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Long Range Scattering and Modified Wave Operators for the Wave-Schrödinger System*

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Abstract. We study the theory of scattering for the system consisting of a Schrödinger equation and a wave equation with a Yukawa type coupling in space dimension 3. We prove in particular the existence of modified wave operators for that system with no size restriction on the data and we determine the asymptotic behaviour in time of solutions in the range of the wave operators. The method consists in solving the wave equation, substituting the result into the Schrödinger equation, which then becomes both nonlinear and nonlocal in time, and treating the latter by the method previously used for a family of generalized Hartree equations with long range interactions.

1 Introduction

This paper is devoted to the theory of scattering and more precisely to the existence of modified wave operators for the Wave-Schrödinger (WS) system

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u - Au & (1.1) \\ \square A = |u|^2 & (1.2) \end{cases}$$

where u and A are respectively a complex valued and a real valued function defined in space time $\mathbb{R}^{3+1}$, Δ is the Laplacian in $\mathbb{R}^3$ and $\square = \partial_t^2 - \Delta$ is the d'Alembertian in $\mathbb{R}^{3+1}$. That system is Lagrangian with Lagrangian density

$$\mathcal{L} = i\,(\bar{u}\ \partial_t\ u - u\ \partial_t\ \bar{u}) - \frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t A)^2 - \frac{1}{2}|\nabla A|^2 + A|u|^2\ . \tag{1.3}$$

Formally, the L^2 norm of u is conserved, as well as the energy

$$E(u, A) = \int dx \Big\{ \frac{1}{2}\left(|\nabla u|^2 + (\partial_t A)^2 + |\nabla A|^2\right) - A|u|^2 \Big\}\ . \tag{1.4}$$

The Cauchy problem for the WS system (1.1) (1.2) is known to be globally well posed in the energy space $X_e = H^1 \oplus \dot{H}^1 \oplus L^2$ for $(u, A, \partial_t A)$ [1] [2] [4] [15].

A large amount of work has been devoted to the theory of scattering for nonlinear equations and systems centering on the Schrödinger equation, in particular for nonlinear Schrödinger (NLS) equations, Hartree equations, Klein-Gordon

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Schrödinger (KGS) and Maxwell-Schrödinger (MS) systems. As in the case of the linear Schrödinger equation, one must distinguish the short range case from the long range case. In the former case, ordinary wave operators are expected and in a number of cases proved to exist, describing solutions where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation. In the latter case, ordinary wave operators do not exist and have to be replaced by modified wave operators including a suitable phase in their definition. In that respect, the WS system (1.1) (1.2) in $\mathbb{R}^{3+1}$ belongs to the borderline (Coulomb) long range case, because of the t^{-1} decay in L^∞ norm of solutions of the wave equation. Such is the case also for the Hartree equation with $|x|^{-1}$ potential. Both are simplified models for the more complicated Maxwell-Schrödinger system in $\mathbb{R}^{3+1}$, which belongs to the same case, as well as the KGS system in $\mathbb{R}^{2+1}$.

Whereas a well developed theory of long range scattering exists for the linear Schrödinger equation (see [3] for a recent treatment and for an extensive bibliography), there exist only few results on nonlinear long range scattering. The existence of modified wave operators in the borderline Coulomb case has been proved for the NLS equation in space dimension $n = 1$ [19]. That result has been extended to the NLS equation in dimensions $n = 2, 3$ and to the Hartree equation in dimension $n \geq 2$ [5], to the derivative NLS equation in dimension $n = 1$ [14], to the KGS system in dimension 2 [20] and to the MS system in dimension 3 [22]. All those results are restricted to the case of small data.

In a recent series of papers, [6] [7] [8], we proved the existence of modified wave operators for a family of Hartree type equations with general (not only Coulomb) long range interactions and without any size restriction on the data. The method is strongly inspired by a previous series of papers by Hayashi et al [9] [10] [11] [12] [13] on the Hartree equation. In the latter papers it is proved first in the borderline Coulomb case and then in the whole long range case, that the global solutions of the Hartree equation with small initial data exhibit an asymptotic behaviour for large time that is typical of long range scattering and includes in particular the expected relevant phase factor.

The present paper is devoted to the extension of the results of [6] [7] [8] to the WS system and in particular to the proof of the existence of modified wave operators for that system without any size restriction on the data. The method consists in eliminating the wave equation by solving it for A in terms of u and substituting the result into the Schrödinger equation, thereby obtaining a new Schrödinger equation which is both nonlinear and nonlocal in time. The latter is then treated as the Hartree equation in [6] [7] [8], namely u is expressed in terms of an amplitude w and a phase φ satisfying an auxiliary system similar to that introduced in [11]. Wave operators are constructed first for that auxiliary system, and then used to construct modified wave operators for the original system (1.1). The detailed construction is too complicated to allow for a more precise description at this stage, and will be described in heuristic terms in Section 2 below. In subsequent papers, the results of the present one will be extended to the case of the MS system.

We now give a brief outline of the contents of this paper. A more detailed description of the technical parts will be given at the end of Section 2. After collecting some notation and preliminary estimates in Section 3, we study the asymptotic dynamics for the auxiliary system in Section 4 and uncover some difficulties due to the different propagation properties of solutions of the wave and Schrödinger equations. As a preparation for the general case, we construct in Section 5 the wave operators associated with the simplified linear system obtained by replacing (1.2) by the free wave equation $\square A = 0$. We then solve the local Cauchy problem at infinity for the auxiliary system in Sections 6 and 7, which contain the main technical results of this paper. We finally come back from the auxiliary system to the original one (1.1) (1.2) and construct the modified wave operators for the latter in Section 8, where the final result is stated in Proposition 8.1.

We conclude this section with some general notation which will be used freely throughout this paper. We denote by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^3)$ and we define $\delta(r) = 3/2 - 3/r$. For any interval I and any Banach space X, we denote by $\mathcal{C}(I, X)$ (resp. $\mathcal{C}_w(I, X)$) the space of strongly (resp. weakly) continuous functions from I to X and by $L^\infty(I, X)$ (resp. $L^\infty_{loc}(I, X)$) the space of measurable essentially bounded (resp. locally essentially bounded) functions from I to X. For real numbers a and b, we use the notation $a \vee b = \mathrm{Max}(a, b)$, and $a \wedge b = \mathrm{Min}(a, b)$. Furthermore, we define

$$\begin{aligned} [a \vee b] &= a \vee b && \text{if } \ a \neq b \\ &= a + \varepsilon && \text{for some } \ \varepsilon > 0 \ \text{ if } \ a = b \ , \\ [a \wedge b] &= a + b - [a \vee b] && \text{and } \ [a]_+ = [a \vee 0] \ . \end{aligned}$$

For any interval $I \subset \mathbb{R}^+$, we denote by $\bar{I}$ the closure of I in $\mathbb{R}^+ \cup \{\infty\}$ and for any interval $I = [a, b)$ we denote by I_+ the interval $I_+ = [a, \infty)$. In the estimates of solutions of the relevant equations, we shall use the letter C to denote constants, possibly different from an estimate to the next, depending on various parameters, but not on the solutions themselves or on their initial data. We shall use the notation $C(a_1, a_2, \cdots)$ for estimating functions, also possibly different from an estimate to the next, depending in addition on suitable norms $a_1, a_2, \cdots$ of the solutions or of their initial data. Additional notation will be given in Section 3.

2 Heuristics

In this section, we discuss in heuristic terms the construction of the modifed wave operators for the system (1.1) (1.2), as it will be performed in this paper. We refer to Section 2 of [6] [7] for general background and for a similar discussion adapted to the case of the Hartree equation. The problem that we want to address is that of classifying the possible asymptotic behaviours in time of the solutions of (1.1) (1.2) by relating them to a set of model functions $\mathcal{V} = \{v = v(v_+)\}$ parametrized by some data v_+ and with suitably chosen and preferably simple asymptotic behaviour in time. For each $v \in \mathcal{V}$, one tries to construct a solution

(u, A) of (1.1) (1.2) such that $(u, A)(t)$ behaves as $v(t)$ when $t \to \infty$ in a suitable sense. We then define the wave operator as the map $\Omega : v_+ \to (u, A)$ thereby obtained. A similar question can be asked for $t \to -\infty$. We restrict our attention to positive time. The more standard definition of the wave operator is to define it as the map $v_+ \to (u, A)(0)$, but what really matters is the solution (u, A) in the neighborhood of infinity in time, namely in some interval $[T, \infty)$, and continuing such a solution down to $t = 0$ is a somewhat different question which we shall not touch here.

In cases such as (1.1) (1.2) where the system of interest is a perturbation of a simple linear system, hereafter called the free system, a natural candidate for $\mathcal{V}$ is the set of solutions of the free system, parametrized by the initial data v_+ at time $t = 0$ for the Cauchy problem for that system. In the case of the system (1.1) (1.2) one is therefore tempted to consider the Cauchy problem

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\Delta u & u(0) = u_+ \\ \square A = 0 & A(0) = A_+ \ , \ \partial_t A(0) = \dot{A}_+ \ , \end{cases} \tag{2.1}$$

to take $v_+ = (u_+, A_+, \dot{A}_+)$ and to take for $v(v_+)$ the solution (u, A) of (2.1). Cases where such a procedure yields an adequate set $\mathcal{V}$ are called short range cases. They require that the perturbation has sufficient decay in time or equivalently in space. This is the case for instance for the linear Schrödinger equation or for the Hartree equation with potential $V(x) = |x|^{-\gamma}$ for $\gamma > 1$. Such is not the case however for the system (1.1) (1.2). This shows up through the fact that the solution A of the wave equation $\square A = 0$ decays at best as t^{-1} (in L^∞ norm), which is the borderline case of nonintegrability in time. That situation corresponds to the limiting case $\gamma = 1$ (the Coulomb case in space dimension $n = 3$) for the linear Schrödinger and for the Hartree equation. A similar situation prevails for the KGS system in space dimension 2 and for the MS system in space dimension 3. In the present case, which is the borderline long range case, the set of solutions of the Cauchy problem (2.1) is inadequate, and one of the tasks that will be performed in this paper (see especially Sections 7 and 8) will be to construct a better set $\mathcal{V}$ of model asymptotic functions.

Constructing the wave operators essentially amounts to solving the Cauchy problem with infinite initial time. The system (1.1) (1.2) in this form is not well suited for that purpose and we shall now perform a number of transformations leading to an auxiliary system for which that problem can be handled. For additional flexibility we shall first of all allow for imposing initial data at two different initial times t_0 and t_1 for the Schrödinger and wave equations respectively. With the aim of letting t_1 and t_0 tend to infinity in that order, we shall take $t_0 \le t_1$. We shall then eliminate the wave equation by solving it and substituting the result into the Schrödinger equation. We define

$$\omega = (-\Delta)^{1/2} \quad , \quad K(t) = \omega^{-1} \sin \omega t \quad , \quad \dot{K}(t) = \cos \omega t$$

and we replace the wave equation (1.2) by its solution

$$A = A_0 + A_1^{t_1}(|u|^2) \tag{2.2}$$

where

$$A_0 = \dot{K}(t)\ A_+ + K(t)\ \dot{A}_+ \tag{2.3}$$

$$A_1^{t_1}(|u|^2) = \int_{t_1}^{t} dt'\ K(t-t')\ |u(t')|^2\ . \tag{2.4}$$

Here A_0 is a solution of the free wave equation, with initial data $(A_+, \dot{A}_+)$ at time $t = 0$. For $t_1 = \infty$, $(A_+, \dot{A}_+)$ is naturally interpreted as the asymptotic state for A, in keeping with the previous discussion.

The Cauchy problem for the system (1.1) (2.2) with initial data $u(t_0) = u_0$ is no longer a usual PDE Cauchy problem because A_1 depends on u nonlocally in time. A convenient way to handle that difficulty is to first replace that problem by a partly linearized form thereof, namely

$$\begin{cases} i\partial_t u' = -\frac{1}{2}\Delta u' - Au'\ , \qquad u'(t_0) = u_0 \\ A = A_0 + A_1(|u|^2)\ . \end{cases} \tag{2.5}$$

For given u, (2.5) is an ordinary (linear) Cauchy problem for u'. Solving that problem for u' defines a map $\Gamma : u \to u'$, and solving the original problem then reduces to finding a fixed point of Γ, which in favourable cases can be done for instance by contraction. We shall make use of that linearization method, not for the equation for u, but for the auxiliary system to be defined below.

Aside from the nonlocality in time of the nonlinear interaction term, which can be handled by the previous linearization, the system (1.1) (2.2) is rather similar to the Hartree type equations considered in [6] [7] [8], and we next perform the same change of variables, which is well adapted to the study of the asymptotic behaviour in time. The unitary group

$$U(t) = \exp(i(t/2)\Delta) \tag{2.6}$$

which solves the free Schrödinger equation can be written as

$$U(t) = M(t)\ D(t)\ F\ M(t) \tag{2.7}$$

where $M(t)$ is the operator of multiplication by the function

$$M(t) = \exp\left(ix^2/2t\right)\ , \tag{2.8}$$

F is the Fourier transform and $D(t)$ is the dilation operator

$$(D(t)f)(x) = (it)^{-n/2}\ f(x/t) \tag{2.9}$$

normalized to be unitary in L^2. We shall also need the operator $D_0(t)$ defined by

$$(D_0(t)f)(x) = f(x/t) \ . \tag{2.10}$$

We now parametrize u in terms of an amplitude w and of a real phase φ as

$$u(t) = M(t)\ D(t) \exp[-i\varphi(t)] w(t) \ . \tag{2.11}$$

Substituting (2.11) into (1.1) yields an evolution equation for (w, φ), namely

$$\left\{ i\partial_t + (2t^2)^{-1}\Delta - i(2t^2)^{-1}(2\nabla\varphi \cdot \nabla + \Delta\varphi) + t^{-1}B + \partial_t\varphi - (2t^2)^{-1}|\nabla\varphi|^2 \right\} w = 0 \tag{2.12}$$

where we have expressed A in terms of a new function B by

$$A = t^{-1}\ D_0\ B \ . \tag{2.13}$$

Corresponding to the decomposition (2.2) of A, we decompose

$$B = B_0 + B_1^{t_1}(w, w) \tag{2.14}$$

where $A_0 = t^{-1} D_0 B_0$ and $A_1^{t_1} = t^{-1} D_0 B_1^{t_1}$. One computes easily

$$B_1^{t_1}(w_1, w_2) = \int_1^{t_1/t} d\nu\ \nu^{-3}\ \omega^{-1} \sin((\nu - 1)\omega) D_0(\nu)(\mathrm{Re}\ \bar{w}_1 w_2)(\nu t) \ . \tag{2.15}$$

As in the case of the Hartree equation, we have only one evolution equation (2.12) for two functions (w, φ). We arbitrarily impose a second equation, namely a Hamilton-Jacobi (or eikonal) equation for the phase φ, thereby splitting the equation (2.12) into a system of two equations, the other one of which being a transport type equation for the amplitude w. For that purpose, we split $B_1^{t_1}$ into a short range and a long range parts

$$B_1^{t_1} = B_S^{t_1} + B_L^{t_1} \ . \tag{2.16}$$

in the following way. We take $0 < \beta < 1$ and we define

$$\begin{cases} \left(FB_S^{t_1}\right)(t, \xi) = \chi(|\xi| > t^\beta) FB_1^{t_1}(t, \xi) \\ \left(FB_L^{t_1}\right)(t, \xi) = \chi(|\xi| \le t^\beta) FB_1^{t_1}(t, \xi) \end{cases} \tag{2.17}$$

where $\chi(|\xi| \lesseqgtr t^\beta)$ is the characteristic function of the set $\{(t, \xi) : |\xi| \lesseqgtr t^\beta\}$. The parameter β will satisfy various conditions which will appear later, all of which will be compatible with $\beta = 1/2$. We then split the equation (2.12) into the following system of two equations

$$\begin{cases} \partial_t w = i(2t^2)^{-1}\Delta w + t^{-2} Q(\nabla\varphi, w) + it^{-1}(B_0 + B_S^{t_1}(w, w))w \\ \partial_t\varphi = (2t^2)^{-1}|\nabla\varphi|^2 - t^{-1}\ B_L^{t_1}(w, w) \end{cases} \tag{2.18}$$

where we have defined

$$Q(s,w) = s \cdot \nabla w + (1/2)(\nabla \cdot s)w \tag{2.19}$$

for any vector field s. The first equation of (2.18) is the transport type equation for the amplitude w, while the second one is the Hamilton-Jacobi type equation for the phase φ. Since the right-hand sides of (2.18) contain φ only through its gradient, we can obtain from (2.18) a closed system for w and $s = \nabla\varphi$ by taking the gradient of the second equation, namely

$$\begin{cases} \partial_t w = i(2t^2)^{-1}\Delta w + t^{-2}Q(s,w) + it^{-1}(B_0 + B_S^{t_1}(w,w))w \\ \partial_t s = t^{-2}s \cdot \nabla s - t^{-1}\nabla B_L^{t_1}(w,w)\ . \end{cases} \tag{2.20}$$

Once the system (2.20) is solved for (w,s), one recovers φ easily by integrating the second equation of (2.18) over time. We refer to [6] for details. The system (2.20) will be referred to as the auxiliary system and will play an essential role in this paper. For the same reason as was explained for the partly resolved system (1.1) (2.2), we shall use at intermediate stages a partly linearized version of the system (2.20), namely

$$\begin{cases} \partial_t w' = i(2t^2)^{-1}\Delta w' + t^{-2}Q(s,w') + it^{-1}(B_0 + B_S^{t_1}(w,w))w' \\ \partial_t s' = t^{-2}s \cdot \nabla s' - t^{-1}\nabla B_L^{t_1}(w,w) \end{cases} \tag{2.21}$$

to be considered as a system of equations for (w', s') for given (w,s). The first question to be considered is whether the auxiliary system (2.20) defines a dynamics for large time, namely whether the Cauchy problem for that system is locally well posed in a neighborhood of infinity in time, more precisely has a unique solution defined up to infinity in time for sufficiently large t_1 and sufficiently large initial time t_0, possibly depending on the size of the initial data. This property was satisfied by the corresponding auxiliary system associated with the Hartree equation and considered in [6] [7]. Here however we encounter serious difficulties associated with the difference of propagation properties of solutions of the Schrödinger and wave equations. In fact a typical solution of the free Schrödinger equation behaves asymptotically in time as

$$(U(t)u_+)(x) \sim (MDFu_+)(x) = \exp(ix^2/2t)(it)^{-3/2}\ Fu_+(x/t)$$

namely spreads by dilation by t in all directions in the support of Fu_+, while by the Huyghens principle A_0 remains concentrated in a neighborhood of the light cone, more precisely within a distance R of the latter if the initial data $(A_+, \dot{A}_+)$ are supported in a ball of radius R. When switching to the new variables (w, B), w tends to a limit when $t \to \infty$ whereas B_0 concentrates in a neighborhood of the unit sphere, within a distance R/t of the latter in the previous case of compactly supported data. Note however that for $t_1 = \infty$, B_1^∞ is expected to tend to a limit like w and not to concentrate like B_0, as can be guessed from (2.15).

We shall treat the auxiliary system (2.20) by energy methods, and in particular look for w in spaces of the type $\mathcal{C}([T,\infty), H^k)$ where H^k is the usual Sobolev space based on L^2. In order to treat the nonlinear term $B_1(w,w)$, we shall need a minimal regularity, in practice $k > 1$. However, when taking H^k norms of B_0, the previous concentration phenomenon implies

$$\| B_0; H^k \| \sim O\left(t^{k-1/2}\right)$$

which has worse and worse asympotic behaviour in time as k increases. This difficulty manifests itself in the following way:

(i) If $t_0 = t_1 < \infty$, the available estimates for the sytem (2.20) do not prevent finite time blow up after t_0, even if $A_0 = 0$.

This encourages us to take $t_1 > t_0$, and actually the situation becomes slightly better in that case. Nevertheless

(ii) the available estimates do not prevent finite time blow up after t_1, which is the same fact as (i) with t_0 replaced by t_1, and

(iii) if $A_0 \neq 0$ and if t_1 is sufficiently large, the available estimates do not prevent blow up before t_1.

A definite improvement occurs however if $A_0 = 0$.

(iv) If $A_0 = 0$, the available estimates allow for a proof of existence of solutions in $[t_0, t_1]$ for t_0 sufficiently large and arbitrary $t_1 > t_0$, possibly $t_1 = \infty$. In particular for $t_1 = \infty$, the solutions are defined up to infinity in time. Furthermore, for those solutions, $w(t)$ has a limit w_+ as $t \to \infty$.

The last case brings us in the same situation as that encountered for the Hartree equation in [6] [7] and could be taken as the starting point for the construction of partial modified wave operators (restricted to the case of vanishing $(A_+, \dot{A}_+)$) by the same method as in [6] [7]. We shall however refrain from performing that construction and turn directly to the case of nonvanishing $(A_+, \dot{A}_+)$. In that case, the need to use H^k norms with $k > 1$ for w makes the treatment of A_0 nontrivial, even if one drops the interaction term A_1. As a preparation for the general case, we shall therefore first construct the wave operators at the same level of regularity for the simplified system

$$\begin{cases} i\partial_t u = -(1/2)\Delta u - A_0 u \\ \square A_0 = 0 \end{cases} \tag{2.22}$$

namely for a linear Schrödinger equation with time dependent potential A_0 satisfying the free wave equation. After the appropriate change of variables

$$u = MDw \quad , \quad A_0 = t^{-1}\, D_0\, B_0 \tag{2.23}$$

the Schrödinger equation becomes

$$R(w) \equiv \partial_t w - i(2t^2)^{-1}\Delta w - it^{-1} B_0 w = 0 \ . \tag{2.24}$$

The construction of the wave operators for that equation in L^2, either in the form (2.22) or (2.24) can be easily performed by a simple variant of Cook's method, and the construction of the wave operators at the level of H^k becomes a regularity problem for the previous wave operators. Solving that problem for $k \geq 1$ (in fact for $k > 1/2$) requires special assumptions on the asymptotic states $(w_+, A_+, \dot{A}_+)$, to the effect that the product $B_0 w_+$ decays faster in time in the relevant norms than what would naturally follow from factorized estimates. Those assumptions can be ensured for instance by imposing support properties of w_+, to the effect that $w_+ = 0$ on the unit sphere, and suitable decay of $(A_+, \dot{A}_+)$ at infinity in space. They will be needed again in the treatment of the general problem.

The construction of the modified wave operators in the general case follows the same pattern as for the Hartree equation. The aim is to construct solutions of the auxiliary system (2.20) with suitably prescribed asymptotic behaviour at infinity, and in particular with $w(t)$ tending to a limit w_+ as $t \to \infty$. That asymptotic behaviour will be imposed in the form of a suitably chosen pair (W, ϕ) and therefore (W, S) with $S = \nabla\phi$, with $W(t)$ tending to w_+ as $t \to \infty$. For fixed (W, S), we make a change of variables in the system (2.18) from (w, φ) to (q, ψ) defined by

$$(q, \psi) = (w, \varphi) - (W, \phi) \tag{2.25}$$

or equivalently a change of variables in the system (2.20) from (w, s) to (q, σ) defined by

$$(q, \sigma) = (w, s) - (W, S)\ , \tag{2.26}$$

and instead of looking for a solution (w, s) of the system (2.20) with (w, s) behaving asymptotically as (W, S), we look for a solution (q, σ) of the transformed system with (q, σ) (and also ψ) tending to zero as $t \to \infty$. Actually for technical reasons, we need to modify the auxiliary system slightly, in the following way. When expanding $w = W + q$ in $B_1^{t_1}(w, w)$, we shall replace that quantity by

$$B_1^{t_1,\infty}(w, w) \equiv B_1^{\infty}(W, W) + 2B_1^{t_1}(W, q) + B_1^{t_1}(q, q)\ . \tag{2.27}$$

We furthermore define the remainders

$$R_1(W, S) = \partial_t W - i(2t^2)^{-1}\Delta W - t^{-2}Q(S, W) - it^{-1}(B_0 + B_S^{\infty}(W, W))W \tag{2.28}$$

$$R_2(W, S) = \partial_t S - t^{-2} S \cdot \nabla S + t^{-1}\nabla B_L^{\infty}(W, W)\ . \tag{2.29}$$

Performing the change of variables (2.26) and including the previous technical modification in the system (2.20) yields the modified auxiliary system for the new variables (q, σ).

$$\begin{cases} \partial_t q = i(2t^2)^{-1}\Delta q + t^{-2}(Q(s, q) + Q(\sigma, W)) + it^{-1}B_0 q \\ +it^{-1}B_S^{t_1,\infty}(w, w)q + it^{-1}\left(2B_S^{t_1}(W, q) + B_S^{t_1}(q, q)\right)W - R_1(W, S) \\ \partial_t \sigma = t^{-2}(s \cdot \nabla\sigma + \sigma \cdot \nabla S) - t^{-1}\nabla\left(2B_L^{t_1}(W, q) + B_L^{t_1}(q, q)\right) - R_2(W, S). \end{cases} \tag{2.30}$$

Note that changing $B^{t_1}(w,w)$ to $B^{t_1,\infty}(w,w)$ changes A by a solution of the free wave equation, so that we are still solving the original system (1.1) (1.2), with however a slightly different A_0 as compared with (2.2).

For the same reason as for the partly resolved system (1.1) (2.2) and for the auxiliary system (2.20), we shall use at intermediate stages a partly linearized version of the system (2.30), namely

$$\begin{cases}\partial_t q' = i(2t^2)^{-1}\Delta q' + t^{-2}(Q(s,q') + Q(\sigma,W)) + it^{-1}B_0 q' \\ +it^{-1}B_S^{t_1,\infty}(w,w)q' + it^{-1}\left(2B_S^{t_1}(W,q) + B_S^{t_1}(q,q)\right)W - R_1(W,S) \\ \partial_t\sigma' = t^{-2}(s\cdot\nabla\sigma' + \sigma\cdot\nabla S) - t^{-1}\nabla\left(2B_L^{t_1}(W,q) + B_L^{t_1}(q,q)\right) - R_2(W,S).\end{cases} \quad (2.31)$$

The construction of solutions (q,σ) tending to zero at infinity for the system (2.30) with $t_1 = \infty$ proceeds in several steps. We assume first that (W,S) and B_0 satisfy suitable boundedness properties and that the remainders $R_1(W,S)$ and $R_2(W,S)$ satisfy suitable decay in time. We solve the linearized system (2.31) for (q',σ') for given (q,σ), both with finite and infinite time t_1 and initial time t_0. We then solve (2.30) by proving that the map $\Gamma : (q,\sigma) \to (q',\sigma')$ is a contraction in suitable norms. We also prove that the solution of (2.30) with $t_0 = t_1 < \infty$ converges to the solution with $t_0 = t_1 = \infty$ when $t_0 \to \infty$, a property which is natural in the framework of scattering theory. There remains the task of constructing (W,S) with $W(t)$ tending to w_+ as $t \to \infty$, and satisfying the required boundedness and decay properties. This is done by solving the auxiliary system (2.20) with $t_1 = \infty$ approximately by iteration. We restrict our attention to the second iteration, which is sufficient to cover the range $1 < k < 2$. The pair (W,S) or equivalently (W,ϕ) thereby obtained depends only on the asymptotic state w_+. Solving the auxiliary system (2.30) with that (W,S) and with $t_0 = t_1 = \infty$ yields a solution (w,s) of the system (2.20) and therefore a solution (w,φ) of the system (2.18) with prescribed asymptotic behaviour characterized by (W,S) or (W,ϕ). That solution depends on $(w_+, A_+, \dot{A}_+)$. Plugging that solution with $w_+ = Fu_+$ into (2.11) and substituting u thereby obtained into (2.2) (2.4) with $t_1 = \infty$ yields a solution (u,A) of the system (1.1) (1.2) with prescribed asymptotic behaviour in time explicitly expressed in terms of the asymptotic state $(u_+, A_+, \dot{A}_+)$. More precisely, that asymptotic behaviour is obtained or rather defined by replacing (w,φ) by (W,ϕ) and $|u|^2 = |Dw|^2$ by $|DW|^2$ in (2.11) and (2.2) (2.4) with $t_1 = \infty$, so that actually (u,A) behaves asymptotically as $(MD\exp(-i\phi)W,\ A_0 + A_1^{\infty}(|DW|^2))$, which plays the role of modified free solution for the system (1.1) (1.2). As a by product of that construction, we can define the map $\Omega : (u_+, A_+, \dot{A}_+) \to (u,A)$, which is the required modified wave operator for the system (1.1) (1.2).

The main result of this paper, namely the construction of solutions of the system (1.1) (1.2) defined for large time and with prescribed asymptotic behaviour as described above, is stated in full mathematical detail in Proposition 8.1 below. Since however that detail is rather cumbersome, we give here a heuristic description thereof, which can serve as a reader's guide for that proposition. One starts with asymptotic states $(u_+, A_+, \dot{A}_+)$ which are sufficiently regular in the sense that (i) $w_+ \equiv Fu_+ \in H^{k_+}$ for sufficiently large k_+, (ii) the solution A_0 of the free wave equation generated by $(A_+, \dot{A}_+)$ according to (2.3) satisfies the optimal time decay associated with that equation in suitable norms (see (8.9)≡(3.15)) and satisfies an additional joint time decay with w_+, needed to damp light cone interferences (see (8.10)). One then constructs model asymptotic functions (W, S) for the auxiliary system (2.20), depending only on w_+, by solving a truncated version of that system approximately by iteration to second order (see (7.3) (7.5) (7.7)). The main technical result is that one can construct a unique solution (w, s) of the auxiliary system (2.20), defined for large time, and asymptotic to (W, S) in suitable norms (see (8.11) (8.12) (8.13)). One then defines the phases φ and ϕ corresponding to s and S according to $s = \nabla\varphi$ and $S = \nabla\phi$ and one reconstructs (u, A) from (w, φ) by (2.11) and (2.2) (2.4) with $t_1 = \infty$. Then (u, A) is a solution of the system (1.1) (1.2), defined for large time, and (u, A) is asymptotic to the modified free solution $(MD\exp(-i\phi)W, A_0 + A_1^\infty(|DW|^2))$ in suitable norms (see (8.15)–(8.23)).

The auxiliary system (2.18) satisfies a gauge invariance property similar to that of the corresponding system for the Hartree equation used in [6] [7], and the construction of the intermediate wave operator for that system can be made in a gauge covariant way. For brevity we shall refrain from discussing that question in this paper.

We now describe the contents of the technical parts of this paper, namely Sections 3–8. In Section 3, we introduce some notation, define the relevant function spaces and collect a number of preliminary estimates. In Section 4, we study the Cauchy problem for large time for the auxiliary system (2.20). We solve the Cauchy problem with finite initial time for the linearized system (2.21) (Proposition 4.1), we prove a number of uniqueness results for the system (2.20) (Proposition 4.2), we prove the existence of a limit $w(t)$ of w_+ for suitably bounded solutions of the system (2.20) (Proposition 4.3), we discuss in more quantitative terms the possible occurrence of blow up mentioned above, and we finally solve the Cauchy problem for the system (2.20) with $t_1 = \infty$ and large t_0 in the special case $A_0 = 0$ (Proposition 4.4).

In Section 5, as a preparation for the construction of the wave operators for the system (2.20) with $A_0 \neq 0$, we study the existence of wave operators for the linear problem (2.22) in the form (2.24). In particular we prove the existence of L^2-wave operators by a variant of Cook's method (Proposition 5.2), we prove the H^k regularity of those wave operators under suitable decay assumptions of $R(W)$ for the model function W (Proposition 5.3) and we finally reduce those decay properties to conditions on the asymptotic state $(w_+, A_+, \dot{A}_+)$. In Section 6 and 7, we study the Cauchy problem at infinity in the general case $A_0 \neq 0$ for the auxiliary system (2.20) in the difference form (2.30). Under suitable boundedness assumptions on (W, S) and decay assumptions on $R_1(W, S)$ andf $R_2(W, S)$ we prove the existence of solutions for t_0 and t_1 finite and infinite, first for the linearized system (2.31) (Propositions 6.1 and 6.2) and then for the nonlinear system (2.30) (Proposition 6.3). We then choose appropriate (W, S), prove that they satisfy the required assumptions (Lemmas 7.1 and 7.2) and finally state the result on the Cauchy problem at infinity for the system (2.30) in H^k for $1 < k < 2$ (Proposition 7.1). Finally in Section 8, we construct the wave operators for the system (1.1) (1.2) from the results previously obtained for the system (2.30) and we derive the asymptotic estimates for the solutions (u, A) in their range that follow from the previous estimates (Proposition 8.1).

3 Notation and preliminary estimates

In this section we introduce some additional notation and we collect a number of estimates which will be used throughout this paper. We shall use the Sobolev spaces H_r^k defined for $1 \le r \le \infty$ by

$$H_r^k = \left\{ u : \parallel u; H_r^k \parallel \equiv \parallel < \omega >^k u \parallel_r < \infty \right\}$$

where $< \cdot >= (1 + |\cdot|^2)^{1/2}$. The subscript r will be omitted if $r = 2$.

We shall look for solutions of the auxiliary system (2.20) in spaces of the type $\mathcal{C}(I, X^{k,\ell})$ where I is an interval and

$$X^{k,\ell} = H^k \oplus \omega^{-1}\ H^\ell$$

namely

$$X^{k,\ell} = \left\{ (w, s) : w \in H^k\ ,\ \nabla s \in H^\ell \right\} \tag{3.1}$$

where it is understood that $\nabla s \in L^2$ includes the fact that $s \in L^6$, and we shall use the notation

$$\parallel w; H^k \parallel = |w|_k\ . \tag{3.2}$$

We shall use extensively the following Sobolev inequalities, stated here in $\mathbb{R}^n$, but to be used only for $n = 3$.

Lemma 3.1 *Let $1 < q,\ r < \infty$, $1 < p \le \infty$ and $0 \le j < k$. If $p = \infty$, assume that $k - j > n/r$. Let σ satisfy $j/k \le \sigma \le 1$ and*

$$n/p - j = (1 - \sigma)n/q + \sigma(n/r - k)\ .$$

Then the following inequality holds

$$\parallel \omega^j u \parallel_p \le C \parallel u \parallel_q^{1-\sigma}\ \parallel \omega^k u \parallel_r^\sigma\ . \tag{3.3}$$

The proof follows from the Hardy-Littlewood-Sobolev (HLS) inequality ([16], p. 117) (from the Young inequality if $p = \infty$), from Paley-Littlewood theory and interpolation.

We shall also use extensively the following Leibnitz and commutator estimates.

Lemma 3.2 *Let $1 < r, r_1, r_3 < \infty$ and*

$$1/r = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4 \ .$$

Then the following estimates hold

$$\| \omega^m (uv) \|_r \le C (\| \omega^m u \|_{r_1} \| v \|_{r_2} + \| \omega^m v \|_{r_3} \| u \|_{r_4}) \tag{3.4}$$

for $m \ge 0$, and

$$\| [\omega^m, u] v \|_r \le C \left(\| \omega^m u \|_{r_1} \| v \|_{r_2} + \| \omega^{m-1} v \|_{r_3} \| \nabla u \|_{r_4}\right) \tag{3.5}$$

for $m \ge 1$, where [,] denotes the commutator.

The proof of those estimates is given in [17] [18] with ω replaced by $< \omega >$ and follows therefrom by a scaling argument.

We shall also need the following consequence of Lemma 3.2.

Lemma 3.3 *Let $m \ge 0$ and $1 < r < \infty$. Then the following estimate holds*

$$\| \omega^m (e^\varphi - 1) \|_r \le \| \omega^m \varphi \|_r \exp (C \| \varphi \|_\infty) \ . \tag{3.6}$$

Proof. For any integer $n \ge 2$, we estimate

$$\begin{aligned} a_n &\equiv \| \omega^m \ \varphi^n \|_r \le C \left(\| \omega^m \varphi \|_r \| \varphi \|_\infty^{n-1} + \| \omega^m \varphi^{n-1} \|_r \| \varphi \|_\infty\right) \\ &= C \left(a_1 \ b^{n-1} + a_{n-1} \ b\right) \end{aligned} \tag{3.7}$$

by (3.4), where $b = \| \varphi \|_\infty$ and we can assume $C \ge 1$ without loss of generality. It follows easily from (3.7) that

$$a_n \le n (Cb)^{n-1} \ a_1$$

for all $n \ge 1$, from which (3.6) follows by expanding the exponential. □

We next give some estimates of $B_1^{t_1}$, $B_S^{t_1}$ and $B_L^{t_1}$ defined by (2.15) (2.17). It follows immediately from (2.17) that

$$\| \omega^m B_S^{t_1} \|_2 \le t^{\beta(m-p)} \| \omega^p B_S^{t_1} \|_2 \le t^{\beta(m-p)} \| \omega^p B_1^{t_1} \|_2 \tag{3.8}$$

for $m \le p$ and similarly

$$\| \omega^m B_L^{t_1} \|_2 \le t^{\beta(m-p)} \| \omega^p B_L^{t_1} \|_2 \le t^{\beta(m-p)} \| \omega^p B_1^{t_1} \|_2 \tag{3.9}$$

for $m \ge p$. On the other hand it follows from (2.15) that

$$\| \omega^{m+1} B_1^{t_1} (w_1, w_2) \|_2 \le I_m^{t_1} (\| \omega^m (w_1 \bar{w}_2) \|_2) \tag{3.10}$$

where $I_m^{t_1}$ is defined by

$$\left(I_m^{t_1}(f)\right)(t) = \left| \int_1^{t_1/t} d\nu \ \nu^{-m-3/2} \ f(\nu t) \right| \tag{3.11}$$

or equivalently

$$\left(I_m^{t_1}(f)\right)(t) = t^{m+1/2} \left| \int_t^{t_1} dt' \ t'^{-m-3/2} \ f(t') \right|$$

for $t > 0$, $t_1 > 0$. Most of the subsequent estimates of $B_1^{t_1}$ will follow from (3.8) (3.9) (3.10) and from an estimate of $\| \omega^m (w_1 \bar{w}_2) \|_2$. The latter follows from the HLS inequality if $-3/2 < m < 0$ and from (3.4) if $m \geq 0$. For future reference, we quote the following special case, which will occur repeatedly

$$\| \omega^{2k-1/2} B_1^{t_1}(w,w) \|_2 \leq C \ I_{2k-3/2}^{t_1} \left(\| \omega^k w \|_2^2 \right) \tag{3.12}$$

and which holds for $0 < k < 3/2$. The required estimate

$$\| \omega^{2k-3/2} |w|^2 \|_2 \leq C \ \| \omega^k w \|_2^2 \tag{3.13}$$

follows from the HLS inequality if $2k < 3/2$ and from (3.4) if $2k \geq 3/2$, as mentioned above, and from Sobolev inequalities.

We next give a special estimate of the long range part $B_L^{t_1}$ of $B_1^{t_1}$.

Lemma 3.4 *Let $m > -3/2$. Then*

$$\| \omega^{m+1} B_L^{t_1}(w_1, w_2) \|_2 \leq C \ t^{\beta(m+3/2)} I_{-3/2}^{t_1} \left(\| w_1 \|_2 \ \| w_2 \|_2 \right) \ . \tag{3.14}$$

Proof. Let $f = D_0(\nu) \mathrm{Re} \ w_1 \bar{w}_2$. From (2.15) (2.17), we estimate

$$\| \omega^{m+1} B_L^{t_1}(w_1, w_2) \|_2 \leq \left| \int_1^{t_1/t} d\nu \ \nu^{-3} \ \ \| \chi(|\xi| \leq t^\beta) |\xi|^m \ Ff(\xi) \|_2 \right|$$

$$\leq \left| \int_1^{t_1/t} d\nu \ \nu^{-3} \ \ \| \chi(|\xi| \leq t^\beta) |\xi|^m \|_2 \ \| Ff \|_\infty \right|$$

$$\leq C \left| \int_1^{t_1/t} d\nu \ t^{\beta(m+3/2)} \ \| (w_1 \ \bar{w}_2)(\nu t) \|_1 \right|$$

which implies (3.14). □

We finally collect some estimates of the solution of the free wave equation $\square A_0 = 0$ with initial data $(A_+, \dot{A}_+)$ at time zero, given by (2.3).

Lemma 3.5 *Let $m \geq 0$. Let $\omega^m A_+ \in L^2$, $\omega^{m-1} \dot{A}_+ \in L^2$, $\nabla^2 \omega^m A_+ \in L^1$ and $\nabla \omega^m \dot{A}_+ \in L^1$. Then the following estimate holds*

$$\| \omega^m A_0 \|_r \leq b_0 \ t^{-1+2/r} \quad \textit{for } 2 \leq r \leq \infty \tag{3.15}$$

for all $t > 0$.

Proof. It suffices to prove (3.15) for $m = 0$, for $r = 2$ and $r = \infty$. For $r = 2$, it follows from (2.3) that

$$\| A_0 \|_2 \leq \| A_+ \|_2 + \| \omega^{-1} \dot{A}_+ \|_2 \tag{3.16}$$

for all $t \in \mathbb{R}$. For $r = \infty$, the result follows from the divergence theorem applied to the standard representation of solutions of the free wave equation in terms of spherical means [21]. □

The time decay expressed by (3.15) is known to be optimal, and we shall always consider solutions A_0 of the free wave equation satisfying those estimates for suitable m. In the applications, we shall use the estimates (3.15) in the equivalent form expressed in terms of B_0 defined by (2.13), namely

$$\| \omega^m \; B_0 \|_r \leq \; b_0 \; t^{m-1/r} \quad \text{for } 2 \leq r \leq \infty \; . \tag{3.17}$$

4 Cauchy problem and preliminary asymptotics for the auxiliary system

In this section, we study the Cauchy problem for the auxiliary system (2.20) and we derive some preliminary asymptotic properties of its solutions. This section illustrates both the method of solution with the help of the linearized version (2.21) of that system and the difficulties arising from the different propagation properties of the Schrödinger and wave equations. In particular we are able to prove the existence of solutions up to infinity in time only if $A_0 = 0$. This section could be the starting point for the construction of partial wave operators with vanishing asymptotic states for the field A, a construction which would be very similar to that of the wave operators for the Hartree equation performed in [6] [7], but which we shall refrain from performing here. The general case of non-vanishing asymptotic states for A will be treated by a similar but more complicated method in Section 6 below.

The basic tool of this section consists of a priori estimates for suitably regular solutions of the linearized system (2.21). Those estimates can be proved by a regularisation and limiting procedure and hold in the integrated form at the available level of regularity. For brevity, we shall state them in differential form and we shall restrict the proof to the formal computation.

We first estimate a single solution of the linearized system (2.21) at the level of regularity where we shall eventually solve the auxiliary system (2.20).

Lemma 4.1 *Let $1 < k \leq \ell$, $\ell > 3/2$ and $\beta > 0$. Let $I \subset [1, \infty)$ be an interval and let $t_1 \in \bar{I}$. Let B_0 satisfy the estimates (3.17) for $0 \leq m \leq k$. Let $(w, s), (w', s') \in \mathcal{C}(I, X^{k,\ell})$ with $w \in L^\infty(I, H^k)$ and let (w', s') be a solution of the system (2.21) in I. Then the following estimates hold for all $t \in I$:*

$\| w' \|_2 =$ *const.*

$$|\partial_t |w'|_k| \leq C\ b_0 \left\{ \| w' \|_2^{1/k}\ \ |w'|_k^{1-1/k} + t^{k-1-\delta/3}\ \ \| w' \|_2^{1-\delta/k}\ \ |w'|_k^{\delta/k} \right\} + C \left\{ t^{-2} |\nabla s|_\ell + t^{-1-\beta_1}\ I_{m_1}^{t_1}(|w|_k^2) \right\} |w'|_k \tag{4.1}$$

$$|\partial_t |\nabla s'|_\ell| \leq C\ t^{-2} |\nabla s|_\ell |\nabla s'|_\ell + C\ t^{-1+\beta_2} \Big(I_{m_1-k}^{t_1}(\| w \|_2\ |w|_k) + I_{m_1}^{t_1}(|w|_k^2) \Big) \tag{4.2}$$

where $0 < \delta \leq [k \wedge 3/2]$,

$$\beta_1 = \beta[1 \wedge 2(k-1)] = \beta(1 - 2[3/2-k]_+)\ , \tag{4.3}$$

$$m_1 = [k \wedge (2k - 3/2)] = k - [3/2-k]_+\ , \tag{4.4}$$

$$\beta_2 = \beta(\ell + 1 - k + [3/2-k]_+)\ . \tag{4.5}$$

Proof. We omit the superscript t_1 in all the proof. We first estimate w'. It is clear from (2.21) that $\| w' \|_2 =$ const. We next estimate

$$\left| \partial_t \| \omega^k w' \|_2 \right| \leq t^{-1} \| [\omega^k, B_0] w' \|_2 + t^{-2} \Big\{ \| [\omega^k, s] \cdot \nabla w' \|_2 + \| (\nabla \cdot s) \omega^k w' \|_2 + \| \omega^k ((\nabla \cdot s) w') \|_2 \Big\} + t^{-1} \| [\omega^k, B_S(w,w)] w' \|_2\ \ . \tag{4.6}$$

The contribution of B_0 is estimated by Lemma 3.2 and (3.17) as

$$\begin{aligned} \| [\omega^k, B_0] w' \|_2 &\leq C \left(\| \nabla B_0 \|_\infty \| \omega^{k-1} w' \|_2 + \| \omega^k B_0 \|_{3/\delta} \| w' \|_r \right) \\ &\leq C\ b_0 \Big(t \| \omega^{k-1} w' \|_2 + t^{k-\delta/3} \| w' \|_r \Big) \end{aligned} \tag{4.7}$$

with $0 < \delta = \delta(r) < k \wedge 3/2$. This yields the first term in the RHS of (4.1) by Sobolev inequalities and interpolation. We next estimate by Lemma 3.2

$$\begin{aligned} &\| [\omega^k, s] \cdot \nabla w' \|_2\ +\ \| (\nabla \cdot s) \omega^k w' \|_2\ +\ \| \omega^k ((\nabla \cdot s) w') \|_2 \\ &\leq C \left(\| \nabla s \|_\infty \| \omega^k w' \|_2 + \| \omega^k s \|_{3/\delta} \| \nabla w' \|_r + \| \omega^k (\nabla \cdot s) \|_{3/\delta'} \| w' \|_{r'} \right) \end{aligned}$$

where $0 < \delta = \delta(r) \leq [(k-1) \wedge 3/2]$ and $0 < \delta' = \delta(r') \leq [k \wedge 3/2]$. Choosing $\delta = [(k-1) \wedge 1/2]$ and $\delta' = [k \wedge 3/2]$ and using Sobolev inequalities, we continue the previous estimate by

$$\cdots \leq C \Big(\| \nabla s \|_\infty \| \omega^k w' \|_2 + \| \omega^{[k \vee 3/2]} \nabla s \|_2 \| \omega^{[k \wedge 3/2]} w' \|_2 + \chi(k > 3/2) \| \omega^k \nabla s \|_2 \| w' \|_\infty \Big) \leq C\ |\nabla s|_\ell\ |w'|_k\ . \tag{4.8}$$

We next estimate the contribution of B_S to (4.6). By Lemma 3.2 and Sobolev inequalities, we estimate

$$\begin{aligned} &\| [\omega^k, B_S(w,w)] w' \|_2 \\ &\leq C \left\{ \| \nabla B_S(w,w) \|_3 \| \omega^{k-1} w' \|_6 + \| \omega^k B_S(w,w) \|_{3/\delta} \| w' \|_r \right\} \\ &\leq C \Big\{ \| \omega^{3/2} B_S(w,w) \|_2 \| \omega^k w' \|_2 + \| \omega^{k+3/2-\delta} B_S(w,w) \|_2 \| w' \|_r \Big\} \end{aligned} \tag{4.9}$$

where $0 < \delta = \delta(r) \leq 3/2$. We choose $\delta = [k \wedge 3/2]$ and continue (4.9) as follows:

If $k < 3/2$, so that $\delta = k$,

$$\begin{aligned} \cdots &\le C \parallel \omega^{3/2} B_S(w,w) \parallel_2 \parallel \omega^k w' \parallel_2 \\ &\le C\, t^{-2\beta(k-1)} \parallel \omega^{2k-1/2} B_1(w,w) \parallel_2 \parallel \omega^k w' \parallel_2 \\ &\le C\, t^{-2\beta(k-1)} I_{2k-3/2} \left(\parallel \omega^k w \parallel_2^2 \right) \parallel \omega^k w' \parallel_2 \\ &\le C\, t^{-\beta_1} I_{m_1} \left(|w|_k^2 \right) |w'|_k \end{aligned} \tag{4.10}$$

by Sobolev inequalities, by (3.8) (3.12) and by the definitions (4.3) (4.4).

If $k = 3/2$, so that $\delta = 3/2 - \varepsilon$,

$$\begin{aligned} \cdots &\le C \parallel \omega^{3/2+\varepsilon} B_S(w,w) \parallel_2 \parallel \omega^{3/2-\varepsilon} w' \parallel_2 \\ &\le C\, t^{-\beta(1-2\varepsilon)} \parallel \omega^{5/2-\varepsilon} B_1(w,w) \parallel_2 \parallel \omega^{3/2-\varepsilon} w' \parallel_2 \\ &\le C\, t^{-\beta(1-2\varepsilon)} I_{3/2-\varepsilon} \Big(\parallel \omega^{(3-\varepsilon)/2} w \parallel_2^2 \Big) \parallel \omega^{3/2-\varepsilon} w' \parallel_2 \\ &\le C\, t^{-\beta_1} I_{m_1} \left(|w|_k^2 \right) |w'|_k \end{aligned} \tag{4.11}$$

by Sobolev inequalities, by (3.8), by (3.12) with $k = (3-\varepsilon)/2$ and by (4.3) (4.4).

If $k > 3/2$, so that $\delta = 3/2$ and $r = \infty$,

$$\begin{aligned} \cdots &\le C \parallel \omega^{k+1} B_1(w,w) \parallel_2 \Big\{ t^{-\beta(k-1/2)} \parallel \omega^k w' \parallel_2 + t^{-\beta} \parallel w' \parallel_\infty \Big\} \\ &\le C\, t^{-\beta}\, I_k \left(\parallel \omega^k w \parallel_2 \parallel w \parallel_\infty \right) \left(\parallel \omega^k w' \parallel_2 + \parallel w' \parallel_\infty \right) \\ &\le C\, t^{-\beta_1} I_{m_1} \left(|w|_k^2 \right) |w'|_k \end{aligned} \tag{4.12}$$

by (3.8) (3.10), Lemma 3.2, Sobolev inequalities and (4.3) (4.4). Substituting (4.7) (4.8) (4.10) (4.11) (4.12) into (4.6) yields (4.1).

We now turn to the estimate of s', namely to the proof of (4.2). For $0 \le m \le \ell$, we estimate

$$\begin{aligned} \left| \partial_t \parallel \omega^{m+1} s' \parallel_2 \right| \le\; & t^{-2} \left\{ \parallel [\omega^{m+1}, s] \cdot \nabla s' \parallel_2 + \parallel (\nabla \cdot s) \omega^{m+1}\, s' \parallel_2 \right\} \\ & + t^{-1} \parallel \omega^{m+2} B_L(w,w) \parallel_2 \ . \end{aligned} \tag{4.13}$$

The first bracket in the RHS of (4.13) is estimated by Lemma 3.2 as

$$\begin{aligned} \{\cdot\} &\le C \left(\parallel \nabla s \parallel_\infty \parallel \omega^{m+1} s' \parallel_2 + \parallel \omega^{m+1} s \parallel_2 \parallel \nabla s' \parallel_\infty \right) \\ &\le C |\nabla s|_\ell \, |\nabla s'|_\ell \end{aligned} \tag{4.14}$$

by Sobolev inequalities.

The contribution of B_L for $m = \ell$ is estimated by (3.9) (3.10) (3.12) and Lemma 3.2 as

$$\begin{aligned} \parallel \omega^{\ell+2} B_L(w,w) \parallel_2 \le\; & C\, t^{\beta(\ell+5/2-2k)}\, I_{2k-3/2} \left(\parallel \omega^k w \parallel_2^2 \right) \quad \text{for } k < 3/2\ , \\ & C\, t^{\beta(\ell+1-k)}\, I_k \left(\parallel \omega^k w \parallel_2 \parallel w \parallel_\infty \right) \quad \text{for } k > 3/2\ , \\ & C\, t^{\beta(\ell-1/2+\varepsilon)}\, I_{3/2-\varepsilon} \Big(\parallel \omega^{(3-\varepsilon)/2} w \parallel_2^2 \Big) \quad \text{for } k = 3/2\ , \\ & \le C\, t^{\beta_2}\, I_{m_1} \left(|w|_k^2 \right) \end{aligned} \tag{4.15}$$

in all cases. The contribution of B_L for $m = 0$ is estimated similarly as

$$\begin{aligned}\| \nabla^2 B_L(w,w) \|_2 \quad \leq \quad & C\ t^{\beta(5/2-k)}\ I_{k-3/2}\ \left(\| \omega^k w \|_2 \| w \|_2\right) & \text{for } k < 3/2\ , \\ & C\ t^{\beta}\ I_0\left(\| w \|_\infty \| w \|_2\right) & \text{for } k > 3/2\ , \\ & C\ t^{\beta(1+\varepsilon)}\ I_{-\varepsilon}\left(\| \omega^{3/2-\varepsilon} w \|_2 \| w \|_2\right) & \text{for } k = 3/2\ , \\ & \leq C\ t^{\beta'_2}\ I_{m_1-k}\left(\| w \|_2\ \ |w|_k\right) & (4.16)\end{aligned}$$

in all cases, with

$$\beta'_2 = \beta\,(1 + [3/2 - k]_+) \leq \beta_2 \tag{4.17}$$

since $\ell \geq k$.

Collecting (4.14) (4.15) (4.16) yields (4.2). □

We next estimate the difference of two solutions of the linearized system (2.21) corresponding to two different choices of (w, s). We estimate that difference at a lower level of regularity than the solutions themselves.

Lemma 4.2 *Let $1 < k \leq \ell$, $\ell > 3/2$ and $\beta > 0$. Let $I \subset [1,\infty)$ be an interval and let $t_1 \in \bar{I}$. Let B_0 be sufficiently regular, for instance $B_0 \in \mathcal{C}(I, H_3^k)$. Let $(w_i, s_i), (w'_i, s'_i) \in \mathcal{C}(I, X^{k,\ell})$ with $w_i \in L^\infty(I, H^k)$, $i = 1, 2$, and let (w'_i, s'_i) be solutions of the system (2.21) associated with (w_i, s_i). Define $(w_\pm, s_\pm) = (1/2)(w_1 \pm w_2, s_1 \pm s_2)$ and $(w'_\pm, s'_\pm) = 1/2(w'_1 \pm w'_2, s'_1 \pm s'_2)$. Then the following estimates hold for all $t \in I$:*

$$\left|\partial_t \| w'_- \|_2\right| \leq C\ t^{-2} |\nabla s_-|_{\ell_0}\ |w'_+|_k + C t^{-1-\beta_1}\ I^{t_1}_{m_1-k}\left(|w_+|_k \| w_- \|_2\right) |w'_+|_k \tag{4.18}$$

$$\begin{aligned}\left|\partial_t |\nabla s'_-|_{\ell_0}\right| \quad \leq \quad & C\ t^{-2}\left(|\nabla s_+|_\ell\ |\nabla s'_-|_{\ell_0} + |\nabla s_-|_{\ell_0}\ |\nabla s'_+|_\ell\right) \\ + \quad & C\ t^{-1+\beta_2}\ I^{t_1}_{m_1-k}\left(|w_+|_k\ \ \| w_- \|_2\right) & (4.19)\end{aligned}$$

where β_1, m_1 and β_2 are defined by (4.3) (4.4) (4.5) and where

$$[3/2 - k]_+ \leq \ell_0 \leq \ell - k\ . \tag{4.20}$$

Proof. We again omit the superscript t_1 in the proof. Taking the difference of the system (2.21) for (w'_i, s'_i), we obtain the following system for (w'_-, s'_-):

$$\left\{\begin{aligned} & \partial_t w'_- = i(2t^2)^{-1}\Delta w'_- + t^{-2}(Q(s_+, w'_-) + Q(s_-, w'_+)) + it^{-1}B_0 w'_- \\ & \qquad + it^{-1}\left\{\left(B_S(w_+, w_+) + B_S(w_-, w_-)\right) w'_- + 2B_S(w_+, w_-)w'_+\right\} \\ & \partial_t s'_- = t^{-2}\left(s_+ \cdot \nabla s'_- + s_- \cdot \nabla s'_+\right) - 2t^{-1}\nabla B_L(w_+, w_-)\ .\end{aligned}\right. \tag{4.21}$$

We first estimate w'_-. From (4.21) we obtain

$$\left|\partial_t \| w'_- \|_2\right| \leq t^{-2} \| Q(s_-, w'_+) \|_2 + 2t^{-1} \| B_S(w_+, w_-)w'_+ \|_2 \tag{4.22}$$

where only those terms appear that do not preserve the L^2-norm. We estimate the first norm in the RHS by Hölder and Sobolev inequalities as follows:

If $k < 3/2$,

$$\begin{aligned}\| Q(s_-, w'_+) \|_2 &\leq C \left(\| s_- \|_{3/(k-1)} + \| \nabla \cdot s_- \|_{3/k}\right) \| \omega^k w'_+ \|_2 \\ &\leq C \| \omega^{3/2-k} \nabla s_- \| \, \| \omega^k w' \|_2 \ .\end{aligned}$$

If $k = 3/2$,

$$\| Q(s_-, w'_+) \|_2 \leq C \| \omega^\varepsilon \nabla s_- \|_2 \, \| \omega^{3/2-\varepsilon} \ w'_+ \|_2 \ .$$

If $k > 3/2$,

$$\| Q(s_-, w'_+) \|_2 \leq C \| \nabla s_- \|_2 \left(\| \nabla w'_+ \|_3 + \| w'_+ \|_\infty\right) \ ,$$

and in all cases

$$\| Q(s_-, w'_+) \|_2 \leq C |\nabla s_-|_{\ell_0} \ |w'_+|_k \tag{4.23}$$

provided $\ell_0 \geq [3/2 - k]_+$.

We estimate the second norm in the RHS of (4.22) by (3.8) (3.10), by Lemma 3.2 and by the Hölder and Sobolev inequalities as follows:

If $k < 3/2$,

$$\begin{aligned}&\| B_S(w_+, w_-) w'_+ \|_2 \leq C \| \omega^{3/2-k} \ B_S(w_+, w_-) \|_2 \, \| \omega^k w'_+ \|_2 \\ &\leq C \ t^{-2\beta(k-1)} \| \omega^{k-1/2} \ B_1(w_+, w_-) \|_2 \, \| \omega^k w'_+ \|_2 \\ &\leq C \ t^{-2\beta(k-1)} \ I_{k-3/2} \left(\| \omega^k w_+ \|_2 \, \| w_- \|_2\right) \| \omega^k w'_+ \|_2 \ .\end{aligned}$$

If $k = 3/2$,

$$\| B_S(w_+, w_-) w'_+ \|_2 \leq C \ t^{-\beta(1-2\varepsilon)} \ I_{-\varepsilon} \left(\| \omega^{3/2-\varepsilon} w_+ \|_2 \, \| w_- \|_2\right) \| \omega^{3/2-\varepsilon} w'_+ \|_2 \ .$$

If $k > 3/2$,

$$\begin{aligned}&\| B_S(w_+, w_-) w'_+ \|_2 \leq t^{-\beta} \| \nabla B_S(w_+, w_-) \|_2 \, \| w'_+ \|_\infty \\ &\leq C \ t^{-\beta} \ I_0 \left(\| w_+ \|_\infty \, \| w_- \|_2\right) \| w'_+ \|_\infty\end{aligned}$$

and in all cases

$$\| B_S(w_+, w_-) w'_+ \|_2 \leq C \ t^{-\beta_1} I_{m_1-k} \left(|w_+|_k \, \| w_- \|_2\right) |w'_+|_k \ . \tag{4.24}$$

with β_1 and m_1 defined by (4.3) (4.4). Substituting (4.23) (4.24) into (4.22) yields (4.18).

We now turn to the estimate of s'_-, namely to the proof of (4.19). From (4.21) we estimate for $m \geq 0$

$$\begin{aligned}\partial_t \| \omega^{m+1} s'_- \|_2 \leq t^{-2} \Big\{ &\| [\omega^{m+1}, s_+] \cdot \nabla s'_- \|_2 + \| (\nabla \cdot s_+) \omega^{m+1} s'_- \|_2 \\ + &\| \omega^{m+1} (s_- \cdot \nabla s'_+) \|_2 \Big\} + 2t^{-1} \| \omega^{m+2} B_L(w_+, w_-) \|_2 \ .\end{aligned} \tag{4.25}$$

If $m = 0$ (a case which has to be self-estimating if $\ell_0 = 0$, which is allowed if $k > 3/2$), we estimate the bracket in the RHS of (4.25) directly as

$$\begin{aligned} \{(m=0)\} &\le \| \nabla s_+ \|_\infty \| \nabla s'_- \|_2 + \| \nabla s_- \|_2 \left(\| \nabla s'_+ \|_\infty + \| \nabla^2 s'_+ \|_3 \right) \\ &\le C \left(|\nabla s_+|_\ell \| \nabla s'_- \|_2 + \| \nabla s_- \|_2 \ |\nabla s'_+|_\ell \right) \end{aligned} \tag{4.26}$$

since $\ell > 3/2$.

If $m > 0$, we estimate that bracket by Lemma 3.2 and Sobolev inequalities as

$$\begin{aligned} \{\cdot\} \le C \Big\{ & \| \nabla s_+ \|_\infty \| \omega^{m+1} s'_- \|_2 + \| \omega^{m+1} s_+ \|_{3/\delta} \| \nabla s'_- \|_r \\ & + \| \omega^{m+1} s_- \|_2 \| \nabla s'_+ \|_\infty + \| s_- \|_{r'} \| \omega^{m+2} s'_+ \|_{3/\delta'} \Big\} \end{aligned} \tag{4.27}$$

where $0 < \delta = \delta(r) \le 3/2$, $0 < \delta' = \delta(r') \le 3/2$. The first and third term in the RHS of (4.27) are readily controlled by the corresponding terms in (4.19) for $0 < m \le \ell_0$ and $\ell > 3/2$. The remaining two terms are similarly controlled through Sobolev inequalities provided

$$\begin{aligned} &0 < \delta \le [\ell_0 \wedge 3/2] \quad , \quad m + 3/2 - \delta \le \ell \ . \\ &1 \le \delta' \le [(\ell_0 + 1) \wedge 3/2] \quad , \quad m + 5/2 - \delta' \le \ell \ . \end{aligned}$$

Those conditions are easily seen to be compatible in δ and δ' for all m, $0 < m \le \ell_0$, provided $\ell \ge [(\ell_0 + 1) \vee 3/2]$, which follows from $\ell > 3/2$ and $\ell \ge \ell_0 + k$.

We finally estimate the contribution of $B_L(w_+, w_-)$ by

$$\| \omega^{m+2} B_L(w_+, w_-) \|_2 \le C \ t^{\beta m} \| \nabla^2 B_L(w_+, w_-) \|_2$$

by (3.9) and we estimate the last norm in exactly the same way as in (4.16), thereby obtaining

$$\| \omega^{m+2} B_L(w_+, w_-) \|_2 \le C \ t^{\beta'_2 + \beta m} \ I_{m_1 - k} \left(|w_+|_k \| w_- \|_2 \right) \ . \tag{4.28}$$

Collecting (4.25) (4.26), (4.27) and the discussion that follows, and (4.28) and noting that $\beta'_2 + \beta m \le \beta_2$ for $m \le \ell_0 \le \ell - k$, we obtain (4.19). □

With the estimates of Lemma 4.1 and 4.2 available, it is an easy matter to solve the Cauchy problem globally in time for the linearized system (2.21).

Proposition 4.1 *Let $1 < k \le \ell$, $\ell > 3/2$ and $\beta > 0$. Let $I \subset [1, \infty)$ be an interval and let $t_1 \in \bar{I}$. Let B_0 satisfy the estimates (3.17) for $0 \le m \le k$. Let $(w, s) \in \mathcal{C}(I, X^{k,\ell})$ with $w \in L^\infty(I, H^k)$. Let $t_0 \in I$ and let $(w'_0, s'_0) \in X^{k,\ell}$. Then the system (2.21) has a unique solution $(w', s') \in \mathcal{C}(I, X^{k,\ell})$ with $(w', s')(t_0) = (w'_0, s'_0)$. That solution satisfies the estimates (4.1) (4.2) for all $t \in I$. Two such solutions (w'_i, s'_i) associated with (w_i, s_i), $i = 1, 2$, satisfy the estimates (4.18) (4.19) for all $t \in I$.*

Proof. The proof proceeds in the same way as that of Proposition 4.1 of [6], through a parabolic regularization and a limiting procedure, with the simplification that the system (2.21) is linear. We define $U_1(t) = U(1/t)$, $\widetilde{w}'(t) = U_1(t)w'(t)$. We first consider the case $t \geq t_0$. The system (2.21) with a parabolic regularization added is rewritten in terms of the variables $(\widetilde{w}', s')$ as

$$\begin{cases} \partial_t \widetilde{w}' & = \eta\Delta\widetilde{w}' + t^{-2}U_1 Q(s, U_1^* \widetilde{w}') + it^{-1}U_1(B_0 + B_S(w,w))U_1^*\widetilde{w}' \\ & \equiv \eta\Delta\widetilde{w}' + F(\widetilde{w}') \\ \partial_t s' & = \eta\Delta s' + t^{-2}s\cdot\nabla s' - t^{-1}\nabla B_L(w,w) \equiv \eta\Delta s' + G(s') \end{cases} \tag{4.29}$$

where the parametric dependence of F, G on (w, s) has been omitted. The Cauchy problem for the system (4.29) can be recast in the integral form

$$\begin{pmatrix} \widetilde{w}' \\ s' \end{pmatrix}(t) = V_\eta(t - t_0)\begin{pmatrix} \widetilde{w}'_0 \\ s'_0 \end{pmatrix} + \int_{t_0}^t dt'\ V_\eta(t - t')\begin{pmatrix} F(\widetilde{w}') \\ G(s') \end{pmatrix}(t') \tag{4.30}$$

where $V_\eta(t) = \exp(\eta t\Delta)$. The operator $V_\eta(t)$ is a contraction in $X^{k,\ell}$ and satisfies the bound

$$\| \nabla V_\eta(t); \mathcal{L}(X^{k,\ell}) \| \leq C(\eta t)^{-1/2}\ .$$

From those facts and from estimates on F, G similar to and mostly contained in those of Lemma 4.1, it follows by a contraction argument that the system (4.30) has a unique solution $(\widetilde{w}'_\eta, s'_\eta) \in \mathcal{C}([t_0, t_0 + T], X^{k,\ell})$ for some $T > 0$ depending only on $|w'_0|_k$, $|s'_0|_\ell$ and η. That solution satisfies the estimates (4.1) and (4.2) and can therefore be extended to $I_+ = I \cap \{t : t \geq t_0\}$ by a standard globalisation argument using Gronwall's inequality.

We next take the limit $\eta \to 0$. Let η_1, $\eta_2 > 0$ and let $(w'_i, s'_i) = (w'_{\eta_i}, s'_{\eta_i})$, $i = 1, 2$ be the corresponding solutions. Let $(w'_-, s'_-) = (1/2)(w'_1 - w'_2, s'_1 - s'_2)$. By estimates similar to, but simpler than those of Lemma 4.2, since in particular $(w_-, s_-) = 0$, we obtain

$$\begin{cases} \partial_t \| w'_- \|_2^2 \leq |\eta_1 - \eta_2| \left(\| \nabla w'_1 \|_2^2 + \| \nabla w'_2 \|_2^2 \right) \\ \partial_t \| \nabla s'_- \|_2^2 \leq |\eta_1 - \eta_2| \left(\| \nabla^2 s'_1 \|_2^2 + \| \nabla^2 s'_2 \|_2^2 \right) + Ct^{-2} \| \nabla s_+ \|_\infty \| \nabla s'_- \|_2^2\ . \end{cases}$$

Those estimates imply that (w'_η, s'_η) converges in $X^{0,0}$ uniformly in time in the compact subintervals of I_+, to a solution of the original system. It follows then by a standard compactness argument using the estimates (4.1) (4.2) that the limit belongs to $\mathcal{C}(I_+, X^{k,\ell})$. This completes the proof for $t \geq t_0$. The case $t \leq t_0$ is treated similarly. □

We now turn to the Cauchy problem for the auxiliary system (2.20). Because of the difficulties described in Section 2, the problem of existence of solutions is scattered with pitfalls, as the discussion below will show. On the other hand, the uniqueness problem of suitably bounded solutions is a rather easy matter and we consider that problem first. The proof relies entirely on Lemma 4.2 and therefore does not require any a priori estimate on B_0. The snag of course is that it is difficult to prove the existence of solutions with the required boundedness properties.

Proposition 4.2 *Let $1 < k \leq \ell$, $\ell > 3/2$ and $\beta > 0$. Let $I \subset [1, \infty)$ be an interval and let $t_1 \in \bar{I}$. Let B_0 be sufficiently regular, for instance $B_0 \in \mathcal{C}(I, H_3^k)$.*

(1) *Let $t_0 = t_1 < \infty$ and let $(w_0, s_0) \in X^{k,\ell}$. Then the system (2.20) has at most one solution $(w, s) \in \mathcal{C}(I, X^{k,\ell})$ with $w \in L^\infty(I, H^k)$ and $(w, s)(t_0) = (w_0, s_0)$.*
Let now $\beta_2 < 1$, where β_2 is defined by (4.5), let ℓ_0 satisfy (4.20). Let (w_i, s_i), $i = 1, 2$ be two solutions of the system (2.20) in I such that $(w_i, t^{\eta-1}s_i) \in (\mathcal{C} \cap L^\infty)(I, X^{k,\ell})$ for some $\eta > 0$ and let

$$\| w_i, L^\infty(H^k) \| \leq a \qquad , \qquad \| t^{\eta-1}\nabla s_i; L^\infty(H^\ell) \| \leq b \; . \tag{4.31}$$

(2) *Let $t_0 \in I$, $t_0 \leq t_1$, $t_0 < \infty$ and assume that $(w_1, s_1)(t_0) = (w_2, s_2)(t_0)$. Then there exists $c = c(a, b)$ such that if*

$$\left(t_0^{-(1-\beta_2)} \vee t_0^{-\beta_1} \right) (1 - (t_0/t_1)^\alpha) \leq c(a, b) \tag{4.32}$$

where $\alpha = [k \wedge 3/2] - 1$, then $(w_1, s_1) = (w_2, s_2)$. In particular there exists $T_0 = T_0(a, b)$ such that if $t_0 \geq T_0$, then $(w_1, s_1) = (w_2, s_2)$.

(3) *Let $t_1 = \infty$. Assume that $\| w_1 - w_2 \|_2 \, t^{\beta_2}$ and $|\nabla(s_1 - s_2)|_{\ell_0}$ tend to zero when $t \to \infty$. Then $(w_1, s_1) = (w_2, s_2)$.*

Proof. If (w_i, s_i), $i = 1, 2$ are two solutions of the system (2.20) in $\mathcal{C}(I, X^{k,\ell})$, then they satisfy the estimates (4.18) (4.19) with $(w'_i, s'_i) = (w_i, s_i)$, which we denote (4.18=) (4.19=) and refrain from rewriting for brevity. The proof consists in exploiting those estimates to prove that $(w_1, s_1) = (w_2, s_2)$. We define $y = \| w_- \|_2$ and $z = |\nabla s_-|_{\ell_0}$.

Part (1). With $t_0 = t_1 < \infty$, the estimates (4.18=) (4.19=) take the general form

$$|\partial_t y| \leq f_1(t) z + g_1(t) \int_{t_0}^t dt' \; h_1(t') \; y(t') \tag{4.33}$$

$$|\partial_t z| \leq f_2(t) z + g_2(t) \int_{t_0}^t dt' \; h_2(t') \; y(t') \tag{4.34}$$

for suitable continuous nonnegative functions f_1, g_1, h_1, f_2, g_2, h_2 (actually $h_1 = h_2$, but that is irrelevant). Furthermore $y(t_0) = z(t_0) = 0$. We shall reduce the system (4.33) (4.34) to a standard form where Gronwall's inequality is applicable. We restrict our attention to the case $t \geq t_0$ for definiteness. The case $t \leq t_0$ can be treated similarly. Defining $\widetilde{z}$ by

$$z(t) = E(t) \; \widetilde{z}(t) = \exp \left\{ \int_{t_0}^t dt' \; f_2(t') \right\} \widetilde{z}(t) \; ,$$

we reduce the system (4.33) (4.34) for (y, z) to a similar system for $(y, \widetilde{z})$, where f_2, g_2 and f_1 are replaced by 0, $E^{-1}g_2$ and Ef_1. We can therefore assume that

$f_2 = 0$. Then

$$z(t) \le \int_{t_0}^{t} dt'' \ g_2(t'') \int_{t_0}^{t''} dt' \ h_2(t') \ y(t') \le G_2(t) \int_{t_0}^{t} dt' \ h_2(t') \ y(t')$$

where

$$G_2(t) = \int_{t_0}^{t} dt'' \ g_2(t'')$$

so that

$$\begin{aligned} \partial_t y \quad &\le \quad f_1(t) G_2(t) \int_{t_0}^{t} dt' \ h_2(t') y(t') + g_1(t) \int_{t_0}^{t} dt' \ h_1(t') \ y(t') \\ &\le \quad (f_1 \ G_2 + g_1) \int_{t_0}^{t} dt' (h_1 \vee h_2)(t') \ y(t') \end{aligned}$$

which is of the same form as (4.33) with $f_1 = 0$. Integrating the latter yields

$$y \le \int_{t_0}^{t} dt'' \ g_1(t'') \int_{t_0}^{t''} dt' \ h_1(t') \ y(t') \le G_1(t) \int_{t_0}^{t} dt' \ h_1(t') \ y(t')$$

where

$$G_1(t) = \int_{t_0}^{t} dt' \ g_1(t') \ ,$$

which together with $y(t_0) = 0$ implies $y(t) = 0$ for all t by an easy variant of Gronwall's inequality. Substituting that result into (4.34) (with $f_2 = 0$) yields $z = 0$ and therefore $(w_1, s_1) = (w_2, s_2)$.

We now turn to the proof of Parts (2) and (3). Introducing the assumption and notation (4.31), changing the variable from ν to $t' = \nu t$ in the definition of $I_m^{t_1}$, and omitting an absolute overall constant, we can rewrite (4.18=) (4.19=) in the form

$$|\partial_t y| \le t^{-2} \ az + t^{-1-\beta_1+\alpha} \ a^2 \int_{t}^{t_1} dt' \ t'^{-1-\alpha} \ y(t') \tag{4.35}$$

$$|\partial_t z| \le t^{-1-\eta} \ bz + t^{-1+\beta_2+\alpha} \ a \int_{t}^{t_1} dt' \ t'^{-1-\alpha} \ y(t') \tag{4.36}$$

where $\alpha = [k \wedge 3/2] - 1 > 0$, and the goal is to prove that (4.35) (4.36) with suitable initial conditions imply $y = z = 0$.

Part (2). Let $Y = \| \ y; L^\infty([t_0, t_1]) \ \|$. Then

$$\begin{aligned} t^\alpha \int_{t}^{t_1} dt' \ t'^{-1-\alpha} \ y(t') \quad &\le \quad Y \ \alpha^{-1} \left(1 - (t/t_1)^\alpha\right) \\ &\le \quad Y \ \alpha^{-1} \left(1 - t_0/t_1)^\alpha\right) \equiv \bar{Y} \ . \end{aligned} \tag{4.37}$$

Substituting (4.37) into (4.36) and integrating with $z(t_0) = 0$ yields

$$z \le \exp\left(b\ \eta^{-1}\ t_0^{-\eta}\right)\ a\ \bar{Y}\ \beta_2^{-1}\ t^{\beta_2}\ . \tag{4.38}$$

Substituting (4.37) (4.38) into (4.35), integrating with $y(t_0) = 0$ and taking the Supremum over t in $[t_0, t_1]$ yields

$$Y \le \left\{\exp\left(b\ \eta^{-1}\ t_0^{-\eta}\right)(1-\beta_2)^{-1}\ t_0^{-(1-\beta_2)} + \beta_1^{-1}\ t_0^{-\beta_1}\right\} a^2 \bar{Y}$$

which implies $Y = 0$ and therefore $y = z = 0$ provided

$$a^2 \left\{\exp\left(b\ \eta^{-1}\ t_0^{-\eta}\right)(1-\beta_2)^{-1}\ t_0^{-(1-\beta_2)} + \beta_1^{-1}\ t_0^{-\beta_1}\right\} \alpha^{-1}\left(1-(t_0/t_1)^{\alpha}\right) < 1\ , \tag{4.39}$$

a condition which follows from (4.32) for suitable $c(a, b)$.

Part (3). We now take $t_1 = \infty$. The term bz in (4.36) can be exponentiated as in the proof of Part (2). Since in addition the statement does not involve conditions on a and b, we can and shall assume without loss of generality that $b = 0$ and $a = 1$. Let

$$\varepsilon(t) = \mathop{\mathrm{Sup}}_{t' \ge t} t'^{\beta_2} y(t')\ . \tag{4.40}$$

Then $\varepsilon(t)$ is nonincreasing in t and tends to zero as $t \to \infty$. Furthermore for any $t_0 \in I$

$$\int_{t_0}^{\infty} dt'\ t'^{-1-\alpha}\ y(t') \le \varepsilon(t_0)(\alpha+\beta_2)^{-1}\ t_0^{-(\alpha+\beta_2)}\ . \tag{4.41}$$

Let now $t_0 \in I$ (t_0 will eventually tend to ∞), $y_0 = y(t_0)$ and $z_0 = z(t_0)$. We estimate y and z for $t \le t_0$ by integrating (4.35) (4.36) (with $t_1 = \infty$, $a = 1$ and $b = 0$) between t and t_0. Integrating (4.36) yields

$$\begin{aligned} z(t) &\le z_0 + (\alpha+\beta_2)^{-2}\varepsilon(t_0) + \int_t^{t_0} dt''\ t''^{-1+\beta_2+\alpha} \int_{t''}^{t_0} dt'\ t'^{-1-\alpha}\ y(t') \\ &\le \cdots + \int_t^{t_0} dt'\ t'^{-1-\alpha}\ y(t') \int_t^{t'} dt'\ t''^{-1+\beta_2+\alpha} \\ &\le z_0 + (\alpha+\beta_2)^{-2}\ \varepsilon(t_0) + (\alpha+\beta_2)^{-1}\ Y(t) \end{aligned} \tag{4.42}$$

where we have used (4.41) and where

$$Y(t) = \int_t^{t_0} dt'\ t'^{-1+\beta_2}\ y(t')\ . \tag{4.43}$$

Substituting (4.42) into (4.35), integrating and using the fact that $Y(t)$ is decreasing in t, we obtain

$$y(t) \le y_0 + t^{-1}\left(z_0 + (\alpha+\beta_2)^{-2}\varepsilon(t_0)\right) + (\alpha+\beta_2)^{-1}\ t^{-1} Y(t) + y_1(t) \tag{4.44}$$

where

$$y_1(t) = \int_t^{t_0} dt''\ t''^{-1-\beta_1+\alpha} \int_{t''}^{\infty} dt'\ t'^{-1-\alpha}\ y(t')\ .$$

Substituting (4.44) into (4.43) yields

$$\begin{aligned} Y(t) \quad &\le \quad y_0\ \beta_2^{-1}\ t_0^{\beta_2} + \big(z_0 + (\alpha+\beta_2)^{-2}\varepsilon(t_0)\big)\,(1-\beta_2)^{-1}\ t^{-(1-\beta_2)} \\ &+ \quad (\alpha+\beta_2)^{-1} \int_t^{t_0} dt'\ t'^{-2+\beta_2}\ Y(t') + Y_1(t) \end{aligned} \tag{4.45}$$

where

$$\begin{aligned} &Y_1(t) = \int_t^{t_0} dt'''\ t'''^{-1+\beta_2}\ y_1(t''') \\ &= \int_t^{t_0} dt''\ t''^{-1-\beta_1+\alpha} \int_t^{t''} dt'''\ t'''^{-1+\beta_2} \int_{t''}^{\infty} dt'\ t'^{-1-\alpha}\ y(t') \\ &\le \beta_2^{-1} \int_t^{t_0} dt''\ t''^{-1-\beta_1+\alpha+\beta_2} \int_{t''}^{\infty} dt'\ t'^{-1-\alpha}\ y(t') \\ &\le \beta_2^{-1}\ t^{-\beta_1}(\alpha+\beta_2)^{-2}\ \varepsilon(t_0) + \beta_2^{-1} \int_t^{t_0} dt''\ t''^{-1-\beta_1} \int_{t''}^{t_0} dt'\ t'^{-1+\beta_2}\ y(t') \\ &\le \beta_2^{-1}\ t^{-\beta_1}(\alpha+\beta_2)^{-2}\ \varepsilon(t_0) + \beta_2^{-1} \int_t^{t_0} dt''\ t''^{-1-\beta_1}\ Y(t'')\ . \end{aligned} \tag{4.46}$$

Substituting (4.46) into (4.45) yields the following inequality for $Y(t)$:

$$Y(t) \le f(t) + \int_t^{t_0} dt'\ g(t')\ Y(t') \tag{4.47}$$

where

$$f(t) = \beta_2^{-1}\varepsilon(t_0) + z_0(1-\beta_2)^{-1}\ t^{-(1-\beta_2)} + (\alpha+\beta_2)^{-2}\varepsilon(t_0)$$

$$\Big((1-\beta_2)^{-1}t^{-(1-\beta_2)} + \beta_2^{-1}t^{-\beta_1}\Big)\, g(t) = (\alpha+\beta_2)^{-1}\ t^{-2+\beta_2} + \beta_2^{-1}\ t^{-1-\beta_1}\ .$$

Note that f and g are decreasing in t and that g is integrable at infinity. Let

$$G(t) = \int_t^{\infty} g(t')\ dt'\ ,$$

$$\bar{Y}(t) = \int_t^{t_0} dt'\ g(t')\ Y(t')\ .$$

Then (4.47) can be rewritten as

$$\partial_t \bar{Y} = -gY \ge -gf - g\bar{Y}$$

which is readily integrated with $\bar{Y}(t_0) = 0$ to yield

$$\bar{Y}(t) \leq \int_t^{t_0} dt'\ g(t')\ f(t') \exp\left\{ \int_t^{t'} dt''\ g(t'') \right\} \leq f(t)\ G(t) \exp(G(t))$$

and therefore by (4.47) again

$$Y(t) \leq f(t) \left\{ 1 + G(t) \exp(G(t)) \right\} \ .$$

Now G is independent of t_0 while f tends to zero when $t_0 \to \infty$ for fixed t under the assumptions made. Letting $t_0 \to \infty$ then shows that

$$Y(t) = \int_t^{t_0} dt'\ t'^{-1+\beta_2}\ y(t') \longrightarrow 0 \quad \text{when } t_0 \to \infty$$

which implies that $y = 0$, from which it follows easily that $z = 0$, and therefore $(w_1, s_1) = (w_2, s_2)$. □

Remark 4.1 The necessity of some condition of the type (4.32) in Part (2) is easily understood on the simpler example

$$|\partial_t y| \leq a^2 \int_t^{t_1} dt'\ y(t') \tag{4.48}$$

with $t_0 = 0$ and $y(0) = 0$. Defining $Y = \parallel y; L^\infty([0, t_1]) \parallel$ we obtain

$$|\partial_t y| \leq a^2 (t_1 - t) Y$$

and therefore by integration

$$Y \leq a^2 Y \mathop{\mathrm{Sup}}_{0 \leq t \leq t_1} \int_0^t dt' (t_1 - t') = (1/2) a^2\ t_1^2\ Y$$

which implies $Y = 0$ if $a t_1 < \sqrt{2}$. However if $a t_1 = \pi/2$, (4.48) admits the nonvanishing solution $y = \sin at$.

We next prove another property which follows easily from estimates similar to those of Lemma 4.2, namely the fact that for suitably bounded solutions (w, s) of the auxiliary system, $w(t)$ tends to a limit w_+ when $t \to \infty$.

Proposition 4.3 *Let $k > 1$, $\ell_0 \leq [3/2 - k]_+$ and $\beta > 0$. Let $T \geq 1$, $t_1 = \infty$ and $I = [T, \infty)$. Let B_0 satisfy the estimate (3.17) for $m = 0$. Let $(w, s) \in \mathcal{C}(I, X^{k,\ell_0})$ with $(w, t^{\eta-1} s) \in L^\infty(I, X^{k,\ell_0})$ for some $\eta > 0$ and let (w, s) satisfy the first equation of the system (2.20). Then there exists $w_+ \in H^k$ such that $w(t)$ tends to w_+ weakly in H^k and strongly in $H^{k'}$ for $0 \leq k' < k$ when $t \to \infty$. Furthermore the following estimate holds for all $t \in I$:*

$$\parallel \widetilde{w}(t) - w_+ \parallel_2 \leq C\ t^{-\alpha_1} \tag{4.49}$$

where $\widetilde{w}(t) = U(1/t)w$, *and*

$$\alpha_1 = \eta \wedge [1/2 \wedge k/3] \wedge (\beta_1 + \beta k)$$

with β_1 *defined by (4.3).*

Proof. Let $t_0 \in I$, $\widetilde{w}_0 = \widetilde{w}(t_0)$ and

$$a = \| w; L^\infty(I, H^k) \| \quad , \quad b = \| t^{\eta-1} \nabla s; L^\infty(I, H^{\ell_0}) \| \ .$$

The first equation of the system (2.20) can be rewritten as

$$\partial_t (\widetilde{w}(t) - \widetilde{w}_0) = t^{-2}\, U(1/t)\, Q(s,w) + it^{-1}\, U(1/t)\, (B_0 + B_S(w,w))\, w$$

where we have omitted the superscript ∞ in B_S, and therefore

$$\partial_t \| \widetilde{w}(t) - \widetilde{w}_0 \|_2 \le t^{-2} \| Q(s,w) \|_2 + t^{-1} \| B_0 w \|_2 + t^{-1} \| B_S(w,w) w \|_2 \ . \tag{4.50}$$

By exactly the same estimate as in (4.23), we obtain

$$\| Q(s,w) \|_2 \le C |\nabla s|_{\ell_0}\ |w|_k \le C\ ab\ t^{1-\eta} \ . \tag{4.51}$$

We next estimate

$$\| B_0 w \|_2 \le \| B_0 \|_{3/\delta} \| w \|_r \le\ C\ ab_0\ t^{-[1/2 \wedge k/3]} \tag{4.52}$$

with $\delta = \delta(r) = [k \wedge 3/2]$, by (3.17) and Sobolev inequalities, and

$$\begin{aligned} \| B_S(w,w) w \|_2 &\le \| B_S(w,w) \|_{3/\delta} \| w \|_r \\ &\le C \| \omega^{[3/2-k]_+}\ B_S(w,w) \|_2\ |w|_k \end{aligned} \tag{4.53}$$

with the same δ. The last norm of B_S in (4.53) is estimated exactly as in the proof of Lemma 4.1 (see (4.10) (4.11) (4.12)) as

$$\| \omega^{[3/2-k]_+}\ B_S(w,w) \|_2 \le C\ t^{-\beta_1 - \beta k}\ I_{m_1}^\infty(|w|_k^2) \le C\ a^2\ t^{-\beta_1-\beta k} \ . \tag{4.54}$$

Substituting (4.51) (4.52) (4.53) (4.54) into (4.50) and integrating between t_0 and t yields

$$\| \widetilde{w}(t) - \widetilde{w}(t_0) \|_2 \le C \left\{ (t \wedge t_0)^{-\eta} ab + (t \wedge t_0)^{-[1/2 \wedge k/3]} ab_0 + (t \wedge t_0)^{-\beta_1 - \beta k} a^3 \right\}$$

from which it follows that $\widetilde{w}(t)$ and therefore also $w(t)$ has a strong limit w_+ in L^2 when $t \to \infty$ and that (4.49) holds. Since in addition $w(t)$ is bounded in H^k, it follows by a standard compactness argument that $w_+ \in H^k$ and that $w(t)$ tends to w_+ in the other topologies stated in the Proposition. □

We now turn to the problem of existence of solutions of the auxiliary system (2.20), with the aim of proving that that system defines an asymptotic dynamics for large times and preferably up to infinity in time. Here however, we encounter the difficulties described in Section 2 and arising from the different propagation properties of the Schrödinger and wave equations. First of all for $t_1 = t_0$, even if $B_0 = 0$, the estimates of Lemma 4.1 are insufficient to prevent blow up of the solutions in a finite time after t_0, independently of the size of t_0 and of the initial data for (w, s) at t_0. In fact, if in the estimates (4.1=) (4.2=) we set $b_0 = 0$, omit the second inequality and take $s = 0$ in the first one, we obtain the following stronger estimate for $y = |w'|_k = |w|_k$

$$\partial_t \; y \leq C \; t^{-1-\beta_1} \; y \int_{t_0/t}^{1} d\nu \; \nu^{-1-m} \; y(\nu t)^p \tag{4.55}$$

where $m = m_1 + 1/2 > \beta_1$ for $\beta \leq 1$, and $p = 2$. We shall prove that (4.55) does not prevent finite time blow up by showing that equality in (4.55) implies such a blow up. Taking y^p instead of y as the unknown function and rescaling t and y, we can take $p = 1$, $t_0 = 1$ and $C = 1$ without loss of generality. We are therefore led to consider the equation

$$\partial_t \; y = t^{-1-\beta_1} \; y \int_{1/t}^{1} d\nu \; \nu^{-1-m} \; y(\nu t) \tag{4.56}$$

or equivalently

$$\partial_t \; y = t^{-1-\beta_1+m} \; y \int_{1}^{t} dt' \; t'^{-1-m} \; y(t') \; . \tag{4.57}$$

Warning 4.1 *Let $0 < \beta_1 < m$. Then the solution of the equation (4.57) with initial data $y(1) = y_0 > 0$ blows up in a finite time.*

The proof will be given in Appendix A.

The previous result encourages us to take $t_1 > t_0$ and actually the situation improves in that case and in particular we shall prove the existence of solutions defined in $[t_0, t_1]$ if $B_0 = 0$ in Proposition 4.4 below. Of course for $t_1 < \infty$, by the previous argument, we shall be unable to exclude finite time blow up after t_1. On the other hand, if $B_0 \neq 0$, we cannot exclude finite time blow up between t_0 and t_1 if t_1 is sufficiently large. Actually, we shall show that equality in a stronger version of (4.1) implies such a blow up. We again drop the inequality (4.2) and take $s = 0$ in (4.1). Omitting in addition the second term with b_0, we are left with

$$\partial_t \; y = C \left\{ y^{1-1/k} + t^{-1-\beta_1} \; y \int_{1}^{t_1/t} \quad \nu^{-1-m} \; y(\nu t)^2 \right\} . \tag{4.58}$$

Since the solution of (4.58) is increasing in time for $t \geq t_0$, blow up for (4.58) is implied by blow up for the equation

$$\partial_t \; y = C \left\{ y^{1-1/k} + t^{-1-\beta_1} \; y^3 m^{-1} \left(1 - (t/t_1)^m\right) \right\} \; . \tag{4.59}$$

Now if blow up occurs for $t \le T^*$ for the equation

$$\partial_t\ y = C\left(y^{1-1/k} + t^{-1-\beta_1}\ y^3 m^{-1}(1-2^{-m})\right) \tag{4.60}$$

then a fortiori blow up will occur for the equation (4.59) if $t_1 \ge 2T^*$. It is therefore sufficient to prove blow up for (4.60), which after rescaling can be rewritten as

$$\partial_t\ y = k\ y^{1-1/k} + t^{-1-\beta_1}\ y^3\ . \tag{4.61}$$

Warning 4.2 *Let $2k > \beta_1$. Then the solution of the equation (4.61) with initial data $y(t_0) > 0$ at time $t_0 \ge 1$ blows up in a finite time.*

The proof will be given in Appendix A. The condition $2k > \beta_1$ is always satisfied in the present situation.

We now prove the main result of this section, namely the existence of solutions of the auxiliary system (2.20) defined up to t_1, possibly with $t_1 = \infty$, for $B_0 = 0$ and for initial data given at sufficiently large $t_0 < t_1$.

Proposition 4.4 *Let $B_0 = 0$. Let $1 < k \le \ell$, $\ell > 3/2$ and $0 < \beta < 1$. Let $\beta_2 < 1$, where β_2 is defined by (4.5). Let $(w_0, \widetilde{s}_0) \in X^{k,\ell}$ and let $y_0 = |w_0|_k$ and $\widetilde{z}_0 = |\nabla\widetilde{s}_0|_\ell$. Then there exists $T_0 < \infty$ depending on $(y_0, \widetilde{z}_0)$ such that for all $t_0 \ge T_0$, there exists $T < t_0$, depending on $(y_0, \widetilde{z}_0)$ and on t_0, such that for all t_1, $t_0 \le t_1 \le \infty$, the auxiliary system (2.20) with initial data $(w, s)(t_0) = (w_0, t_0^{\beta_2}\widetilde{s}_0)$ has a unique solution (w, s) in the interval $I = [T, t_1)$ such that $(w, t^{-\beta_2}s) \in (\mathcal{C}\cap L^\infty)(I, X^{k,\ell})$. One can take*

$$T_0 = C\left\{\left(\widetilde{z}_0 + y_0^2\right)^{1/(1-\beta_2)} \vee y_0^{2/\beta_1}\right\}\ , \tag{4.62}$$

$$T = t_0^{\beta_2}\ T_0^{1-\beta_2}\ , \tag{4.63}$$

and the solution (w, s) is estimated for all $t \in I$ by

$$|w(t)|_k \le 2y_0\ , \tag{4.64}$$

$$|\nabla s(t)|_\ell \le \left(2\widetilde{z}_0 + C\ y_0^2\right)(t_0 \vee t)^{\beta_2}\ . \tag{4.65}$$

Proof. The proof consists in exploiting the estimates of Lemmas 4.1 and 4.2 in order to show that the map $\Gamma : (w, s) \to (w', s')$, where (w', s') is defined from (w, s) by Proposition 4.1, is a contraction of a suitable subset of $\mathcal{C}(I, X^{k,\ell})$ for a suitably time rescaled norm of $L^\infty(I, X^{0,\ell_0})$. We first consider the interval $I = [t_0, t_1)$ and we define the set

$$\mathcal{R} = \left\{(w, s) \in \mathcal{C}(I, X^{k,\ell}) : \parallel w; L^\infty(I, H^k) \parallel \le Y, \parallel t^{-\beta_2}\nabla s; L^\infty(I, H^\ell) \parallel \le Z\right\}\ ,$$

for $Y > 0$, $Z > 0$. Let $(w, s) \in \mathcal{R}$ and $(w', s') = \Gamma(w, s)$. Let $y = |w(t)|_k$, $y' = |w'(t)|_k$, $z = |\nabla s(t)|_\ell$ and $z' = |\nabla s'(t)|_\ell$. From Lemma 4.1, namely (4.1) (4.2) with

$b_0 = 0$ and with an overall constant omitted, we obtain

$$\begin{cases} \partial_t\ y' \le t^{-2+\beta_2}\ Zy' + t^{-1-\beta_1}\ Y^2 y' \\ \partial_t\ z' \le t^{-2+\beta_2}\ Zz' + t^{-1+\beta_2}\ Y^2\ . \end{cases} \tag{4.66}$$

Integrating from t_0 to t with $(y', z')(t_0) = (y, z)(t_0) = (y_0, z_0)$ where $z_0 = \widetilde{z}_0 t_0^{\beta_2}$, we estimate

$$Y' = \| y'; L^\infty(I) \| \quad , \quad Z' = \| t^{-\beta_2} z'; L^\infty(I) \| \tag{4.67}$$

by

$$\begin{cases} Y' \le y_0 \exp\left\{ (1-\beta_2)^{-1}\ t_0^{-1+\beta_2}\ Z + \beta_1^{-1}\ t_0^{-\beta_1}\ Y^2 \right\} \\ Z' \le \left(\widetilde{z}_0 + \beta_2^{-1}\ Y^2 \right) \exp\left\{ (1-\beta_2)^{-1}\ t_0^{-1+\beta_2}\ Z \right\}\ . \end{cases} \tag{4.68}$$

We now impose

$$\begin{cases} (1-\beta_2)\ t_0^{1-\beta_2} \ge 2(\ell n 2)^{-1} Z \\ \beta_1\ t_0^{\beta_1} \ge 2(\ell n 2)^{-1}\ Y^2 \end{cases} \tag{4.69}$$

and choose

$$Y = 2y_0 \quad , \quad Z = \sqrt{2}\left(\widetilde{z}_0 + 4\beta_2^{-1}\ y_0^2 \right) \tag{4.70}$$

thereby ensuring that $Y' \le Y$, $Z' \le Z$, so that the set $\mathcal{R}$ is mapped into itself by Γ. The conditions (4.69) can be rewritten as

$$\begin{cases} (1-\beta_2)\ t_0^{1-\beta_2} \ge 2\sqrt{2}(\ell n 2)^{-1} \left(\widetilde{z}_0 + 4\beta_2^{-1}\ y_0^2 \right) \\ \beta_1\ t_0^{\beta_1} \ge 8(\ell n 2)^{-1}\ y_0^2 \end{cases} \tag{4.71}$$

and hold for all $t_0 \ge T_0$ for T_0 satisfying (4.62) with suitable C.

We next show that the map Γ is a contraction on $\mathcal{R}$. We use the notation of Lemma 4.2 and in addition

$$\begin{cases} y_- = \| w_-(t) \|_2 \quad , \quad z_- = |\nabla s_-(t)|_{\ell_0}\ , \\ Y_- = \| y_-; L^\infty(I) \| \quad , \quad Z_- = \| t^{-\beta_2}\ z_-; L^\infty(I) \| \end{cases} \tag{4.72}$$

and a similar notation for primed quantities. From Lemma 4.2, in particular (4.18) (4.19), and again with an overall constant omitted, we obtain

$$\begin{cases} \partial_t\ y'_- \le t^{-2}\ Y\ z_- + t^{-1-\beta_1}\ Y^2 Y_- \\ \partial_t\ z'_- \le t^{-2+\beta_2}\ Z(z_- + z'_-) + t^{-1+\beta_2}\ Y\ Y_- \end{cases} \tag{4.73}$$

and by integration with $(y'_-, z'_-)(t_0) = 0$,

$$\begin{cases} Y'_- \le (1-\beta_2)^{-1}\ t_0^{-1+\beta_2} Y Z_- + \beta_1^{-1}\ t_0^{-\beta_1}\ Y^2\ Y_- \\ Z'_- \le \exp\left((1-\beta_2)^{-1}\ t_0^{-1+\beta_2} Z \right) \left\{ (1-\beta_2)^{-1}\ t_0^{-1+\beta_2}\ Z Z_-\ + \beta_2^{-1}\ Y Y_- \right\}\ . \end{cases} \tag{4.74}$$

The second inequality in (4.74) reduces to

$$Z'_- \leq \sqrt{2}\left\{(1-\beta_2)^{-1}\ t_0^{-1+\beta_2}\ ZZ_- + \beta_2^{-1}\ YY_-\right\} \tag{4.75}$$

under the first condition in (4.69) imposed previously.

We now ensure that the map Γ is a contraction for the norms defined by (4.72) in the form

$$\begin{cases} Y'_- \leq (c^{-1}\ Z_- + Y_-)/4 \\ Z'_- \leq (Z_- + c\ Y_-)/4 \end{cases} \tag{4.76}$$

which implies

$$Z'_- + c\ Y'_- \leq (Z_- + c\ Y_-)/2 \tag{4.77}$$

by taking $c = 8\beta_2^{-1}Y$ and imposing the conditions

$$\begin{cases} (1-\beta_2)t_0^{1-\beta_2} \geq 8Z \quad , \quad \beta_1 t_0^{\beta_1} \geq 4Y^2 \\ (1-\beta_2)\ t_0^{1-\beta_2} \geq 4c\ Y = 32\beta_2^{-1}\ Y^2 \end{cases}$$

which follow again from (4.62) for all $t_0 \geq T_0$.

We have proved that Γ maps $\mathcal{R}$ into itself and is a contraction for the norms (4.72). By a standard compactness argument, $\mathcal{R}$ is easily shown to be closed for the latter norms. Therefore Γ has a unique fixed point in $\mathcal{R}$, which completes the proof for $t \geq t_0$.

We now turn to the case $t \leq t_0$, namely we consider the interval $I = [T, t_0]$ for some $T < t_0$. The proof proceeds in exactly the same way, with however slightly different norms. In addition, one has to take into account the following fact: the various integrals $I_m^{t_1}$ that occur in (4.1) (4.2) and (4.18) (4.19) involve w and w_1, w_2 up to time t_1. In the subinterval $[t_0, t_1]$, one takes $w = w_1 = w_2 =$ the solution constructed at the previous step (in particular $w_- = 0$ for $t \geq t_0$, so that actually no contribution from the interval $[t_0, t_1]$ occurs in (4.18) (4.19)). In (4.1) (4.2) the contribution of the interval $[t_0, t_1]$ is taken into account by using the fact that all the integrals over time in the relevant $I_m^{t_1}$ are convergent at infinity and that we shall eventually use the same ansatz $|w(t)|_k \leq Y = 2y_0$ both for $t \leq t_0$ and $t \geq t_0$. With this in mind, we complete the proof by simply giving the computational details. We consider the set

$$\mathcal{R}_< = \left\{(w,s) \in \mathcal{C}(I, X^{k,\ell}); \parallel w; L^\infty(I, H^k) \parallel \leq Y, \parallel \nabla s; L^\infty(I, H^\ell) \parallel \leq Z\right\}\ .$$

For $(w, s) \in \mathcal{R}_<$, $(w', s') = \Gamma(w, s)$ and y, y', z and z' defined as previously, we estimate by Lemma 4.1, again with an overall constant omitted

$$\begin{cases} |\partial_t\ y'| \leq t^{-2}\ Zy' + t^{-1-\beta_1}\ Y^2 y' \\ |\partial_t\ z'| \leq t^{-2}\ Zz' + t^{-1+\beta_2}\ Y^2 \end{cases} \tag{4.78}$$

and by integration from t to t_0 with $(y', z')(t_0) = (y, z)(t_0) = (y_0, z_0)$, we obtain $y'(t) \le Y'$, $z'(t) \le Z'$ with

$$\begin{cases} Y' \le y_0 \exp\left(t^{-1} Z + \beta_1^{-1}\ t^{-\beta_1}\ Y^2\right) \\ Z' \le \left(z_0 + \beta_2^{-1}\ Y^2\ t_0^{\beta_2}\right) \exp(t^{-1} Z)\ . \end{cases} \tag{4.79}$$

We now impose

$$\begin{cases} t \ge 2(\ell n\ 2)^{-1}\ Z \\ \beta_1\ t^{\beta_1} \ge 2(\ell n\ 2)^{-1}\ Y^2 \end{cases} \tag{4.80}$$

and choose

$$Y = 2y_0 \quad , \quad Z = \sqrt{2}\left(z_0 + 4\beta_2^{-1}\ y_0^2\ t_0^{\beta_2}\right) \tag{4.81}$$

thereby ensuring that $Y' \le Y$, $Z' \le Z$ so that the set $\mathcal{R}_<$ is mapped into itself by Γ. The conditions (4.80) can be rewritten as

$$\begin{cases} \beta_1\ t^{\beta_1} \ge 8(\ell n\ 2)^{-1}\ y_0^2 \\ t \ge 2\sqrt{2}(\ell n\ 2)^{-1} \left(\widetilde{z}_0 + 4\beta_2^{-1}\ y_0^2\right) t_0^{\beta_2} \end{cases} \tag{4.82}$$

and hold for all $t \ge T$ with T defined by (4.63) and T_0 satisfying (4.62) for suitable C.

We next prove that Γ is a contraction on $\mathcal{R}_<$ for the norm in $L^\infty(I, X^{0,\ell_0})$. With the notation of Lemma 4.2 and in addition

$$\begin{cases} y_- = \parallel w_-(t) \parallel_2 \quad , \quad z_- = |\nabla s_-(t)|_{\ell_0} \\ Y_- = \parallel y_-; L^\infty(I) \parallel \quad , \quad Z_- = \parallel z_-; L^\infty(I) \parallel \end{cases} \tag{4.83}$$

and a similar notation for primed quantities, we obtain from (4.18) (4.19)

$$\begin{cases} |\partial_t\ y'_-| \le t^{-2}\ Y z_- + t^{-1-\beta_1}\ Y^2\ Y_- \\ |\partial_t\ z'_-| \le t^{-2}\ Z(z_- + z'_-) + t^{-1+\beta_2}\ YY_-\ . \end{cases} \tag{4.84}$$

By integration between t and t_0, we deduce therefrom

$$\begin{cases} Y'_- \le t^{-1} Y Z_- + \beta_1^{-1}\ t^{-\beta_1}\ Y^2 Y_- \\ Z'_- \le \left(t^{-1} Z Z_- + \beta_2^{-1}\ t_0^{\beta_2}\ YY_-\ \right) \exp(t^{-1} Z) \end{cases} \tag{4.85}$$

thereby ensuring the contraction in the form (4.76) which implies (4.77) by taking $c = 8\beta_2^{-1} Y t_0^{\beta_2}$ and imposing

$$t \ge 8Z \quad , \quad \beta_1 t^{\beta_1} \ge 4Y^2 \quad , \quad t \ge 4c\ Y = 32\beta_2^{-1}\ t_0^{\beta_2}\ Y^2$$

which hold for all $t \ge T$ with the choice (4.81) under the conditions (4.62) (4.63). With the previous estimates available, the proof proceeds as in the case $t \ge t_0$. $\square$

5 Asymptotics and wave operators for the linear system

In this section we study the asymptotic properties of solutions of the linear equation (2.22) in the form (2.24) at the level of regularity of H^k with $k \geq 1$ for w. In particular we solve the Cauchy problem at infinity, thereby constructing the wave operators in H^k. For the linear equation (2.24), the wave operators in L^2 can be easily constructed by a variant of Cook's method and the construction of the wave operators in H^k reduces to a regularity problem for the L^2 wave operators thereby obtained. As a preliminary to that study, we shall first solve the Cauchy problem for the equation (2.24) with finite initial time. We emphasize the fact that in this section we do not strive after any kind of optimality in the treatment of the linear equation, since we are mainly interested in a form of that treatment that can be incorporated in that of the fully interacting system.

Proposition 5.1 *Let $I = [1, \infty)$, let $k \geq 1$ and let B_0 satisfy the estimates (3.17) for $0 \leq m \leq k$. Let $t_0 \in I$ and $w_0 \in H^k$. Then the equation (2.24) has a unique solution $w \in \mathcal{C}(I, L^2)$ with $w(t_0) = w_0$. Furthermore $w \in \mathcal{C}(I, H^k) \cap L^\infty(I, L^2)$ and w satisfies the conservation law*

$$\| w(t) \|_2 = \text{const.}$$

and the estimate

$$|w(t)|_k \leq \left(1 + C_k |t - t_0| (t \vee t_0)^{\bar{k}-1} \right) |w_0|_k \tag{5.1}$$

where $\bar{k} = k$ for integer k and $\bar{k} = k + \varepsilon$ with $\varepsilon > 0$ for noninteger k.

Proof. It follows easily from standard arguments and from Lemma 3.2 that w exists and satisfies the properties stated except possibly the estimate (5.1), and we concentrate on the proof of the latter, assuming without loss of generality that $|w_0|_k = 1$. We first prove (5.1) by induction for integer $k \geq 1$. Let $0 \leq j \leq k-1$ and $y_j = \| \omega^j w \|_2$. From (2.24) and from the Leibnitz formula and Sobolev inequalities, we obtain

$$|\partial_t \ y_{j+1}| \leq C \ t^{-1} \Big\{ \ \| \nabla B_0 \|_\infty \ \| \omega^j w \|_2 + \sum_{|\alpha|=j+1} \| \partial^\alpha B_0 \|_\infty \ \| w \|_2 \ \Big\}$$

and therefore by (3.17)

$$|\partial_t \ y_{j+1}| \leq C \ b_0 \left(y_j + t^j \right) \ . \tag{5.2}$$

Substituting the induction assumption for y_j and integrating (5.2) between t_0 and t, we obtain

$$\begin{aligned} y_{j+1} &\leq 1 + C \ b_0 \left(1 + (C_j + 1)(t \vee t_0)^j \right) |t - t_0| \\ &\leq 1 + C_{j+1} |t - t_0| (t \vee t_0)^j \end{aligned}$$

with $C_{j+1} = C b_0 (C_j + 2)$. This completes the proof for integer k.

Let now $k = k_0 + \theta$ with integer $k_0 \geq 1$ and $0 < \theta < 1$. We estimate

$$\begin{aligned} \left|\partial_t \parallel \omega^k w \parallel_2\right| &\leq t^{-1} \parallel [\omega^k, B_0]w \parallel_2 \\ &\leq Ct^{-1} \left\{\parallel \nabla B_0 \parallel_\infty \parallel \omega^{k-1} w \parallel_2 + \parallel \omega^k B_0 \parallel_{3/\delta} \parallel w \parallel_r\right\} \end{aligned} \tag{5.3}$$

by Lemma 3.2, with $0 < \delta = \delta(r) \leq 1$,

$$\cdots \leq C\ b_0 \left(\parallel \omega^{k-1} w \parallel_2 + t^{k-1-\delta/3} \parallel \omega^\delta w \parallel_2\right)$$

by (3.17) and Sobolev inequalities. We next interpolate and obtain

$$\cdots \leq C\ b_0 \left(\parallel \omega^{k_0-1} w \parallel_2^{1-\theta} \parallel \omega^{k_0} w \parallel_2^\theta + t^{k-1-\delta/3} \parallel w \parallel_2^{1-\delta} \parallel \nabla w \parallel_2^\delta\right) .$$

We finally substitute the estimate (5.1) for the integer values $k_0 - 1$, k_0, and 1 and integrate between t_0 and t, thereby obtaining

$$\parallel \omega^k w \parallel_2 \leq 1 + C\ b_0 |t - t_0| (t \vee t_0)^{k-1+2\delta/3}$$

which yields (5.1) with $\varepsilon = 2\delta/3$. □

The fact that a direct H^k estimate of the solution does not prevent its H^k norm to increase as a power of t is a warning of the fact that the construction of the wave operators at that level of regularity is not trivial. The same fact appeared already in Section 4 above in Warning 4.2 and compelled us to assume $B_0 = 0$ in Proposition 4.4.

We next construct the L^2-wave operators for (2.24).

Proposition 5.2 *Let $I = [1, \infty)$ and let B_0 satisfy the estimates (3.17) for $m = 0$.*

(1) Let $W \in \mathcal{C}(I, L^2)$ with $U(1/t)W \in \mathcal{C}^1(I, L^2)$, satisfying

$$\parallel R(W) \parallel_2 \leq\ c_0\ t^{-1-\lambda_0} \tag{5.4}$$

for some $\lambda_0 > 0$ and for all $t \in I$. Then there exists a unique solution $w \in \mathcal{C}(I, L^2)$ of the equation (2.24), such that $w(t) - W(t)$ tends to zero strongly in L^2 when $t \to \infty$. Furthermore, for all $t \in I$,

$$\parallel w(t) - W(t) \parallel_2 \leq\ c_0\ \lambda_0^{-1}\ t^{-\lambda_0} . \tag{5.5}$$

The solution w is the norm limit in $L^\infty(I, L^2)$ as $t_0 \to \infty$ of the solution w_{t_0} of the equation (2.24) with initial condition $w_{t_0}(t_0) = W(t_0)$ obtained in Proposition 5.1, and the following estimate holds for all $t \in I$:

$$\parallel w_{t_0}(t) - w(t) \parallel_2 \leq\ c_0\ \lambda_0^{-1}\ t_0^{-\lambda_0} . \tag{5.6}$$

(2) Let in addition $W \in L^\infty(I, H^k)$ for some k, $0 < k < 3/2$. Then there exists $w_+ \in H^k$ such that $W(t)$ tends to w_+ strongly in L^2 and weakly in H^k when $t \to \infty$, and the following estimate holds for all $t \in I$:

$$\parallel W(t) - w_+ \parallel_2 \leq\ C \left(t^{-\lambda_0} + t^{-k/3}\right) . \tag{5.7}$$

Conversely let $w_+ \in H^k$ *for some* k, $0 < k < 3/2$, *and let* $W_1 = U^*(1/t)w_+$. *Then* W_1 *satisfies the assumptions of Part (1) with* $\lambda_0 = k/3$.

Let W, w_+ *and* W_1 *be related as above. Then the solutions of the equation (2.24) constructed in Part (1) from* W *and* W_1 *coincide.*

(3) Let $w_+ \in L^2$. *Then the equation (2.24) has a unique solution* $w \in \mathcal{C}(I, L^2)$ *such that* $w(t)$ *tends to* w_+ *strongly in* L^2 *when* $t \to \infty$.

Proof. Part (1). Following the sketch of Section 2, we look for w in the form $w = W + q$, so that q satisfies the equation

$$\partial_t\ q = i(2t^2)^{-1}\Delta q + i\ t^{-1}\ B_0\ q - R(W) \tag{5.8}$$

and therefore the a priori estimate

$$|\partial_t \parallel q \parallel_2 | \le \parallel R(W) \parallel_2 \le c_0\ \lambda_0^{-1}\ t^{-1-\lambda_0}\ . \tag{5.9}$$

Define w_{t_0} as in Part (1) and let $w_{t_0} = W + q_{t_0}$ so that $q_{t_0}(t_0) = 0$. Integrating (5.9) between t_0 and t yields

$$\parallel q_{t_0}(t) \parallel_2 \le c_0\ \lambda_0^{-1}|t^{-\lambda_0} - t_0^{-\lambda_0}| \tag{5.10}$$

and therefore, by L^2 norm conservation for the difference of two solutions

$$\parallel q_{t_0}(t) - q_{t_1}(t) \parallel_2 = \parallel q_{t_0}(t_1) \parallel_2 \le c_0\ \lambda_0^{-1}|t_1^{-\lambda_0} - t_0^{-\lambda_0}| \tag{5.11}$$

for any t_0 and t_1, $1 \le t_0$, $t_1 < \infty$. This proves convergence of q_{t_0} and therefore of w_{t_0} in norm in $L^\infty(I, L^2)$. Let w be the limit of w_{t_0}. Taking the limit $t_0 \to \infty$ in (5.10) yields (5.5), while taking the limit $t_1 \to \infty$ in (5.11) yields (5.6). Clearly w satisfies the equation (2.24).

Part (2). W satisfies the equation

$$\partial_t\ U(1/t)\ W = i\ t^{-1}\ U(1/t)\ B_0\ W + U(1/t)\ R(W)\ . \tag{5.12}$$

From (3.17) and Sobolev inequalities, we obtain

$$\parallel B_0 W \parallel_2 \le \parallel B_0 \parallel_{3/k} \parallel W \parallel_r \le C\ ab_0\ t^{-k/3} \tag{5.13}$$

where $a = \parallel W; L^\infty(I, H^k) \parallel$ and $k = \delta(r)$. Integrating (5.12) between t_1 and t_2 and using (5.4) and (5.13), we obtain

$$\parallel U(1/t_1)W(t_1) - U(1/t_2)W(t_2) \parallel_2 \le C \left\{ |t_1^{-k/3} - t_2^{-k/3}| + |t_1^{-\lambda_0} - t_2^{-\lambda_0}| \right\} \tag{5.14}$$

for any t_1 and t_2, $1 \le t_1$, $t_2 < \infty$. Therefore $U(1/t)W(t)$ and therefore also $W(t)$ has a strong limit w_+ in L^2, and

$$\parallel U(1/t)\ W(t) - w_+ \parallel_2 \le C \left(t^{-k/3} + t^{-\lambda_0} \right)\ , \tag{5.15}$$

from which (5.7) follows. Furthermore by a standard compactness argument, $w_+ \in H^k$ with $|w_+|_k \le a$ and $W(t)$ tends to w_+ weakly in H^k.

Let now $w_+ \in H^k$ and $W_1 = U^*(1/t)w_+$. Then

$$R(W_1) = -it^{-1}\ B_0\ U^*(1/t)\ w_+ \tag{5.16}$$

so that

$$\| R(W_1) \|_2 \le C\ b_0 |w_+|_k\ t^{-k/3} \tag{5.17}$$

by (5.13). The last statement follows from the fact that W and W_1 have the same limit w_+ in L^2 and from L^2 norm conservation for the equation (2.24).

Part (3) follows from Parts (1) and (2) by a standard density argument. □

We next prove that the solutions with asymptotic properties in L^2 obtained in Proposition 5.2 exhibit similar asymptotic properties in H^k under suitable additional assumptions.

Proposition 5.3 *Let $I = [1, \infty)$, let $k \ge 1$ and let B_0 satisfy the estimates (3.17) for $0 \le m \le k$. Let $\lambda > 0$ and $\lambda_0 > \lambda + k$ and let $U(1/t)W \in \mathcal{C}^1(I, H^k)$ satisfy the estimates (5.4) and*

$$\| \omega^k\ R(W) \|_2 \le c_1\ t^{-1-\lambda} \tag{5.18}$$

for all $t \in I$.

(1) Let w be the solution of the equation (2.24) obtained in Proposition 5.2 part (1). Then $w \in \mathcal{C}(I, H^k)$ and w satisfies the estimates (5.5) and

$$\| \omega^k (w(t) - W(t)) \|_2 \le C\ t^{-\lambda} \tag{5.19}$$

for all $t \in I$.

(2) Let w_{t_0} be the solution of the equation (2.24) defined in Proposition 5.2 part (1). When $t_0 \to \infty$, w_{t_0} converges to w strongly in $L^\infty([1, \bar{T}], H^{k'})$ for $0 \le k' < k$ and in the weak $*$ sense in $L^\infty([1, \bar{T}], H^k)$ for any $\bar{T} < \infty$.

Proof. Part (1) will be proved together with the limiting properties stated in Part (2). We know from Proposition 5.1 that $w_{t_0} \in \mathcal{C}(I, H^k)$. The main point of the proof consists in estimating $q_{t_0} = w_{t_0} - W$ in H^k uniformly in t_0 for $t \le t_0$. We know already from (5.10) that

$$\| q_{t_0}(t) \|_2 \le Y_0\ t^{-\lambda_0} \tag{5.20}$$

for $t \le t_0$, with $Y_0 = c_0 \lambda_0^{-1}$. We next estimate $y \equiv \| \omega^k q_{t_0} \|_2$. From (5.8) we obtain

$$\left| \partial_t \| \omega^k q_{t_0} \|_2 \right| \le t^{-1} \| [\omega^k, B_0] q_{t_0} \|_2 \ + \ \| \omega^k R(W) \|_2 \tag{5.21}$$

so that by Lemma 3.2, in the same way as in (4.7),

$$\begin{aligned} \left| \partial_t \| \omega^k q_{t_0} \|_2 \right| \quad \le \quad & C t^{-1} \left(\| \nabla B_0 \|_\infty \| \omega^{k-1} q_{t_0} \|_2 \ + \ \| \omega^k B_0 \|_{3/\delta} \| q_{t_0} \|_r \right) \\ & + \| \omega^k\ R(W) \|_2 \end{aligned} \tag{5.22}$$

with $0 < \delta = \delta(r) < k \wedge 3/2$. Using the estimates (3.17), Sobolev inequalities and interpolation with the help of (5.20) and the assumption (5.18), we obtain

$$\begin{aligned} |\partial_t y| \quad \leq \quad & C\ b_0 \Big\{ Y_0^{1-\delta/k}\ y^{\delta/k}\ t^{k-1-\delta/3-\lambda_0(1-\delta/k)} \\ & + Y_0^{1/k}\ y^{1-1/k}\ t^{-\lambda_0/k} \Big\} + c_1\ t^{-1-\lambda}\ . \end{aligned} \tag{5.23}$$

We now define $Y \equiv \parallel t^\lambda y; L^\infty([1, t_0]) \parallel$, substitute that definition into the RHS of (5.23), integrate the latter between t and t_0, and obtain

$$\begin{aligned} y \leq C b_0 \left(Y_0^{1-\delta/k}\ Y^{\delta/k} (\lambda + \nu_1)^{-1}\ t^{-\lambda-\nu_1} + Y_0^{1/k}\ Y^{1-1/k} (\lambda + \nu_2)^{-1}\ t^{-\lambda-\nu_2} \right) \\ + c_1\ \lambda^{-1}\ t^{-\lambda} \end{aligned} \tag{5.24}$$

where

$$\nu_1 = (\lambda_0 - \lambda)(1 - \delta/k) - k + \delta/3 \tag{5.25}$$

$$\nu_2 = (\lambda_0 - \lambda)/k - 1 \tag{5.26}$$

provided $\lambda + \nu_1 > 0$ and $\lambda + \nu_2 > 0$. We impose in addition $\nu_1 \geq 0$, $\nu_2 \geq 0$, multiply (5.24) by t^λ, take the Supremum over t and obtain

$$Y \leq \lambda^{-1}\ C\ b_0 \left(Y_0^{1-\delta/k}\ Y^{\delta/k} + Y_0^{1/k}\ Y^{1-1/k} \right) + \lambda^{-1}\ c_1 \tag{5.27}$$

which is uniform in t_0. The condition $\nu_1 \geq 0$ reduces to

$$\lambda_0 \geq \lambda + k + 2\delta k/3(k - \delta) \tag{5.28}$$

and can be satisfied for $\lambda_0 > \lambda + k$ by taking δ sufficiently small. It implies $\nu_2 > 0$. Changing the notation to $x = Y Y_0^{-1}$, $b = \lambda^{-1} C b_0$ and $c = \lambda^{-1} c_1 Y_0^{-1}$, we rewrite (5.27) as

$$x \leq b \left(x^{\delta/k} + x^{1-1/k} \right) + c\ . \tag{5.29}$$

Assuming $\delta \leq k - 1$ without loss of generality and using

$$x^\theta \leq \varepsilon x + \varepsilon^{-\theta/(1-\theta)}$$

for $x > 0$, $\varepsilon > 0$ and $0 < \theta < 1$, we obtain from (5.29)

$$x \leq 2b \left(\varepsilon x + \varepsilon^{1-k} \right) + c$$

and for $\varepsilon = (4b)^{-1}$

$$x \leq (4b)^k + 2c$$

or equivalently

$$Y \leq \left(4C\lambda^{-1} b_0 \right)^k Y_0 + 2\lambda^{-1}\ c_1\ , \tag{5.30}$$

which completes the proof of the estimate of q_{t_0} in H^k uniformly in t_0 for $t \leq t_0$.

Let now $\bar{T} < \infty$ and $J = [1, \bar{T}]$. We know from (5.30) that w_{t_0} is estimated in $L^\infty(J, H^k)$ uniformly in t_0 for $t_0 \geq \bar{T}$ and that w_{t_0} converges to w in norm in $L^\infty(J, L^2)$ by Proposition 5.2 part (1). It follows therefrom by a standard compactness argument that $w \in (L^\infty \cap \mathcal{C}_w)(J, H^k)$, that w also satisfies the estimate (5.30) and that w_{t_0} converges to w in the topologies described in Part (2). Strong continuity of w in H^k follows from Proposition 5.1. □

In order to complete the construction of the wave operators in H^k, we now have to construct model functions W satisfying the assumptions (5.4) and (5.18). In view of Proposition 5.2 part (2), we restrict our attention to W of the form

$$W(t) = U^*(1/t) w_+ \tag{5.31}$$

for some fixed $w_+ \in H^{k_+}$ and we take $k_+ > 3/2$. With that choice, we obtain

$$R(W) = R(U^*(1/t)W) = -i\, t^{-1}\, B_0\, U^*(1/t) w_+ \ . \tag{5.32}$$

However, with no further assumptions on w_+, we are restricted to $\lambda_0 \leq 1/2$ and consequently to $k < 1/2$. In fact, from (3.17) we obtain

$$\parallel R(W) \parallel_2 \leq t^{-1} \parallel B_0 \parallel_2 \parallel U^*(1/t) w_+ \parallel_\infty \leq C b_0\, t^{-3/2} |w_+|_{k_+} \ . \tag{5.33}$$

Furthermore, from Lemma 3.2 and (3.17) we obtain for $k < 1/2$

$$\begin{aligned} &\parallel \omega^k R(W) \parallel_2 \\ &\leq C\, t^{-1} \left\{ \parallel \omega^k B_0 \parallel_2 \parallel U^*(1/t) w_+ \parallel_\infty + \parallel B_0 \parallel_r \parallel \omega^k U^*(1/t) w_+ \parallel_{3/\delta} \right\} \\ &\leq C b_0\, t^{-3/2+k}\, |w_+|_{k_+} \end{aligned} \tag{5.34}$$

with $0 < \delta = \delta(r) \leq 3k < 3/2$. Together with an extension of Proposition 5.3 to $k \leq 1/2$, which we have not performed, the estimates (5.33) (5.34) would allow us to complete the construction of the wave operators for $0 < k < 1/2$, with $\lambda_0 = 1/2$ and $0 < \lambda < 1/2 - k$.

In order to cover higher values of k, and in particular for $k > 1$, as will be needed for the nonlinear system (1.1) (1.2), we shall need additional conditions on w_+ and B_0. We first exhibit a set of local sufficient conditions in the form of joint decay estimates for w_+, and B_0, where the nonlocal operator $U^*(1/t)$ no longer appears.

Lemma 5.1 *Let $\lambda_0 > 0$ and let $\bar{m}$ be a nonnegative integer. Let B_0 satisfy the estimates (3.17) for $0 \leq m \leq \bar{m}$. Let $w_+ \in H^{k_+}$ with $k_+ \geq 2\lambda_0 \vee \bar{m}$ and let $a_+ = |w_+|_{k_+}$. Assume that B_0 and w_+ satisfy the estimates*

$$\parallel (\partial^{\alpha_1} B_0)\, (\partial^{\alpha_2} w_+) \parallel_2 \leq b_1\, t^{-\lambda_0 + |\alpha_1| + |\alpha_2|/2} \tag{5.35}$$

for all multi-indices α_1, α_2 with $0 \leq |\alpha_1| \leq \bar{m}$ and $0 \leq |\alpha_2| < 2\lambda_0$, and for all $t \geq 1$. Then the following estimates hold for all m, $0 \leq m \leq \bar{m}$, and for all $t \geq 1$

$$\begin{aligned} \parallel \omega^m R(U^*(1/t) w_+) \parallel_2 &= t^{-1} \parallel \omega^m (B_0\, U^*(1/t) w_+) \parallel_2 \\ &\leq C\, (b_1 + b_0 a_+)\, t^{-1-\lambda_0+m} \ . \end{aligned} \tag{5.36}$$

Proof. By interpolation, it is sufficient to prove (5.36) for integer m. Let α be a multi-index with $|\alpha| = m$. We estimate

$$\| \partial^{\alpha} (B_0\ U^*(1/t)w_+) \|_2 \le C \sum_{\alpha_1+\alpha_3=\alpha} \| (\partial^{\alpha_1} B_0)\, U^*(1/t)\ \partial^{\alpha_3} w_+ \|_2\ . \tag{5.37}$$

If $|\alpha_3| < 2\lambda_0$, we expand $U^*(1/t)$ through the relation

$$\left| e^{ix} - \sum_{j\le p} (j!)^{-1}(ix)^j \right| \le 2(p!)^{-1}\ |x|^{p+\theta}$$

with $p + \theta = \lambda_0 - |\alpha_3|/2$ and $0 < \theta \le 1$, so that

$$\begin{aligned} &\| (\partial^{\alpha_1} B_0)\, U^*(1/t)\partial^{\alpha_3} w_+ \|_2 \le C \sum_{j<\lambda_0-|\alpha_3|/2} t^{-j}\ \| (\partial^{\alpha_1} B_0)\, \Delta^j\ \partial^{\alpha_3} w_+ \|_2 \\ &+C \ \| \partial^{\alpha_1} B_0 \|_\infty\ \ t^{-(\lambda_0-|\alpha_3|/2)}\ \| \omega^{2\lambda_0} w_+ \|_2 \\ &\le C(b_1 + b_0 a_+)t^{-\lambda_0+|\alpha_1|+|\alpha_3|/2} = C(b_1 + b_0 a_+)t^{-\lambda_0+m-|\alpha_3|/2} \end{aligned} \tag{5.38}$$

by (5.35) and (3.17), which proves (5.36) in this case.

If $|\alpha_3| \ge 2\lambda_0$, the last norm in (5.37) is estimated by the use of (3.17) as

$$\begin{aligned} &\| (\partial^{\alpha_1} B_0)\, U^*(1/t)\partial^{\alpha_3} w_+ \|_2 \le C \ \| \partial^{\alpha_1} B_0 \|_\infty\ \| \partial^{\alpha_3} w_+ \|_2 \\ &\le C\ b_0\ a_+\ t^{|\alpha_1|} \le C b_0\ a_+\ t^{m-2\lambda_0} \end{aligned} \tag{5.39}$$

since $|\alpha_1| = m - |\alpha_3| \le m - 2\lambda_0$, which completes the proof of (5.36). □

We shall apply Lemma 5.1 with $\bar{m} = \{k\}$, the smallest integer $\ge k$. Then (5.36) with $m = 0$ and $m = k$ reduces to (5.4) and (5.18) with $\lambda = \lambda_0 - k$ respectively. For $\lambda_0 > k \ge 1$, one can take $k_+ = 2\lambda_0$.

We now give sufficient conditions that ensure the assumption (5.35). We first remark that (5.35) is trivially satisfied under suitable support properties of w_+ and of the initial data $(A_+, \dot{A}_+)$ of the scalar field A_0 at time $t = 0$ (see (2.3)). In fact, assume that

$$\text{Supp}\ (A_+, \dot{A}_+) \subset \{x : |x| \le R\}\ . \tag{5.40}$$

Then, by the Huyghens principle

$$\text{Supp}\ A_0 \subset \{(x,t) : ||x| - t| \le R\} \tag{5.41}$$

so that

$$\text{Supp}\ B_0 \subset \{(x,t) : ||x| - 1| \le R/t\}\ . \tag{5.42}$$

If on the other hand

$$\text{Supp}\ w_+ \subset \{x : ||x| - 1| \ge \eta\} \tag{5.43}$$

for some η, $0 < \eta < 1$, then $(\partial^{\alpha_1} B_0)\partial^{\alpha_2} w_+ = 0$ for $t \ge R/\eta$ for any multi-indices α_1 and α_2, which ensures (5.35) in a trivial way.

We shall now give more general conditions that ensure (5.35), keeping the support condition (5.43) on w_+, and relaxing the support condition (5.40) on $(A_+, \dot{A}_+)$ to space decay conditions.

Lemma 5.2 *Let $\lambda_0 > 0$, $k_+ \geq 2\lambda_0$ and let $k_+ > 3/2$. Let $w_+ \in H^{k_+}$. Let α_1 be a multi-index.*

(1) Let B_0 satisfy (3.17) for $m = |\alpha_1|$ and in addition

$$\| \chi_0 \ \partial^{\alpha_1} B_0 \|_2 \leq b_2 \ t^{-\lambda_0 + |\alpha_1|} \tag{5.44}$$

where χ_0 is the characteristic function of the support of w_+. Then (5.35) holds for any multi-index α_2, $0 \leq |\alpha_2| \leq 2\lambda_0$. The constant in (5.35) can be taken as $b_1 = C(b_0 \vee b_2)a_+$.

(2) Let w_+ satisfy the support property (5.43) and let $(A_+, \dot{A}_+)$ satisfy the following conditions for all $R \geq R_0$ for some $R_0 > 0$:

$$\| \chi(|x| \geq R)\partial^{\alpha_1} A_+ \|_2 \leq C \ R^{-\lambda_0 + 1/2} \ , \tag{5.45}$$

$$\| \chi(|x| \geq R)\dot{A}_+ \|_{6/5} \leq C \ R^{-\lambda_0 + 1/2} \quad \textit{if } \alpha_1 = 0 \ , \tag{5.46}$$

$$\| \chi(|x| \geq R)\partial^{\alpha'_1}\dot{A}_+ \|_2 \leq C \ R^{-\lambda_0 + 1/2} \quad \textit{if } \alpha_1 \neq 0 \ , \tag{5.47}$$

where α'_1 is a multi-index satisfying $\alpha'_1 \leq \alpha_1$, $|\alpha'_1| = |\alpha_1| - 1$, and where $\chi(|x| \geq R)$ is the characteristic function of $\{x : |x| \geq R\}$. Then (5.44) holds.

Proof. Part (1). We estimate by the Hölder inequality and interpolation between (3.17) and (5.44)

$$\begin{aligned} &\| (\partial^{\alpha_1} B_0) \, (\partial^{\alpha_2} w_+) \|_2 \leq \| \chi_0 \, (\partial^{\alpha_1} B_0) \|_r \ \| \partial^{\alpha_2} w_+ \|_{3/\delta} \\ &\leq (b_0 \vee b_2) t^{-\lambda_0 + |\alpha_1| + 2\lambda_0\delta/3} \ \| \partial^{\alpha_2} w_+ \|_{3/\delta} \\ &= (b_0 \vee b_2) t^{-\lambda_0 + |\alpha_1| + |\alpha_2|/2} \ \| \partial^{\alpha_2} w_+ \|_{3/\delta} \end{aligned} \tag{5.48}$$

where $\delta = \delta(r) = 3|\alpha_2|/4\lambda_0$, so that $0 \leq \delta \leq 3/2$. The last norm in (5.48) is estimated by $|w_+|_{k_+}$ through Sobolev inequalities since $|\alpha_2| + 3/2 - \delta \equiv 3/2 + |\alpha_2|(1 - 3/4\lambda_0)$ ranges from $3/2$ to $2\lambda_0$ when $|\alpha_2|$ ranges from 0 to $2\lambda_0$.

Part (2). Using the support properties of w_+ and returning to the variable A_0, we see that (5.44) is implied by

$$\| \chi(||x| - t| \geq \eta t)\partial^{\alpha_1} A_0(t) \|_2 \leq b_2 \ t^{-\lambda_0 + 1/2} \ . \tag{5.49}$$

Let now $R > 0$, let $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^3)$, $0 \leq \chi_1 \leq 1$, $\chi_1(x) = 0$ for $|x| \leq 1$, $\chi_1(x) = 1$ for $|x| \geq 2$ and let $\chi_R(x) = \chi(x/R)$. Let $\widetilde{A}_R$ be the solution of the wave equation $\square \widetilde{A}_R = 0$ with initial data

$$\left(\widetilde{A}_R, \partial_t \widetilde{A}_R\right)(0) = \left(\chi_R \ \partial^{\alpha_1} A_+, \chi_R \ \partial^{\alpha_1} \dot{A}_+\right) \quad \text{at } t = 0 \ .$$

By the Huyghens principle, $\widetilde{A}_R(t) = \partial^{\alpha_1} A_0(t)$ for $||x| - t| \geq 2R$ so that

$$\| \chi(||x|-t| \geq 2R)\partial^{\alpha_1} A_0(t) \|_2 = \| \chi(||x|-t| \geq 2R)\widetilde{A}_R(t) \|_2 \leq \| \widetilde{A}_R(t) \|_2 \ . \tag{5.50}$$

It follows now from (3.16) that

$$\| \widetilde{A}_R(t) \|_2 \leq \| \chi_R \ \partial^{\alpha_1} A_+ \|_2 \ + \ \| \omega^{-1} \ \chi_R \ \partial^{\alpha_1} \dot{A}_+ \|_2 \ . \tag{5.51}$$

If $\alpha_1 = 0$, we estimate the last norm in (5.51) by

$$\| \omega^{-1} \ \chi_R \ \dot{A}_+ \|_2 \leq C \ \| \chi_R \ \dot{A}_+ \|_{6/5} \ . \tag{5.52}$$

If $\alpha_1 \neq 0$, we rewrite $\partial^{\alpha_1} = \partial_j \partial^{\alpha'_1}$ and estimate

$$\begin{aligned} \| \omega^{-1}\chi_R\partial^{\alpha_1}\dot{A}_+ \|_2 \quad &\leq \quad \| \omega^{-1}\partial_j\chi_R\partial^{\alpha'_1}\dot{A}_+ \|_2 + \| \omega^{-1}(\partial_j\chi_R)\partial^{\alpha'_1}\dot{A}_+ \|_2 \\ &\leq \quad \| \chi_R\partial^{\alpha'_1}\dot{A}_+ \|_2 + C \ \| \partial_j\chi_1 \ \|_3 \| \ \chi(|x| \geq R)\partial^{\alpha'_1}\dot{A}_+ \|_2 \end{aligned} \tag{5.53}$$

by the Sobolev and Hölder inequalities. Collecting (5.50)–(5.53) and using the assumption (5.45)–(5.47), we obtain

$$\| \chi(||x| - t| \geq 2R)\partial^{\alpha_1} A_0(t) \|_2 \leq C \ R^{-\lambda_0+1/2}$$

from which (5.49) follows by taking $2R = \eta t$. □

Collecting the previous results essentially yields the wave operators in H^k for the equation (2.24) in the form of Proposition 5.4 below. In that proposition, we have kept the assumptions on B_0 in the implicit form of the estimates (3.17) and (5.35). If so desired, those assumptions can be replaced by sufficient conditions on $(w_+, A_+, \dot{A}_+)$ by the use of Lemmas 3.5 and 5.2.

Proposition 5.4 *Let $k \geq 1$, $k_+ > 2k$, let λ_0 and λ satisfy $\lambda > 0$, $k+\lambda < \lambda_0 \leq k_+/2$. Let $w_+ \in H^k$, let B_0 satisfy the estimates (3.17) for $0 \leq m \leq k$, and let (w_+, B_0) satisfy the estimates (5.35) for all multi-indices α_1, α_2 with $0 \leq |\alpha_1| \leq \bar{k}$ and $0 \leq |\alpha_2| < 2\lambda_0$, where $\bar{k}$ is the smallest integer $\geq k$. Then the equation (2.24) has a unique solution $w \in \mathcal{C}([1,\infty), L^2)$ such that*

$$\| w(t) - w_+ \|_2 \to 0 \quad \textit{when } t \to \infty \ .$$

Furthermore $w \in \mathcal{C}([1,\infty), H^k)$ and w satisfies the estimates

$$\| w(t) - U^*(1/t)w_+ \|_2 \leq C \ t^{-\lambda_0} \tag{5.54}$$

$$\| \omega^k(w(t) - U^*(1/t)w_+) \|_2 \leq C \ t^{-\lambda} \tag{5.55}$$

for all $t \geq 1$.

Proof. The results follow from Propositions 5.2 and 5.3 and from Lemma 5.1. □

The existence of the wave operators for u in the usual sense at the corresponding level of regularity is an easy consequence of Proposition 5.4. We refrain from giving a formal statement at this stage. The same question will be considered in Section 8 in the case of the interacting system (1.1) (1.2).

6 Cauchy problem at infinity for the auxiliary system

In this section we begin the construction of the wave operators for the auxiliary system (2.20) by solving the Cauchy problem at infinity for that system in the difference form (2.30), for large or infinite initial time, and for a given choice of (W, S) satisfying a number of a priori estimates. The construction of (W, S) satisfying those estimates is deferred to the next section. In the same spirit as in Section 4, we solve the system (2.30) in two steps. We first solve the linearized version of that system (2.31), thereby defining a map $\Gamma : (q, \sigma) \to (q', \sigma')$. We then show that this map is a contraction in suitable norms on a suitable set.

The basic tool of this section again consists of a priori estimates for suitably regular solutions of the linearized system (2.31). In order to handle efficiently a non-vanishing B_0, those estimates have to be much more elaborate than those of Section 4.

We first estimate a single solution of the linearized system (2.31) at the level of regularity where we shall eventually solve the auxiliary system (2.30).

Lemma 6.1 *Let $1 < k \le \ell$, $\ell > 3/2$ and $\beta > 0$. Let $I \subset [1, \infty)$ be an interval and let $t_1 \in \bar{I}$. Let B_0 satisfy the estimates (3.17) for $0 \le l \le k$. Let $(U(1/t))W, S) \in \mathcal{C}(I_+, X^{k+1,\ell+1}) \cap \mathcal{C}^1(I_+, X^{k,\ell})$ with $W \in L^\infty(I_+, H^{k+1})$ and let*

$$a = \| W; L^\infty(I_+, H^{k+1}) \| \quad . \tag{6.1}$$

Let (q, σ), $(q', \sigma') \in \mathcal{C}(I, X^{k,\ell})$ with $q \in L^\infty(I, H^k) \cap L^2(I, L^2)$ if $t_1 = \infty$, and let (q', σ') be a solution of the system (2.31) in I. Then the following estimates hold for all $t \in I$:

$$\begin{aligned} |\partial_t \| q' \|_2| \quad \le \quad & C\Big\{ t^{-2} a \| \nabla\sigma \|_2 + t^{-1-\beta}\ a^2\ I_0 (\| q \|_2) \\ & + t^{-1}\ a\ I_{-1} (\| q \|_2 \| q \|_3) \Big\} + \ \| R_1(W, S) \|_2 \ , \end{aligned} \tag{6.2}$$

$$\begin{aligned} & \left| \partial_t \| \omega^k q' \|_2 \right| \le C \Big\{ b_0 \left(\| \omega^{k-1} q' \|_2 + t^{k-1-\delta/3} \ \| q' \|_r \right) \\ & + t^{-2}\ a \left(\| \omega^k \nabla\sigma \|_2 + \| \sigma \|_\infty + \| \nabla\sigma \|_3 \right) \\ & t^{-2} \Big(\| \nabla s \|_\infty \| \omega^k q' \|_2 + \| \omega^{[k\vee 3/2]} \nabla s \|_2 \| \omega^{[k\wedge 3/2]} q' \|_2 \\ & + \chi(k > 3/2) \| \omega^k \nabla s \|_2 \| q' \|_\infty \big) \\ & + t^{-1} a^2 \left(I_{k-1} \left(\| \omega^{k-1} q \|_2 + \| q \|_2 \right) + I_0 (\| q \|_2) + \ \| \omega^{k-1} q' \|_2 + \| q' \|_2 \right) \\ & + t^{-1} a \left(I_{k-1} \left(\| \omega^k q \|_2 \| q \|_3 \right) + I_0 (\| \nabla q \|_2 \| q \|_3) \right. \\ & + I_{1/2} \Big(\| \omega^{1/2} q \|_2 \Big) \| \omega^k q' \|_2 + I_{k-1/2} \Big(\| \omega^{k-1/2} q \|_2 + \| q \|_2 \Big) \| \nabla q' \|_2 \Big) \\ & + t^{-1} \left(I_{1/2} \left(\| \nabla q \|_2^2 \right) \| \omega^k q' \|_2 + I_{k-1/2} \left(\| \omega^k q \|_2 \| \nabla q \|_2 \right) \| \nabla q' \|_2 \right) \Big\} \\ & + \| \omega^k R_1(W, S) \|_2 \end{aligned} \tag{6.3}$$

where $s = S + \sigma$ *and* $0 < \delta = \delta(r) \leq [k \wedge 3/2]$,

$$|\partial_t \parallel \omega^m \nabla \sigma' \parallel_2| \leq C\ t^{-2} \Big\{ \parallel \nabla s \parallel_\infty \parallel \omega^m \nabla \sigma' \parallel_2 \ + \ \parallel \omega^m \nabla s \parallel_2 \parallel \nabla \sigma' \parallel_\infty$$
$$\parallel \omega^m \nabla \sigma \parallel_2 \parallel \nabla S \parallel_\infty \ + \ \parallel \sigma \parallel_\infty \parallel \omega^m \nabla^2 S \parallel_2 \Big\}$$
$$+ C \Big\{ t^{-1+\beta(m+1)} a\ I_0 (\parallel q \parallel_2) + t^{-1+\beta(m+5/2)}\ I_{-3/2} \left(\parallel q \parallel_2^2 \right) \Big\}$$
$$+ \ \parallel \omega^m \nabla R_2(W, S) \parallel_2 \tag{6.4}$$

for $0 \leq m \leq \ell$,

$$|\partial_t \parallel \nabla \sigma' \parallel_2| \leq C t^{-2} \Big\{ \parallel \nabla s \parallel_\infty \parallel \nabla \sigma' \parallel_2 + \parallel \nabla \sigma \parallel_2 \Big(\parallel \nabla S \parallel_\infty \ + \parallel \omega^{3/2} \nabla S \parallel_2 \Big) \Big\}$$
$$+ C \Big\{ t^{-1+\beta} a\ I_0 (\parallel q \parallel_2) + t^{-1+5\beta/2} I_{-3/2} \left(\parallel q \parallel_2^2 \right) \Big\} + \parallel \nabla R_2(W, S) \parallel_2 \tag{6.4_0}$$

where the time parameter is t_1 *in all the estimating functions* I_m*, and the superscript* t_1 *is omitted for brevity.*

Remark 6.1. All the norms of (q, σ) and (q', σ') that appear in (6.2)–(6.4) are controlled by the norms in $X^{k,\ell}$ through Sobolev inequalities. Furthermore all the integrals I_m are convergent if $t_1 = \infty$. This follows from boundedness of q in H^k in all cases where $m > -1/2$, namely in all cases but two. The exceptions are

$$I_{-3/2} \left(\parallel q \parallel_2^2 \right) = \int_1^\infty d\nu \parallel q(\nu t) \parallel_2^2$$

in (6.4) and

$$I_{-1} \left(\parallel q \parallel_2 \parallel q \parallel_3 \right) \leq C \int_1^\infty d\nu\ \nu^{-1/2} \ \parallel q \parallel_2^{3/2} \parallel \nabla q \parallel_2^{1/2}$$
$$\leq C \parallel \nabla q; L^\infty([t, \infty), L^2) \parallel^{1/2} \parallel q; L^2((t, \infty), L^2) \parallel^{3/2}$$

in (6.2), both of which are controlled under the additional assumption that $q \in L^2(I, L^2)$.

Finally it is easy to see by estimates similar to, but simpler than, those of Lemma 4.1 that all the norms of the remainders R_1 and R_2 that occur in (6.2)–(6.4) are finite under the assumptions made on (W, S).

Proof of Lemma 6.1 In all the proof, the time superscript in B_S, B_L and in the various I_m is omitted, except in dubious cases. That time superscript is in general t_1, except in $B_S(W, W)$ where it is ∞.

Proof of (6.2). From (2.31), we estimate

$$|\partial_t \parallel q' \parallel_2| \leq$$
$$t^{-2} \parallel Q(\sigma, W) \parallel_2 \ + t^{-1} \parallel (B_S(q, q) + 2B_S(W, q))\, W \parallel_2 \ + \ \parallel R_1(W, S) \parallel_2 \ . \tag{6.5}$$

We next estimate by (3.8) (3.10) and Sobolev inequalities

$$\| Q(\sigma, W) \|_2 \le C \| \nabla\sigma \|_2 (\| \nabla W \|_3 + \| W \|_\infty) \le Ca \| \nabla\sigma \|_2, \quad (6.6)$$

$$\| B_S(W,q)W \|_2 \le Ct^{-\beta} \| W \|_\infty \ I_0 (\| W \|_\infty \| q \|_2) \le Ca^2 t^{-\beta} I_0 (\| q \|_2), \quad (6.7)$$

$$\| B_S(q,q)W \|_2 \le C \| W \|_\infty I_{-1} (\| q \|_2 \| q \|_3) \le CaI_{-1} (\| q \|_2 \| q \|_3). \quad (6.8)$$

Substituting (6.6) (6.7) (6.8) into (6.5) yields (6.2).

Proof of (6.3). From (2.31), we estimate

$$\begin{aligned} &\left|\partial_t \| \omega^k q' \|_2\right| \le t^{-1} \| [\omega^k, B_0]q' \|_2 \ + t^{-2} \big(\| [\omega^k, s] \cdot \nabla q' \|_2 \\ &+ \| (\nabla \cdot s)\omega^k q' \|_2 \ + \ \| \omega^k ((\nabla \cdot s)q') \|_2 \ + \ \| \omega^k Q(\sigma, W) \|_2\big) \\ &t^{-1} \big(\| [\omega^k, B_S(w,w)]q' \|_2 \ + \ \| \omega^k (B_S(q,q) + 2B_S(q,W)) W \|_2\big) \\ &+ \ \| \omega^k \ R_1(W,S) \|_2 \end{aligned} \quad (6.9)$$

and we estimate the various terms in the RHS successively.

The contribution of B_0 is estimated by Lemma 3.2 and by (3.17) exactly as in Section 4 (see (4.7)) and yields

$$\| [\omega^k, B_0]q' \|_2 \le \ C \ b_0 \left(t \| \omega^{k-1} q' \|_2 \ + t^{k-\delta/3} \ \| q' \|_r\right) . \quad (6.10)$$

The contribution of $Q(s, q')$ is estimated by (4.8) as

$$\begin{aligned} &\| [\omega^k, s] \cdot \nabla q' \|_2 \ + \ \| (\nabla \cdot s)\omega^k q' \|_2 \ + \ \| \omega^k ((\nabla \cdot s)q') \|_2 \\ &\le C \Big\{ \| \nabla s \|_\infty \| \omega^k q' \|_2 \ + \ \| \omega^{[k\vee 3/2]} \nabla s \|_2 \| \omega^{[k\wedge 3/2]} q' \|_2 \\ &+ \chi(k > 3/2) \| \omega^k \nabla s \|_2 \| q' \|_\infty \Big\} . \end{aligned} \quad (6.11)$$

The contribution of $Q(\sigma, W)$ is estimated by Lemma 3.2 and Sobolev inequalities as

$$\begin{aligned} &\| \omega^k Q(\sigma, W) \|_2 \le C \Big\{ \| \sigma \|_\infty \| \omega^k \nabla W \|_2 \ + \ \| \omega^k \sigma \|_6 \| \nabla W \|_3 \\ &+ \ \| \omega^k \nabla\sigma \|_2 \| W \|_\infty \ + \ \| \nabla\sigma \|_3 \| \omega^k W \|_6 \Big\} \\ &\le C \ a \Big\{ \| \omega^k \nabla\sigma \|_2 \ + \ \| \sigma \|_\infty \ + \ \| \nabla\sigma \|_3 \Big\} . \end{aligned} \quad (6.12)$$

The contribution of B_S with $w = W + q$ yields a number of terms which we order by increasing number of q or q' occurring therein. We first expand

$$B_S^{t_1,\infty}(w,w) = B_S^{\infty}(W,W) + 2B_S^{t_1}(W,q) + B_S^{t_1}(q,q) \ .$$

By Lemma 3.2 and Sobolev inequalities, we estimate

$$\begin{aligned} &\| [\omega^k, B_S(W,W)]q' \|_2 \le C \Big\{ \| \nabla B_S(W,W) \|_\infty \| \omega^{k-1} q' \|_2 \\ &+ \ \| \omega^{k+3/2-\delta} B_1(W,W) \|_2 \| q' \|_r \Big\} \end{aligned} \quad (6.13)$$

where we take $0 < \delta = \delta(r) = (k-1) \wedge 1/2$ so that

$$\| q' \|_r \le C \left(\| \omega^{k-1} q' \|_2 \ + \ \| q' \|_2 \right) \ .$$

Furthermore

$$\| \nabla B_S(W,W) \|_\infty \le C \ \| \nabla^2 B_1(W,W) \|_2^{1-\theta} \ \| \omega^{k+3/2} \ B_1(W,W) \|_2^\theta$$

with $\theta = 1/(2k-1)$, and by (3.10) and Lemma 3.2

$$\| \omega^{m+1} B_1(W,W) \|_2 \le C \ I_m \left(\| \omega^m W \|_2 \ \| W \|_\infty \right) \le C \ a^2$$

which we use with $m = 1$, $k + 1/2$ and $k + 1/2 - \delta$. Substituting those estimates into (6.13) yields

$$\| [\omega^k, B_S(W,W)] q' \|_2 \le C \ a^2 \left(\| \omega^{k-1} q' \|_2 \ + \ \| q' \|_2 \right) \ . \tag{6.14}$$

In a similar way, we estimate

$$\| [\omega^k, B_S(W,q)] q' \|_2 \le C \Big\{ \ \| \omega^{3/2} B_1(W,q) \|_2 \ \| \omega^k q' \|_2 + \ \| \omega^{k+1/2} B_1(W,q) \|_2 \ \| \nabla q' \|_2 \ \Big\} \ . \tag{6.15}$$

By Lemma 3.2 again and by (3.10)

$$\| \omega^{m+1} B_1(W,q) \|_2 \le I_m \left(\| \omega^m W q \|_2 \right) \ ,$$
$$\| \omega^m W q \|_2 \le C \left(\| W \|_\infty \ \| \omega^m q \|_2 \ + \ \| \omega^{m+3/2-\delta} W \|_2 \ \| q \|_r \right)$$

with $0 < \delta = \delta(r) = m \wedge 1/2$, so that for $m \le k - 1/2$

$$\| \omega^{m+1} B_1(W,q) \|_2 \le C \ a \ I_m \left(\| \omega^m q \|_2 \ + \ \| q \|_2 \right) \ , \tag{6.16}$$

where $\| q \|_2$ can be omitted for $m \le 1/2$. Substituting (6.16) with $m = 1/2$ and $m = k - 1/2$ into (6.15) yields

$$\| [\omega^k, B_S(W,q)] q' \|_2 \le C \ a \Big\{ I_{1/2} \left(\| \omega^{1/2} q \|_2 \right) \| \omega^k q' \|_2 + I_{k-1/2} \left(\| \omega^{k-1/2} q \|_2 \ + \ \| q \|_2 \right) \| \nabla q' \|_2 \ \Big\} \ . \tag{6.17}$$

By Lemma 3.2 and Sobolev inequalities again, we next estimate

$$\| [\omega^k, B_S(q,q)] q' \|_2 \le C \Big\{ \ \| \omega^{3/2} B_1(q,q) \|_2 \ \| \omega^k q' \|_2 + \| \omega^{k+1/2} B_1(q,q) \|_2 \ \| \nabla q' \|_2 \ \Big\} \tag{6.18}$$

followed by (see also (3.10))

$$\| \omega^{3/2} B_1(q,q) \|_2 \le C I_{1/2} \left(\| \nabla q \|_2^2 \right)$$
$$\| \omega^{k+1/2} B_1(q,q) \|_2 \le C I_{k-1/2} \left(\| \omega^k q \|_2 \ \| \nabla q \|_2 \right)$$

so that

$$\| [\omega^k, B_S(q,q)]q' \|_2 \le C\Big\{ I_{1/2} \left(\| \nabla q \|_2^2 \right) \| \omega^k q' \|_2$$
$$+ I_{k-1/2} \left(\| \omega^k q \|_2 \| \nabla q \|_2 \right) \| \nabla q' \|_2 \Big\} . \qquad (6.19)$$

We now turn to the second contribution of B_S to (6.9). By Lemma 3.2 and Sobolev inequalities again

$$\| \omega^k (B_S(W,q)W) \|_2 \le C\Big\{ \| \omega^k B_1(W,q) \|_2 \| W \|_\infty$$
$$+ \| \nabla B_1(W,q) \|_2 \| \omega^{k+1/2} W \|_2 \Big\} \qquad (6.20)$$

and by (6.16) with $m = k-1$ and $m = 0$

$$\| \omega^k (B_S(W,q)W) \|_2 \le C\ a^2 \left(I_{k-1} \left(\| \omega^{k-1} q \|_2 + \| q \|_2 \right) + I_0 \left(\| q \|_2 \right) \right) . \qquad (6.21)$$

Similarly

$$\| \omega^k (B_S(q,q)W) \|_2 \le C\Big\{ \| \omega^k B_1(q,q) \|_2 \| W \|_\infty$$
$$+ \| \nabla B_1(q,q) \|_2 \| \omega^{k+1/2} W \|_2 \Big\} \qquad (6.22)$$

followed by (see the proof of (6.19))

$$\| \omega^k B_1(q,q) \|_2 \le C\ I_{k-1} \left(\| \omega^k q \|_2 \| q \|_3 \right)$$
$$\| \nabla B_1(q,q) \|_2 \le C\ I_0 \left(\| \nabla q \|_2 \| q \|_3 \right)$$

yields

$$\| \omega^k (B_S(q,q)W) \|_2 \le C\ a \left(I_{k-1} \left(\| \omega^k q \|_2 \| q \|_3 \right) + I_0 \left(\| \nabla q \|_2 \| q \|_3 \right) \right) . \qquad (6.23)$$

Substituting (6.10) (6.11) (6.12) (6.14) (6.17) (6.19) (6.21) and (6.23) into (6.9) and reordering the contributions of B_S by increasing powers of (q, q') yields (6.3).

Proof of (6.4). From (2.31), we estimate

$$\left| \partial_t \| \omega^{m+1} \sigma' \|_2 \right| \le t^{-2} \left(\| [\omega^{m+1}, s] \cdot \nabla \sigma' \|_2 + \| (\nabla \cdot s) \omega^{m+1} \sigma' \|_2 \right.$$
$$\left. + \| \omega^{m+1} (\sigma \cdot \nabla S) \|_2 \right) + t^{-1} \| \omega^{m+2} \left(B_L(q,q) + 2 B_L(W,q) \right) \|_2$$
$$+ \| \omega^{m+1}\ R_2(W,S) \|_2 . \qquad (6.24)$$

We next estimate by Lemma 3.2 again

$$\| [\omega^{m+1}, s] \cdot \nabla \sigma' \|_2 \le C \left(\| \nabla s \|_\infty \| \omega^{m+1} \sigma' \|_2 + \| \omega^{m+1} s \|_2 \| \nabla \sigma' \|_\infty \right) \qquad (6.25)$$

$$\| \omega^{m+1} (\sigma \cdot \nabla S) \|_2 \le C \left(\| \omega^{m+1} \sigma \|_2 \| \nabla S \|_\infty + \| \sigma \|_\infty \| \omega^{m+1} \nabla S \|_2 \right) \qquad (6.26)$$

while by (3.9) and Lemma 3.4

$$\begin{aligned} \| \omega^{m+2} B_L(W,q) \|_2 &\leq t^{\beta(m+1)}\ I_0 \left(\| W \|_\infty \| q \|_2\right) \\ &\leq t^{\beta(m+1)}\ a\ I_0 \left(\| q \|_2\right) \ , \end{aligned} \tag{6.27}$$

$$\| \omega^{m+2} B_L(q,q) \|_2 \leq C\ t^{\beta(m+5/2)}\ I_{-3/2} \left(\| q \|_2^2\right) \ . \tag{6.28}$$

Substituting (6.25) (6.26) (6.27) (6.28) into (6.24) yields (6.4). For $m = 0$, the term $\| \omega^m \nabla s \|_2 \| \nabla \sigma' \|_\infty$ can be omitted, and the term $\sigma \cdot \nabla S$ can be estimated in a sllightly different way, thereby leading to $(6.4)_0$. □

We next estimate the difference of two solutions of the linearized system (2.31) corresponding to two different choices of (q,σ), but to the same choice of (W,S). As in Section 4, we estimate that difference at a lower level of regularity than the solutions themselves.

Lemma 6.2 *Let $1 < k \leq \ell$, $\ell > 3/2$ and $\beta > 0$. Let $I \subset [1,\infty)$ be an interval and let $t_1 \in \bar{I}$. Let B_0 be sufficiently regular, for instance $B_0 \in \mathcal{C}(I, H_3^k)$. Let (W,S) satisfy the assumptions of Lemma 6.1. Let (q_i,σ_i), $(q'_i,\sigma'_i) \in \mathcal{C}(I, X^{k,\ell})$ with $q_i \in L^\infty(I,H^k) \cap L^2(I,L^2)$, $i = 1,2$, if $t_1 = \infty$, and let (q'_i,σ'_i) be solutions of the system (2.31) associated with (q_i,σ_i) and (W,S). Define $(q_\pm,\sigma_\pm) = (1/2)(q_1 \pm q_2, \sigma_1 \pm \sigma_2)$ and $(q'_\pm,\sigma'_\pm) = (1/2)(q'_1 \pm q'_2, \sigma'_1 \pm \sigma'_2)$. Then the following estimates hold for all $t \in I$.*

$$\begin{aligned} \left|\partial_t \| q'_- \|_2\right| \leq C \Big\{ & t^{-2}\ a \| \nabla \sigma_- \|_2 + t^{-2} \Big(\| \omega^{[3/2-k]_+} \nabla \sigma_- \|_2 \| \omega^{[k \wedge 3/2]} q'_+ \|_2 \\ & + \chi(k > 3/2) \| \nabla \sigma_- \|_2 \| q'_+ \|_\infty \Big) + t^{-1-\beta}\ a^2\ I_0 \left(\| q_- \|_2\right) \\ & + t^{-1}\ a \left(\| q'_+ \|_3\ \ I_0 \left(\| q_- \|_2\right) + I_{-1} \left(\| q_+ \|_3 \| q_- \|_2\right)\right) \\ & + t^{-1} \| q'_+ \|_6\ \ I_{-1/2} \left(\| q_+ \|_6 \| q_- \|_2\right) \Big\} \ , \end{aligned} \tag{6.29}$$

$$\begin{aligned} \left|\partial_t \| \omega^m \nabla \sigma'_- \|_2\right| \leq C\ t^{-2} \Big\{ & \| \nabla s_+ \|_\infty \| \omega^m \nabla \sigma'_- \|_2 \ + \ \| \nabla s'_+ \|_\infty \| \omega^m \nabla \sigma_- \|_2 \\ & + \| \omega^{m-m'+3/2} \nabla s_+ \|_2 \| \omega^{m'} \nabla \sigma'_- \|_2 \ + \ \| \omega^{m+5/2-\delta} \nabla s'_+ \|_2 \| \sigma_- \|_r \Big\} \\ & + C \Big\{ t^{-1+\beta(m+1)}\ a\ I_0 \left(\| q_- \|_2\right) + t^{-1+\beta(m+5/2)} I_{-3/2} \left(\| q_+ \|_2 \| q_- \|_2\right) \Big\} \ , \end{aligned} \tag{6.30}$$

$$\begin{aligned} \left|\partial_t \| \nabla \sigma'_- \|_2\right| \leq C\ t^{-2} \Big\{ & \| \nabla s_+ \|_\infty \| \nabla \sigma'_- \|_2 \\ & + \ \| \nabla \sigma_- \|_2 \left(\| \nabla s'_+ \|_\infty + \| \omega^{3/2} \nabla s'_+ \|_2 \right) \Big\} \\ + C \Big\{ & t^{-1+\beta}\ a\ I_0 \left(\| q_- \|_2\right) + t^{-1+5\beta/2} I_{-3/2} \left(\| q_+ \|_2 \| q_- \|_2\right) \Big\} \end{aligned} \tag{6.30_0}$$

where $s_+ = S + \sigma_+$, $s'_+ = S + \sigma'_+$,

$$\begin{aligned} & 0 \leq m \leq \ell_0 \ , \ m' = m \wedge 1/2 \ , \ \delta = \delta(r) = [(m+1) \wedge 3/2] \ , \\ & [3/2 - k]_+ \leq \ell_0 \leq \ell - 1 \ , \end{aligned} \tag{6.31}$$

and the superscript t_1 is again omitted in the estimating functions I_m.

Remark 6.2 Under the assumptions made, all the norms in the first part of the RHS of (6.30) are controlled by $|\nabla s_+|_\ell$, $|\nabla s'_+|_\ell$, $|\nabla \sigma_-|_{\ell_0}$ and $|\nabla \sigma'_-|_{\ell_0}$.

Proof. Taking the difference of the system (2.31) for (q'_i, σ'_i), or equivalently and more simply rewriting (4.21) with $(w_-, s_-) = (q_-, \sigma_-)$, $(w'_-, s'_-) = (q'_-, \sigma'_-)$, and accounting for the replacement of $B_S^{t_1}(w,w)$ by $B_S^{t_1,\infty}(w,w)$, we obtain

$$\left\{ \begin{array}{l} \partial_t q'_- = i(2t^2)^{-1}\Delta q'_- + t^{-2}\Big\{Q(s_+, q'_-) + Q(\sigma_-, w'_+)\Big\} + it^{-1}B_0 q'_- \\ +it^{-1}\Big\{ \left(B_S^{t_1,\infty}(w_+,w_+) + B_S^{t_1}(q_-,q_-)\right) q'_- + 2B_S^{t_1}(w_+,q_-)w'_+ \Big\} \\ \partial_t \sigma'_- = t^{-2}\left(s_+ \cdot \nabla \sigma'_- + \sigma_- \cdot \nabla s'_+\right) - 2t^{-1}\nabla B_L^{t_1}(w_+, q_-) \ . \end{array} \right. \tag{6.32}$$

We first estimate q'_-. From (6.32) we obtain

$$\Big|\partial_t \parallel q'_- \parallel_2\Big| \le t^{-2} \parallel Q(\sigma_-, w'_+) \parallel_2 \ +2t^{-1} \parallel B_S(w_+, q_-)w'_+ \parallel_2 \ . \tag{6.33}$$

We expand (6.33) by using

$$(w_+, s_+) = (W,S) + (q_+, \sigma_+) \ , \ (w'_+, s'_+) = (W,S) + (q'_+, \sigma'_+)$$

and we estimate the various terms successively.

From (6.6) we obtain

$$\parallel Q(\sigma_-, W) \parallel_2 \le C\ a \parallel \nabla\sigma_- \parallel_2 \ . \tag{6.34}$$

By the same estimates as in the proof of (4.23), we next obtain

$$\begin{aligned} \parallel Q(\sigma_-, q'_+) \parallel_2 \ \le \ & C\Big\{ \parallel \omega^{[3/2-k]_+}\nabla\sigma_- \parallel_2 \parallel \omega^{[k\wedge 3/2]} q'_+ \parallel_2 \\ & +\chi(k > 3/2) \parallel \nabla\sigma_- \parallel_2 \parallel q'_+ \parallel_\infty \Big\} \ . \end{aligned} \tag{6.35}$$

We next estimate by (6.7) (6.8)

$$\parallel B_S(W, q_-)W \parallel_2 \le C\ a^2\ t^{-\beta}\ I_0 \left(\parallel q_- \parallel_2\right) \ , \tag{6.36}$$

$$\parallel B_S(q_+, q_-)W \parallel_2 \le C\ a\ I_{-1} \left(\parallel q_+ \parallel_3 \parallel q_- \parallel_2\right) \ . \tag{6.37}$$

The remaining terms are new. Using (3.10) and Sobolev inequalities, we obtain successively

$$\parallel B_S(W, q_-)q'_+ \parallel_2 \le C\ a \parallel q'_+ \parallel_3\ \ I_0 \left(\parallel q_- \parallel_2\right) \ , \tag{6.38}$$

$$\begin{aligned} \parallel B_S(q_+, q_-)q'_+ \parallel_2 \ & \le \ C \parallel B_S(q_+, q_-) \parallel_3 \parallel q'_+ \parallel_6 \\ & \le \ C\ I_{-1/2} \left(\parallel q_+ \parallel_6 \parallel q_- \parallel_2\right) \parallel q'_+ \parallel_6 \ . \end{aligned} \tag{6.39}$$

Substituting (6.34)–(6.39) into (6.33) yields (6.29).

We next estimate σ'_-. From (6.32) we obtain

$$\begin{aligned}\left|\partial_t \parallel \omega^{m+1}\sigma'_- \parallel_2\right| &\le t^{-2}\left(\parallel [\omega^{m+1}, s_+]\cdot\nabla\sigma'_- \parallel_2 \ + \ \parallel (\nabla\cdot s_+)\omega^{m+1}\sigma'_- \parallel_2\right.\\ &\left.+ \parallel \omega^{m+1}(\sigma_-\cdot\nabla s'_+) \parallel_2\right) + 2t^{-1} \parallel \omega^{m+2}(B_L(W,q_-) + B_L(q_+,q_-)) \parallel_2\end{aligned} \tag{6.40}$$

and we estimate the various terms successively.

From Lemma 3.2 and Sobolev inequalities, we obtain

$$\begin{aligned}&\parallel [\omega^{m+1}, s_+]\cdot\nabla\sigma'_- \parallel_2 \le C\left(\parallel \nabla s_+ \parallel_\infty \parallel \omega^{m+1}\sigma'_- \parallel_2\right.\\ &\left.+ \parallel \omega^{m+1}s_+ \parallel_{3/m'} \parallel \nabla\sigma'_- \parallel_{r'}\right)\\ &\le C\left(\parallel \nabla s_+ \parallel_\infty \parallel \omega^m\nabla\sigma'_- \parallel_2 \ + \ \parallel \omega^{m-m'+3/2}\nabla s_+ \parallel_2 \parallel \omega^{m'}\nabla\sigma'_- \parallel_2\right)\end{aligned} \tag{6.41}$$

with $m' = \delta(r') = m \wedge 1/2$, and

$$\begin{aligned}&\parallel \omega^{m+1}(\sigma_-\nabla s'_+) \parallel_2 \le C\left(\parallel \nabla s'_+ \parallel_\infty \parallel \omega^m\nabla\sigma_- \parallel_2 + \parallel \omega^{m+1}\nabla s'_+ \parallel_{3/\delta} \parallel \sigma_- \parallel_r\right)\\ &\le C\left(\parallel \nabla s'_+ \parallel_\infty \parallel \omega^m\nabla\sigma_- \parallel_2 + \parallel \omega^{m+5/2-\delta}\nabla s'_+ \parallel_2 \parallel \sigma_- \parallel_r\right)\end{aligned} \tag{6.42}$$

with $\delta = \delta(r) = [(m+1) \wedge 3/2]$.

The contribution of B_L to (6.40) is estimated exactly as in the proof of (6.4) (see (6.27) and (6.28)) by

$$\begin{aligned}&\parallel \omega^{m+2}\left(B_L(W,q_-) + B_L(q_+,q_-)\right) \parallel_2 \le C\left(a\ t^{\beta(m+1)}\ I_0\left(\parallel q_- \parallel_2\right)\right.\\ &\left.+t^{\beta(m+5/2)}\ I_{-3/2}\left(\parallel q_+ \parallel_2 \parallel q_- \parallel_2\right)\right)\ .\end{aligned} \tag{6.43}$$

Substituting (6.41) (6.42) (6.43) into (6.40) yields (6.30) and (6.30_0), where one term from (6.41) can be omitted. □

We now begin the study of the Cauchy problem for the auxiliary system in the difference form (2.30) and for that purpose we first study that problem for the linearized version of that system. For finite initial time t_0, that problem is solved by a minor modification of Proposition 4.1. The following proposition is simply a compilation of that result and of Lemmas 6.1 and 6.2.

Proposition 6.1 *Let $1 < k \le \ell$, $\ell > 3/2$ and $\beta > 0$. Let $I \subset [1,\infty)$ be an interval and let $t_1 \in \bar{I}$. let B_0 satisfy the estimates (3.17) for $0 \le m \le k$. Let $(U(1/t)W, S) \in \mathcal{C}(I_+, X^{k+1,\ell+1}) \cap \mathcal{C}^1(I_+, X^{k,\ell})$ with $W \in L^\infty(I_+, H^{k+1})$ and define a by (6.1). Let $(q,\sigma) \in \mathcal{C}(I, X^{k,\ell})$ with $q \in L^\infty(I, H^k) \cap L^2(I, L^2)$ if $t_1 = \infty$. Let $t_0 \in I$ and let $(q'_0, \sigma'_0) \in X^{k,\ell}$. Then the system (2.31) has a unique solution $(q',\sigma') \in \mathcal{C}(I, X^{k,\ell})$ with $(q',\sigma')(t_0) = (q'_0,\sigma'_0)$. That solution satisfies the estimates (6.2) (6.3) (6.4) of Lemma 6.1 for all $t \in I$. Two such solutions (q'_i,σ'_i) associated with (q_i,σ_i), $i = 1,2$ and with the same (W,S) satisfy the estimates (6.29) (6.30) of Lemma 6.2 for all $t \in I$.*

We shall be eventually interested in solving the Cauchy problem for the auxiliary system (2.30) with infinite initial time t_0. As a preliminary, we need to solve the same problem for the linearized system (2.31). This is done in the following proposition, which of course requires much stronger assumptions on the asymptotic behaviour in time of (W, S) and (q, σ). With the study of the nonlinear system in view, we already make the assumptions that will be needed for that purpose, although they could be slightly weakened for the linear problem. Since we want to take $t_0 = \infty$, we also take $t_1 = \infty$.

Proposition 6.2 *Let $1 < k \leq \ell$, $\ell > 3/2$. Let β, λ_0 and λ satisfy*

$$0 < \beta < 1 \quad , \quad \lambda > 0 \quad , \quad \lambda_0 > \lambda + k \quad , \quad \lambda_0 > \beta(\ell + 1) \ . \tag{6.44}$$

Let $t_1 = \infty$, let $1 \leq T < \infty$, and $I = [T, \infty)$. Let B_0 satisfy the estimates (3.17) for $0 \leq m \leq k$. Let (W, S) satisfy the assumptions of Proposition 6.1 with

$$|W|_{k+1} \leq a \ , \tag{6.45}$$

$$\| \omega^m \nabla S \|_2 \leq b \ t^{1-\eta+\beta(m-3/2)} \tag{6.46}$$

for some $\eta > 0$ and for $0 \leq m \leq \ell + 1$,

$$\| R_1(W, S) \|_2 \leq c_0 \ t^{-1-\lambda_0} \quad , \quad \| \omega^k R_1(W, S) \|_2 \leq c_1 \ t^{-1-\lambda} \ , \tag{6.47}$$

$$\| \omega^m \nabla R_2(W, S) \|_2 \leq c_2 \ t^{-1-\lambda_0+\beta(m+1)} \quad \text{for } 0 \leq m \leq \ell \ , \tag{6.48}$$

for all $t \in I$. Let $(q, \sigma) \in \mathcal{C}(I, X^{k,\ell})$ satisfy

$$\| q \|_2 \leq Y_0 \ t^{-\lambda_0} \quad , \quad \| \omega^k q \|_2 \leq Y \ t^{-\lambda} \ , \tag{6.49}$$

$$\| \omega^m \nabla \sigma \|_2 \leq Z \ t^{-\lambda_0+\beta(m+1)} \quad \text{for } 0 \leq m \leq \ell \ , \tag{6.50}$$

for all $t \in I$. Then the system (2.31) has a (unique) solution $(q', \sigma') \in \mathcal{C}(I, X^{k,\ell})$ satisfying

$$\| q' \|_2 \leq Y'_0 \ t^{-\lambda_0} \quad , \quad \| \omega^k q' \|_2 \leq Y' \ t^{-\lambda} \ , \tag{6.51}$$

$$\| \omega^m \nabla \sigma' \|_2 \leq Z' \ t^{-\lambda_0+\beta(m+1)} \quad \text{for } 0 \leq m \leq \ell \ , \tag{6.52}$$

for some Y'_0, Y', Z' depending on k, ℓ, β, λ_0, λ, a, b, c_0, c_1, c_2, Y_0, Y, Z and T, for all $t \in I$. That solution satisfies the estimates (6.2) (6.3) (6.4) of Lemma 6.1 for all $t \in I$. Two such solutions (q'_i, σ'_i) associated with (q_i, σ_i), $i = 1, 2$, satisfy the estimates (6.29) (6.30) of Lemma 6.2 for all $t \in I$. The solution (q', σ') is actually unique in $\mathcal{C}(I, X^{k,\ell})$ under the condition that (q', σ') tends to zero in $X^{0,0}$ norm when $t \to \infty$.

Proof. The proof consists in showing that the solution $(q'_{t_0}, \sigma'_{t_0})$ of the linearized system (2.31) with $t_1 = \infty$ and with initial data $(q'_{t_0}, \sigma'_{t_0})(t_0) = 0$ for finite t_0, obtained from Proposition 6.1, satisfies the estimates (6.51) (6.52) uniformly in t_0 for $t \leq t_0$ (namely with Y'_0, Y' and Z' independent of t_0), and that when $t_0 \to \infty$, that solution converges on the compact subintervals of I uniformly in suitable norms.

We first derive the estimates (6.51) (6.52) for that solution, omitting the subscript t_0 for brevity in that part of the proof. Let

$$y'_0 = \| q' \|_2 \quad , \quad y' = \| \omega^k q' \|_2 \quad , \quad z'_m = \| \omega^m \nabla \sigma' \|_2 \ . \tag{6.53}$$

We first estimate y'_0. Substituting (6.47) (6.49) (6.50) into (6.2), and omitting an overall constant, we obtain

$$\begin{aligned} |\partial_t \ y'_0| \quad &\leq \quad a \ Z \ t^{-2-\lambda_0+\beta} + a^2 \ Y_0 \ t^{-1-\beta-\lambda_0} \\ &\quad + a \left(Y_0^3 \bar{Y} \right)^{1/2} \ t^{-1-2\lambda_0+(\lambda_0-\lambda)/2k} + c_0 \ t^{-1-\lambda_0} \end{aligned} \tag{6.54}$$

where $\bar{Y} = Y \vee Y_0$. Integrating (6.54) from t_0 to t with $y'_0(t_0) = 0$, using the fact that $\lambda_0 > 1$ and $\lambda_0 - (\lambda_0 - \lambda)/2k > k - 1/2 + \lambda$, and defining

$$Y'_0 = \| \ t^{\lambda_0} \ y'_0 ; L^\infty([T, t_0]) \ \| \ , \tag{6.55}$$

we obtain

$$Y'_0 \leq c_0 + a \ Z \ T^{-(1-\beta)} + a^2 \ Y_0 \ T^{-\beta} + a \left(Y_0^3 \bar{Y} \right)^{1/2} \ T^{-(k-1/2+\lambda)} \ . \tag{6.56}$$

That estimate is manifestly uniform in t_0.

We next estimate y', wasting part of the time decay in order to alleviate the computation. In particular when estimating $s = S + \sigma$, we use the fact that the time decay of σ is better than that of S by at least a power $1-\eta$. Furthermore in the contributions coming from B_S, we eliminate Y_0 and λ_0 by using $Y_0 \leq \bar{Y} = Y \vee Y_0$ and $\lambda_0 > \lambda + k$. In particular we estimate

$$\begin{aligned} \| \ \omega^m q \ \|_2 \quad &\leq \quad \| \ q \ \|_2^{1-m/k} \ \| \ \omega^k q \ \|_2^{m/k} \\ &\leq \quad \bar{Y} \ t^{-\lambda_0(1-m/k)-\lambda m/k} \leq \bar{Y} \ t^{-\lambda+m-k} \end{aligned} \tag{6.57}$$

for $0 \leq m \leq k$, and similarly

$$\begin{aligned} &\| \ \omega^m q' \ \|_2 \leq y'^{m/k} \ \left(Y'_0 \ t^{-\lambda_0} \right)^{1-m/k} \\ &\leq t^{m-k} \ y'^{m/k} \left(Y'_0 \ t^{-\lambda} \right)^{1-m/k} \leq t^{m-k} \left(y' + Y'_0 \ t^{-\lambda} \right) \ . \end{aligned} \tag{6.58}$$

Substituting (6.47) (6.49) (6.50) into (6.3), using (6.57) (6.58) and omitting an overall constant, we obtain

$$\begin{aligned} &|\partial_t y'| \leq b_0 \left\{ {y'}_0^{1/k} \ {y'}^{1-1/k} + {y'}_0^{1-\delta/k} \ {y'}^{\delta/k} \ t^{k-1-\delta/3} \right\} \\ &+ a \ Z \ t^{-2-\lambda-k+\beta(k+1)} + t^{-1} \left(b t^{-\eta} + Z t^{-1} \right) \left(y' + Y'_0 \ t^{-\lambda} \right) \\ &+ a^2 \ t^{-2} \left(y' + Y'_0 \ t^{-\lambda} + \bar{Y} \ t^{-\lambda} \right) + a \ t^{-k-1/2-\lambda} \left(y' + Y'_0 \ t^{-\lambda} + \bar{Y} \ t^{-\lambda} \right) \bar{Y} \\ &+ t^{1-2k-2\lambda} \ \bar{Y}^2 \left(y' + Y'_0 \ t^{-\lambda} \right) + c_1 \ t^{-1-\lambda} \ . \end{aligned} \tag{6.59}$$

The terms linear in y' in the RHS of (6.59) can be eliminated by changing variables from y' to $y' \exp(-E(t))$, where

$$E(t) = b\eta^{-1}\ t^{-\eta} + Zt^{-1} + a^2t^{-1} + (k-1/2+\lambda)^{-1} a\bar{Y} t^{-(k-1/2+\lambda)} + (2(k-1+\lambda))^{-1}\ \bar{Y}^2\ t^{-2(k-1+\lambda)} \qquad (6.60)$$

so that it is sufficient to estimate y' from (6.59) with those terms omitted and to multiply the end result by $\exp(E(t))$. With those terms omitted, and with the help of the estimate of y'_0, (6.59) can be rewritten as

$$|\partial_t y'| \ \le\ b_0 \Big\{ {Y'}_0^{1-\delta/k}\ {y'}^{\delta/k}\ t^{k-1-\delta/3-\lambda_0(1-\delta/k)} + {Y'}_0^{1/k}\ {y'}^{1-1/k}\ t^{-\lambda_0/k} \Big\} + t^{-1-\lambda}\ C_1(t) \qquad (6.61)$$

where

$$C_1(t) = a\ Z\ t^{-(1-\beta)(k+1)} + \left(b\ t^{-\eta} + Zt^{-1}\right) Y'_0 + \left(a^2 t^{-1} + a\bar{Y} t^{-(k-1/2+\lambda)}\right)\left(Y'_0 + \bar{Y}\right) + \bar{Y}^2 Y'_0 t^{-2(k-1+\lambda)} + c_1\ . \qquad (6.62)$$

In particular $C_1(t)$ is decreasing in t.

The inequality (6.61) is essentially identical with (5.24), up to notational change and replacement of c_1 by $C_1(t)$. Proceeding as in Section 5, defining

$$Y' = \|\ t^\lambda y'; L^\infty([T,t_0])\ \| \qquad (6.63)$$

and reintroducing the factor $\exp(E(t))$, we obtain (see (5.30))

$$Y' \le \exp(E(T)) \Big\{ \left(4\lambda^{-1} b_0\right)^k\ Y'_0 + 2\lambda^{-1}\ C_1(T) \Big\}\ , \qquad (6.64)$$

an estimate which is again manifestly uniform in t_0. This completes the proof of (6.51).

We next estimate z'_m for $0 \le m \le \ell$. By interpolation, it suffices to estimate z'_0 and z'_ℓ. We define

$$Z'_m = \|\ t^{\lambda_0 - \beta(m+1)}\ z'_m; L^\infty([T,t_0])\ \| \qquad (6.65)$$

and

$$Z' = \mathop{\mathrm{Sup}}_{0\le m\le \ell} Z'_m = Z'_0 \vee Z'_\ell\ . \qquad (6.66)$$

We first estimate z'_0. Substituting (6.48) (6.49) (6.50) into $(6.4)_0$ and omitting an overall constant, we obtain

$$|\partial_t\ z'_0| \le \left(b\ t^{-1-\eta} + Zt^{-2}\right) z'_0 + bZt^{-1-\eta-\lambda_0+\beta} + a\ Y_0\ t^{-1-\lambda_0+\beta} + Y_0^2\ t^{-1-2\lambda_0+5\beta/2} + c_2\ t^{-1-\lambda_0+\beta}\ . \qquad (6.67)$$

Integrating (6.67) from t to t_0, we obtain

$$Z'_0 \leq \exp\left(b\ \eta^{-1}\ T^{-\eta} + Z\ T^{-1}\right) \Big\{ (\eta + \lambda_0 - \beta)^{-1}\ \ b\ Z\ t^{-\eta} + (\lambda_0 - \beta)^{-1} \left(a\ Y_0 + Y_0^2\ T^{-\beta} + c_2\right) \Big\} \tag{6.68}$$

where we have used again the fact that $\lambda_0 > 5\beta/2$.

We next estimate z'_ℓ. Using the inequality

$$\begin{aligned} \| \nabla\sigma' \|_\infty &\leq C \parallel \omega^\ell \nabla\sigma' \|_2^{3/2\ell} \parallel \nabla\sigma' \|_2^{1-3/2\ell} \\ &\leq C\ t^{3\beta/2} \left(t^{-\beta\ell} \parallel \omega^\ell \nabla\sigma' \|_2\ +\ \parallel \nabla\sigma' \|_2\right)\ , \end{aligned} \tag{6.69}$$

substituting (6.48) (6.49) (6.50) into (6.4) with $m = \ell$, and omitting again an overall constant, we obtain

$$\begin{aligned} &|\partial_t\ z'_\ell| \leq \left(b\ t^{-1-\eta} + Zt^{-2}\right) \left(z'_\ell + Z'_0\ t^{-\lambda_0+\beta(\ell+1)}\right) \\ &+ b\ Z\ t^{-1-\eta-\lambda_0+\beta(\ell+1)} + a\ Y_0\ t^{-1-\lambda_0+\beta(\ell+1)} \\ &+ Y_0^2\ t^{-1-2\lambda_0+\beta(\ell+5/2)} + c_2\ t^{-1-\lambda_0+\beta(\ell+1)}\ . \end{aligned} \tag{6.70}$$

Integrating (6.70) as before, we obtain

$$Z'_\ell \leq \exp\left(b\ \eta^{-1}\ T^{-\eta} + Z\ T^{-1}\right) \Big\{ b(Z + Z'_0)\eta^{-1}\ T^{-\eta} + ZZ'_0\ T^{-1} + \nu^{-1} \left(a\ Y_0 + Y_0^2\ T^{-\beta} + c_2\right) \Big\} \tag{6.71}$$

where $\nu = \lambda_0 - \beta(\ell + 1) > 0$, which together with (6.68) completes the proof of (6.52).

We have proved that the solution $(q'_{t_0}, \sigma'_{t_0})$ of the system (2.31), vanishing at t_0, satisfies the estimates (6.51) (6.52) for $t \in [T, t_0]$, with Y'_0, Y', Z' satisfying (6.56) (6.64) (6.66) (6.68) (6.71), which are uniform in t_0. We now prove that $(q'_{t_0}, \sigma'_{t_0})$ tends to a limit when $t_0 \to \infty$. For that purpose, we first let (q'_i, σ'_i), $i = 1, 2$, be two solutions of the system (2.31) corresponding to the same (q, σ) and defined in an interval $[T, t_0)$ for some $t_0 > T$. Let $(q'_-, \sigma'_-) = (1/2)(q'_1 - q'_2, \sigma'_1 - \sigma'_2)$. L^2 norm conservation for q'_- implies

$$\| q'_-(t) \|_2 = \| q'_-(t_0) \|_2 \qquad \text{for all } t \in [T, t_0]\ . \tag{6.72}$$

Furthermore, the simple case $q_- = 0$, $\sigma_- = 0$ of $(6.30)_0$ implies

$$\partial_t \parallel \nabla\sigma'_- \|_2 \leq\ C\ t^{-1} \left(b\ t^{-\eta} + Z\ t^{-1}\right) \parallel \nabla\sigma'_- \|_2 \tag{6.73}$$

and therefore

$$\| \nabla\sigma'_-(t) \|_2 \leq \exp\left(C\left(\eta^{-1}\ b\ t^{-\eta} + Z\ t^{-1}\right)\right) \parallel \nabla\sigma'_-(t_0) \|_2\ . \tag{6.74}$$

Let now $T < t'_1 < t'_2 < \infty$ and let $(q'_i, \sigma'_i) = (q'_{t'_i}, \sigma'_{t'_i})$, $i = 1, 2$. Then

$$\| q'_-(t) \|_2 = \| q'_-(t'_1) \|_2 = (1/2) \| q'_2(t'_1) \|_2 \le Y'_0 \ t_1{}'^{-\lambda_0} \quad \text{for all } t < t'_1 \tag{6.75}$$

by (6.72) with $t_0 = t'_1$ and (6.51) for $q' = q'_2$ and $t = t'_1$ Similarly, by (6.52) with $\sigma' = \sigma'_2$ and $t = t'_1$,

$$\| \nabla\sigma'_-(t'_1) \| = (1/2) \| \nabla\sigma'_2(t'_1) \| \le Z'_0 \ t'^{-\lambda_0+\beta}_1$$

so that by (6.74) with $t_0 = t'_1$,

$$\| \nabla\sigma'_-(t) \| \le \exp\left(C\left(\eta^{-1} \ b \ t^{-\eta} + Z \ t^{-1}\right)\right) Z'_0 \ t'^{-\lambda_0+\beta}_1 \quad \text{for all } t < t'_1 \ . \tag{6.76}$$

From (6.75) (6.76), it follows that $(q'_{t_0}, \sigma'_{t_0})$ converges to a limit $(q', \sigma') \in \mathcal{C}(I, X^{0,0})$ uniformly on the compact subintervals of I. From the uniform estimates (6.51) (6.52) and from Lemma 6.1, it then follows by a standard compactness argument that $(q', \sigma') \in \mathcal{C}(I, X^{k,\ell})$ and that (q', σ') also satisfies the estimates (6.51) (6.52). Clearly (q', σ') satisfies the system (2.31). This completes the existence part of the proof.

The uniqueness statement follows immediately from (6.72) (6.74) by letting $t_0 \to \infty$. □

We now turn to the main result of this section, namely the fact that for T sufficiently large (depending on (W, S)), the auxiliary system in difference form (2.30) has a solution (q, σ) defined for all $t \ge T$ and decaying at infinity in a suitable sense. In the same spirit as for Proposition 4.4, this will be done by showing that the map $\Gamma : (q, \sigma) \to (q', \sigma')$ defined by Proposition 6.2 is a contraction in suitable circumstances. According to our intuition of scattering, another natural route towards the same result would be to construct first the solution (q_{t_0}, σ_{t_0}) of the auxiliary system (2.30) vanishing at t_0 and to take the limit of that solution as $t_0 \to \infty$. That route can also be followed, but it is slightly more complicated than the previous one. One of the complications comes from the fact that the system (2.30) depends on t_1. In view of Warning 4.2, for finite t_0, we expect difficulties if we take $t_1 > t_0$. This prompts us to take $t_1 = t_0$. The comparison of two solutions (q_{t_0}, σ_{t_0}) corresponding to different values of t_0 is then complicated by the fact that they do not solve exactly the same system, so that Lemma 6.2 is not directly applicable and additional terms occur in the comparison. On the other hand, for $B_0 \ne 0$, the construction of the solution (q_{t_0}, σ_{t_0}) of (2.30) is expected to meet difficulties for $t \ge t_0$ because of Warning 4.1. We shall therefore undertake it for $t \le t_0$ only, which is sufficient anyway to take the limit $t_0 \to \infty$. That construction proceeds again by a contraction starting from the solutions obtained for the linearized system. The corresponding proof for $t_0 < \infty$ is not significantly simpler than for $t_0 = \infty$, which is another reason why the second method is more complicated than the first one, since in addition to that construction, a limiting procedure is needed.

We now state the main result and formalize the previous heuristic discussion in the following proposition.

Proposition 6.3 *Let $1 < k \le \ell$ and $\ell > 3/2$. Let β, λ_0 and λ satisfy (6.44) and in addition*

$$1 + \lambda > \beta(5/2 - k) \ .$$

Let B_0 satisfy the estimates (3.17) for $0 \le m \le k$. Let (W, S) satisfy the assumptions of Proposition 6.2 in $[1, \infty)$. Then there exists T, $1 \le T < \infty$, and positive constants Y_0, Y and Z, depending on k, ℓ, β, λ_0, λ, a, b, c_0, c_1 and c_2, such that the following holds.

(1) *For all t_0, $T \le t_0 < \infty$, the system (2.30) with $t_1 = t_0$ has a unique solution $(q, \sigma) \in \mathcal{C}(I, X^{k,\ell})$ with $I = [T, t_0]$ and $(q, \sigma)(t_0) = 0$. That solution satisfies the estimates (6.49) (6.50) for all $t \in I$.*

(2) *The system (2.30) with $t_1 = \infty$ has a unique solution $(q, \sigma) \in \mathcal{C}(I, X^{k,\ell})$, where $I = [T, \infty)$ satisfying the estimates (6.49) (6.50) for all $t \in I$.*

(3) *Let (q_{t_0}, σ_{t_0}) be the solution defined in Part (1) for $t_0 < \infty$ and let (q, σ) be the solution defined in Part (2) for $t_0 = \infty$. When $t_0 \to \infty$, (q_{t_0}, σ_{t_0}) converges to (q, σ) strongly in $L^\infty(J, X^{k',\ell'})$ for $0 \le k' < k$, $0 \le \ell' < \ell$, and in the weak-$*$ sense in $L^\infty(J, X^{k,\ell})$ for any interval $J = [T, \bar{T}]$ with $\bar{T} < \infty$.*

Proof. Parts (1) and (2). We prove Parts (1) and (2) together, because the proof is exactly the same for both. It consists in showing that the map $\Gamma : (q, \sigma) \to (q', \sigma')$ defined by solving the linearized system (2.31) is a contraction on a suitable subset of $\mathcal{C}(I, X^{k,\ell})$ in the lower norms used in Lemma 6.2. For $t_0 < \infty$, the map Γ is defined by Proposition 6.1, restricted to those (q, σ) satisfying $(q, \sigma)(t_0) = 0$, with the initial data $(q', \sigma')(t_0) = 0$. For $t_0 = \infty$, the map Γ is defined by Proposition 6.2. The relevant estimates on Γ are those derived in the proof of Proposition 6.2 in the case $t_1 = \infty$. The same estimates also apply to the case $t_0 = t_1 < \infty$, which is relevant for Part (1) of this proposition. They are independent of t_0.

We define the set

$$\begin{aligned}\mathcal{R} = \Big\{ &(q, \sigma) \in \mathcal{C}(I, X^{k,\ell}) : (q, \sigma)(t_0) = 0 \quad \text{if } t_0 < \infty \ , \\ &\| \, t^{\lambda_0} q; L^\infty(I, L^2) \, \| \le Y_0 \ , \ \| \, t^{\lambda} \omega^k q; L^\infty(I, L^2) \, \| \le Y \ , \\ &\mathop{\mathrm{Sup}}_{0 \le m \le \ell} \| \, t^{\lambda_0 - \beta(m+1)} \omega^m \nabla \sigma; L^\infty(I, L^2) \, \| \le Z \Big\} \ . \end{aligned} \tag{6.77}$$

We first show that $\mathcal{R}$ is stable under Γ for suitable Y_0, Y, Z and for sufficiently large T. Let $(q, \sigma) \in \mathcal{R}$ and $(q', \sigma') = \Gamma(q, \sigma)$. Then (q', σ') satisfies the estimates (6.56) (6.64) (6.68) (6.71) where Y_0', Y', Z_m' are defined by (6.53) (6.55) (6.63) (6.65) and/or their extension to $t_0 < \infty$. It is therefore sufficient to ensure that the RHS of (6.56) (6.64) (6.68) (6.71) are not larger than Y_0, Y, Z, and Z respectively.

For that purpose, it is sufficient to choose

$$\left\{\begin{array}{l} Y_0 = 2c_0 \quad , Z = 2e\nu^{-1}(c_2 + 4ac_0) \\ Y = e\Big\{(4\lambda^{-1}b_0)^k 2c_0 + 4\lambda^{-1}(c_1 + c_0)\Big\} \end{array}\right. \tag{6.78}$$

and to take T sufficiently large in the sense that

$$\left\{\begin{array}{l} a\ Z\ T^{-(1-\beta)} + a^2\ Y_0\ T^{-\beta} + a\left(Y_0^3\ \bar{Y}\right)^{1/2} T^{-(k-1/2+\lambda)} \le c_0\ , \\ \eta\ T^\eta \ge 8b \quad , \quad T \ge 4eZ \quad , \quad a\ T^\beta \ge Y_0\ , \\ E(T) \le 1 \quad , \quad C_1(T) \le 2(c_1 + c_0)\ . \end{array}\right. \tag{6.79}$$

The conditions (6.79) are lower bounds on T expressed in terms of the parameters listed in the Proposition, after substitution of (6.78).

We next show that Γ is a contraction on $\mathcal{R}$ in the norms considered in Lemma 6.2. Let $(q_i, \sigma_i) \in \mathcal{R}$ and $(q'_i, \sigma'_i) = \Gamma(q_i, \sigma_i)$, $i = 1, 2$, and define $(q_\pm, \sigma_\pm)$ and $(q'_\pm, \sigma'_\pm)$ as in Lemma 6.2. We define in addition

$$y_- = \parallel q_- \parallel_2 \quad , \quad z_{-m} = \parallel \omega^m \nabla \sigma_- \parallel_2 \tag{6.80}$$

$$Y_- = \parallel t^{\lambda_0} y_-; L^\infty(I) \parallel \quad , \quad Z_- = \mathop{\mathrm{Sup}}_{0 \le m \le \ell_0} \parallel t^{\lambda_0 - \beta(m+1)} z_{-m}; L^\infty(I) \parallel \tag{6.81}$$

and we make similar definitions for the primed quantities. We take $\ell_0 = [3/2 - k]_+$ and estimate y'_- and z'_{-m} by (6.29) (6.30), taking advantage of the fact that $m' = m$ in (6.30) for that choice of ℓ_0. Using the fact that Γ maps $\mathcal{R}$ into itself and omitting again overall constants, we obtain

$$\begin{aligned} \left|\partial_t y'_-\right| &\le a\ Z_-\ t^{-2-\lambda_0+\beta} + \bar{Y}\ Z_-\ t^{-2-\lambda_0+\beta(\ell_0+1)-\lambda} + a^2\ Y_-\ t^{-1-\beta-\lambda_0} \\ &+ a\ \bar{Y} Y_-\ t^{-1-2\lambda_0+(\lambda_0-\lambda)/2k} + \bar{Y}^2\ Y_-\ t^{-1-3\lambda_0+2(\lambda_0-\lambda)/k} \end{aligned} \tag{6.82}$$

where $\bar{Y} = Y \vee Y_0$,

$$\begin{aligned} \left|\partial_t z'_{-m}\right| &\le \left(b\ t^{-1-\eta} + Z\ t^{-2}\right)\left(z'_{-m} + Z_-\ t^{-\lambda_0+\beta(m+1)}\right) \\ &+ a\ Y_-\ t^{-1-\lambda_0+\beta(m+1)} + Y_0 Y_-\ t^{-1-2\lambda_0+\beta(m+5/2)} \end{aligned} \tag{6.83}$$

for $0 \le m \le \ell_0$. Integrating (6.82) (6.83) from t to t_0 with $(y'_-, z'_{-m})(t_0) = 0$ and using again the fact that $\lambda_0 > \lambda + k > 1$ and $\lambda_0 > \beta(\ell + 1) > \beta((\ell_0 + 2) \vee 5/2)$, we obtain

$$\begin{aligned} Y'_- &\le a\ Z_-\ T^{-(1-\beta)} + \bar{Y}\ Z_-\ T^{-1-\lambda+\beta(\ell_0+1)} \\ &+ a^2\ Y_-\ T^{-\beta} + a\ \bar{Y}\ Y_-\ T^{-(k-1/2+\lambda)} + \bar{Y}^2\ Y_-\ T^{-2(k-1+\lambda)}\ , \end{aligned} \tag{6.84}$$

$$\begin{aligned} Z'_- &\le \exp\left(b\ \eta^{-1}\ T^{-\eta} + Z\ T^{-1}\right)\Big\{\left(b\ \eta^{-1}\ T^{-\eta} + Z\ T^{-1}\right) Z_- \\ &+ \beta^{-1}\left(a\ Y_- + T^{-\beta} Y_0\ Y_-\right)\Big\}\ . \end{aligned} \tag{6.85}$$

We now ensure that the map Γ is a contraction for the norms defined by (6.80) (6.81) in the form

$$\begin{cases} Y'_- \leq \left(c^{-1}\, Z_- + Y_-\right)/4 \\ Z'_- \leq \left(Z_- + c\, Y_-\right)/4 \end{cases} \tag{6.86}$$

which imply

$$Z'_- + c\, Y'_- \leq \left(Z_- + c\, Y_-\right)/2 \tag{6.87}$$

by taking $c = 8\beta^{-1}a$ and T sufficiently large, depending on the parameters listed in the proposition, in part explicitly and in part through Y_0, Y and Z defined by (6.78). (It is only at this point that we need the condition $1 + \lambda > \beta(5/2 - k)$, in order to ensure that the power of T in the second term in the RHS of (6.84) is negative).

We have proved that for sufficiently large T, the map Γ maps $\mathcal{R}$ defined by (6.77) into itself and is a contraction for the norms (6.81). By a standard compactness argument, $\mathcal{R}$ is closed for the latter norms, and therefore Γ has a unique fixed point in $\mathcal{R}$, which completes the existence part of the proof of Parts (1) and (2).

The uniqueness statement of Part (1) is a special case of Proposition 4.2 part (1), while the uniqueness statement of Part (2) follows from Proposition 4.2 part (3) and from the fact that $\lambda_0 > 1 > \beta_2$.

Part (3). Let $T < t'_1 < t'_2 < \infty$ and let (q_i, σ_i), $i = 1, 2$, be the solutions of the ssytem (2.30) obtained in part (1) and corresponding to $t_0 = t_1 = t'_i$ respectively. Those solutions satisfy the estimates (6.49) (6.50) for $t \leq t'_i$. Define as before $(q_\pm, \sigma_\pm) = (1/2)(q_1 \pm q_2, \sigma_1 \pm \sigma_2)$. We shall estimate (q_-, σ_-) for $t \leq t'_1$ in the norms considered in Lemma 6.2. In order to alleviate the notation, we omit the prime on t_1, t_2 in the rest of the proof. By (6.49) (6.50), we estimate

$$\begin{cases} \parallel q_-(t_1) \parallel_2 = (1/2) \parallel q_2(t_1) \parallel_2 \leq (1/2) Y_0\; t_1^{-\lambda_0} \\ \parallel \omega^m \nabla \sigma_-(t_1) \parallel_2 = (1/2) \parallel \omega^m \nabla \sigma_2(t_1) \parallel_2 \leq (1/2) Z t_1^{-\lambda_0 + \beta(m+1)} \end{cases} \tag{6.88}$$

for $0 \leq m \leq \ell$. On the other hand (q_-, σ_-) satisfies a system closely related to (6.32) where however $(q'_\pm, \sigma'_\pm) = (q_\pm, \sigma_\pm)$ and where additional terms appear because of the different values t_1 and t_2 occuring in B_S and B_L. More precisely

$$\begin{aligned} \partial_t q_- &= i(2t^2)^{-1} \Delta q_- + t^{-2} \Big\{ Q(s_+, q_-) + Q(\sigma_-, w_+) \Big\} + i t^{-1} B_0 q_- \\ &+ i t^{-1} \Big\{ \left(B_S^{t_1, \infty}(w_+, w_+) + B_S^{t_1}(q_-, q_-) \right) q_- + 2 B_S^{t_1}(w_+, q_-) w_+ \Big\} \\ &- i(2t)^{-1} \left(B_S^{t_2} - B_S^{t_1} \right) (q_2, q_2 + 2W) \,(q_2 + W) \end{aligned} \tag{6.89}$$

$$\begin{aligned} \partial_t \sigma_- = t^{-2} \left(s_+ \cdot \nabla \sigma_- + \sigma_- \cdot \nabla s_+ \right) - 2t^{-1} \nabla B_L^{t_1}(w_+, q_-) \\ + t^{-1} \nabla \left(B_L^{t_2} - B_L^{t_1} \right) (q_2, q_2 + 2W)\ . \end{aligned}$$

We first estimate the additional terms in (6.89) as compared with (6.32). From (6.36)–(6.39) we obtain

$$\begin{aligned}&\| \left(B_S^{t_2} - B_S^{t_1}\right)(q_2, q_2 + 2W)(q_2 + W) \|_2 \\ &\le C\Big\{ \left(at^{-\beta} + \ \| q_2 \|_3\right) a\ I_0\left(\| q_2 \|_2\right) + a\ I_{-1}\left(\| q_2 \|_2 \| q_2 \|_3\right) \\ &+ \| q_2 \|_6 \ \ I_{-1/2}\left(\| q_2 \|_2 \| q_2 \|_6\right)\Big\} \end{aligned} \tag{6.90}$$

where the various I_m's are taken in the interval $[t_1, t_2]$. From (6.49) and Sobolev inequalities, we obtain

$$I_0\left(\| q_2 \|_2\right) \le Y_0\ t^{1/2} \int_{t_1}^{t_2} dt'\ t'^{-3/2-\lambda_0} \le Y_0\ t^{1/2}\ t_1^{-1/2-\lambda_0} \tag{6.91}$$

$$\begin{aligned} I_{-1}\left(\| q_2 \|_2 \| q_2 \|_3\right) &\le C\ Y_0\ \bar{Y}\ t^{-1/2} \int_{t_1}^{t_2} dt'\ t'^{-1/2-2\lambda_0+(\lambda_0-\lambda)/2k} \\ &\le C\ Y_0\ \bar{Y}\ t^{-1/2}\ t_1^{1/2-2\lambda_0+(\lambda_0-\lambda)/2k} \end{aligned} \tag{6.92}$$

$$\begin{aligned}&\| q_2 \|_6\ I_{-1/2}\left(\| q_2 \|_2 \| q_2 \|_6\right) \le C Y_0 \bar{Y}^2 t^{-\lambda_0+(\lambda_0-\lambda)/k} \\ &\times \int_{t_1}^{t_2} dt' t'^{-1-2\lambda_0+(\lambda_0-\lambda)/k} \le C Y_0 \bar{Y}^2 t^{-\lambda_0+(\lambda_0-\lambda)/k} t_1^{-2\lambda_0+(\lambda_0-\lambda)/k}\end{aligned} \tag{6.93}$$

and therefore for $t \le t_1$

$$\begin{aligned}&\| \left(B_S^{t_2} - B_S^{t_1}\right)(q_2, q_2 + 2W)(q_2 + W) \|_2 \le C\ t^{-\lambda_0+1/2}\Big\{a^2\ Y_0\ t_1^{-\beta-1/2} \\ &+ a\ Y_0\ \bar{Y}\ t_1^{-\lambda_0+(\lambda_0-\lambda)/2k-1/2} + Y_0\ \bar{Y}^2\ t_1^{-2\lambda_0+2(\lambda_0-\lambda)/k-1/2}\Big\}\ .\end{aligned} \tag{6.94}$$

Similarly using (6.43) and

$$I_{-3/2}\left(\| q_2 \|_2^2\right) \le Y_0^2\ t^{-1} \int_{t_1}^{t_2} dt'\ t'^{-2\lambda_0} \le Y_0^2\ t^{-1}\ t_1^{1-2\lambda_0}$$

we estimate for $t \le t_1$

$$\| \omega^{m+2} \left(B_L^{t_2} - B_L^{t_1}\right)(q_2, q_2 + 2W) \|_2 \le C\ \left(a\ Y_0 + Y_0^2 t^{3\beta/2-1}\ t_1^{1-\lambda_0}\right) t^{\beta(m+1)} t_1^{-\lambda_0}. \tag{6.95}$$

We define y_- and z_{-m} by (6.80), we take again $\ell_0 = [3/2 - k]_+$, we choose λ_0' satisfying

$$1 \vee (\lambda_0 - 1/2) \vee \beta\left(\ell_0 + 1\right) < \lambda_0' < \lambda_0\ , \tag{6.96}$$

we define (see (6.81))

$$Y_- = \| t^{\lambda_0'} y_-; L^\infty([T, t_1]) \|\ ,\ Z_- = \mathop{\mathrm{Sup}}_{0 \le m \le \ell_0} \| t^{\lambda_0'-\beta(m+1)} z_{-m}; L^\infty([T, t_1]) \| \tag{6.97}$$

and we estimate those quantities in the same way as in the proof of Parts (1) and (2). From (6.89), we obtain differential inequalities for y_-, z_{-m}, very similar to (6.82) (6.83) with $y'_- = y_-$, $z'_{-m} = z_{-m}$ with however additional terms estimated by (6.94) (6.95).

We integrate those inequalities from t to t_1, with initial condition at t_1 estimated by (6.88). We then substitute the result in (6.97) and omitting an overall constant, we obtain finally (see (6.84) (6.85))

$$\begin{aligned} Y_- &\le aZ_-T^{-(1-\beta)} + \bar{Y} Z_- T^{-1-\lambda+\beta(\ell_0+1)} + a^2 Y_- T^{-\beta} \\ &+ a\bar{Y} Y_- T^{-(k-1/2+\lambda)} + \bar{Y}^2 Y_- T^{-2(k-1+\lambda)} \\ &+ \Big\{ Y_0 + a^2 Y_0 t_1^{-\beta} + a Y_0 \bar{Y} t_1^{-(k-1/2+\lambda)} + Y_0 \bar{Y}^2 t_1^{-2(k-1+\lambda)} \Big\} t_1^{-(\lambda_0-\lambda'_0)} \end{aligned} \quad (6.98)$$

$$\begin{aligned} Z_- &\le \exp\left(b\,\eta^{-1}\,T^{-\eta} + Z\,T^{-1}\right) \Big\{ \left(b\,\eta^{-1}\,T^{-\eta} + Z\,T^{-1}\right) Z_- \\ &+ \beta^{-1}\left(a\,Y_- + T^{-\beta} Y_0\,Y_-\right) + \Big(Z + a\,Y_0 + Y_0^2\,t_1^{-\beta}\Big)\,t_1^{-(\lambda_0-\lambda'_0)} \Big\} . \end{aligned} \quad (6.99)$$

Proceeding as above, we deduce therefrom that for T sufficiently large and for a suitable constant c

$$Y_- + cZ_- \le O\left(t_1^{-(\lambda_0-\lambda'_0)}\right) . \quad (6.100)$$

From (6.100) it follows that (q_{t_0}, σ_{t_0}) tends to a limit uniformly in compact subintervals of $[T,\infty)$ in the norms (6.80). By a standard compactness argument, that limit belongs to $\mathcal{C}([T,\infty), X^{k,\ell})$ and satisfies (6.49) (6.50). One sees easily that the limit satisfies the system (2.30) with $t_1 = \infty$, and therefore coincides with the solution obtained in Part (2). Actually, as mentioned before, Part (3) provides an alternative (more complicated) proof of Part (2). □

7 Choice of (W, S) and remainder estimates

In this section, we construct approximate solutions (W,S) of the system (2.20) satisfying the assumptions needed for Propositions 6.2 and 6.3 and in particular the remainder estimates (6.47) (6.48), thereby allowing for the applicability of Proposition 6.3, namely for the construction of solutions of the system (2.30). More general (W,S) also suitable for the same purpose, could also be constructed by exploiting the gauge invariance of the system (2.20).

We rewrite the remainders as

$$R_1(W,S) = U^*(1/t)\partial_t(U(1/t)W) - t^{-2}Q(S,W) - it^{-1}(B_0 + B_S(W,W))W \quad (7.1)$$

$$R_2(W,S) = \partial_t S - t^{-2} S \cdot \nabla S + t^{-1} \nabla B_L(W,W) . \quad (2.29) \equiv (7.2)$$

We recall that $t_1 = \infty$ in R_1, R_2, and we omit t_1 from the notation. We construct (W,S) by solving the system (2.20) approximately by iteration. The n-th iteration should be sufficient to cover the case $\lambda_0 < n$. Here we need $\lambda_0 > k > 1$, and we must therefore use at least the second iteration, which will allow for $\lambda_0 < 2$. For simplicity, we shall not go any further here. Accordingly we take

$$W = w_0 + w_1 \qquad , \quad S = s_0 + s_1 \tag{7.3}$$

where

$$\begin{cases} \partial_t \ U(1/t)w_0 = 0 & w_0(\infty) = w_+ \ , \\ \partial_t \ s_0 = -t^{-1}\nabla B_L(w_0, w_0) & s_0(1) = 0 \ , \end{cases} \tag{7.4}$$

so that

$$\begin{cases} w_0 = U^*(1/t)w_+ \ , \\ s_0(t) = -\displaystyle\int_1^t dt' \ t'^{-1}\nabla B_L(w_0(t'), w_0(t')) \end{cases} \tag{7.5}$$

and

$$\begin{cases} \partial_t(U(1/t)w_1) = t^{-2} \ U(1/t) \ Q(s_0, w_0) & w_1(\infty) = 0 \ , \\ \partial_t \ s_1 = t^{-2}s_0 \cdot \nabla s_0 - 2t^{-1}\nabla B_L(w_0, w_1) & s_1(\infty) = 0 \ , \end{cases} \tag{7.6}$$

so that

$$\begin{cases} w_1(t) = -U^*(1/t)\displaystyle\int_t^\infty dt' \ t'^{-2} \ U(1/t') \ Q(s_0(t'), w_0(t')) \\ s_1(t) = -\displaystyle\int_t^\infty dt' \ t'^{-2}s_0(t') \cdot \nabla s_0(t') + 2\int_t^\infty dt' \ t'^{-1}\nabla B_L(w_0(t'), w_1(t')) \ . \end{cases} \tag{7.7}$$

The remainders then become

$$\begin{cases} R_1(W, S) = -t^{-2}\Big\{Q(s_0, w_1) + Q(s_1, w_0) + Q(s_1, w_1)\Big\} \\ \qquad\qquad -it^{-1}(B_0 + B_S(W, W))W \ , \\ R_2(W, S) = -t^{-2}\Big\{s_0 \cdot \nabla s_1 + s_1 \cdot \nabla s_0 + s_1 \cdot \nabla s_1\Big\} + t^{-1}\nabla B_L(w_1, w_1) \ . \end{cases} \tag{7.8}$$

Note that the term with $B_0 + B_S(W, W)$ in (7.1) is regarded as short range and not included in the definition of (W, S).

We now turn to the derivation of the estimates (6.45)–(6.48). The regularity properties of (W, S) used in Section 6 follow from similar but simpler estimates. We first estimate all the terms not containing B_0.

Lemma 7.1 *Let $0 < \beta < 1$, $k_+ \geq 3$, $w_+ \in H^{k_+}$ and $a_+ = |w_+|_{k_+}$. Then the following estimates hold:*

$$|w_0|_{k_+} \ \leq a_+ \tag{7.9}$$

$$\parallel \omega^m \ s_0 \parallel_2 \leq \begin{cases} C \ a_+^2 \ \ell n \ t & \text{for } 0 \leq m \leq k_+ \\ C \ a_+^2 \ t^{\beta(m-k_+)} & \text{for } m > k_+ \ , \end{cases} \tag{7.10}$$

$$|w_1|_{k_+-1} \le C\ a_+^3\ t^{-1}(1+\ell n\ t)\ , \tag{7.11}$$

$$\parallel \omega^m\ s_1 \parallel_2 \le \begin{cases} C\ a_+^4\ t^{-1}(1+\ell n\ t)^2 & \textit{for } 0 \le m \le k_+ - 1 \\ C\ a_+^4\ t^{-1+\beta(m+1-k_+)}(1+\ell n\ t) & \\ \quad \textit{for } k_+ - 1 < m < k_+ - 1 + \beta^{-1}\ , & \end{cases} \tag{7.12}$$

$$\parallel \omega^m\ R_2(W,S) \parallel_2 \le \begin{cases} C(a_+)\ t^{-3}(1+\ell n\ t)^3 & \textit{for } 0 \le m \le k_+ - 2 \\ C(a_+)\ t^{-3+\beta(m+2-k_+)}(1+\ell n\ t)^2 & \\ \quad \textit{for } k_+ - 2 < m < k_+ - 2 + \beta^{-1}\ , & \end{cases} \tag{7.13}$$

$$\parallel \omega^m (Q(S,w_1) + Q(s_1,w_0)) \parallel_2 \le C(a_+)\ t^{-1}(1+\ell n\ t)^2 \qquad \textit{for } 0 \le m \le k_+ - 2 \tag{7.14}$$

$$\parallel \omega^m (B_S(W,W)W) \parallel_2 \le C(a_+)\ t^{-\beta(k_+-m+1)} \parallel \qquad \textit{for } 0 \le m \le k_+ - 1\ . \tag{7.15}$$

Proof. (7.9) is trivial.

(7.10). By (3.10), Lemma 3.2 and (3.9) we estimate

$$\parallel \omega^m\ s_0 \parallel_2 \le \int_1^t dt'\ t'^{-1} \parallel \omega^{m+1}\ B_L(w_0(t'), w_0(t')) \parallel_2$$

$$\le \begin{cases} C\displaystyle\int_1^t dt'\ t'^{-1}\ I_m\left(\parallel \omega^m w_0(t') \parallel_2 \parallel w_0(t') \parallel_\infty\right) \le C a_+^2 \ell n\ t \\ \hfill \text{for } 0 \le m \le k_+ \\ C\displaystyle\int_1^t dt'\ t'^{-1+\beta(m-k_+)}\ I_{k_+}\left(\parallel \omega^{k_+} w_0(t') \parallel_2 \parallel w_0(t') \parallel_\infty\right) \le C a_+^2\ t^{\beta(m-k_+)} \\ \hfill \text{for } m > k_+\ . \end{cases}$$

(7.11). By Lemma 3.2 and (7.10), we estimate

$$\parallel Q(s_0,w_0) \parallel_2 \le C \parallel \nabla s_0 \parallel_2 \left(\parallel \omega^{3/2} w_0 \parallel_2\ +\ \parallel w_0 \parallel_\infty\right) \le C\ a_+^3 \ell n\ t$$

$$\parallel \omega^{k_+-1} Q(s_0,w_0) \parallel_2 \le C\Big\{ \parallel \omega^{k_+} s_0 \parallel_2 \left(\parallel \omega^{3/2} w_0 \parallel_2\ +\ \parallel w_0 \parallel_\infty\right)$$

$$+ \left(\parallel \omega^{3/2} s_0 \parallel_2\ +\ \parallel s_0 \parallel_\infty\right) \parallel \omega^{k_+} w_0 \parallel_2 \Big\} \le C\ a_+^3 \ell n\ t$$

from which (7.11) follows by integration.

(7.12). By Lemma 3.2 and (7.10) we estimate

$$\parallel \omega^m (s_0 \cdot \nabla s_0) \parallel_2 \le C \parallel \omega^{m+1}\ s_0 \parallel_2 \left(\parallel \omega^{3/2} s_0 \parallel_2\ +\ \parallel s_0 \parallel_\infty\right)$$

$$\le \begin{cases} C a_+^4 (\ell n\ t)^2 & \text{for } m \le k_+ - 1 \\ C a_+^4\ t^{\beta(m+1-k_+)} \ell n\ t & \text{for } m > k_+ - 1\ . \end{cases} \tag{7.16}$$

On the other hand

$$\| \omega^{m+1} B_L(w_0, w_1) \|_2$$

$$\leq \begin{cases} CI_m \left(\| \omega^m w_0 \|_2 \| w_1 \|_\infty + \| \omega^m w_1 \|_2 \| w_0 \|_\infty\right) \leq Ca_+^4 t^{-1}(1 + \ell n\ t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } m \leq k_+ - 1 \\ Ct^{\beta(m+1-k_+)} I_{k_+-1} \left(\| \omega^{k_+-1} w_0 \|_2 \| w_1 \|_\infty + \| \omega^{k_+-1} w_1 \|_2 \| w_0 \|_\infty\right) \\ \leq Ca_+^4\ t^{\beta(m+1-k_+)-1}(1 + \ell n\ t) \qquad \text{for } m > k_+ - 1\ . \end{cases} \tag{7.17}$$

(7.12) now follows from (7.16) and (7.17) by integration provided $\beta(m+1-k_+) < 1$.

(7.13). By Lemma 3.2 again, and by (7.10) (7.12) we estimate

$$\| \omega^m (s_0 \cdot \nabla s_1 + s_1 \cdot \nabla s_0 + s_1 \cdot \nabla s_1) \|_2 \leq$$

$$C\Big\{ \| \omega^{m+1} s_0 \|_2 \left(\| \omega^{3/2} s_1 \|_2 + \| s_1 \|_\infty\right) + \| \omega^{m+1} s_1 \|_2 \Big(\| \omega^{3/2} s_0 \|_2 + \| s_0 \|_\infty$$

$$+ \| \omega^{3/2} s_1 \|_2 + \| s_1 \|_\infty\Big) \Big\}$$

$$\leq \begin{cases} C(a_+)\ t^{-1}(1 + \ell n\ t)^3 & \text{for } m \leq k_+ - 2 \\ C(a_+)\ t^{-1+\beta(m+2-k_+)}(1 + \ell n\ t)^2 & \text{for } k_+ - 2 < m < k_+ - 2 + \beta^{-1}\ . \end{cases} \tag{7.18}$$

On the other hand

$$\| \omega^{m+1} B_L(w_1, w_1) \|_2$$

$$\leq \begin{cases} CI_m \left(\| \omega^m w_1 \|_2 \| w_1 \|_\infty\right) \leq Ca_+^6 t^{-2}(1 + \ell n\ t)^2 \qquad \text{for } m \leq k_+ - 1 \\ Ct^{\beta(m+1-k_+)} I_{k_+-1} \left(\| \omega^{k_+-1} w_1 \|_2 \| w_1 \|_\infty\right) \leq Ca_+^6 t^{\beta(m+1-k_+)-2}(1 + \ell n t)^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } m > k_+ - 1. \end{cases} \tag{7.19}$$

(7.13) now follows from (7.8) (7.18) (7.19).

(7.14). By Lemma 3.2 again, and by (7.10) (7.11) (7.12) we estimate

$$\| \omega^m (Q(s_0, w_1) + Q(s_1, w_0) + Q(s_1, w_1)) \|_2$$

$$\leq C\Big\{ \| \omega^{m+1} s_0 \|_2 \left(\| \omega^{3/2} w_1 \|_2 + \| w_1 \|_\infty\right) + \| \omega^{m+1} w_1 \|_2 \Big(\| \omega^{3/2} s_0 \|_2$$

$$+ \| s_0 \|_\infty + \| \omega^{3/2} s_1 \|_2 + \| s_1 \|_\infty\Big) + \| \omega^{m+1} s_1 \|_2 \Big(\| \omega^{3/2} w_0 \|_2 + \| w_0 \|_\infty$$

$$+ \| \omega^{3/2} w_1 \|_2 + \| w_1 \|_\infty\Big) + \| \omega^{m+1} w_0 \|_2 \left(\| \omega^{3/2} s_1 \|_2 + \| s_1 \|_\infty\right) \Big\}$$

$$\leq Ca_+^5\ t^{-1}(1 + \ell n\ t)^2(1 + a^2\ t^{-1}(1 + \ell n\ t)) \tag{7.20}$$

from which (7.14) follows.

(7.15). As previously, we estimate for $0 \leq m \leq k_+$

$$\| \omega^m B_S(W, W) \|_2 \leq \| \omega^m B_S(w_0, w_0) \|_2 + 2 \| \omega^m B_S(w_0, w_1) \|_2 + \| \omega^m B_S(w_1, w_1) \|_2$$

$$\leq Ct^{\beta(m-k_+)}\Big\{t^{-\beta}I_{k_+}\left(\| \omega^{k_+}w_0 \|_2 \| w_0 \|_\infty\right) + I_{k_+-1}\left(\| \omega^{k_+-1}w_0 \|_2 \| w_1 \|_\infty\right.$$

$$\left. + \| \omega^{k_+-1}w_1 \|_2 \left(\| w_0 \|_\infty + \| w_1 \|_\infty\right)\right)\Big\}$$

$$\leq Ct^{\beta(m-k_+)}\Big\{t^{-\beta}\ a_+^2 + t^{-1}\ a_+^4(1+\ell n\ t) + t^{-2}\ a_+^6(1+\ell n\ t)^2\Big\} \qquad (7.21)$$

by (7.11). Therefore

$$\| B_S(W,W)W \|_2 \leq \| B_S(W,W) \|_2 \ \| W \|_\infty \leq C(a_+)\ t^{-\beta(k_++1)}$$

$$\| \omega^{k_+-1}(B_S(W,W)W) \|_2 \leq C\Big\{ \| \omega^{k_+-1}B_S(W,W \|_2 \ \| W \|_\infty$$

$$+ \ \| B_S(W,W) \|_\infty \ \| \omega^{k_+-1}W \|_2 \Big\} \leq C(a_+)\ t^{-2\beta} \qquad (7.22)$$

which yields (7.15) by interpolation. □

We next estimate the terms in R_1 containing B_0.

Lemma 7.2 *Let $0 < \beta < 1$. Let $1/2 < \lambda_0 < 2$ and $k_+ \geq 2\lambda_0 \vee 3$. Let B_0 satisfy the estimates (3.17) for $0 \leq m \leq 2$, let $w_+ \in H^{k_+}$ and assume that B_0 and w_+ satisfy the estimate (5.35) for all multi-indices α_1, α_2 with $0 \leq |\alpha_1| \leq 2$ and $0 \leq |\alpha_2| < 2\lambda_0$. Then the following estimate holds for all m, $0 \leq m \leq 2$, and all $t \geq 1$.*

$$\| \omega^m(B_0\ W) \|_2 \leq C\ t^{-\lambda_0+m}\ . \qquad (7.23)$$

Proof. The contribution of w_0 to (7.23) is estimated by Lemma 5.1 with $\bar{m} = 2$. In order to estimate the contribution of w_1, we decompose $w_1 = w_1' + w_1''$ where

$$w_1' = -\int_t^\infty dt'\ t'^{-2}\ Q(s_0(t'), w_+)$$

$$w_1''(t) = (1-U^*(1/t))\int_t^\infty dt'\ t'^{-2}U(1/t')Q(s_0(t'), w_0(t'))$$

$$+ \int_t^\infty dt'\ t'^{-2}\Big\{ (1-U(1/t'))Q(s_0(t'), w_0(t')) + Q(s_0(t'), (1-U(1/t'))w_+) \Big\}.$$

We first consider

$$B_0(t)w_1'(t) = -\int_t^\infty dt'\ t'^{-2}\Big\{s_0(t')\cdot B_0(t)\nabla w_+ + (1/2)(\nabla\cdot s_0)(t')B_0(t)w_+\Big\}$$

We estimate

$$\| B_0(t)w_1'(t) \|_2 \leq \int_t^\infty dt'\ t'^{-2}\Big\{ \| s_0(t') \|_\infty \ \| B_0(t)\nabla w_+ \|_2$$

$$+ \ \| (\nabla\cdot s_0)(t') \|_\infty \ \| B_0(t)w_+ \|_2 \Big\} \leq C\ t^{-\lambda_0+1/2}\int_t^\infty dt'\ t'^{-2}\ell n\ t'$$

$$\leq C\ t^{-\lambda_0-1/2}(1+\ell n\ t) \qquad (7.24)$$

by (5.35) and (7.10).

Similarly, we estimate

$$\| \Delta(B_0(t)w_1'(t)) \|_2 \leq C \int_t^\infty dt'\ t'^{-2} \Big\{ \| \Delta s_0 \|_3 \| B_0 \nabla w_+ \|_6$$

$$+ \| \nabla s_0 \|_\infty \| \nabla(B_0 \nabla w_+) \|_2 + \| s_0 \|_\infty \| \Delta(B_0 \nabla w_+) \|_2 + \| \Delta\nabla \cdot s_0 \|_2 \| B_0 w_+ \|_\infty$$

$$+ \| \nabla^2 s_0 \|_6 \| \nabla(B_0 w_+) \|_3 + \| \nabla \cdot s_0 \|_\infty \| \Delta(B_0 w_+) \|_2 \Big\} \qquad (7.25)$$

where $s_0 = s_0(t')$ and $B_0 = B_0(t)$, and therefore by (7.10)

$$\| \Delta(B_0\ w_1') \|_2 \leq C\ a_+^2\ t^{-1}(1 + \ell n\ t) \Big\{ (\| B_0 \|_\infty + \| \nabla B_0 \|_3)\, a_+$$

$$+ \| (\Delta B_0) \nabla w_+ \|_2 + \| (\Delta B_0) w_+ \|_2 \Big\}$$

$$\leq C\ a_+^2 \left(a_+\ b_0\ t^{-1/3} + b_1\ t^{-\lambda_0 + 3/2} \right) (1 + \ell n\ t) \qquad (7.26)$$

by (3.17) and (5.35).

We next estimate the contribution of w_1''. By the same estimates as for w_1 (see the proof of (7.11)) we obtain

$$|w_1''|_{k_+ - 1} \leq C\ a_+^3\ t^{-1}(1 + \ell n\ t)$$

$$|w_1''|_{k_+ - 3} \leq C\ a_+^3\ t^{-2}(1 + \ell n\ t)$$

where we have used the fact that the factors $(1 - U^{(*)}(1/t))$ can be replaced by $t^{-1}\Delta$ for the purpose of the second estimate, and therefore

$$\| \omega^m w_1'' \|_2 \leq C\ a_+^3\ t^{-2+m/2}(1 + \ell n\ t) \quad \text{for } 0 \leq m \leq k_+ - 1 \qquad (7.27)$$

by interpolation. By Lemma 3.2 and (3.17) we then obtain

$$\| \omega^m (B_0\ w_1'') \|_2 \leq C\, (\| \omega^m B_0 \|_\infty \| w_1'' \|_2 + \| B_0 \|_\infty \| \omega^m w_1'' \|_2)$$

$$\leq C\ b_0\ a_+^3\ t^{-2+m}\ (1 + \ell n\ t)\ . \qquad (7.28)$$

Collecting (7.24) (7.26) (7.28) and the estimates of $B_0 w_0$ coming from Lemma 5.1 yields (7.23) for $0 \leq m \leq 2$. □

We can now collect Proposition 6.3 and Lemmas 7.1 and 7.2 to obtain the main technical result on the Cauchy problem for the auxiliary system in the difference form (2.30). We again keep the assumptions on B_0 in the implicit form of the estimates (3.17) and (5.35), which can however be replaced by sufficient conditions on $(w_+, A_+, \dot{A}_+)$ by the use of Lemmas 3.5 and 5.2.

Proposition 7.1 *Let $1 < k \leq \ell$ and $\ell > 3/2$. Let β, λ_0 and λ satisfy*

$$0 < \beta < 2/3 \quad , \quad \lambda > 0 \quad , \quad \lambda + k < \lambda_0 < 2 \quad , \quad \lambda_0 > \beta(\ell + 1)\ . \qquad (7.29)$$

Let k_+ satisfy

$$k_+ \geq k+2 \quad , \quad k_+ \geq 2\lambda_0 \quad , \quad \beta(k_+ + 1) \geq \lambda_0 \quad , \quad \beta(\ell + 3 - k_+) < 1 \ . \tag{7.30}$$

Let $w_+ \in H^{k_+}$, let B_0 satisfy the estimates (3.17) for $0 \leq m \leq k$ and let (w_+, B_0) satisfy the estimates (5.35) for all multi-indices α_1, α_2 with $0 \leq |\alpha_1| \leq 2$ and $0 \leq |\alpha_1| < 2\lambda_0$. Let (W, S) be defined by (7.3) (7.5) (7.7). Then

(1) *(W, S) satisfy the estimates (6.45) (6.46) (6.47) (6.48), with $0 < \eta < 1 - 3\beta/2$ in (6.46).*

(2) *All the statements of Proposition 6.3 hold.*

Proof. It follows from Lemmas 7.1 and 7.2 that all the assumptions of Proposition 6.3, and in particular the estimates (6.45)–(6.48), are satisfied. □

8 Wave operators and asymptotics for (u, A)

In this section we complete the construction of the wave operators for the system (1.1) (1.2) and we derive asymptotic properties of solutions in their range. The construction relies in an essential way on Proposition 7.1. So far we have worked with the system (2.20) for (w, s) and the first task is to reconstruct the phase φ. Corresponding to $S = s_0 + s_1$, we define $\phi = \varphi_0 + \varphi_1$ where

$$\varphi_0 = -\int_1^t dt' \ t'^{-1} \ B_L^\infty(w_0(t'), w_0(t')) \tag{8.1}$$

$$\varphi_1 = -\int_t^\infty dt' (2t'^2)^{-1} |s_0(t')|^2 + 2\int_t^\infty dt' \ t'^{-1} \ B_L^\infty(w_0(t'), w_1(t')) \tag{8.2}$$

so that $s_0 = \nabla \varphi_0$ and $s_1 = \nabla \varphi_1$.

Let now (q, σ) be the solution of the system (2.30) constructed in Proposition 6.3 part (2) and let $(w, s) = (W, S) + (q, \sigma)$. We define

$$\begin{aligned} \psi = &-\int_t^\infty dt' (2t'^2)^{-1} \left(\sigma \cdot (\sigma + 2S) + s_1 \cdot (s_1 + 2s_0)\right)(t') \\ &+ \int_t^\infty dt' \ t'^{-1} \left(B_L^\infty(q, q) + 2B_L^\infty(W, q) + B_L^\infty(w_1, w_1)\right)(t') \end{aligned} \tag{8.3}$$

which is taylored to ensure that $\nabla \psi = \sigma$, given the fact that s_0, s_1 and σ are gradients. The integral converges in $\dot{H}^1$, as follows from (6.49) (6.50) and from the estimate (see the proof of (6.4))

$$\begin{aligned} &\partial_t \parallel \sigma \parallel_2 \leq t^{-2} \parallel \nabla \sigma \parallel_2 (\parallel s \parallel_\infty \ + \ \parallel \nabla S \parallel_3) + t^{-1} \ a \ I_0(\parallel q \parallel_2) \\ &+ t^{-1+3\beta/2} \ I_{-3/2}\left(\parallel q \parallel_2^2\right) + \ \parallel R_2(W, S) \parallel_2 \\ &\leq C \left(t^{-2-\lambda_0+\beta}(1 + \ell n \ t) + t^{-1-\lambda_0} + t^{-1-2\lambda_0+3\beta/2} + t^{-3}(1 + \ell n \ t)^3\right) \\ &\leq C \ t^{-1-\lambda_0} \ . \end{aligned} \tag{8.4}$$

Furthermore, this implies that

$$\parallel \nabla\psi \parallel_2 = \parallel \sigma \parallel_2 \leq C\ t^{-\lambda_0}\ . \tag{8.5}$$

Finally we define $\varphi = \phi + \psi$ so that $\nabla\varphi = s$, and (w, φ) solves the system (2.18). For more details on the reconstruction of φ from s, we refer to Section 7 of [6].

We can now define the wave operators for the system (1.1) (1.2) as follows. We start from the asymptotic state $(u_+, A_+, \dot{A}_+)$ for (u, A). We define $w_+ = Fu_+$, we define (W, S) by (7.3) (7.5) (7.7) and B_0 by (2.3) (2.13), namely

$$A_0 = \dot{K}(t)\ A_+ + K(t)\ \dot{A}_+ = t^{-1}\ D_0\ B_0\ .$$

We next solve the system (2.30) with $t_1 = \infty$ and with initial time $t_0 = \infty$ for (q, σ) by Proposition 6.3, part (2), we define $(w, s) = (W, S) + (q, \sigma)$ and we reconstruct φ from s as explained above, namely $\varphi = \varphi_0 + \varphi_1 + \psi$ with φ_0, φ_1 and ψ defined by (8.1) (8.2) (8.3). We finally substitute (w, φ) thereby obtained into (2.11) (2.2), thereby obtaining a solution (u, A) of the system (1.1) (1.2). The wave operator is defined as the map $\Omega : (u_+, A_+, \dot{A}_+) \to (u, A)$.

In order to state the regularity properties of u that follow in a natural way from the previous construction, we introduce appropriate function spaces. In addition to the operators $M = M(t)$ and $D = D(t)$ defined by (2.8) (2.9), we introduce the operator

$$J = J(t) = x + it\ \nabla\ , \tag{8.6}$$

the generator of Galilei transformations. The operators M, D, J satisfy the commutation relation

$$i\ M\ D\ \nabla = J\ M\ D\ . \tag{8.7}$$

For any interval $I \subset [1, \infty)$ and any $k \geq 0$, we define the space

$$\mathcal{X}^k(I) = \left\{u : D^* M^* u \in \mathcal{C}(I, H^k)\right\}$$

$$= \left\{u : < J(t) >^k\ \ u \in \mathcal{C}(I, L^2)\right\} \tag{8.8}$$

where $< \lambda >= (1+\lambda^2)^{1/2}$ for any real number or self-adjoint operator λ and where the second equality follows from (8.7).

We now collect the information obtained for the solutions of the system (1.1) (1.2) and state the main result of this paper as follows.

Proposition 8.1 *Let $1 < k \leq \ell$, $\ell > 3/2$. Let β, λ_0 and λ satisfy*

$$0 < \beta < 2/3 \quad , \quad \lambda > 0 \quad , \quad \lambda + k < \lambda_0 < 2 \quad , \quad \lambda_0 > \beta(\ell + 1)\ . \tag{7.29}$$

Let k_+ satisfy

$$k_+ \geq k + 2 \quad , \quad k_+ \geq 2\lambda_0 \quad , \quad \beta(k_+ + 1) \geq \lambda_0 \quad , \quad \beta(\ell + 3 - k_+) < 1\ . \tag{7.30}$$

Let $u_+ \in FH^{k_+}$, *let* $w_+ = Fu_+$ *and* $a_+ = |w_+|_{k_+}$. *Let* $(A_+, \dot{A}_+) \in H^k \oplus H^{k-1}$. *Let* A_0 *defined by (2.3) satisfy the estimates*

$$\| \omega^m \ A_0(t) \|_r \leq b_0 \ t^{2/r-1} \qquad (3.15) \equiv (8.9)$$

for $0 \leq m \leq k$, $2 \leq r \leq \infty$ *and all* $t \geq 1$, *and the estimates*

$$\| (\partial^{\alpha_1} A_0) \left((\partial^{\alpha_2} w_+) (x/t) \right) \|_2 \leq b_1 \ t^{-\lambda_0 + (1+|\alpha_2|)/2} \qquad (8.10)$$

for all multi-indices α_1, α_2 *with* $0 \leq |\alpha_1| \leq 2$ *and* $0 \leq |\alpha_2| < 2\lambda_0$. *Let* (W, S) *be defined by (7.3) (7.5) (7.7). Then*

(1) *There exists* T, $1 \leq T < \infty$ *such that the auxiliary system (2.20) with* $t_1 = \infty$ *has a unique solution* $(w, s) \in \mathcal{C}([T, \infty), X^{k,\ell})$ *satisfying*

$$\| w(t) - W(t) \|_2 \leq C \ t^{-\lambda_0} \qquad (8.11)$$

$$\| \omega^k (w(t) - W(t)) \|_2 \leq C \ t^{-\lambda} \qquad (8.12)$$

$$\| \omega^m (s(t) - S(t)) \|_2 \leq C \ t^{-\lambda_0 + \beta m} \quad \text{for } 0 \leq m \leq \ell + 1 \ . \qquad (8.13)$$

(2) *Let* $\phi = \varphi_0 + \varphi_1$ *be defined by (8.1) (8.2), let* $\varphi = \phi + \psi$ *with* ψ *defined by (8.3) and* $(q, \sigma) = (w, s) - (W, S)$. *Let*

$$u = MD \exp(-i\varphi) w \qquad (2.11) \equiv (8.14)$$

and define A *by (2.2) (2.3) (2.4) with* $t_1 = \infty$. *Then* $u \in \mathcal{X}^k([T, \infty))$, $(A, \partial_t A) \in \mathcal{C}([T, \infty), H^k \oplus H^{k-1})$, (u, A) *solves the system (1.1) (1.2) and* u *behaves asymptotically in time as* $MD \exp(-i\phi) W$ *in the sense that it satisfies the following estimates:*

$$\| u(t) - M(t) \ D(t) \exp(-i\phi(t)) W(t) \|_2 \leq C(a_+, b_0, b_1) t^{-\lambda_0} \qquad (8.15)$$

$$\| \, |J(t)|^k \left(\exp(i\phi(t, x/t)) u(t) - M(t) \ D(t) \ W(t) \right) \|_2 \leq C(a_+, b_0, b_1) t^{-\lambda} \qquad (8.16)$$

$$\| u(t) - M(t) \ D(t) \exp(-i\phi(t)) W(t) \|_r \leq C(a_+, b_0, b_1) t^{-\lambda_0 + (\lambda_0 - \lambda)\delta(r)/k} \qquad (8.17)$$

for $0 \leq \delta(r) = (3/2 - 3/r) \leq [k \wedge 3/2]$.

Define in addition

$$A_2 = A - A_0 - A_1^\infty(|DW|^2) \ . \qquad (8.18)$$

Then A *behaves asymptotically in time as* $A_0 + A_1^\infty(|DW|^2)$ *in the sense that* A_2 *satisfies the following estimates:*

$$\| A_2(t) \|_2 \leq C(a_+, b_0, b_1) \ t^{-\lambda_0 + 1/2} \ . \qquad (8.19)$$

Furthermore, for $3/2 < k (< 2)$:

$$\| \nabla A_2(t) \|_2 \leq C(a_+, b_0, b_1) \ t^{-\lambda_0 - 1/2} \qquad (8.20)$$

$$\| \omega^k \nabla A_2(t) \|_2 \leq \ C(a_+, b_0, b_1) \ t^{-\lambda - k - 1/2} \ , \qquad (8.21)$$

while for $(1 <)k < 3/2$*:*

$$\| \nabla A_2(t) \|_2 \leq C(a_+, b_0, b_1) \left(t^{-\lambda_0 - 1/2} + t^{-2\lambda_0 - 1/2 + (\lambda_0 - \lambda)3/2k} \right) \tag{8.22}$$

$$\| \omega^{2k-1/2} A_2(t) \|_2 \leq C(a_+, b_0, b_1) t^{-\lambda - 2k + 1} \left(t^{-\lambda} + t^{k-3/2} \right) . \tag{8.23}$$

A similar result holds for $k = 3/2$ *with a* t^{ε} *loss in the decay.*

Proof. The proof follows from Propositions 6.3 part (2) and from Proposition 7.1, supplemented with the reconstruction of φ described above in this section, except for the estimates (8.15)–(8.17) on u and (8.19)–(8.23) on A. In particular the estimate (8.10) is nothing but the estimate (5.35) expressed in terms of A_0 instead of B_0 while the estimates (8.11) (8.12) (8.13) are essentially (6.49) (6.50) supplemented with (8.4) (8.5).

We next prove the estimates (8.15) - (8.17) on u. From (8.14) with $\varphi = \phi + \psi$ and from (8.7), it follows that

$$\| |J|^m (\exp(i\ D_0\ \phi) u - MDW) \|_2 = \| \omega^m \left(w\ e^{-i\psi} - W \right) \|_2 \tag{8.24}$$

For $m = 0$, we estimate

$$\begin{aligned} \| w \exp(-i\psi) - W \|_2 &\leq \| w (\exp(-i\psi) - 1) \|_2 + \| w - W \|_2 \\ &\leq \| w \|_3 \| \psi \|_6 + \| q \|_2 \leq C\ t^{-\lambda_0} \end{aligned}$$

by (8.5), a Sobolev inequality and (8.11). This proves (8.15). For $m = k$, we estimate by Lemma 3.2

$$\begin{aligned} &\| \omega^k (\exp(-i\psi) w - W) \|_2 \leq C \Big\{ \| \omega^k (\exp(-i\psi) - 1) \|_3 \| w \|_6 \\ &+ \| \exp(-i\psi) - 1 \|_\infty \| \omega^k w \|_2 + \| \omega^k (w - W) \|_2 \\ &\leq C \ \| \omega^{k-1/2} \sigma \|_2 \ \exp (C \| \psi \|_\infty) \| \nabla w \|_2 + (\| \sigma \|_2 \| \nabla \sigma \|_2)^{1/2} \ \| \omega^k w \|_2 \\ &+ \| \omega^k q \|_2 \leq C \left(t^{-\lambda_0 + \beta(k-1/2)} + t^{-\lambda_0 + \beta/2} + t^{-\lambda} \right) \leq C\ t^{-\lambda} \end{aligned}$$

by Lemma 3.3, by the Sobolev inequality

$$\| \psi \|_\infty \leq C (\| \sigma \|_2 \| \nabla \sigma \|_2)^{1/2}$$

and by (8.12) (8.13). This proves (8.16).

The estimate (8.17) follows immediately from (8.15) (8.16) and from the inequality

$$\begin{aligned} \| f \|_r = t^{-\delta(r)} \ \| D^* M^* f \|_r &\leq C\ t^{-\delta(r)} \| \omega^{\delta(r)} D^* M^* f \|_2 \\ &= C\ t^{-\delta(r)} \ \| |J(t)|^{\delta(r)} f \|_2 \ . \end{aligned}$$

We finally prove the estimates (8.19)–(8.23) on A. It follows from the definitions (2.2) (2.3) (2.4) (8.18) and from (2.13) (2.14) that

$$A_2 = t^{-1}\ D_0\ B_1^{\infty}(q, q + 2W)\ . \tag{8.25}$$

It is therefore sufficient to estimate $B_1^{\infty}(q, q+2W)$. We omit the superscript ∞ for brevity. We first estimate by (3.10)

$$\begin{aligned}\| B_1(q, q+2W) \|_2 &\le C\ I_{-1}\left(\| \omega^{-1}(q(q+2W)) \|_2\right)\\ &\le C\ I_{-1}\left(\| q \|_2\ \| q+2W \|_3\right) \le C\ t^{-\lambda_0}\end{aligned} \tag{8.26}$$

by Sobolev inequalities and by (8.11), since $q+2W$ is bounded in H^k and a fortiori in L^3. This proves (8.19).

For $k > 3/2$, we estimate similarly

$$\| \nabla B_1(q, q+2W) \|_2 \le I_0\left(\| q \|_2\ \| q+2W \|_\infty\right) \le C\ t^{-\lambda_0} \tag{8.27}$$

by (3.10) and (8.11), since $q + 2W$ is bounded in L^∞ in that case. Furthermore, by (3.10), Lemma 3.2 and Sobolev inequalities

$$\begin{aligned}\| \omega^{k+1}\ B_1(q, q+2W) \|_2 &\le C\ I_k\Big(\| \omega^k q \|_2\left(\| q \|_\infty\ +\ \| W \|_\infty\right)\\ &\quad + \| \nabla q \|_2\ \| \omega^{k+1/2} W \|_2\Big) \le C\ t^{-\lambda}\end{aligned} \tag{8.28}$$

by (8.11) (8.12). The last two inequalities imply (8.20) and (8.21) respectively.

For $k < 3/2$, we must estimate $B_1(q, q)$ and $B_1(q, W)$ separately because q is no longer controlled in L^∞. We estimate as before

$$\| \nabla B_1(q, W) \|_2 \le I_0\left(\| q \|_2\ \| W \|_\infty\right) \le C\ t^{-\lambda_0}$$

by (8.11), while

$$\| \nabla B_1(q, q) \|_2 \le I_0\left(\| q \|_4^2\ \right) \le C\ t^{-2\lambda_0 + (\lambda_0 - \lambda)3/2k}$$

by (8.11) (8.12), which together imply (8.22).

We next estimate by (3.12) and (8.12)

$$\| \omega^{2k-1/2}\ B_1(q, q) \|_2 \le C\ I_{2k-3/2}\left(\| \omega^k q \|_2^2\right) \le C\ t^{-2\lambda} \tag{8.29}$$

while by (3.10) and Lemma 3.2

$$\begin{aligned}\| \omega^{2k-1/2}\ B_1(q, W) \|_2 &\le C\ I_{2k-3/2}\Big(\| \omega^{2k-3/2} q \|_2\ \| W \|_\infty\\ &\quad + \| q \|_r\ \| \omega^{2k-3/2} W \|_{3/\delta}\Big)\\ &\le C\ I_{2k-3/2}\Big(\| \omega^{2k-3/2} q \|_2\left(\| W \|_\infty\ +\ \| \omega^{3/2} W \|_2\right)\Big)\end{aligned}$$

by Sobolev inequalities, with $1/2 < \delta = \delta(r) = 2k - 3/2 < 3/2$,

$$\cdots \le C\ t^{-\lambda - (\lambda_0 - \lambda)(3/2k - 1)} \le C\ t^{-\lambda - 3/2 + k} \tag{8.30}$$

by interpolation between (8.11) and (8.12). Now (8.23) follows from (8.29) and (8.30). $\square$

We conclude this section with some remarks on variations which can be made or attempted in the formulation of Proposition 8.1.

Remark 8.1 We have stated the assumptions on $(A_+, \dot{A}_+)$ in an implicit way in the form of conditions on the solution A_0 of the free wave equation that they generate. Sufficient conditions for (8.9) and (8.10) to hold directly expressed in terms of $(A_+, \dot{A}_+)$ and possibly w_+ can be found in Lemma 3.5 and Lemma 5.2, but those conditions are far from being optimal (especially those of Lemma 5.2).

Remark 8.2 The available regularity for A is stronger than stated, as follows from the assumption (8.9) on A_0, from the simple estimate on $A_1^\infty(|DW|^2)$

$$\| \omega^m A_1^\infty(|DW|^2) \|_2 \leq C(a_+)\ t^{-m+1/2}$$

for $0 \leq m \leq k_+$, and from the remainder estimates (8.19)–(8.23).

Remark 8.3 The asymptotic behaviour in time of the scalar field A differs in an important way from that of a solution of the free wave equation. In fact A behaves asymptotically in time as

$$A \sim A_0 + A_1^\infty(|DW|^2)\ .$$

Replacing W by w_+ as a first approximation in the last term, one obtains

$$A_1^\infty(|Dw_+|^2) = t^{-1}\ D_0\ B_1^\infty(w_+, w_+)$$

with $B_1^\infty(w_+, w_+)$ constant in time. This yields a contribution to A which spreads by dilation by t and decays as t^{-1} in L^∞ norm. That contribution can in no obvious sense be regarded as small as compared with A_0.

Remark 8.4 One might be tempted to look for simpler asymptotic forms for u and for A by replacing for instance W by w_+ in (8.15) (8.16) (8.18) and/or by omitting a few factors $U^{(*)}(1/t)$ in (7.5) (7.7). This however would introduce errors at least $O(t^{-1})$ and spoil the $t^{-\lambda_0}$ decay in (8.11) (8.15) (8.19) (8.20) (8.22).

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Appendix A

In this appendix, we prove Warnings 4.1 and 4.2.

Proof of Warning 4.1 One sees easily that (4.57) with $y(1) = y_0 > 0$ has a unique maximal increasing solution $y \in \mathcal{C}^1([1, T^*), \mathbb{R}^+)$ for some $T^* > 1$. We shall argue by contradiction by showing that if T^* is sufficiently large, then $y(t)$ is infinite for some $t < T^*$. By integration, (4.57) with $y(1) = y_0$ is converted into the integral equation

$$y(t) = y_0 \exp\left\{(m - \beta_1)^{-1}\ t^{-\beta_1} \int_{1/t}^{1} d\nu\ y(\nu t) \left(\nu^{-1-m} - \nu^{-1-\beta_1}\right)\right\}\ . \qquad \text{(A.1)}$$

We shall prove by induction that (A.1) implies a sequence of lower bounds $y(t) \geq a_n\ t^{\alpha_n}$ with α_n rapidly growing and a_n not too small. We start with $a_0 = y_0$, $\alpha_0 = 0$. Substituting that lower bound into (A.1) yields

$$y \geq y_0 \exp\{y_0\ h(t)\}$$

where

$$\begin{aligned} h(t) &= (m-\beta_1)^{-1}\ t^{-\beta_1} \int_{1/t}^{1} d\nu \left(\nu^{-1-m} - \nu^{-1-\beta_1}\right) \\ &= (m-\beta_1)^{-1} \left(m^{-1}\ t^{m-\beta_1} - \beta_1^{-1}\right) + m^{-1}\ \beta_1^{-1}\ t^{-\beta_1} \end{aligned}$$

so that $y \geq a_1 t^{\alpha_1}$ provided

$$\ell n\ a_1/y_0 \leq (m-\beta_1)^{-1} \left(-\alpha_1\ \ell n\ \tau + y_0(m^{-1}\tau - \beta_1^{-1})\right) \tag{A.2}$$

where $\tau = t^{m-\beta_1} \geq 1$. The minimum of the RHS is attained for $\tau = m\alpha_1/y_0$, which we take > 1, and we can then take

$$a_1 = y_0 \exp\left\{(m-\beta_1)^{-1}\ \beta_1^{-1}\ y_0\right\}(e\ y_0/m\ \alpha_1)^{\alpha_1/(m-\beta_1)}\ . \tag{A.3}$$

Here α_1 is an arbitrary fixed parameter, which we take large. In particular we impose $\alpha_1 > (m^{-1}y_0 \vee 2\beta_1)$.

At the following steps of the iteration, it will be sufficient to replace (A.1) by the lower bound obtained by letting m decrease to β_1, namely

$$y(t) \geq y_0 \exp\left\{t^{-\beta_1} \int_{1/t}^{1} d\nu\ y(\nu t)\ \nu^{-1-\beta_1} |\ell n\ \nu|\right\} \tag{A.4}$$

or equivalently

$$y(t) \geq y_0 \exp\left\{\int_1^t dt'\ \ y(t')\ t'^{-1-\beta_1}\ \ell n(t/t')\right\}\ . \tag{A.5}$$

We now describe the determination of $(a_{n+1}, \alpha_{n+1}) = (a', \alpha')$ from $(a_n, \alpha_n) = (a, \alpha)$. Substituting the induction assumption into (A.5), we obtain for $\alpha > \beta_1$ (a condition that will be ensured below)

$$\begin{aligned} y &\geq y_0 \exp\left\{a \int_1^t dt'\ t'^{-1-\beta_1+\alpha}\ \ell n(t/t')\right\} \\ &= y_0 \exp\left\{a(\alpha-\beta_1)^{-2}(\tau - 1 - \ell n\ \tau)\right\} \end{aligned}$$

where $\tau = t^{\alpha-\beta_1} > 1$. This implies $y \geq a' t^{\alpha'}$ provided

$$\begin{aligned} \ell n\ a'/y_0 &\leq a(\alpha-\beta_1)^{-2}(\tau - 1 - \ell n\ \tau) - \alpha'(\alpha-\beta_1)^{-1}\ell n\ \tau \\ &= \theta\left(\widetilde{a}(\tau-1) - (\widetilde{a}+1)\ell n\ \tau\right) \end{aligned} \tag{A.6}$$

where

$$\theta = \alpha'/(\alpha - \beta_1) \quad , \quad \widetilde{a} = a/\alpha'(\alpha - \beta_1) \ . \tag{A.7}$$

The minimum over τ of the last member of (A.6) is attained for $\widetilde{a}\tau = \widetilde{a} + 1$, and it suffices to impose

$$\ell n(a'/y_0) \le \theta(1 - (\widetilde{a}+1)\ell n(\widetilde{a}+1)/\widetilde{a}) \equiv \theta(\ell n\ \widetilde{a} - f(\widetilde{a}))$$

which allows us to take $a' = y_0 \widetilde{a}^\theta$ provided

$$f(\widetilde{a}) \equiv (\widetilde{a}+1)\ell n(\widetilde{a}+1) - \widetilde{a}\ \ell n\ \widetilde{a} - 1 \le 0$$

a condition which is easily seen to hold for $\widetilde{a} \le 1/2$.

Finally we can take

$$\alpha' = \theta(\alpha - \beta_1) \qquad , \quad a' = y_0 \left(a/\theta(\alpha - \beta_1)^2 \right)^\theta \tag{A.8}$$

provided

$$a \le \theta(\alpha - \beta_1)^2/2 \ . \tag{A.9}$$

So far θ is a free parameter. For definiteness we choose $\theta = 2$, so that after coming back to the original notation, (A.8) (A.9) become

$$\alpha_{n+1} = 2(\alpha_n - \beta_1) \ , \tag{A.10}$$

$$a_{n+1} = y_0\ a_n^2/4(\alpha_n - \beta_1)^4 \ , \tag{A.11}$$

$$a_n \le (\alpha_n - \beta_1)^2 \ . \tag{A.12}$$

(A.10) is readily solved by

$$\alpha_n = 2\beta_1 + 2^{n-1}(\alpha_1 - 2\beta_1) \ .$$

(A.12) is harmless and holds for all n if it holds for $n = 1$ and if $y_0 \le 4(\alpha_1 - \beta_1)^2$, which can be arranged by taking α_1 sufficiently large. (A.11) can be rewritten as

$$\frac{a_{n+1}\ y_0}{64(\alpha_{n+1} - \beta_1)^4} = \frac{y_0^2\ a_n^2}{64^2(\alpha_n - \beta_1)^8} \left(\frac{2(\alpha_n - \beta_1)}{\alpha_{n+1} - \beta_1} \right)^4 \ge \left\{ \frac{a_n\ y_0}{64(\alpha_n - \beta_1)^4} \right\}^2 \tag{A.13}$$

by (A.10). Let now $t \ge 1$ and define

$$u_n = a_n\ t^{\alpha_n - 2\beta_1}\ y_0/64(\alpha_n - \beta_1)^4 \ .$$

It follows from (A.10) (A.13) that $u_{n+1} \ge u_n^2$ and in particular $u_n \ge 1$ for all n if $u_1 \ge 1$, namely if t is sufficiently large in the sense that

$$t^{\alpha_1 - 2\beta_1} \ge (a_1 y_0)^{-1}\ 64(\alpha_1 - \beta_1)^4 \ . \tag{A.14}$$

For such t, the condition $u_n \ge 1$ can be rewritten as

$$\begin{aligned} y(t) \ge a_n\ t^{\alpha_n} &\ge t^{2\beta_1}\ y_0^{-1}\ 64(\alpha_n - \beta_1)^4 \\ &\ge 4t^{2\beta_1}\ y_0^{-1}\ 2^{4n}(\alpha_1 - 2\beta_1)^4 \ . \end{aligned} \tag{A.15}$$

Since the last member of (A.15) tends to infinity with n, such a t cannot be smaller than T^*, which proves finite time blow up. □

Remark A1. Since the RHS of (4.56) and (4.57) is decreasing in β_1 and increasing in m, blow up in finite time for (β_1, m) implies blow up in finite time for (β_1', m) with $\beta_1' \leq \beta_1$ and for (β_1, m') with $m' \geq m$, while the opposite situation prevails as regards the existence of global solutions. Actually it is easy to see that (4.56) or (4.57) admits global solutions for small data if $\beta_1 > 0$ and $m \leq \beta_1$. When coming back to the original equation (4.55), the condition of small data becomes a condition of large t_0.

Proof of Warning 4.2. We want to prove finite time blow up for (4.61) with $y(t_0) = y_0 > 0$. Omitting the second term in the RHS and integrating the remaining inequality, we obtain

$$y \geq \left(y_0^{1/k} + t - t_0\right)^k \geq (t - t_0)^k \ . \tag{A.16}$$

We next keep (A.16), omit the first term in the RHS of (4.61) and change t to $t + t_0$. It is then sufficient to prove blow up for

$$\begin{cases} y \geq t^k \\ \partial_t y \geq (t + t_0)^{-1-\beta_1}\ y^3 \ . \end{cases} \tag{A.17}$$

For that purpose, we show by induction that y satisfies

$$y(t) \geq y_n(t) \geq a_n\ t^{\alpha_n} (t + t_0)^{-(1+\beta_1)\gamma_n} \ , \tag{A.18}$$

starting with $a_0 = 1$, $\alpha_0 = k$ and $\gamma_0 = 0$ given by (A.17). We obtain

$$\begin{aligned} y_{n+1} &= \int_0^t dt'(t_0 + t')^{-1-\beta_1}\ y_n^3(t') \\ &\geq a_n^3 \int_0^t dt'(t_0 + t')^{-(1+\beta_1)(3\gamma_n+1)}\ t'^{3\alpha_n} \\ &\geq a_n^3 (t_0 + t)^{-(1+\beta_1)(3\gamma_n+1)}\ t^{3\alpha_n+1}(3\alpha_n + 1)^{-1} \end{aligned}$$

thereby ensuring (A.18) at the level $n + 1$ if we choose

$$\alpha_{n+1} = 3\alpha_n + 1 \quad , \quad \gamma_{n+1} = 3\gamma_n + 1 \ , \tag{A.19}$$

$$a_{n+1} = a_n^3/(3\alpha_n + 1) \ . \tag{A.20}$$

(A.19) is readily solved by

$$\alpha_n = 3^n(k + 1/2) - 1/2 \quad , \quad \gamma_n = (3^n - 1)/2 \tag{A.21}$$

so that

$$a_{n+1} \geq a_n^3 (k + 1/2)^{-1}\ 3^{-(n+1)} \tag{A.22}$$

or equivalently

$$b_{n+1} \geq b_n^3 \tag{A.23}$$

where

$$b_n = a_n 3^{-n/2-3/4} \ (k+1/2)^{-1/2} \ . \tag{A.24}$$

Let now $t > 0$ and

$$u_n = b_n \ t^{\alpha_n+1/2} \ (t_0+t)^{-(1+\beta_1)(\gamma_n+1/2)} \ . \tag{A.25}$$

It follows from (A.19) (A.23) that $u_{n+1} \geq u_n^3$ and in particular that $u_n \geq 1$ for all n if $u_0 \geq 1$. The condition $u_0 \geq 1$ reduces to

$$t^{2k+1}(t_0+t)^{-(1+\beta_1)} \geq 3^{3/2}(k+1/2) \tag{A.26}$$

and holds for t sufficiently large if $2k > \beta_1$. For such a t, by (A.18)

$$y \geq a_n \ t^{\alpha_n}(t_0+t)^{-(1+\beta_1)\gamma_n} \geq (k+1/2)^{1/2} \ 3^{n/2+3/4} \ t^{-1/2}(t_0+t)^{(1+\beta_1)/2} \ . \tag{A.27}$$

Since the last member of (A.27) tends to infinity with n, such a t cannot be smaller than the maximal time T^* of existence of the solution y of (4.61), which proves finite time blow up. □

References

[1] A. Bachelot, Problème de Cauchy pour des systèmes hyperboliques semi-linéaires, *Ann. IHP (Anal. non lin.)*, **1**, 453–478 (1984).

[2] J. B. Baillon and J. M. Chadam, The Cauchy problem for the coupled Schrödinger-Klein-Gordon equations, in "Contemporary Developments in Continuum Mechanics and Partial Differential Equations", G. M. de La Penha and L. A. Medeiros eds, North-Holland, Amsterdam, 1978.

[3] J. Derezinski and C. Gérard, Scattering theory of classical and quantum N-particle systems, Springer, Berlin, 1997.

[4] I. Fukuda and M. Tsutsumi, On coupled Klein-Gordon-Schrödinger equations II, *J. Math. Anal. Appl.*, **66**, 358–378 (1978).

[5] J. Ginibre and T. Ozawa, Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$, *Commun. Math. Phys.*, **151**, 619–645 (1993).

[6] J. Ginibre and G. Velo, Long range scattering and modified wave operators for some Hartree type equations I, *Rev. Math. Phys.*, **12**, 361–429 (2000).

[7] J. Ginibre and G. Velo, Long range scattering and modified wave operators for some Hartree type equations II, *Ann. H.P.*, **1**, 753–800 (2000).

[8] J. Ginibre and G. Velo, Long range scattering and modified wave operators for some Hartree type equations III, Gevrey spaces and low dimensions, *J. Diff. Eq.*, **175**, 415–501 (2001).

[9] N. Hayashi, E. I. Kaikina and P. I. Naumkin, On the scattering theory for the cubic nonlinear Schrödinger and Hartree type equations in one space dimension, *Hokkaïdo Math. J.*, **27**, 651–667 (1998).

[10] N. Hayashi and P. I. Naumkin, Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations, *Am. J. Math.*, **120**, 369–389 (1998).

[11] N. Hayashi and P. I. Naumkin, Scattering theory and large time asymptotics of solutions to Hartree type equations with a long range potential, preprint, 1997.

[12] N. Hayashi and P. I. Naumkin, Remarks on scattering theory and large time asymptotics of solutions to Hartree type equations with a long range potential, *SUT J. of Math.*, **34**, 13–24 (1998).

[13] N. Hayashi, P. I. Naumkin and T. Ozawa, Scattering theory for the Hartree equation, *SIAM J. Math. Anal.*, **29**, 1256–1267 (1998).

[14] N. Hayashi and T. Ozawa, Modified wave operators for the derivative nonlinear Schrödinger equation, *Math. Ann.*, **298**, 557–576 (1994).

[15] N. Hayashi and W. von Wahl, On the global strong solutions of coupled Klein-Gordon-Schrödinger equations, *J. Math. Soc. Japan*, **39**, 489–497 (1987).

[16] L. Hörmander, The Analysis of Linear Partial Differential Operators, Vol I, Springer, Berlin, 1983.

[17] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, **41**, 891–907 (1988).

[18] C. Kenig, G. Ponce and L. Vega, The initial value problem for a class of nonlinear dispersive equations, in Functional-Analytic Methods for Partial Differential Equations, *Lect. Notes Math.*, **1450**, 141–156 (1990).

[19] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension, *Commun. Math. Phys.*, **139**, 479–493 (1991).

[20] T. Ozawa and Y. Tsutsumi, Asymptotic behaviour of solutions for the coupled Klein-Gordon-Schrödinger equations, in Spectral and Scattering Theory and Applications, Adv. Stud. in Pure Math., Jap. Math. Soc., **23**, 295–305 (1994).

[21] W. Strauss, Nonlinear Wave Equations, CMBS Lecture notes 73, Am. Math. Soc., Providence, 1989.

[22] Y. Tsutsumi, Global existence and asymptotic behaviour of solutions for the Maxwell-Schrödinger system in three space dimensions, *Commun. Math. Phys.*, **151**, 543–576 (1993).

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