u(r,\cdot)\rangle - \langle V'\phi u(r,\cdot), \phi u(r,\cdot)\rangle$$
$$-2\varepsilon \mathrm{Im}\,\langle \phi u(r,\cdot), (\phi u)'(r,\cdot)\rangle - 2\lambda \mathrm{Im}\,\langle \phi P u(r,\cdot), \mathcal{D}_r(\phi u)(r,\cdot)\rangle$$
$$-2\lambda \mathrm{Im}\,\langle [P,\phi]u(r,\cdot), \phi\mathcal{D}_r u(r,\cdot)\rangle - 2\lambda \mathrm{Im}\,\langle [P,\phi]u(r,\cdot), [\mathcal{D}_r,\phi]u(r,\cdot)\rangle.$$

Since

$$[P,\phi] = [\mathcal{D}_r^2,\phi] = -\lambda^{-2}\phi'' - 2i\lambda^{-1}\phi'\mathcal{D}_r,$$

we obtain in view of (2.19),

$$E'(r) \geq \frac{C}{r}\langle L_r(\phi u)(r,\cdot), \phi u(r,\cdot)\rangle - \varepsilon\lambda\left(\|\phi u(r,\cdot)\|^2 + \|\mathcal{D}_r(\phi u)(r,\cdot)\|^2\right)$$
$$-O(\gamma) r^{-2s}\left(\|\phi u(r,\cdot)\|^2 + \|\mathcal{D}_r(\phi u)(r,\cdot)\|^2\right)$$
$$-O(\lambda^{-1})\left(\|\phi_1 u(r,\cdot)\|^2 + \|\phi_1\mathcal{D}_r u(r,\cdot)\|^2\right)$$
$$+4\phi\phi'\|\mathcal{D}_r u(r,\cdot)\|^2 - O_\gamma(\lambda^2) r^{2s}\|Pu(r,\cdot)\|^2.$$

Since $\phi\phi' \geq 0$, we deduce

$$E'(r) \geq \frac{C}{r}\langle L_r(\phi u)(r,\cdot), \phi u(r,\cdot)\rangle - \varepsilon\lambda\left(\|\phi u(r,\cdot)\|^2 + \|\mathcal{D}_r(\phi u)(r,\cdot)\|^2\right)$$
$$-O(\gamma) r^{-2s}\left(\|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2\right) - O_\gamma(\lambda^2) r^{2s}\|Pu(r,\cdot)\|^2. \tag{2.25}$$

Integrating (2.25) from $t \geq a$ to $+\infty$ and using that $L_r \geq 0$ and (2.24), we get

$$E(t) \leq O(\gamma)\|r^{-s}u\|^2_{H^1(X_{1,a})} + O_\gamma(\lambda^2)\|r^s Pu\|^2_{L^2(X_{1,a})} - 3\lambda^{-1}\mathrm{Im}\,\langle u', u\rangle_{L^2(\partial X_{1,a})}, \tag{2.26}$$

$\forall\gamma > 0$. Multiplying (2.26) by t^{-2s} and integrating from a to $+\infty$ yield (with a constant $C > 0$):

$$\int_a^\infty r^{-2s}E(r)dr \leq O(\gamma)\|r^{-s}u\|^2_{H^1(X_{1,a})}$$
$$+O_\gamma(\lambda^2)\|r^s Pu\|^2_{L^2(X_{1,a})} - C\lambda^{-1}\mathrm{Im}\,\langle u', u\rangle_{L^2(\partial X_{1,a})}, \tag{2.27}$$

$\forall\gamma > 0$. On the other hand, multiplying (2.25) by r^{1-2s}, integrating from a to $+\infty$, using (2.23), (2.24) and the identity

$$\int_a^\infty r^{1-2s}E'(r)dr = (2s-1)\int_a^\infty r^{-2s}E(r)dr,$$

we obtain (with a new constant $C > 0$):

$$\|r^{-s}L_r^{1/2}(\phi u)\|^2_{L^2(X_{1,a})} \le O(\gamma)\|r^{-s}u\|^2_{H^1(X_{1,a})}$$
$$+O_\gamma(\lambda^2)\|r^sPu\|^2_{L^2(X_{1,a})} - C\lambda^{-1}\mathrm{Im}\,\langle u',u\rangle_{L^2(\partial X_{1,a})}, \tag{2.28}$$

$\forall\gamma > 0$. Combining (2.23), (2.27) and (2.28), we get (with possibly a new constant $C > 0$):

$$\|r^{-s}\phi u\|^2_{H^1(X_{1,a})} \le O(\gamma)\|r^{-s}u\|^2_{H^1(X_{1,a})}$$
$$+O_\gamma(\lambda^2)\|r^sPu\|^2_{L^2(X_{1,a})} - C\lambda^{-1}\mathrm{Im}\,\langle u',u\rangle_{L^2(\partial X_{1,a})}, \tag{2.29}$$

$\forall 0 < \gamma \ll 1$, which clearly implies (2.22). □

Let $u \in H^2(X_1)$, $u = 0$, $\partial_r u = 0$ on ∂X_1, be such that $r^sPu \in L^2(X_1)$. Choose a function $\chi \in C^\infty(X_1)$ such that $\chi = 1$ on $X_1 \setminus X_{1,a+2}$, $\chi = 0$ on $X_{1,a+3}$. Applying Proposition 2.3 to the function $e^{\lambda\varphi}\chi u$ (with a replaced by $a+3$), we get

$$\|e^{\lambda\varphi}u\|^2_{H^1(X_1\setminus X_{1,a+2})}$$
$$\le O(\lambda^2)\|e^{\lambda\varphi}Pu\|^2_{L^2(X_1\setminus X_{1,a+3})} + O(1)\|e^{\lambda\varphi}u\|^2_{H^1(X_{1,a+2}\setminus X_{1,a+3})}. \tag{2.30}$$

Since $1 \le e^{\lambda(\varphi(r)-\varphi(a))} \le Const$ for $a \le r \le a+3$, we deduce

$$\|e^{\lambda(\varphi(r)-\varphi(a))}u\|^2_{H^1(X_1\setminus X_{1,a})} + \|u\|^2_{H^1(X_{1,a}\setminus X_{1,a+2})}$$
$$\le O(\lambda^2)\|e^{\lambda(\varphi(r)-\varphi(a))}Pu\|^2_{L^2(X_1\setminus X_{1,a})}$$
$$+O(\lambda^2)\|Pu\|^2_{L^2(X_{1,a}\setminus X_{1,a+3})} + O(1)\|u\|^2_{H^1(X_{1,a+2}\setminus X_{1,a+3})}. \tag{2.31}$$

It is easy to see that (2.13) follows from combining (2.22) and (2.31).

3 Uniform a priori estimates on X_2

The purpose of this section is to prove the following

Proposition 3.1 *Let $u \in H^2(X_2, d\mathrm{Vol}_g)$, $u = 0$, $\partial_r u = 0$ on ∂X_2. Then $\forall\delta > 0$, $0 < \varepsilon \le 1$, we have*

$$\|r^{-1-\delta}e^{\lambda r^{-2\delta}}u\|_{H^1(X_2,d\mathrm{Vol}_g)} \le C\lambda^{-3/2}\|e^{\lambda r^{-2\delta}}(\Delta_{X_2} - \lambda^2 + i\varepsilon)u\|_{L^2(X_2,d\mathrm{Vol}_g)}, \tag{3.1}$$

for $\lambda \ge \lambda_0$ with constants $C, \lambda_0 > 0$ independent of λ, ε and u but depending on δ.

Proof. Define the spaces $L^2(X_2)$ and $H^1(X_2)$ analogously to $L^2(X_1)$ and $H^1(X_1)$ introduced in the previous section. Denote $\varphi(r) = r^{-2\delta}$, $w = e^{\lambda\varphi}u$, and

$$P := p_2^{1/2}(\lambda^{-2}\Delta_{X_2} - 1 + i\varepsilon)p_2^{-1/2} = \mathcal{D}_r^2 + L_r - 1 + V + i\varepsilon,$$
$$P_\varphi = e^{\lambda\varphi}Pe^{-\lambda\varphi} = P - \varphi'(r)^2 + \lambda^{-1}\varphi''(r) + 2i\varphi'(r)\mathcal{D}_r,$$

where $0 < \varepsilon = O(\lambda^{-2})$, $L_r = \lambda^{-2}\Lambda_{2,r}$, $V = \lambda^{-2}q_2$. Note that (1.3) implies

$$[\partial_r, L_r] \geq \frac{C}{r} L_r, \quad C > 0. \tag{3.2}$$

Clearly, (3.1) is equivalent to the estimate

$$\|r^{-1-\delta}w\|_{H^1(X_2)} \leq O(\lambda^{1/2})\|P_\varphi w\|_{L^2(X_2)}. \tag{3.3}$$

Denote by P_φ^* the adjoint operator of P_φ with respect to the scalar product in $L^2(X_2)$, and set $\operatorname{Re} P_\varphi = \frac{P_\varphi + P_\varphi^*}{2}$, $\operatorname{Im} P_\varphi = \frac{P_\varphi - P_\varphi^*}{2i}$. We have

$$\operatorname{Re} P_\varphi = \mathcal{D}_r^2 + L_r - 1 - \varphi'(r)^2 + V, \quad \operatorname{Im} P_\varphi = \varphi'(r)\mathcal{D}_r + \mathcal{D}_r\varphi'(r) + \varepsilon.$$

In view of (1.2) and (3.2), and taking into account that

$$\varphi'(r) = -2\delta r^{-2\delta-1}, \ \varphi''(r) = 2\delta(2\delta{+}1)r^{-2\delta-2}, \ \varphi'''(r) = -2\delta(2\delta{+}1)(2\delta{+}2)r^{-2\delta-3},$$

it is easy to see that we have, in view of (3.2) and (1.2),

$$\begin{aligned}
&\lambda\|P_\varphi w\|^2_{L^2(X_2)} \\
&= \lambda\|(\operatorname{Re} P_\varphi)w\|^2_{L^2(X_2)} + \lambda\|(\operatorname{Im} P_\varphi)w\|^2_{L^2(X_2)} + i\lambda\langle[\operatorname{Re} P_\varphi, \operatorname{Im} P_\varphi]w, w\rangle_{L^2(X_2)} \\
&\geq i\lambda\langle[\operatorname{Re} P_\varphi, \operatorname{Im} P_\varphi]w, w\rangle_{L^2(X_2)} \geq 2\langle\varphi''\mathcal{D}_r w, \mathcal{D}_r w\rangle_{L^2(X_2)} \\
&\qquad + 4\langle -\varphi'[\partial_r, L_r]w, w\rangle_{L^2(X_2)} + 4\langle\varphi'^2\varphi'' w, w\rangle_{L^2(X_2)} - 2\langle\varphi' V' w, w\rangle_{L^2(X_2)} \\
&\qquad - O(\lambda^{-1})\left(\|r^{-1-\delta}\mathcal{D}_r w\|_{L^2(X_2)} + \|r^{-1-\delta}w\|_{L^2(X_2)}\right) \\
&\geq C\|r^{-1-\delta}\mathcal{D}_r w\|^2_{L^2(X_2)} + C\|r^{-1-\delta}L_r^{1/2}w\|^2_{L^2(X_2)} - O(\lambda^{-1})\|r^{-1-\delta}w\|^2_{H^1(X_2)}.
\end{aligned} \tag{3.4}$$

On the other hand, integrating by parts leads to the identity

$$\operatorname{Re}\langle r^{-2-2\delta}P_\varphi w, w\rangle_{L^2(X_2)} = \|r^{-1-\delta}\mathcal{D}_r w\|^2_{L^2(X_2)}$$

$$+\langle r^{-2-2\delta}(L_r{-}1{+}V{-}\varphi'^2{-}4\delta(\delta{+}1)\lambda^{-1}r^{-2-2\delta}{-}(\delta{+}1)(2\delta{+}3)\lambda^{-2}r^{-2})w, w\rangle_{L^2(X_2)},$$

and hence

$$\begin{aligned}
&\frac{1}{2}\|r^{-1-\delta}w\|^2_{L^2(X_2)} \\
&\leq \|r^{-1-\delta}\mathcal{D}_r w\|^2_{L^2(X_2)} + \|r^{-1-\delta}L_r^{1/2}w\|^2_{L^2(X_2)} + \left|\langle r^{-2-2\delta}P_\varphi w, w\rangle_{L^2(X_2)}\right|.
\end{aligned}$$

Since

$$\left|\langle r^{-2-2\delta}P_\varphi w, w\rangle_{L^2(X_2)}\right| \leq \frac{1}{4}\|r^{-1-\delta}w\|^2_{L^2(X_2)} + \|P_\varphi w\|^2_{L^2(X_2)},$$

we conclude

$$\frac{1}{4}\|r^{-1-\delta}w\|^2_{L^2(X_2)} \leq \|r^{-1-\delta}\mathcal{D}_r w\|^2_{L^2(X_2)} + \|r^{-1-\delta}L_r^{1/2}w\|^2_{L^2(X_2)} + \|P_\varphi w\|^2_{L^2(X_2)}. \tag{3.5}$$

Now (3.3) follows from (3.4) and (3.5). □

4 Proof of Theorem 1.1

Let (M_0, g_0) be a compact, connected Riemannian manifold with a C^∞-smooth boundary ∂M_0 and a metric g_0 of class $C^\infty(\overline{M}_0)$. Denote by Δ_{M_0} the (positive) Laplace-Beltrami operator on (M_0, g_0) and let $U \subset M_0$, $U \neq \emptyset$, be an arbitrary open domain such that $\partial U \cap \partial M_0 = \emptyset$. Suppose that $\partial M_0 = \Gamma \cup \widetilde{\Gamma}$, $\Gamma \neq \emptyset$, $\widetilde{\Gamma} \neq \emptyset$, $\Gamma \cap \widetilde{\Gamma} = \emptyset$, and given $0 < \varepsilon_0 \ll 1$ denote $M_{0,\varepsilon_0} = M_0 \setminus \{x \in M_0 : \mathrm{dist}(x, \partial M_0) \leq \varepsilon_0\}$, $\widetilde{M}_{0,\varepsilon_0} = M_0 \setminus \{x \in M_0 : \mathrm{dist}(x, \widetilde{\Gamma}) \leq \varepsilon_0\}$. Let $U \subset M_{0,2\varepsilon_0}$. The following proposition is proved in [8] using the interpolation inequalities of Lebeau-Robbiano [4], [5] (see Theorem 3.2 of [8]) and this is why we omit the proof.

Proposition 4.1 *Let $u \in H^2(M_0)$ satisfy either Dirichlet or Neumann boundary conditions on Γ. Then, $\forall \beta > 0$ $\exists C_\beta, \gamma_\beta > 0$ (independent of u and λ below but depending on U) so that we have*

$$\|u\|_{H^1(\widetilde{M}_{0,\varepsilon_0})} \leq C_\beta e^{\gamma_\beta|\lambda|} \|(\Delta_{M_0} - \lambda^2)u\|_{L^2(M_0)}$$

$$+C_\beta e^{\gamma_\beta|\lambda|} \|u\|_{H^1(U)} + e^{-\beta|\lambda|} \|u\|_{H^1(M_0 \setminus \widetilde{M}_{0,\varepsilon_0})}, \quad \lambda \in \mathbf{C},\ |\lambda| \gg 1. \tag{4.1}$$

Let $u \in D(G)$ be such that $\chi_{s_1,s_2}^{-1} u \in L^2(M, d\mathrm{Vol}_g)$, where s_1 and s_2 are as in Theorem 1.1. Let $\chi_2 \in C^\infty(\overline{M})$, $\chi_2 = 0$ on $M \setminus X_{2,r_2+1}$, $\chi_2 = 1$ on X_{2,r_2+2}. Applying Proposition 3.1 (with $\delta = s_2 - 1$) to $\chi_2 u$ yields

$$\|r^{-s_2} u\|^2_{H^1(X_{2,r_2+2}, d\mathrm{Vol}_g)} \leq e^{c_0\lambda} \|(\Delta_g - \lambda^2 + i\varepsilon)u\|^2_{L^2(X_2, d\mathrm{Vol}_g)}$$

$$+e^{c_0\lambda} \|u\|^2_{H^1(X_{2,r_2+1} \setminus X_{2,r_2+2}, d\mathrm{Vol}_g)}. \tag{4.2}$$

Let $\chi_1 \in C^\infty(\overline{M})$, $\chi_1 = 1$ on $M \setminus X_{1,r_1+2}$, $\chi_1 = 0$ on X_{1,r_1+3}. By Proposition 4.1 applied to the function $\chi_1 u$ we get

$$\|\chi_1 u\|^2_{H^1(M \setminus X_{2,r_2+2}, d\mathrm{Vol}_g)} \leq C_\beta e^{\gamma_\beta\lambda} \|(\Delta_g - \lambda^2 + i\varepsilon)\chi_1 u\|^2_{L^2(M \setminus X_{2,r_2+3}, d\mathrm{Vol}_g)}$$

$$+e^{-\beta\lambda} \|u\|^2_{H^1(X_{2,r_2+2} \setminus X_{2,r_2+3}, d\mathrm{Vol}_g)}, \tag{4.3}$$

$\forall \beta > 0$ with $C_\beta, \gamma_\beta > 0$ independent of λ, ε and u. Hence,

$$\|u\|^2_{H^1(M \setminus (X_{1,r_1+2} \cup X_{2,r_2+2}), d\mathrm{Vol}_g)}$$

$$\leq C_\beta e^{\gamma_\beta\lambda} \|(\Delta_g - \lambda^2 + i\varepsilon)u\|^2_{L^2(M \setminus (X_{1,r_1+3} \cup X_{2,r_2+3}), d\mathrm{Vol}_g)}$$

$$+C'_\beta \lambda^2 e^{\gamma_\beta\lambda} \|u\|^2_{H^1(X_{1,r_1+2} \setminus X_{1,r_1+3}, d\mathrm{Vol}_g)} + e^{-\beta\lambda} \|u\|^2_{H^1(X_{2,r_2+2} \setminus X_{2,r_2+3}, d\mathrm{Vol}_g)}, \tag{4.4}$$

$\forall \beta > 0$.

Combining (4.2) and (4.4), for $\lambda \gg 1$, we obtain

$$\begin{aligned} \|r^{-s_2}u\|^2_{H^1(X_{2,r_2+2},d\mathrm{Vol}_g)} + \|u\|^2_{H^1(M\setminus(X_{1,r_1+2}\cup X_{2,r_2+2}),d\mathrm{Vol}_g)} \\ \le e^{c_0\lambda}\|(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(X_2,d\mathrm{Vol}_g)} \\ +e^{2\gamma_1\lambda}\|(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(M\setminus(X_{1,r_1+3}\cup X_{2,r_2+3}),d\mathrm{Vol}_g)} \\ +e^{2\gamma_1\lambda}\|u\|^2_{H^1(X_{1,r_1+2}\setminus X_{1,r_1+3},d\mathrm{Vol}_g)}, \end{aligned} \tag{4.5}$$

with a constant $\gamma_1 > 0$ independent of λ, ε and u. Let $r_1 < b_1 < b_2 < r_1 + 1$ be such that $\varphi(b_1) < \varphi(b_2) < 0$ and choose $\widetilde{\chi}_1 \in C^\infty(\overline{M})$, $\widetilde{\chi}_1 = 0$ on $M \setminus X_{1,b_1}$, $\widetilde{\chi}_1 = 1$ on X_{1,b_2}. By Theorem 2.2 applied to $\widetilde{\chi}_1 u$ (with $\gamma_0 = \gamma_1 + 1$, $s = s_1$), we get

$$\begin{aligned} \|e^{\lambda\varphi}u\|^2_{H^1(X_{1,b_2}\setminus X_{1,a},d\mathrm{Vol}_g)} + e^{2\lambda\varphi(a)}\|r^{-s_1}u\|^2_{H^1(X_{1,a},d\mathrm{Vol}_g)} \\ \le O(\lambda^{-2})\|e^{\lambda\varphi}(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(X_1\setminus X_{1,a},d\mathrm{Vol}_g)} \\ +O(\lambda^{-2})e^{2\lambda\varphi(a)}\|r^{s_1}(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(X_{1,a},d\mathrm{Vol}_g)} \\ -C\lambda^{-1}e^{2\lambda\varphi(a)}\mathrm{Im}\,\langle\partial_r u,u\rangle_{L^2(\partial X_{1,a})} + e^{-c\lambda}\|u\|^2_{H^1(X_{1,b_1}\setminus X_{1,b_2},d\mathrm{Vol}_g)}, \end{aligned} \tag{4.6}$$

with some $c, C > 0$. Since $\varphi(r) \ge \gamma_1 + 1$ for $r \ge r_1 + 2$, by combining (4.5) and (4.6) one can absorb the last terms in the right-hand sides and conclude

$$\begin{aligned} \|r^{-s_2}u\|^2_{H^1(X_{2,r_2+2},d\mathrm{Vol}_g)} + \|u\|^2_{H^1(M\setminus(X_{1,r_1+2}\cup X_{2,r_2+2}),d\mathrm{Vol}_g)} \\ +\|e^{\lambda\varphi}u\|^2_{H^1(X_{1,b_2}\setminus X_{1,a},d\mathrm{Vol}_g)} + e^{2\lambda\varphi(a)}\|r^{-s_1}u\|^2_{H^1(X_{1,a},d\mathrm{Vol}_g)} \\ \le e^{c_0\lambda}\|(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(X_2,d\mathrm{Vol}_g)} \\ +e^{2\gamma_1\lambda}\|(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(M\setminus(X_{1,r_1+3}\cup X_{2,r_2+3}),d\mathrm{Vol}_g)} \\ +O(\lambda^{-2})\|e^{\lambda\varphi}(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(X_1\setminus X_{1,a},d\mathrm{Vol}_g)} \\ +O(\lambda^{-2})e^{2\lambda\varphi(a)}\|r^{s_1}(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(X_{1,a},d\mathrm{Vol}_g)} \\ -C\lambda^{-1}e^{2\lambda\varphi(a)}\mathrm{Im}\,\langle\partial_r u,u\rangle_{L^2(\partial X_{1,a})}. \end{aligned} \tag{4.7}$$

On the other hand, by Green's formula we have

$$\begin{aligned} &-\mathrm{Im}\,\langle\partial_r u,u\rangle_{L^2(\partial X_{1,a})} \\ &= -\mathrm{Im}\,\langle(\Delta_g-\lambda^2+i\varepsilon)u,u\rangle_{L^2(M\setminus X_{1,a},d\mathrm{Vol}_g)} - \varepsilon\|u\|^2_{L^2(M\setminus X_{1,a},d\mathrm{Vol}_g)} \\ &\le e^{-\beta\lambda}\|\rho_{s_2}u\|^2_{L^2(M\setminus X_{1,a},d\mathrm{Vol}_g)} + e^{\beta\lambda}\|\rho_{s_2}^{-1}(\Delta_g-\lambda^2+i\varepsilon)u\|^2_{L^2(M\setminus X_{1,a},d\mathrm{Vol}_g)}, \end{aligned} \tag{4.8}$$

$\forall\beta > 0$, where $\rho_s \in C^\infty(M)$, $\rho_s = r^{-s}$ on X_{2,r_2+1}, $\rho_s = 1$ on $M \setminus X_2$.

Combining (4.7) and (4.8) leads to the estimate

$$\begin{aligned} e^{-c_1\lambda}\|\rho_{s_2}u\|^2_{H^1(M\setminus X_{1,a},d\mathrm{Vol}_g)} + \|r^{-s_1}u\|^2_{H^1(X_{1,a},d\mathrm{Vol}_g)} \\ \le e^{c_2\lambda}\|\rho_{s_2}^{-1}(\Delta_g - \lambda^2 + i\varepsilon)u\|^2_{L^2(M\setminus X_{1,a},d\mathrm{Vol}_g)} \\ +O(\lambda^{-2})\|r^{s_1}(\Delta_g - \lambda^2 + i\varepsilon)u\|^2_{L^2(X_{1,a},d\mathrm{Vol}_g)}, \end{aligned} \tag{4.9}$$

with some constants $c_1, c_2 > 0$. Hence,

$$\|\chi_{s_1,s_2}u\|_{L^2(M,d\mathrm{Vol}_g)} \le Ce^{\gamma\lambda}\|\chi^{-1}_{s_1,s_2}(\Delta_g - \lambda^2 + i\varepsilon)u\|_{L^2(M,d\mathrm{Vol}_g)} \tag{4.10}$$

for $\lambda \ge \lambda_0$, with some constants $C, \lambda_0, \gamma > 0$ independent of λ, ε and u, which implies the existence of the limit in Theorem 1.1 as well as the bound (1.4) (with $z = \lambda^2$).

Let now $(\Delta_g - \lambda^2 + i\varepsilon)u = 0$ in $M \setminus X_{1,a}$. Then (4.9) yields

$$\|r^{-s_1}u\|_{L^2(X_{1,a},d\mathrm{Vol}_g)} \le O(\lambda^{-1})\|r^{s_1}(\Delta_g - \lambda^2 + i\varepsilon)u\|_{L^2(X_{1,a},d\mathrm{Vol}_g)}, \tag{4.11}$$

which clearly implies (1.5).

References

[1] V. Bruneau and V. Petkov, Semiclassical resolvent estimates for trapping perturbations, *Commun. Math. Phys.* **213**, 413–432 (2000).

[2] N. Burq, Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel, *Acta Math.* **180**, 1–29 (1998).

[3] N. Burq, Lower bounds for shape resonances widths of long-range Schrödinger operators, *American J. Math.*, to appear.

[4] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur, *Commun. Partial Diff. Equations* **20**, 335–356 (1995).

[5] G. Lebeau and L. Robbiano, Stabilization de l'équation des ondes par le bord, *Duke Math. J.* **86**, 465–490 (1997).

[6] G. Vodev, On the exponential bound of the cutoff resolvent, *Serdica Math. J.* **26**, 49–58 (2000).

[7] G. Vodev, Exponential bounds of the resolvent for a class of noncompactly supported perturbations of the Laplacian, *Math. Res. Lett.* **7**, 287–298 (2000).

[8] G. Vodev, Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds with cusps, *Commun. Partial Diff. Equations*, **27**, 1437–1465 (2002).

Fernando Cardoso[1]
Universidade Federal de Pernambuco
Departamento de Matemàtica
CEP. 50540-740 Recife-Pe
Brazil
email: fernando@dmat.ufpe.br

Georgi Vodev
Université de Nantes
Département de Mathématiques
UMR 6629 du CNRS
2, rue de la Houssinière, BP 92208,
44072 Nantes Cedex 03
France
email: vodev@math.univ-nantes.fr

Communicated by Bernard Helffer
submitted 13/11/01, accepted 14/05/02

[1]Partially supported by CNPq (Brazil)