

Algebraic Structure of n -Body Systems

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Abstract. A general method to easily build global and relative operators for any number n of elementary systems if they are defined for 2 is presented. It is based on properties of the morphisms valued in the tensor products of algebras of the kinematics and it allows also the generalization to any n of relations demonstrated for two. The coalgebra structures play a peculiar role in the explicit constructions. Three examples are presented concerning the Galilei, Poincaré and deformed Galilei algebras.

1 Introduction

It has been recently found that the renormalization procedure in Quantum Field Theory is intrinsically determined by an Hopf algebra whose essential constituent has been soon recognized to be the set of the parameters of the classical group of Virasoro with their law of composition [1]. The presence and the utility of basic algebraic concepts even in such an elementary problem as the search of the so called “relative variables” in classical and quantum mechanics will be illustrated here. In this note indeed we introduce a method, whose use is based on the coalgebraic structure of canonical commutation relations, which allows for the explicit construction of the collective, *i.e.* global and relative, canonical operators for any number n of elementary systems in a given kinematics once one has been able to operate this transformation for $n = 2$. It is also shown that there are classes of relations between single system operators and the collective ones that once they hold for two of them, then they are straightforward extended to any n . This is done by using the same algorithm which generates the transformation of the operators. The construction is *a priori* possible in any situation in which operators are constructed in terms of single algebra generators and the separation of the “global operators” is necessary. The presentation is self explanatory, very heuristic and constructive. We tacitly suppose the existence of any object necessary for the results.

The examples are thus essential, not only to illustrate the physical utility of the method but also to show that it is rather flexible and not mathematically void. In section 1 we give the general definitions and results. In sections 2, 3 we present examples from usual Galilei and Poincaré kinematics while useful applications are devised for a quantum algebra too in 4. We use always $1d$ algebras, thus avoiding the rotations whose consideration is not essential in exemplifying the method. In 5 there are some concluding remarks.

2 Collective operators

Let us consider an algebra A , whose elements can be represented as hermitian operators on an Hilbert space, endowed with algebra morphism $\Gamma : A \rightarrow A \otimes A$,

$$\begin{aligned} \Gamma a &= \Sigma(a_1)_i \otimes (a_2)_i \quad \forall a \in A, \quad (a_1)_i, (a_2)_i \in A, \\ \Gamma(ab) &= \Gamma a \Gamma b, \quad \forall a, b \in A, \end{aligned} \tag{2.1}$$

With such a morphism Γ one can combine the two operators of the single systems in *collective* ones, we call *global*. The set of operators acting on the same space must be completed to conserve the number of the operators of the original set of the single systems, (such sets include too the Poisson algebra of functions on a symplectic manifold).

This involves the introduction of the aforementioned *relative* operators which complement the global ones in the set of the collective operators.

We thus introduce, by assuming it exists, another homomorphic map δ on A :

$$\delta : A \rightarrow A \otimes A, \quad \delta(ab) = \delta a \delta b, \quad \forall a, b \in A, \quad \Gamma A \oplus \delta A = A \otimes 1 \oplus 1 \otimes A \tag{2.2}$$

Moreover we impose the commutation property

$$\Gamma a \delta b - \delta b \Gamma a \doteq [\Gamma a, \delta b] = 0 \quad \forall a, b \in A \tag{2.3}$$

so that Γ and δ implement exactly a transformation we may call canonical. If the generators of A satisfy canonical relations we may call canonical the operators recovered by the transformation.

As a direct consequence of the more general result below it is possible to produce all the collective canonical operators for n -body by using only Γ and δ and the right tensor multiplication $\otimes 1$.

Indeed let us given n morphisms, $\Gamma_j : A \rightarrow A \otimes A, (j = 0, \dots, n - 1)$, not necessarily different, and a morphism δ satisfying (2.2) and (2.3), with $\Gamma = \Gamma_j, \forall j \in (0, n - 1)$.

Let us now consider the following expressions:

$$\begin{aligned} &\delta a \otimes 1^{\otimes(n-2)}, \\ &\Gamma^{(n-1)} \delta a \otimes 1^{\otimes(n-3)}, \\ &\Gamma^{(n-1)} \Gamma^{(n-2)} \delta a \otimes 1^{\otimes(n-4)}, \\ &\dots\dots\dots \\ &\Gamma^{(n-1)} \dots\dots\dots \Gamma^{(1)} \delta a, \\ &\Gamma^{(n-1)} \dots\dots\dots \Gamma^{(1)} \Gamma^{(0)} a \end{aligned} \tag{2.4}$$

where the following notations have been introduced:

$\Gamma^{(0)} = \Gamma_{\sigma(0)}, \Gamma^{(j)} = \Gamma_{\sigma(j)} \otimes id^{\otimes(j)}$ with $x^{\otimes(j)} \equiv x \otimes \dots \otimes x, (j \text{ factors } x)$, and σ is any permutation of $i, (i = 0, \dots, n - 1)$.

The following *proposition* then holds:

“Each one of the previous expressions realizes the algebra A in the n -fold tensor product $A^{\otimes(n)}$. Moreover the n realizations are all mutually commuting.”

The demonstration is by recurrence. So let us suppose the proposition is true for n and apply to the left of each one of the n previous expressions 2.4 the map $\Gamma^{(n)}$. This map is a morphism $A^{\otimes(n)} \rightarrow A^{\otimes(n+1)}$ so that the new expressions satisfy the same algebraic relations as the previous ones. Let us now complete the set with the new expression $\delta a \otimes 1^{\otimes(n-1)}$: to end the proof it must be shown that it commutes with the n new expressions. By construction the last ones are n -multilinear sums of elements of the form $(\Gamma_{\sigma(n)} a_{(1)}) \otimes a_{(2)} \dots \otimes a_{(n)}$, $a_{(k)} \in A$, $k \in (1, n)$.

The commutators to evaluate are sums of elements:

$$\left[(\Gamma_{\sigma(n)} a_{(1)}) \otimes a_{(2)} \dots \otimes a_{(n)}, \delta a \otimes 1^{\otimes(n-1)} \right] = [\Gamma_{\sigma(n)} a_{(1)}, \delta a] \otimes a_{(2)} \dots \otimes a_{(n)} = 0.$$

The proposition being true for $n = 2$ is thus demonstrated for any n .

If one deals, as it is very probable, with only one morphism $\Gamma = \Gamma_j$, $\forall j \in (0, n-1)$ then the (2.4) give a straightforward procedure to generate the n -body expressions from the two-body ones.

By the way in this case the proposition holds even if we change everywhere in (2.4) Γ with δ .

Let us show now a general implication of the use of algorithmic definitions in generating relations for n -body once they hold for 2.

So suppose that for 2 peculiar elements $a, b \in A$ (a may be equal or not to b) it exists a function R on $A \otimes A \times A \otimes A \times A \otimes A$ with values in $A \otimes A$, which can be extended on the direct product of growing tensor powers of A , which makes explicit a relation between global and the relative operators built on a and b in the form, *e.g.*:

$$\Gamma^{(0)} a = R(a_1, a_2, \delta b) \tag{2.5}$$

where we have introduced the notation:

$$z_i = 1 \otimes 1 \otimes 1 \dots \otimes z \otimes 1 \otimes \dots \dots \otimes 1,$$

with z acting on the i -th space.

Let us now apply to both sides of (2.5) one time the right multiplication $\otimes 1$ and another time the operation from the left $\Gamma^{(1)}$. We thus get by exploiting identities like $f(a \otimes 1) = f(a) \otimes 1$, with $a, f(a) \in A^{\otimes(j)}$, j integer :

$$\Gamma^{(0)} a \otimes 1 = R(a_1, a_2, \delta b \otimes 1), \quad \Gamma^{(1)} \Gamma^{(0)} a = R(\Gamma^{(0)} a \otimes 1, a_3, \Gamma^{(1)} \delta b)$$

where now a_1, a_2 must be read as $a \otimes 1 \otimes 1, 1 \otimes a \otimes 1$ and R is valued in $A \otimes A \otimes A$. The explicit relation between the 3-body collective operators and the 3 single body ones is therefore:

$$\Gamma^{(1)} \Gamma^{(0)} a = R(R(a_1, a_2, \delta b \otimes 1), a_3, \Gamma^{(1)} \delta b)$$

It is now straightforward to find, by iterating $n - 2$ times the two previous operations, that for the n -body the following implicit relation between all the single and all the collective operators derived from a and b (the tensor product domain and codomain of R is extended at each step) holds:

$$\Gamma^{(n-2)} \dots \Gamma^{(1)} \Gamma^{(0)} a = R(\Gamma^{(n-3)} \dots \Gamma^{(1)} \Gamma^{(0)} a \otimes 1, a_n, \Gamma^{(n-2)} \dots \Gamma^{(1)} \delta b), \quad (2.6)$$

and the general solution in the computable form of a recursive function is:

$$\begin{aligned} &\Gamma^{(n-2)} \dots \Gamma^{(1)} \Gamma^{(0)} a \\ &= R\left(R\left(\dots\left(R\left(a_1, a_2, \delta b \otimes 1^{\otimes(n-2)}\right), a_3, \Gamma^{(1)} \delta b \otimes 1^{\otimes(n-3)}\right), \dots\right), a_n, \Gamma^{(n-2)} \dots \Gamma^{(1)} \delta b\right) \end{aligned} \quad (2.7)$$

which eventually recovers an explicit expression by taking into account the concrete form of the 2-body initial relation (2.5). Let us remark now that a could be any expressions of the generators so that there can be interesting cases in which R is simply a primitive recursive function. Moreover the demonstration deals only with the elements a, b and those expressions derived from them by using Γ, δ so that (2.7) could hold even if Γ, δ don't fulfil their defining properties on all A .

As is well known a morphism $\Delta A \rightarrow A \otimes A$ is called *coproduct* when the *coassociativity* holds:

$$(\Delta \otimes id)\Delta a = (id \otimes \Delta)\Delta a, \quad \forall a \in A. \quad (2.8)$$

In this case A is a *coalgebra*, all the Lie and quantum algebras stay in this category. A property to notice in this context is that owing to the coassociativity (2.8) the action of Δ can be univocally iterated to any $A^{\otimes(n)}$. There are also algebras where the coassociativity is fulfilled only modulo some equivalence: the *quasi-coalgebras*, and the quasi-coassociative morphism is the *quasi-coproduct*. When we deal with n representations (*rps* from now on) of A we can, by means of the map Δ and id recover a set of *global* operators on the product space satisfying exactly the original algebra of the single components L_a , independently of the order of Δ and id . In any Lie algebra the coproduct simply reads in algebraic terms:

$$\Delta L_a = L_a \otimes 1 + 1 \otimes L_a.$$

When there is a basis of an algebra A in which Δ gets this form it is called a *primitive* coproduct. If Δ is invariant after the interchange of the two base spaces in the tensor product it is called *cocommutative*, any element built in terms of the generators of a Lie algebra clearly shares this property. The “barycenter formulas” of the classical kinematics are tied to the canonical coassociative coproduct. Thus the starting point in the research of the collective operators must be, if it exists, the coproduct. But one cannot find in general a δ satisfying (2.3) with $\Gamma = \Delta$; clear examples are given by semisimple Lie algebras. A near solution to this problem could exist *e.g.* for nonsemisimple Lie algebras with non null first

class of cohomology, where an arbitrary scalar variation can be given to the action of the global morphism on some generators and a quasi-coassociative Γ results. Sometimes we can thus satisfy (2.3), at the price however of the non univocity of some global operators. Actually to proceed with $\Gamma \neq \Delta$ seems physically reliable only when one deals with global operators of no direct physical meaning. Anyway it must be remarked again that the collective set is completely defined once the morphisms Γ_j, δ , whichever they are, have been done. This will be illustrated from the three examples we present in the next sections which share different degrees of complexity.

3 The Galilei Algebra

Despite its ubiquitous presence in the contemporary Physics, as symmetry of the non relativistic Q.M., the literature on the Galilei group is not huge, and even in general presentations [2] the space devoted to collective coordinates is not large. Moreover in last times physical results and researches mainly concerned the classical and quantum statistical mechanics and the field theory implications of the Galilean invariance [3, 4, 5, 6]. Therefore it maybe that an extensive treatment of the collective position operators in 2-body Galilei kinematics must be searched yet in [7]. Thus it will be instructive to apply firstly our method to the $1d$ Galilei group. The mass is chosen to be a Lie generator, this implies the use of non projective representations with the advantages that the Galilean symmetry is seen from the physicist viewpoint, see *e.g.* [8], and that this is the form necessary to obtain the deformed version [9].

We start thus with the 3 generators Lie algebra $gh(1)$:

$$[B, P] = iM, \quad [M, B] = [M, P] = 0; \quad (3.9)$$

It is the algebra of the purely spatial $1d$ extended Galilean transformations where B is the boost, P the momentum and the central generator M is the mass.

If one defines, by exploiting the localization with respect to the center, the position generator $X = B/M$ one gets:

$$[X, P] = i1, \quad [M, X] = [M, P] = 0; \quad (3.10)$$

where 1 is the identity element of the enveloping algebra $U(gh(1))$.

The first commutator of (3.10) define a couple of Heisenberg canonical operators. But the Lie coalgebraic structure in the Heisenberg commutator is not compatible with X primitive if P is primitive owing to $\Delta 1 \doteq 1 \otimes 1$. Indeed once the momenta have been summed the corresponding positions must be linearly combined with arbitrary coefficients whose sum is 1.

This is recovered by exploiting the algebraic status of M. In fact in the Lie algebra (3.9) the coproduct amounts simply to:

$$\Delta P = P_1 + P_2, \quad \Delta B = B_1 + B_2, \quad \Delta M = M_1 + M_2 . \quad (3.11)$$

Consequently one has for X:

$$\Delta X = \Delta B / \Delta M = (M_1 X_1 + X_2 M_2) / (M_1 + M_2) . \tag{3.12}$$

It is therefore very sensible to think in this case to the Heisenberg canonical set as a coalgebra with three generators.

A good well known map δ is given by

$$\begin{aligned} \delta P &= (P \otimes M - M \otimes P) / (\Delta M) = \frac{P_1 M_2 - M_1 P_2}{M_1 + M_2}, \\ \delta X &= X \otimes 1 - 1 \otimes X = X_1 - X_2, \\ \delta M &= M \otimes M / (\Delta M) = \frac{M_1 M_2}{M_1 + M_2}, \end{aligned} \tag{3.13}$$

The expressions (2.4), with $\Gamma_j = \Delta$, are then the usual canonical Jacobi coordinates.

Anyway the algebra (3.9) is a sub-algebra of the full Galilei Lie algebra g one gets by adding a fourth Lie generator E , the energy, whose non zero commutator is:

$$[B, E] = iP .$$

The center of g is generated besides M even by the quadratic Casimir $C = 2ME - P^2$. It is now obvious that the coproduct of the energy cannot commute with all the relative operators. But the generator E cannot be written as a commutator and we can put $\Gamma E \neq \Delta E$. Thus the set of collective operators can be completed by introducing a *global energy* ΓE and a *relative energy* δE . They can be found by imposing that $2\Gamma M \Gamma E - \Gamma P^2$ and $2\delta M \delta E - \delta P^2$ are Casimir. The result is:

$$\Gamma E = \frac{M_1 E_1 + M_2 E_2 + P_1 P_2}{M_1 + M_2} \tag{3.14}$$

and

$$\begin{aligned} \delta E &= (M \otimes E + E \otimes M) - P \otimes P / (\Delta M) \\ &= \frac{M_1 E_2 + M_2 E_1 - P_1 P_2}{M_1 + M_2} = E_1 + E_2 - \Gamma E . \end{aligned} \tag{3.15}$$

The definition of ΓE is anyway coassociative modulo global Galilei invariant operators.

The application of (2.4) gives the expressions for any n . Let us notice that (3.15) is in a form where the general recursive formula (2.7) is trivially explicitated so that we have for any n :

$$\begin{aligned} \Sigma E_j &= \Gamma^{(n-2)} \dots \Gamma^{(0)} E + \Gamma^{(n-3)} \dots \Gamma^{(1)} \delta E \otimes 1 + \\ &\dots + \Gamma^{(1)} \delta E \otimes 1^{\otimes(n-3)} + \delta E \otimes 1^{\otimes(n-2)} \end{aligned} \tag{3.16}$$

Let us observe also that by choosing $a = b = P^2 / (2M)$ and then recovering from the 2-body that $R(x, y, z) = x + y - z$ the formula (2.7) gives immediately that the

sum of the n single kinetic energies transforms in the identical formal expression in terms of the collective Jacobi set. It must be remarked that the map δ (3.13), doesn't satisfy coassociativity nor it is a *coaction*, (*i.e.*: $(\Delta \otimes id)\delta a$ is not equal to $(id \otimes \delta)\delta a, \forall a \in A$). Indeed it is just the initial support for the action of the globalizing and injecting operations. It can be shown by direct calculation that there is no coassociative δ producing all the previous properties in the Galilei algebra. Of course one can introduce functions of the masses as factors in the definition of δ , as, *e.g.*, in the analysis of the $1d$ integrable many-body Schrödinger equation by McGuire [10]. A δ actually quasi-coassociative can be obtained in this way, with a lack of completeness however as relative and global masses happen to be the same.

It is worth noticing that the analogous coproduct and the same role of the mass hold in the three-dimensional situation, where the Heisenberg set can be derived again by a sub-algebra of the extended Galilei. In this case the expressions of the 2-body collective operators can be much more composite, following the dynamical problems one has to face. But, as shown before, once the collective expressions have been found for 2 the algorithm to give expressions for n is straightforward.

4 The Poincaré Algebra

The proposals about the localization and the canonical operators of the position in special relativity are not univocal, see *e.g.* [11] and references therein. We adopt here the one, firstly studied in [7], based on the Weyl algebra, analyzed and exploited in [12] where the hamiltonian dynamics of 1 and 2 scalar or spinning relativistic particles was written. Coulomb and Schwartzschild type 2-body interactions were covariantly introduced in the mass square and the dynamics of two scalar particles completely solved, with results in very good agreement with field calculations (see [13] also). Some very encouraging quantistic estimates were also done for 2 and 3 interacting scalar particles [14]. Moreover operators with identical expressions, although there the Weyl algebra is included in the conformal one, have been independently rediscovered and proposed as the quantum observables of relativistic spinning particles in many recent papers see [15] and references therein. We exploit the cohomological based possibility of adding a 2-body Weyl invariant operator to the global dilatator defined by the primitive coproduct. Thus the coassociativity holds only modulo Weyl invariant operators and the global operators involving the dilatators are strictly dependent on the order of the Γ and id .

We discuss now the $(1, 1)d$ situation. The analogous in the Poincaré kinematics of the (2.2) is given by the $E(1, 1)$ Lie algebra :

$$[B, P] = iE, \quad [B, E] = iP, \quad [E, P] = 0.$$

However, to get a time operator, the starting point of our procedure must be the Weyl algebra, obtained by adding as fourth Lie generator the dilatator D :

$$[D, P] = -iP, \quad [D, E] = -iE, \quad [D, B] = 0 .$$

We construct then two commuting Heisenberg pairs by defining the two “Lorentz (1,1)-vectors”: (P, E) and (X, T) ,

$$X = M^{-2}(DP + BE), \quad T = M^{-2}(DE + BP) \tag{4.17}$$

where $M^2 = E^2 - P^2$ is the Casimir of $E(1, 1)$ and one has $[X, P] = i$, $[T, E] = -i$, all the other commutators being zero. Those operators are the building blocks of the 1-body. It must be remarked however that the dynamics of such systems must be generated by Hamiltonians conserving the Poincaré invariant mass and that the maximal invariance can be the Poincaré symmetry, not the Weyl one, because in any situation the physical time T at least must change with any evolution parameter: all that is done in quite natural manner in this framework. The projection on the irreducible rps of $E(1, 1)$ is indeed the equivalent of the classical reduction procedure on the fixed mass sub-variety. Let us now discuss the 2-body collective scheme. It reads

$$\Gamma E = E_1 + E_2, \quad \Gamma P = P_1 + P_2, \quad \Gamma B = B_1 + B_2 \quad \text{and} \quad \Gamma D = D_1 + D_2 + c$$

where c is an arbitrary element in the center of the global Weyl in the tensor product, allowed because D never appears on the right member of the commutations relations (this happens in the $(3, 1)d$ case also). We have thus:

$$\Gamma M = ((\Gamma E)^2 - (\Gamma P)^2)^{1/2}$$

and the “quasi-coproduct” of X, T is given by

$$\begin{aligned} \Gamma X &= (\Gamma M)^{-2}(((\mu_1)^2 X_1 + (\mu_2)^2 X_2) - t(P_1 E_2 - P_2 E_1) + (2i + c)(P_1 + P_2)) \\ \Gamma T &= (\Gamma M)^{-2}(((\mu_1)^2 T_1 + (\mu_2)^2 T_2) + r(P_1 E_2 - P_2 E_1) + (2i + c)(E_1 + E_2)) \end{aligned}$$

where $(\mu_A)^2 = (E_A)^2 - (P_A)^2 + (E_1 E_2 - P_1 P_2)$ so that $(\mu_1)^2 + (\mu_2)^2 = (\Gamma M)^2$, and it is

$$\begin{aligned} r &= X_1 - X_2, & t &= T_1 - T_2. \\ q &= (P_1 - P_2)/2, & u &= (E_1 - E_2)/2. \end{aligned}$$

Let us choose $c = -2i - (ut - qr)$: it is straightforward to show that $(\Gamma \otimes id)\Gamma D - (id \otimes \Gamma)\Gamma D$ is again an operator invariant under the global 3-body Weyl algebra. A good set of relative operators is then obtained by adding the definitions

$$\begin{aligned} \delta X = \tilde{r} &= (\Gamma E r - \Gamma P t)/(\Gamma M), & \delta P = \tilde{q} &= (\Gamma E q - \Gamma P u)/(\Gamma M) \\ \delta T = \tilde{r} &= (\Gamma E t - \Gamma P r)/(\Gamma M), & \delta E = \tilde{q} &= (\Gamma E u - \Gamma P q)/(\Gamma M) \end{aligned}$$

Together with ΓX , ΓT , and ΓP , ΓE they give a complete set of canonical and “covariant” (invariant in this $1d$ case) operators as a direct calculation can confirm. The relevant property of this set is the existence of a relation:

$$(\Gamma M)^2 = (((M_1)^2 + (\tilde{q})^2)^{1/2} + ((M_2)^2 + (\tilde{q})^2)^{1/2})^2 \tag{4.18}$$

recovered by eliminating δE from the collective expressions of $(M_1)^2$ and $(M_2)^2$ and solving in $(\Gamma M)^2$. By projecting on definite values $(M_1)^2 = m_1^2$, $(M_2)^2 = m_2^2$ one recovers for the relativistic 2-body a rigorous hamiltonian formulation in terms of one global time, while the relative time $\delta T = \bar{r}$ is ignorable and can be chosen *a posteriori* to reconstruct the dynamics in the higher dimension. At this point one can introduce interactions depending on $|\bar{r}|$. Clearly the physical description is given at this level, the galilean limit too must be checked there.

It is now possible to extend straightforward (4.18) to any number of massive Poincaré representations because it is given explicitly in the form of relation (2.5). The absence of angular momenta in those $(1, 1)d$ models avoid any problem of formal covariance (as opposed to the commutativity of the components of the position, see [12]). It is thus possible to construct recursively, by adopting the formulas (2.4) for the n -body and the corresponding expressions (2.7) with nested square roots, a genuine relativistic hamiltonian system of n interacting particles, with n given masses and one global physical time.

5 The Quantum framework

The definitions (2.4), (2.6) depend on a canonical map and thus they can be in principle applied to any coalgebra. The crucial problem is to find a good map δ for the 2-body system. It is thus interesting to analyze from this view point the operators of the quantum version of the Galilei algebra [9], where (2.3) cannot be completely realized. This deformed algebra has found physical applications directly as kinematical symmetry of many-body quantum dynamics on lattice [9]. Moreover its unitary irreducible rps have been studied by inducing on the non commutative space of parameters and they appear in agreement with those of Heisenberg on the lattice, but the recovering of unitary irreducible rps in the usual way in the common space of the product of two is rather problematic, notwithstanding the algebra has a real form although rather unconventional [17].

Thus let us introduce the coalgebra $gh_a(1)$ having the same 3 generators and algebraic relations as $gh(1)$ and non trivial coproduct of B and M:

$$\begin{aligned} \Delta P &= P \otimes 1 + 1 \otimes P \\ \Delta B &= B \otimes \exp(iaP) + \exp(-iaP) \otimes B \\ \Delta M &= M \otimes \exp(iaP) + \exp(-iaP) \otimes M \end{aligned} \tag{5.19}$$

where the *length* a is the deformation parameter and having, defined again $X = B/M$, one gets

$$\Delta X = \frac{\Delta B}{\Delta M} = \frac{M_1 X_1 + M_2 X_2 \exp(-ia(P_1 + P_2))}{M_1 + M_2 \exp(-ia(P_1 + P_2))} . \tag{5.20}$$

We take obviously $\Gamma = \Delta$ but let us observe that ΔM is a Casimir of the algebra ΔA but it is not a central element of $A \otimes A$. Its expression implies that it is

impossible to get δ such that both δX and δP commute with ΔX , ΔP and even with ΔM . The map δ we define is the following:

$$\begin{aligned} \delta X &= X_1 - X_2 \\ \delta P &= \frac{i}{a} \log \left(\frac{\Gamma M}{(M_1 + M_2)} \right) \\ \delta M &= \frac{M_1 M_2}{(M_1 + M_2)} \end{aligned} \tag{5.21}$$

and we have two couple of commuting canonical operators, although not a direct product of the two triples. Indeed there is a deformed commutator:

$$[\delta X, \Gamma M] = a \Gamma M .$$

By looking at the structure of the expressions (2.4) one sees that in this case they produce n distinct realizations of the algebra which however are not commuting between them. It must be remarked again that ΔM , ΔX are given *a priori* and δX has the form necessary to commute with total momentum while the remaining expressions have correct relations. Thus the previous choice must be accepted and one has to pay the price of a deformation of canonicity, starting from $n = 3$, in the collective formulation.

A quasi-associative energy E completes the Galilean deformed algebra. The resulting nonstandard

$$[B, E] = (i/a) \sin(aP)$$

determines a Casimir $C = ME - (1/a^2)(1 - \cos(aP))$, from which we define the deformed kinetic energy:

$$T = (1/(Ma^2))(1 - \cos(aP)) .$$

It is then straightforward to obtain for the 2-body operators:

$$\begin{aligned} T_1 + T_2 &= (1/(\Delta M a^2))(1 - \cos(a\Delta P)) + (1/(\delta M a^2))(1 - \cos(a \delta P)) \\ &= \Delta T + \delta T . \end{aligned} \tag{5.22}$$

We are again in a situation where an explicit elementary expression of the (2.6) exists and the previous anomalies cannot affect the result given by (2.7). Indeed we are using only the abelian coalgebra generated by P and M , with their coproducts. Therefore we can be sure of the existence of the set of trigonometric identities which state in the deformed case the same theorem about the kinetic energies as in the classical one:

$$\begin{aligned} \Sigma T_j &= \Delta^{(n-2)} \dots \Delta^{(0)} T + \Delta^{(n-3)} \dots \Delta^{(1)} \delta T \otimes 1 + \\ &\dots + \Delta^{(1)} \delta T \otimes 1^{\otimes(n-3)} + \delta T \otimes 1^{\otimes(n-2)} . \end{aligned} \tag{5.23}$$

This is the kinetic part of a lattice Hamiltonian. If one searches for values of observables such that the kinetic energy is given only by the barycenter term the result is that all the relative momenta must be zero, *i.e.*:

$$\begin{aligned} \Delta^{(0)}M &= M_1 + M_2 \\ \Delta^{(j)} \dots \Delta^{(1)}\Delta^{(0)}M &= \Delta^{(j-1)} \dots \Delta^{(1)}\Delta^{(0)}M + M_{j+2}, \quad j \in (1, n-2). \end{aligned} \quad (5.24)$$

It has been demonstrated that when all the masses are equal the system (5.24) gives exactly the Bethe conditions for the momenta of n -magnons bound states of the XXX model and the right spectrum of the energy [9].

It is possible to introduce in the same way as in the classical case the global energy and the relative one δE :

$$\delta E = \frac{m_1 E_2 + m_2 E_1}{m_1 + m_2} + \delta T - \frac{m_1 T_2 + m_2 T_1}{m_1 + m_2}$$

whose non deformed limit is (3.15). The global energy is $E_1 + E_2 - \delta E$, which - like ΔT - doesn't commute with δX . A sum rule formally identical to (3.16) can be written however.

6 Concluding remarks

An intuitive method of constructing collective classical canonical coordinates or quantum mechanical operators for n -body on the ground of their expressions for $n = 2$ has been precisely formulated and demonstrated by means of algebra morphisms, constructed on the basis of the coalgebra of the systems. Examples from Galilei, Poincaré and deformed Galilei are discussed. An interesting result is the ability of writing immediately for n relations calculated for 2. A further point worth to be studied is the way to apply the algorithm in field theory and the possible connection to the integrability suggested by section 4. Preliminary analysis of those problems are *in fieri*.

Concerning the coproduct it must be stressed that its possible substitution by the morphism Γ is essential in allowing a rigorous and physically good description of the many-body relativistic systems in our approach to the Poincaré systems. From this view point the inclusion of the Weyl in the larger conformal algebra as in [15] may generate problems, because in that case there is no space to substitute the coproduct of D with a morphism having $c \neq 0$. This remark leads us again to enhance a very general point sometimes ignored in the practice, owing to the long monopoly of the Lie primitive structures; *i.e.* that a complete knowledge of an algebra can be obtained only by the knowledge of the coalgebra too. All that is very important in those attempts to grasp quantum gravity by means of noncommutative geometries, implied *e.g.* by the introduction of deformed relativistic kinematics, strongly supported in last years by the preliminary astrophysical measures concerning gamma-ray bursts and the possible violation

of the GZK threshold in cosmic rays, see [17, 18] and references therein. A deep analysis of the collective operators connected to the proposed deformations of the Poincaré kinematics could then be very useful in formulating their phenomenological implications. Indeed one exotic relation of dispersion is in itself not enough, but if it is accompanied by the emergence of 2-body spectra deduced from noncommutative coalgebra it will be read as a clear signature of a noncommutative space-time.

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