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Modified Wave Operators for the Hartree Equation with Data, Image and Convergence in the Same Space, II

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Abstract. We study modified wave operators for the Hartree equation with a long-range potential $|x|^{-\nu}$, extending the result in [12] to the whole range of the Dollard type $1/2 < \nu < 1$. We construct the modified wave operators in the whole space of $(1+|x|)^{-s}L^2$. We also have the image, strong continuity and strong asymptotic approximation in the same space. The lower bound $s > 1 - \nu/2$ of the weight is sharp from the scaling argument. Those maps are homeomorphic onto open subsets, which implies in particular asymptotic completeness for small data.

1 Introduction

In this paper, we continue the study in [12] on asymptotic behavior of solutions for the Hartree equation with a long-range potential $|x|^{-\nu}$:

$$2i\dot{u} - \Delta u + V(u)u = 0, \tag{1.1}$$

where

$$V(u) = K_V(x) * |u|^2, \quad K_V(x) = \lambda |x|^{-\nu}, \tag{1.2}$$

$u = u(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}$ ($n \in \mathbb{N}$) is the unknown function and $\lambda \in \mathbb{R}$ is a real constant.

The main goal in [12] was to obtain results for the modified wave operators in the long range case that are as good as those for the ordinary wave operators in the short range case $\nu > 1$, especially as to the domain, the range and the topology of convergence. In fact, such a result was obtained in [12] in the limiting case $\nu = 1$, which is almost the same as in the short range case except the presence of the modification and the exclusion of the scaling critical case. Before that result, the modified wave operators were defined under much stronger assumptions on the data, while the range and the convergence were given in larger spaces or weaker senses than that for the data. Actually, it was rather recent [3, 4] even that those operators were obtained without any smallness assumption on the data and for $\nu < 1$.

However, the argument in [12] strongly depended on the fact that the phase modifier diverges slower than any positive power of t, which might have the readers wonder that the result in [12] for the long range was somewhat special for the borderline case $\nu = 1$ only. For more detail of known results on the modified wave operators, see [3, 4] and the references therein.

In this paper, we show that the same result as in [12] holds actually in the whole range $1/2 < \nu \le 1$ where the Dollard-type first-order modification suffices. Let $U(t) = e^{-i\Delta t/2}$ denote the free propagator and $\mathcal{F}$ the Fourier transform. H^s denotes the usual inhomogeneous Sobolev space based on L^2. The main result of this paper is the following. We do not consider the case $\nu = 1$, which has been solved in [12].

Theorem 1.1 *Let $n \ge 3$, $1/2 < \nu < 1$, $1 - \nu/2 < s < 1$ and $\lambda \in \mathbb{R}$. Then, for any $\psi \in \mathcal{F}H^s$, there exists a unique solution u of* (1.1) *satisfying $U(-t)u(t) \in C(\mathbb{R}; \mathcal{F}H^s)$ and*

$$\mathcal{F}^{-1} \exp\left(\frac{1}{2i} V(\mathcal{F}\psi)\frac{t^{1-\nu}}{\nu - 1}\right) \mathcal{F}U(-t)u(t) \to \psi \tag{1.3}$$

as $t \to \infty$ in $\mathcal{F}H^s$. Thus we have the modified wave operator W defined by

$$W : \psi \mapsto u(0). \tag{1.4}$$

W is a homeomorphism from $\mathcal{F}H^s$ to an open subset of $\mathcal{F}H^s$ in the strong topology. We have the same result for the negative time $t \to -\infty$.

This result is the same as that in [12] except the extension of the range of ν to $1/2 < \nu < 1$ and the restrictions $n \ge 3$ and $s < 1$. The exception of lower dimensions is related to the Sobolev embedding. Actually, the case $n = 2$ is required to be excluded only in one place of the estimates, and that restriction may be hopefully avoidable. However, the one dimensional case looks more different. That is because the homogeneous part $\dot{H}^s$ can not dominate any Lebesgue norm when $s > n/2$, which is always the case when $n = 1$, while we can choose $s < n/2$ when $n \ge 2$. The restriction $s < 1$ is much more technical and hardly essential. It is required just because we estimate the H^s norm mainly by the spatial difference and we consider only the first-order difference for the sake of simplicity.

The basic strategy to construct the modified wave operators is almost the same as in the previous paper [12]; We transform the scattering problem to the initial value problem by the pseudo-conformal inversion, eliminate the diverging oscillation by using the prescribed asymptotic states, and solve the Cauchy problem of thereby modified equation coupled with the evolution equation of the potential term.

The essential novelty is in the estimate for the phase terms, where we will see that the divergence in the phases can be cancelled by each other without losing any regularity or decay in time. Since direct calculations of Fourier transform are not helpful for the phase terms, we estimate them in the physical space by using some decomposition arguments in the frequency. The fractional derivative is not so convenient by the same reason, so we will employ the difference operator instead, which is easy to handle in the phase terms and also in the equations. Here we need only the first order of difference, because we can choose $s < 1$. Other advantages of the difference operator are that we can replace the commutator estimate in

H^s (which played the essential role in [12]) with the trivial chain rule for the difference, and that the difference operator effectively localizes the frequency so that the arguments are indeed free from the particular choice of s and we can obtain some uniform decay estimate for the higher frequency, which will play an important role to obtain the continuity and asymptotic completeness as in [12].

Now we briefly recall our strategy used in the previous paper [12]. We use the well-known transform of the pseudo-conformal inversion:

$$u \mapsto u^* = (it)^{-n/2} e^{|x|^2/(2it)} \overline{u}(1/t, x/t). \tag{1.5}$$

Then the equation (1.1) for u is transformed to the following equation for u^*:

$$2i\dot{u}^* - \Delta u^* + |t|^{\nu-2} V(u^*) u^* = 0, \tag{1.6}$$

and the asymptotic behavior of u as $t \to \pm\infty$ can be described by that of u^* as $t \to \pm 0$, using the relation:

$$\overline{U(-1/t)u^*(1/t)} = (2\pi)^{-n/2} \mathcal{F} U(-t) u(t). \tag{1.7}$$

To eliminate the singularity at $t = 0$ of (1.6), we define the modified field w by

$$u^*(t) = U(t) e^{i\Phi} w(t), \tag{1.8}$$

where

$$\Phi(\phi) = V(\phi) \frac{|t|^{\nu-2} t}{2(\nu - 1)}. \tag{1.9}$$

It is easy to check that $e^{i\Phi(\phi)}\phi$ is the general solution to the ODE given by dropping the non-singular term Δu^* from (1.6). It is also easy to see that the second derivative in Δ will create another singular term if $\nu \le 1/2$, so that the above first-order approximation can be valid only when $\nu > 1/2$. The equation for w is the following.

$$2i\dot{w} + |t|^{\nu-2} e^{-i\Phi} \{U(-t)V(u^*)U(t) - V(\phi)\} e^{i\Phi} w = 0, \tag{1.10}$$

where $U(-t)V(u^*)U(t)$ denotes the operator defined by

$$U(-t)V(u^*)U(t)\varphi := U(-t)(V(u^*)U(t)\varphi). \tag{1.11}$$

As was mentioned in [12], the advantage of our choice of the modification in (1.8) is that we do not encounter any derivative loss as we do if we choose other modifications such as $u^*(t) = e^{i\Phi} w(t)$. But here we have to square up to the disadvantage that the phase factors remain without exact cancellations, which was disposed with in [12] by relying totally on the fact that the divergence of the phase was only $\log t$. Thus our main problem in this paper is to derive the cancellation estimate at $t = 0$ of those phase factors, without losing any (extra) regularity or any decay.

The rest of this paper is organized as follows. First in the next Section 2, we derive the most important estimates with respect to the control of the phase terms. They reduce the necessary estimates on w and V to those for the equation without the phase modifier. In Section 3, we derive several estimates that will be used after that phase elimination. Using those estimates, we can derive some bounds on the difference energy of the modified field in Section 4, and some bounds as well as decay estimates at $t = +0$ for the potential term in Section 5. Combining those bounds and decay estimates, we can solve the Cauchy problem for the modified equation by the iteration argument in Section 6. These argument can yield some uniform decay of the higher frequency in terms of energy. This uniform estimate effectively reduces any convergence problems in H^s to those in L^2. Then we can easily show the continuity properties of the modified wave operator in Section 7, and the openness of it in Section 8.

We conclude this introduction by giving some notations used throughout this paper. H^s, $\dot{H}^s$ and $\dot{B}^s_{p,q}$ denote the inhomogeneous Sobolev space, the homogeneous Sobolev space and the homogeneous Besov space, respectively (see [1] for the definition). We use the following abbreviation for the norm for the potential term:

$$B^s := \dot{B}^{s+n/2}_{2,1}. \tag{1.12}$$

In general, elements in this space can not be uniquely determined as usual distributions when $s > 0$. However, we will use this space only for $s < 1$ and then the elements are uniquely determined up to addition of constants. The readers need not care about this ambiguity, since it will be clear that addition of constants does not matter in each estimate involving this space with $s > 0$. We also use the following dual space:

$$B^s_* := \dot{B}^{s-n/2}_{2,\infty}. \tag{1.13}$$

We will use the following notation to express polynomial bounds.

$$a^{[b,c]} := \max(a^b, a^c). \tag{1.14}$$

For any spatial function u, we denote by $\varphi_I * u$ the Littlewood-Paley projection on $\mathbb{R}^n$ to the frequency of the size $|\xi| \sim I$;

$$\varphi = \sum_{I=2^j, j\in\mathbb{Z}} \varphi_I * \varphi, \quad \operatorname{supp} \mathcal{F}(\varphi_I * u) \subset \{|\xi| \sim I\}. \tag{1.15}$$

$\delta^h\varphi$ denotes the spatial difference

$$\delta^h\varphi(x) := \varphi(x+h) - \varphi(x), \tag{1.16}$$

with a parameter $h \in \mathbb{R}^n$.

For any sequence a and any function F, we denote

$$\begin{aligned} &\delta_k a := a_k - a_{k-1}, \quad F(a_{k+}) := F(a_k) + F(a_{k+1}), \\ &F(a_{k+}, a_{k+}) := \sum_{i,j=k,k+1} F(a_i, a_j), \text{ etc.}, \end{aligned} \tag{1.17}$$

We will frequently use the above notation for $k = 0, 1$, even if a_k is defined only for $k = 0, 1$. In particular, we denote

$$\begin{aligned}&\delta_1 a = a_1 - a_0, \quad F(a_{0+}) = F(a_0) + F(a_1),\\ &\delta_1 F(a_*) := F(a_1) - F(a_0),\\ &\delta_1 F(a_*, b_*) := F(a_1, b_1) - F(a_0, b_0), \text{ etc.}\end{aligned} \tag{1.18}$$

2 Phase estimates

In this section, we derive the most important estimates on the phase terms. We will use the following identity:

$$S(v, w) := (U(t)v)\overline{U(t)w} = U(-t)\mathcal{F}^{-1} \int v(x + t\xi)\overline{w}(x)e^{-ix\xi}dx, \tag{2.1}$$

which is easy to verify by using the explicit formula for $U(t)$:

$$U(t)\varphi = ct^{-n/2} \int e^{|x-y|^2/(2it)}\varphi(y)dy. \tag{2.2}$$

The effect of the phase factor is skimmed into

$$\begin{aligned}S^0(\Phi; v, w) = S^0(v, w) &:= S(e^{i\Phi}v, e^{i\Phi}w) - S(v, w)\\ &= U(-t)\mathcal{F}^{-1} \int \left(e^{i\Phi(x+t\xi)-i\Phi(x)} - 1\right) v(x + t\xi)\overline{w}(x)e^{-ix\xi}dx.\end{aligned} \tag{2.3}$$

This identity suggests that we may expect the phase factors to cancel each other in the order roughly $O(t\xi\|\nabla\Phi\|_{L^\infty})$. However, it is not always possible to estimate the phase term just in L^∞, in particular, when v or w has frequency less than $|\xi|$. Then the following L^p estimate of the phase factor plays an essential role. We denote

$$\Phi'(x, t\xi) := \Phi(x + t\xi) - \Phi(x), \quad \Psi(\Phi; x, t\xi) := e^{i\Phi'(x,t\xi)} - 1. \tag{2.4}$$

The parameter α below will be fixed as $\alpha = s - \nu/2$ in the later applications, though we will ignore it for a while, since it has nothing essential to do with the phase estimates.

Lemma 2.1 *Let $n \in \mathbb{N}$, $m \geq 1$, $0 \leq \alpha < 1$ and $0 \leq \beta \leq n/2$. Assume that*

$$(\alpha - 1)m + \beta < \theta < \alpha + n/2, \tag{2.5}$$

$$-\alpha \leq \theta \leq 1 - \alpha. \tag{2.6}$$

Then we have for $|\xi| \sim N$,

$$\|\varphi_I * \Psi(\Phi)\|_{L_x^{n/\beta}} \lesssim |tN^2|^\theta N^{-\beta} \max_{1\leq d\leq m} (N/I)^{\beta-\theta+d\alpha}(t^\alpha\|\Phi\|_{B^{2\alpha}})^d.$$

Instead of (2.5), *assume that*

$$(\alpha-1)m+\beta+\alpha<\theta+\sigma<\alpha+n/2. \tag{2.7}$$

Then we have for $|\xi|\sim N$,

$$\begin{aligned}\|\varphi_I*(\Psi(\Phi_1)-\Psi(\Phi_0))\|_{L^{n/\beta}_x} \lesssim |tN^2|^\theta N^{\sigma-\beta}&\max_{1\le d\le m+1}(N/I)^{\beta-\theta+d\alpha}\\ &\times(t^\alpha\|\Phi_1-\Phi_0\|_{B^{2\alpha-\sigma}})\\ &\times(t^\alpha\|\Phi_0\|_{B^{2\alpha}}+t^\alpha\|\Phi_1\|_{B^{2\alpha}})^{d-1}.\end{aligned} \tag{2.8}$$

Proof. In this proof, we define the norm in $\mathbb{R}^d$ for $d\in\mathbb{N}$ by

$$|(x_1,\dots,x_d)|:=|x_1|+\cdots+|x_d|. \tag{2.9}$$

Denote $\Psi_i:=\Psi(\Phi_i)$ and $\Psi:=\Psi(\Phi)$. By convexity, it suffices to prove the estimate in the case $m\in\mathbb{N}$. Fix $\xi\in\mathbb{R}^n$ as $|\xi|\sim N$. We estimate Ψ by taking the m-th order difference. By the assumptions (2.5), $\alpha<1$ and $0\le\beta\le n/2$, we can find $\beta'\in(\beta,n/2)$ such that

$$(\alpha-1)m+\beta'<\theta<\alpha+\beta'. \tag{2.10}$$

and in case we have (2.7), then we can find $\beta'\in(\beta,n/2)$ such that

$$(\alpha-1)m+\beta'+\alpha<\theta+\sigma<\alpha+\beta'. \tag{2.11}$$

It is easy to find functions $\chi_k\in\mathcal{S}(\mathbb{R}^n)$, $k=1,\dots,n$, satisfying

$$\varphi_1=\sum_{k=1}^n\chi_k*\varphi_1 \tag{2.12}$$

such that $\mathcal{F}\delta^h=e^{ih\xi}-1$ does not vanish on the support of $\mathcal{F}\chi_k$ when h is the k-th unit vector of $\mathbb{R}^n$. Then δ^h is invertible when restricted on the Fourier support of φ_1 so that φ_1*u can be effectively dominated by $(\delta^h)^m u$ with finitely many h of size 1. By dilation, we have the same conclusion for any I. See [1, Lemma 6.2.6] for more details. Thus we can estimate

$$\|\varphi_I*u\|_{L^{n/\beta'}_x}\lesssim\sup_{|h|\sim 1/I}\|(\delta^h)^m u\|_{L^{n/\beta'}_x}, \tag{2.13}$$

for any function u.

By the chain rule for the difference operator, we have

$$\|(\delta^h)^m\Psi\|_{L^{n/\beta'}}\lesssim\sum_{d=1}^m\sum_{a\in\mathbb{N}^d,|a|=m}\|[(\delta^h)^{a_1}\Phi']\cdots[(\delta^h)^{a_d}\Phi']\|_{L^{n/\beta'}_x}, \tag{2.14}$$

$$\begin{aligned}
&\|(\delta^h)^m \delta_1 \Psi\|_{L^{n/\beta'}} \\
&\lesssim \sum_{d=1}^{m} \sum_{a\in\mathbb{N}^d, |a|=m} \|[(\delta^h)^{a_1}\Phi'_{0+}]\cdots[(\delta^h)^{a_{d-1}}\Phi'_{0+}][(\delta^h)^{a_d}\delta_1\Phi']\|_{L_x^{n/\beta'}} \qquad (2.15)\\
&\quad + \sum_{d=2}^{m+1} \sum_{a\in\mathbb{N}^{d-1}, |a|=m} \|[(\delta^h)^{a_1}\Phi'_{0+}]\cdots[(\delta^h)^{a_{d-1}}\Phi'_{0+}][\delta_1\Phi']\|_{L_x^{n/\beta'}}, \qquad (2.16)
\end{aligned}$$

where we denote, for simplicity,

$$[\varphi] := \sum_{k=0}^{m} |\varphi(x+kh)|. \tag{2.17}$$

Let

$$\mu_k \in [0,1], \quad \tau_k \in (0, a_k), \quad b_k \in [0, n/2], \quad \tau_k + \mu_k = b_k + \gamma_k, \tag{2.18}$$

for $k = 1, \dots, d$, with the exceptional rule that $\tau_d = a_d = 0$ for (2.16). By convexity, we can find such (μ_k, τ_k, b_k) when $|\mu|$, $|\tau|$ and $|b|$ are given from the region

$$0 \le |\mu| \le d, \; 0 < |\tau| < m, \; 0 \le |b| \le nd/2, \; |\mu| + |\tau| = |b| + |\gamma|. \tag{2.19}$$

We set

$$|b| = \beta', \quad |\mu| = \theta + \alpha d, \quad |\tau| = \beta' + |\gamma| - \theta - \alpha d, \tag{2.20}$$

with

$$\gamma_1 = \cdots = \gamma_d = 2\alpha \tag{2.21}$$

for (2.14), and

$$\gamma_1 = \cdots = \gamma_{d-1} = 2\alpha, \quad \gamma_d = 2\alpha - \sigma \tag{2.22}$$

for (2.15) and (2.16). Then the assumptions (2.6), $0 < \beta' < n/2$, (2.10), and (2.11) imply that $(|\mu|, |\tau|, |b|)$ is in the region (2.19) so that we can choose (μ_k, τ_k, b_k) for $k = 1, \dots, d$ satisfying (2.18). Using the difference norm of the Besov spaces, we can dominate the summand in (2.14) by

$$\begin{aligned}
&\prod_{k=1}^{d} |h|^{\tau_k} \|\Phi'\|_{\dot{B}^{\tau_k}_{n/b_k,\infty}} \lesssim \prod_{k=1}^{d} |tN|^{\mu_k} I^{-\tau_k} \|\Phi\|_{\dot{B}^{\tau_k+\mu_k}_{n/b_k,\infty}} \\
&\lesssim |tN|^{|\mu|} I^{-|\tau|} \|\Phi\|^d_{B^{2\alpha}} = |tN|^{\theta+\alpha d} I^{-\beta'-\alpha d+\theta} \|\Phi\|^d_{B^{2\alpha}} \\
&= |tN^2|^{\theta} (N/I)^{\alpha d - \theta} I^{-\beta'} (|t|^{\alpha} \|\Phi\|_{B^{2\alpha}})^d.
\end{aligned} \tag{2.23}$$

We can bound the summand in (2.15) by

$$\begin{aligned} &|h|^{\tau_d}\|\delta_1\Phi'\|_{\dot B^{\tau_d}_{n/b_d,\infty}}\prod_{k=1}^{d-1}|h|^{\tau_k}\|\Phi'_{0+}\|_{\dot B^{\tau_k}_{n/b_k,\infty}} \\ &\lesssim |tN|^{\mu_d}I^{-\tau_d}\|\delta_1\Phi\|_{\dot B^{\tau_d+\mu_d}_{n/b_d,\infty}}\prod_{k=1}^{d-1}|tN|^{\mu_k}I^{-\tau_k}\|\Phi_{0+}\|_{\dot B^{\tau_k+\mu_k}_{n/b_k,\infty}} \\ &\lesssim |tN|^{|\mu|}I^{-|\tau|}\|\delta_1\Phi\|_{B^{2\alpha-\sigma}}\|\Phi_{0+}\|^{d-1}_{B^{2\alpha}} \\ &= |tN^2|^{\theta}(N/I)^{\alpha d-\theta}I^{-\beta'+\sigma}(|t|^{\alpha}\|\delta_1\Phi\|_{B^{2\alpha-\sigma}})(|t|^{\alpha}\|\Phi_{0+}\|_{B^{2\alpha}})^{d-1}. \end{aligned} \tag{2.24}$$

The summand in (2.16) is dominated by

$$\begin{aligned} &\|\delta_1\Phi'\|_{L^{n/b_d}}\prod_{k=1}^{d-1}|h|^{\tau_k}\|\Phi'_{0+}\|_{\dot B^{\tau_k}_{n/b_k,\infty}} \\ &\lesssim |tN|^{\mu_d}\|\delta_1\Phi\|_{\dot B^{\mu_d}_{n/b_d,\infty}}\prod_{k=1}^{d-1}|tN|^{\mu_k}I^{-\tau_k}\|\Phi_{0+}\|_{\dot B^{\tau_k+\mu_k}_{n/b_k,\infty}} \\ &\lesssim |tN|^{|\mu|}I^{-|\tau|}\|\delta_1\Phi\|_{B^{2\alpha-\sigma}}\|\Phi_{0+}\|^{d-1}_{B^{2\alpha}} \\ &= |tN^2|^{\theta}(N/I)^{\alpha d-\theta}I^{-\beta'+\sigma}(|t|^{\alpha}\|\delta_1\Phi\|_{B^{2\alpha-\sigma}})(|t|^{\alpha}\|\Phi_{0+}\|_{B^{2\alpha}})^{d-1}. \end{aligned} \tag{2.25}$$

Thus we obtain

$$\begin{aligned} \|\varphi_I*\Psi\|_{L^{n/\beta}_x} &\lesssim I^{\beta'-\beta}\|\varphi_I*\Psi\|_{L^{n/\beta'}_x} \\ &\lesssim I^{\beta'-\beta}\sum_{d=1}^{m}|tN^2|^{\theta}(N/I)^{\alpha d-\theta}I^{-\beta'}(|t|^{\alpha}\|\Phi\|_{B^{2\alpha}})^d \\ &\lesssim |tN^2|^{\theta}N^{-\beta}\max_{1\le d\le m}(N/I)^{\beta-\theta+d\alpha}(|t|^{\alpha}\|\Phi\|_{B^{2\alpha}})^d \end{aligned} \tag{2.26}$$

$$\begin{aligned} \|\varphi_I*(\delta_1\Psi)\|_{L^{n/\beta}_x} &\lesssim I^{\beta'-\beta}\|\varphi_I*(\delta_1\Psi)\|_{L^{n/\beta'}_x} \\ &\lesssim |tN^2|^{\theta}N^{-\beta+\sigma}\max_{1\le d\le m+1}(N/I)^{\beta-\theta-\sigma+d\alpha} \\ &\qquad\times(|t|^{\alpha}\|\delta_1\Phi\|_{B^{2\alpha-\sigma}})(|t|^{\alpha}\|\Phi_{0+}\|_{B^{2\alpha}})^{d-1} \end{aligned} \tag{2.27}$$

□

Now we proceed to the main estimates on the bilinear operator S^0. We also need to estimate the difference of the phase term:

$$S^{\delta}(\Phi_0,\Phi_1;v,w) := S^0(\Phi_1;v,w) - S^0(\Phi_0;v,w). \tag{2.28}$$

When estimating the evolution of the potential, we need to estimate the following variant which has an additional decay in time:

$$S^{\psi}(v,w) := S^0(\psi v,w) - S^0(v,\overline{\psi}w). \tag{2.29}$$

We need also to estimate the effect of phase change for this operator:

$$S^{\psi,\delta}(\Phi_0, \Phi_1; v, w) := S^{\psi}(\Phi_1; v, w) - S^{\psi}(\Phi_0; v, w). \tag{2.30}$$

The following are the main estimates in this paper.

Lemma 2.2 *Let $0 \le \theta \le \alpha < 1/2$, $\beta < n/2$, $\gamma < n/2$, $0 < \beta + \gamma < n/2$.*

(i) Assume

$$(\alpha - 1)n/2 + \max(\beta, \gamma) < \theta. \tag{2.31}$$

Then we have

$$\|S^0(\Phi; v, w)\|_{\dot{B}^{\beta+\gamma-2\theta-n/2}_{2,1}} \lesssim |t|^\theta d(\Phi) \|v\|_{\dot{H}^\beta} \|w\|_{\dot{H}^\gamma}, \tag{2.32}$$

where we denoted

$$d(\Phi) := (|t|^\alpha \|\Phi\|_{B^{2\alpha}})^m + 1, \tag{2.33}$$

where $m > n/2$ is sufficiently large depending on $\alpha, \beta, \gamma, \theta$.

(ii) Let $\sigma \in [0, 1]$ satisfy

$$(\alpha - 1)n/2 + \max(\beta, \gamma) + \alpha < \theta + \sigma < \alpha + \beta + \gamma, \tag{2.34}$$

$$\alpha \le \theta + \sigma. \tag{2.35}$$

Then we have

$$\begin{aligned} &\|S^\delta(\Phi_0, \Phi_1; v, w)\|_{\dot{B}^{\beta+\gamma-2\theta-\sigma-n/2}_{2,1}} \\ &\lesssim |t|^\theta (d(\Phi_0) + d(\Phi_1)) |t|^\alpha \|\Phi_1 - \Phi_0\|_{B^{2\alpha-\sigma}} \|v\|_{\dot{H}^\beta} \|w\|_{\dot{H}^\gamma}. \end{aligned} \tag{2.36}$$

(iii) Assume (2.31) and let $\theta' \in (0, 1)$ and σ' satisfy

$$0 < \theta' - \sigma' < \beta + \gamma. \tag{2.37}$$

Then we have

$$\begin{aligned} &\|S^\psi(\Phi; v, w)\|_{\dot{B}^{\beta+\gamma-2(\theta+\theta')+\sigma'-n/2}_{2,1}} \\ &\lesssim |t|^{\theta+\theta'} d(\Phi) \|\psi\|_{\dot{B}^{n/2+\sigma'}_{2,\infty}} \|v\|_{\dot{H}^\beta} \|w\|_{\dot{H}^\gamma}. \end{aligned} \tag{2.38}$$

(iv) Assume (2.34), (2.35), (2.37), and

$$\theta + \sigma + \theta' - \sigma' < \alpha + \beta + \gamma. \tag{2.39}$$

Then we have

$$\begin{aligned} &\|S^{\psi,\delta}(\Phi_0, \Phi_1; v, w)\|_{\dot{B}^{\beta+\gamma-2(\theta+\theta')+\sigma'-\sigma-n/2}_{2,1}} \\ &\lesssim |t|^{\theta+\theta'} (d(\Phi_0) + d(\Phi_1)) |t|^\alpha \|\Phi_1 - \Phi_0\|_{B^{2\alpha-\sigma}} \\ &\quad \times \|\psi\|_{\dot{B}^{n/2+\sigma'}_{2,\infty}} \|v\|_{\dot{H}^\beta} \|w\|_{\dot{H}^\gamma}. \end{aligned} \tag{2.40}$$

Since we have to use this lemma with $\beta = \gamma = s$, the assumption $\beta + \gamma < n/2$ completely exclude possible choice of s when $n \le 2$ since $s > 1 - \nu/2 > 1/2$. When $n \ge 3$, we can choose $s > 1 - \nu/2 < 3/4$ satisfying $2s < n/2$. The upper bounds in (2.34) and (2.37) can be identified with (2.39) by regarding unused parameters as 0.

Proof. Denote

$$\begin{aligned} &\Psi^0 := \Psi, \quad \Psi^\delta := \Psi(\Phi_1) - \Psi(\Phi_0), \\ &\Psi^\psi := \Psi^0 \psi', \quad \Psi^{\psi,\delta} := \Psi^\delta \psi', \end{aligned} \tag{2.41}$$

where $\psi'(x, t\xi) := \psi(x + t\xi) - \psi(x)$ and $\Psi(\Phi)$ is defined in (2.4). In this proof, we sometimes use the superscript $(\psi, 0)$ instead of ψ. By (2.1), we have

$$S^*(v, w) = U(-t)\mathcal{F}^{-1} \int \Psi^* v(x + t\xi)\overline{w}(x)e^{-ix\xi}dx, \tag{2.42}$$

for $* = 0, \delta, \psi$ and (ψ, δ). First we use the Littlewood-Paley decomposition for x to localize every function in the frequency. Let $I, J, K, M, N > 0$ be dyadic parameters $\in \{2^j | j \in \mathbb{Z}\}$. For brevity, we denote $v_J := \varphi_J * v$, $w_K := \varphi_K * w$, etc. Then we have

$$\begin{aligned} &S^*(v, w) \\ &\quad = \sum_{I,J,K,N:dyadic} U(-t)\varphi_N * \mathcal{F}^{-1} \int \Psi^*_I(x, t\xi)v_J(x + t\xi)\overline{w}_K(x)e^{-ix\xi}dx, \end{aligned} \tag{2.43}$$

for $* = 0, \delta$. If the operator contains ψ, we have to decompose ψ' also. Then we have

$$\begin{aligned} S^{\psi,*}(v, w) = &\sum_{I,J,K,N,M:dyadic} \\ &\quad U(-t)\varphi_N * \mathcal{F}^{-1} \int \Psi^*_I \psi'_M(x, t\xi)v_J(x + t\xi)\overline{w}_K(x)e^{-ix\xi}dx, \end{aligned} \tag{2.44}$$

for $* = 0, \delta$. In the former case, we denote $\Psi^*_{I,M} := \Psi^*_I$ and let $M = 0$. In the latter case, we denote $\Psi^*_{I,M} := \Psi^*_I \psi'_M$.

The estimate for the summand with the appropriate weights in the dyadic parameters will imply the boundedness of S^0 as

$$\|S^0(v, w)\|_{\dot{B}^{\beta+\gamma-2\theta-n/2}_{2,\infty}} \lesssim |t|^\theta d(\Phi)\|v\|_{\dot{B}^\beta_{2,1}}\|w\|_{\dot{B}^\gamma_{2,1}}. \tag{2.45}$$

Then the desired estimate will follow from this via the bilinear real interpolation[1]. Thus we need only to estimate the L^2_ξ norm of

$$R^* := \tilde{\varphi}_N(\xi) \int \Psi^*_{I,M}(x, t\xi)v_J(x + t\xi)\overline{w}_K(x)e^{-ix\xi}dx, \tag{2.46}$$

for each I, J, K, M, N, at least when $* = 0$ or δ.

[1]When a bilinear operator is bounded from $X_i \times Y_j$ to Z_{i+j} for $(i, j) = (0, 0), (0, 1), (1, 0)$, then it is also bounded from $(X_0, X_1)_{\theta_0,r_0} \times (Y_0, Y_1)_{\theta_1,r_1}$ to $(Z_0, Z_1)_{\theta_0+\theta_1,r}$ for $\theta_0, \theta_1, \theta \in (0, 1)$ with $\theta = \theta_0 + \theta_1$ and $r_0, r_1, r \in [1, \infty]$ with $1/r = 1/r_0 + 1/r_1$. See [1, 3.13.5(b)].

Also in the trilinear case $S^{\psi,*}$, the original estimate will follow from that for $R^{\psi,*}$ via the bilinear real interpolation as

$$\begin{aligned} &(\psi, v, w) \mapsto S^{\psi,*}(v,w), \\ &\dot{B}^*_{2,1} \times \dot{B}^*_{2,1} \times \dot{B}^*_{2,1} \to \dot{B}^*_{2,\infty} \\ \Rightarrow \quad & \\ &\dot{B}^*_{2,\infty} \times \dot{B}^*_{2,2} \times \dot{B}^*_{2,1} \to \dot{B}^*_{2,2} \\ \Rightarrow \quad & \\ &\dot{B}^*_{2,\infty} \times \dot{B}^*_{2,2} \times \dot{B}^*_{2,2} \to \dot{B}^*_{2,1}. \end{aligned} \tag{2.47}$$

Therefore it suffices to estimate the dyadic pieces R^* in any case.

We have to employ different arguments depending on the frequency size of each function. In the following, we always assume that ξ has size N, i.e., $|\xi| \sim N$. Let $\sigma' = \theta' = M = 0$ when considering R^0 or R^δ. Let $\sigma = 0$ when considering R^0 or R^ψ.

Case I: $J \sim K \gtrsim N$ This is the easiest case since the vw part is bounded in L^1. Let $b = \max(\theta - \alpha + \sigma, 0)$. If $b > 0$, then we have

$$\begin{aligned} \|\delta_1 \Psi\|_{L^{n/b}_x} \lesssim \|\delta_1 \Phi'\|_{L^{n/b}_x} &\lesssim (tN)^{\theta+\alpha} \|\delta_1 \Phi\|_{\dot{B}^{\theta+\alpha}_{n/b,1}} \\ &\lesssim |tN^2|^\theta N^{\alpha-\theta} (t^\alpha \|\delta_1 \Phi\|_{B^{2\alpha-\sigma}}), \end{aligned} \tag{2.48}$$

where we used the conditions $0 \le \theta + \alpha \le 1$ and $b \le n/2$. Similarly, we have

$$\|\Psi\|_{L^{n/b}_x} \lesssim |tN^2|^\theta N^{\alpha-\theta} (t^\alpha \|\Phi\|_{B^{2\alpha}}), \tag{2.49}$$

when $\sigma = 0$. If $b = 0$, then we have,

$$\begin{aligned} \|\Psi\|_{L^\infty_x} \lesssim \|\Phi'\|^{\theta/\alpha}_{L^\infty_x} &\lesssim \left\{ |tN|^{2\alpha} \|\Phi\|_{\dot{B}^{2\alpha}_{\infty,1}} \right\}^{\theta/\alpha} \\ &\lesssim |tN^2|^\theta (|t|^\alpha \|\Phi\|_{B^{2\alpha}})^{\theta/\alpha}, \end{aligned} \tag{2.50}$$

where we need $0 \le \theta \le \alpha$. Although we have a similar estimate for $\delta_1 \Psi$, we avoid to use it since it is sublinear for the difference $\delta_1 \Phi$. This is the reason why we assume (2.35) when S^* bears δ.

In both cases, we have

$$\|\Psi\|_{L^{n/b}_x} \lesssim |tN^2|^\theta N^{\sigma-b} d(\Phi), \tag{2.51}$$

when $\sigma = 0$. We estimate the ψ part by the Sobolev embedding as

$$\begin{aligned} \|\psi'\|_{L^{n/(\theta'-\sigma')}_x} \lesssim \|\psi'\|_{B^{\sigma'-\theta'}} &\lesssim (tN)^{\theta'} \|\psi\|_{B^{\sigma'}} \\ &\lesssim |tN^2|^{\theta'} N^{-\theta'} \|\psi\|_{B^{\sigma'}}, \end{aligned} \tag{2.52}$$

where we need $0 \le \theta' - \sigma' \le n/2$. Thus we obtain

$$\|\Psi^*_{I,M}\|_{L_x^{n/(b+\theta'-\sigma')}} \lesssim |tN^2|^{\theta+\theta'} N^{\sigma-b-\theta'} D^*, \tag{2.53}$$

where we put

$$\begin{aligned} &D^0 := d(\Phi), \quad D^\delta := d(\Phi_{0+}) t^\alpha \|\delta_1 \Phi\|_{B^{2\alpha-\sigma}},\\ &D^\psi := \|\psi\|_{B^{\sigma'}} D^0,\ D^{\psi,\delta} := \|\psi\|_{B^{\sigma'}} D^\delta, \end{aligned} \tag{2.54}$$

and we need $0 \le b + \theta' - \sigma' \le n$. Using this estimate and the Hölder and the Sobolev inequalities, we can estimate R^* as

$$\begin{aligned} \|R^*\|_{L^2} &\lesssim N^{n/2} \|\Psi^*_{I,M}(x, t\xi) v_J(x + t\xi) \overline{w}_K(x)\|_{L^\infty_{|\xi|\sim N} L^1_x}\\ &\lesssim N^{n/2} \sup_{|\xi|\sim N} \|\Psi^*_{I,M}\|_{L_x^{n/(b+\theta'-\sigma')}} \|v_J(x + t\xi)\overline{w}_K(x)\|_{L_x^{n/(n-b-\theta'+\sigma')}}\\ &\lesssim N^{n/2} |tN^2|^{\theta+\theta'} N^{\sigma-b-\theta'} D^* J^{-\beta-\gamma+b+\theta'-\sigma'} \|v_J\|_{\dot H^\beta} \|w_K\|_{\dot H^\gamma}\\ &\lesssim N^{n/2-\beta-\gamma+\sigma-\sigma'} |tN^2|^{\theta+\theta'} D^* \|v\|_{\dot H^\beta} \|w\|_{\dot H^\gamma}, \end{aligned} \tag{2.55}$$

where we used that $-\beta - \gamma + b + \theta' - \sigma' \le 0$ and $J \sim K \gtrsim N$. Thus we obtain the desired estimates in this Case I.

Case II: $I + M \gtrsim J, K, N$ This is the case where the phase term has the highest frequency. Then we can use the full strength of the above Lemma 2.1 to have spatial decay of Ψ. However, we have to assume $\beta + \gamma \le n/2$ here to get spatial decay only from the phase factor. Let $\beta' := \max(\beta, 0)$ and $\gamma' := \max(\gamma, 0)$. By the assumptions $\beta < n/2$, $\gamma < n/2$ and $0 < \beta + \gamma < n/2$, we have $0 < \beta' + \gamma' < n/2$. By Hölder's inequality and the Sobolev embedding, we have

$$\begin{aligned} \|R^*\|_{L^2} &\lesssim N^{n/2} \sup_{|\xi|\sim N} \|v_J\|_{\dot H^{\beta'}} \|w_K\|_{\dot H^{\gamma'}} \|\Psi^*_{I,M}\|_{L_x^{n/(\beta'+\gamma')}}\\ &\lesssim N^{n/2} J^{-\beta+\beta'} K^{-\gamma+\gamma'} \|v\|_{\dot H^\beta} \|w\|_{\dot H^\gamma} \|\Psi^*_{I,M}\|_{L^\infty_{|\xi|\sim N} L_x^{n/(\beta'+\gamma')}}\\ &\lesssim N^{n/2} (I+M)^{-\beta+\beta'-\gamma+\gamma'} \|v\|_{\dot H^\beta} \|w\|_{\dot H^\gamma} \|\Psi^*_{I,M}\|_{L^\infty_{|\xi|\sim N} L_x^{n/(\beta'+\gamma')}}. \end{aligned} \tag{2.56}$$

First we consider the case $M \ge I$, which can occur only for R^ψ and $R^{\psi,\delta}$. We estimate

$$\|\Psi^*_I \psi'_M\|_{L_x^{n/(\beta'+\gamma')}} \lesssim \|\Psi^*_I\|_{L_x^{n/b}} \|\psi'_M\|_{L_x^{n/(\beta'+\gamma'-b)}}, \tag{2.57}$$

where we need $\beta' + \gamma' \ge b$. The norm for Ψ is estimated by (2.48) or (2.51), and the norm for ψ is estimated by the Sobolev embedding as

$$\begin{aligned} \|\psi'_M\|_{L_x^{n/(\beta'+\gamma'-b)}} &\lesssim |tN|^{\theta'} \|\psi_M\|_{B^{\theta'}_{n/(\beta'+\gamma'-b),1}}\\ &\lesssim |tN^2|^{\theta'} N^{-\theta'} M^{\theta'-\sigma'-\beta'-\gamma'+b} \|\psi\|_{B^{\sigma'}}, \end{aligned} \tag{2.58}$$

where we need $\beta' + \gamma' - b \le n/2$. Then we obtain

$$\|\Psi^*_{I,M}\|_{L_x^{n/(\beta'+\gamma')}} \lesssim |tN^2|^{\theta+\theta'} N^{\sigma-b-\theta'} M^{\theta'-\sigma'-\beta'-\gamma'+b} D^*. \tag{2.59}$$

After plugging this into (2.56), the exponent of M is $\theta' - \sigma' - \beta - \gamma + b \le 0$, so that we may replace M with N. Then we obtain the desired estimate in this case.

Next we consider the case $I \ge M$. We estimate

$$\|\Psi_I^* \psi'_M\|_{L_x^{n/(\beta'+\gamma')}} \lesssim \|\Psi_I^*\|_{L_x^{n/(\beta'+\gamma')}} \|\psi'_M\|_{L_x^\infty}. \tag{2.60}$$

Then the norm for ψ is estimated by the Sobolev embedding as

$$\begin{aligned} \|\psi'_M\|_{L_x^\infty} &\lesssim (tN)^{\theta'} \|\psi_M\|_{B^{\theta'}} \lesssim |tN^2|^{\theta'} N^{-\theta'} M^{\theta'-\sigma'} \|\psi\|_{B^{\sigma'}}, \\ &\lesssim |tN^2|^{\theta'} N^{-\theta'} I^{\theta'-\sigma'} \|\psi\|_{B^{\sigma'}}, \end{aligned} \tag{2.61}$$

where we used the condition $\theta' - \sigma' \ge 0$. As for the Ψ part, we use Lemma 2.1 with sufficiently large m to obtain

$$\begin{aligned} &\|\delta_1 \Psi_I\|_{L_x^{n/(\beta'+\gamma')}} \\ &\lesssim |tN^2|^\theta N^{\sigma-\beta'-\gamma'} (N/I)^{\beta'+\gamma'-\theta-\sigma+\alpha} d(\Phi_{0+}) t^\alpha \|\delta_1 \Phi\|_{B^{2\alpha-\sigma}}, \end{aligned} \tag{2.62}$$

where we need $\beta' + \gamma' \le n/2$, $\theta + \sigma < \alpha + n/2$ and $\theta + \alpha \in [0,1]$. We have a similar estimate for Ψ_I. Thus we obtain

$$\begin{aligned} &\|\Psi_{I,M}^*\|_{L_x^{n/(\beta'+\gamma')}} \\ &\lesssim |tN^2|^{\theta+\theta'} N^{\sigma-\sigma'-\beta'-\gamma'} (N/I)^{\beta'+\gamma'+\sigma'-\theta-\theta'-\sigma+\alpha} D^*. \end{aligned} \tag{2.63}$$

When we plug this estimate into (2.56), the exponent of I becomes $-\alpha - \beta - \gamma + \theta + \theta' - \sigma' + \sigma \le 0$, so that we may replace I with N. Then we obtain the desired estimate in this case.

Case III: $I, J, M \lesssim N \sim K$ This is the case where only one term has the highest frequency of size N and that term is not the phase term. Then the spatial decay provided by the Sobolev embedding is too weak to treat the lower frequency terms, and so we need further decomposition in the Fourier space. Specifically, we decompose the high frequency term w into functions with Fourier support of size $I + M + J$. More precisely, let $\mathcal{K}$ be the set of disjoint cubes of size $I + M + J$ in $\mathbb{R}^n$ that are parallel to the axes such that the union of those cubes covers the whole $\mathbb{R}^n$. For each $\kappa \in \mathcal{K}$, w^κ denotes the Fourier restriction of w onto κ, and $\tilde{\kappa}$ denotes the cube of size $3(I + M + J)$ consisting 3^n cubes in $\mathcal{K}$ with κ as its center. Then

$$\int \Psi_{I,M}^*(x, t\xi) v_J(x + t\xi) \overline{w}_K^\kappa(x) e^{-ix\xi} dx, \tag{2.64}$$

is supported on $\tilde{\kappa}$, which can be verified by first freezing $t\xi$ as η and putting $\eta = t\xi$ after the Fourier transform. By this support property and the essential orthogonality of $\tilde{\kappa}$, we have

$$\begin{aligned}\|R^*\|_{L^2}^2 &= \left\|\sum_{\kappa\in\mathcal{K}} \int \Psi^*_{I,M}(x,t\xi)v_J(x+t\xi)\overline{w}^\kappa_K(x)e^{-ix\xi}dx\right\|_{L^2}^2 \\ &\lesssim \sum_{\kappa\in\mathcal{K}} \left\|\int \Psi^*_{I,M}v_J(x+t\xi)\overline{w}^\kappa_K(x)e^{-ix\xi}dx\right\|_{L^2}^2 \\ &\lesssim \sum_{\kappa\in\mathcal{K}} (I+M+J)^n\|\Psi^*_{I,M}v_J(x+t\xi)\overline{w}^\kappa_K(x)\|^2_{L^\infty_{|\xi|\sim N}(L^1_x)}.\end{aligned} \tag{2.65}$$

First we consider the case $I+M \le J$. Then the remaining argument is almost the same as in Case I. Indeed, by the same estimate as in (2.55), we have

$$\begin{aligned}&\|\Psi^*_{I,M}v_J(x+t\xi)\overline{w}^\kappa_K(x)\|_{L^1_x} \\ &\lesssim |tN^2|^{\theta+\theta'}N^{\sigma-b-\theta'}D^*J^{\beta'-\beta}\|v_J\|_{\dot{H}^\beta}K^{\gamma'-\gamma}\|w^\kappa_K\|_{\dot{H}^\gamma},\end{aligned} \tag{2.66}$$

where we choose $\beta',\gamma' \in [0,n/2]$ such that $\beta'+\gamma' = b+\theta'-\sigma' \in [0,n]$. Plugging this estimate into the above, we obtain

$$\begin{aligned}\|R^*\|_{L^2} &\lesssim |tN^2|^{\theta+\theta'}N^{\sigma-b-\theta'}D^*J^{n/2+\beta'-\beta}N^{\gamma'-\gamma} \\ &\quad\times \|v\|_{\dot{H}^\beta}\|w^\kappa_K\|_{L^2(\kappa\in\mathcal{K};\dot{H}^\gamma)} \\ &\lesssim |tN^2|^{\theta+\theta'}D^*N^{\sigma-b-\theta'+n/2+\beta'-\beta+\gamma'-\gamma}\|v\|_{\dot{H}^\beta}\|w\|_{\dot{H}^\gamma},\end{aligned} \tag{2.67}$$

where we used that $J \lesssim N \sim K$ and $n/2+\beta'-\beta \ge 0$. Since $\beta'+\gamma' = b+\theta'-\sigma'$, the above is the desired estimate in this case.

Next we consider the case $J \le I+M$. Then the remaining argument is similar to Case II, but not quite the same. Let $\beta' := \max(0,\beta)$ and $\gamma' := \max(0,\gamma)$. Then we have $\beta \le \beta' \le \max(\beta,\gamma)$, since $\beta+\gamma > 0$.

Suppose that $M \ge I$, which is possible only when S^* carries ψ. Then we have $J, I \lesssim M \lesssim N \sim K$. As in (2.56) and (2.59), we estimate

$$\begin{aligned}&\|\Psi^*_{I,M}v_J(x+t\xi)\overline{w}^\kappa_K(x)\|_{L^1_x} \\ &\lesssim \|\Psi^*_{I,M}\|_{L_x^{n/(\beta'+\gamma')}}J^{\beta'-\beta}\|v\|_{\dot{H}^\beta}K^{\gamma'-\gamma}\|w^\kappa_K\|_{\dot{H}^\gamma} \\ &\lesssim |tN^2|^{\theta+\theta'}N^{\sigma-b-\theta'+\gamma'-\gamma}M^{\theta'-\sigma'-\beta'-\gamma'+b}D^*J^{\beta'-\beta}\|v\|_{\dot{H}^\beta}\|w^\kappa_K\|_{\dot{H}^\gamma}.\end{aligned} \tag{2.68}$$

When we put this into (2.65), the power of M becomes $n/2+\theta'-\sigma'-\beta'-\gamma'+b \ge 0$, so that we may replace M with N. We may replace J with N also. Then we obtain the desired result.

Now we proceed to the final remaining case $J, M \lesssim I \lesssim N \sim K$. By the Hölder and the Sobolev inequalities, we have

$$\begin{aligned}&\|\Psi^*_{I,M}v_J(x+t\xi)\overline{w}^\kappa_K(x)\|_{L^1_x} \\ &\lesssim \|\Psi^*_{I,M}\|_{L^\infty_{|\xi|\sim N}L_x^{n/\beta'}}\|v_J\|_{L_x^{2n/(n-2\beta')}}\|w^\kappa_K\|_{L^2} \\ &\lesssim N^{-\gamma}J^{\beta'-\beta}\|\Psi^*_{I,M}\|_{L_x^{n/\beta'}}\|v\|_{\dot{H}^\beta}\|w^\kappa_K\|_{\dot{H}^\gamma}.\end{aligned} \tag{2.69}$$

We estimate the ψ part in L_x^∞, while we use Lemma 2.1 with $m = n/2$ to estimate the Ψ part as

$$\|\delta_1 \Psi_I\|_{L_x^{n/\beta'}} \lesssim |tN^2|^\theta N^{\sigma-\beta'} (N/I)^{\beta'-\theta-\sigma+\alpha n/2} D^\delta, \tag{2.70}$$

where we need $\beta' \le n/2$, $(\alpha-1)n/2 + \beta' + \alpha < \theta + \sigma < \alpha + n/2$ and $\theta + \alpha \in [0,1]$. We have a similar estimate for Ψ_I, but with the weaker lower bound condition $(\alpha-1)n/2 + \beta' < \theta$. Thus we obtain

$$\|\Psi_{I,M}^*\|_{L_x^{n/\beta'}} \lesssim |tN^2|^{\theta+\theta'} N^{\sigma-\sigma'-\beta'} (N/I)^{\beta'-\theta-\theta'+\sigma'-\sigma+\alpha n/2} D^*, \tag{2.71}$$

where we used (2.61). By (2.65), (2.69) and (2.71), we have

$$\begin{aligned} \|R^*\|_{L^2} \lesssim\; & I^{n/2} |tN^2|^{\theta+\theta'} N^{\sigma-\sigma'-\beta'-\gamma} J^{\beta'-\beta} (N/I)^{\beta'-\theta-\theta'+\sigma'-\sigma+\alpha n/2} \\ & \times D^* \|v\|_{\dot H^\beta} \|w_K^\kappa\|_{L^2(\kappa\in\mathcal{K};\dot H^\gamma)}, \end{aligned} \tag{2.72}$$

where the power of I is $n/2(1-\alpha) + \theta - \beta' + \theta' - \sigma' + \sigma \ge 0$ by our assumptions. So we may replace I with N and J with N. Then we obtain the desired estimate in this case.

Case III′: $I, K, M \lesssim N \sim J$ This case is reduced to the previous one by the symmetry of v and w in the operators S^*.

We have exhausted all the cases where $\int \Psi_{I,M}^* v_J w_K e^{-ix\xi} dx$ interacts V_N, which can be easily checked as follows. By the Fourier support property, we have $N \lesssim \max(I,J,K,M)$, and if the maximum is essentially bigger than N, then it must be essentially attained by at least two of I,J,K,M. If $\max(I,J,K,M) < 100(I+M)$, then we are in case II. Otherwise, the maximum is attained by J or K. If it is much bigger than N, we come into the case I. If it is essentially the same size as N, then we arrive at case III or III′, depending whether the maximum is K or J. □

We will use the above lemma with $1/2 < \nu < 1$, $n \ge 3$, $1 - \nu/2 < s < 3/4$ and $\alpha = s - \nu/2$. Then we have

$$1 - \nu < \alpha < 1/2, \tag{2.73}$$

so that $d(\Phi)$ can be bounded. Since we have

$$(\alpha - 1)n/2 < -s, \tag{2.74}$$

the lower bound of θ in (2.31) will never bother us, though it could if we would try the lower dimensional $n \le 2$ case.

3 Phase-free estimates

In this section, we derive a few basic estimates to treat those terms that do not include the phase function. They are actually variants of those which played the central roles in [12]. The main term after the phase elimination can be given by using the following operator

$$T^U(A) := U(-t)AU(t) - A, \tag{3.1}$$

where $U(-t)AU(t)$ is defined by $U(-t)AU(t)\varphi := U(-t)(AU(t)\varphi)$ for any space-time real-valued function A. Then the following multilinear operator is a counterpart of S^ψ for the phase-free part $T^U(A)$.

$$T^\psi(A; v, w) := \langle U(-t)AU(t)v, \overline{\psi} w\rangle_{L^2} - \langle U(-t)AU(t)\psi v, w\rangle_{L^2}. \tag{3.2}$$

Lemma 3.1 *(i) Let $\beta, \gamma < n/2$, $\beta + \gamma > 0$, $\theta_0, \theta_1 \in [0, 1]$ and $\theta = \theta_0 + \theta_1 \le 1$. Then we have*

$$\|uv\|_{\dot B^{\beta+\gamma-n/2}_{2,1/\theta}} \lesssim \|u\|_{\dot B^{\beta}_{2,1/\theta_0}} \|v\|_{\dot B^{\gamma}_{2,1/\theta_1}}. \tag{3.3}$$

(ii) Let $\sigma \le 0$ and $-n/2 - \sigma < \beta < n/2$. Then we have

$$\|Au\|_{\dot H^{\beta+\sigma}} \lesssim \|A\|_{B^\sigma} \|u\|_{\dot H^\beta}. \tag{3.4}$$

(iii) Let $\beta \in \mathbb{R}$ and $\theta \in [0, 1]$. Then we have

$$\|(U(t) - I)u\|_{\dot H^{\beta+2\theta}} \lesssim |t|^\theta \|u\|_{\dot H^\beta}. \tag{3.5}$$

(iv) Let $\theta \in [0, 1]$, $\sigma \le \theta$ and $-n/2 + 2\theta - \sigma < \beta < n/2$. Then we have

$$\|T^U(A)u\|_{\dot H^{\beta-2\theta+\sigma}} \lesssim |t|^\theta \|A\|_{B^\sigma} \|u\|_{\dot H^\beta} \tag{3.6}$$

In case $\sigma < \theta$, we may replace the norm B^σ for A with $\dot B^{n/2+\sigma}_{2,\infty}$.

(v) Let $\theta \in [0, 1]$, $\sigma \le \theta$, $\sigma' < \theta$, $\sigma + \sigma' + \beta + \gamma = 2\theta$, and $\beta, \gamma < n/2$. Then we have

$$|T^\psi(A; v, w)| \lesssim |t|^\theta \|A\|_{B^\sigma} \|\psi\|_{B^{n/2+\sigma'}_{2,\infty}} \|v\|_{\dot H^\beta} \|w\|_{\dot H^\gamma}. \tag{3.7}$$

Proof. (i) : By the bilinear real interpolation, it suffices to consider the dyadic pieces of Littlewood-Paley decomposition. By the Sobolev and the Hölder inequalities, we have

$$\begin{aligned} \|\varphi_K * ((\varphi_I * u)(\varphi_J * v))\|_{L^2} &\lesssim \min(I, J, K)^{n/2} \|\varphi_I * u\|_{L^2} \|\varphi_J * v\|_{L^2} \\ &\lesssim \min(I, J, K)^{n/2} I^{-\beta} J^{-\gamma} \|u\|_{\dot H^\beta} \|v\|_{\dot H^\gamma}. \end{aligned} \tag{3.8}$$

By the Fourier support property, we have $I \lesssim J \sim K$, $J \lesssim K \sim I$ or $K \lesssim I \sim J$. In any case, we have

$$\min(I, J, K)^{n/2} I^{-\beta} J^{-\gamma} \lesssim K^{n/2-\beta-\gamma}, \tag{3.9}$$

since $\beta, \gamma < n/2$ and $\beta + \gamma > 0$. Thus we obtain the desired estimate for any dyadic pieces, from which the original estimate follows.

(ii) : This estimate follows from (i) except the borderline case $\sigma = 0$, for which we give another proof. By duality, it suffices to consider the case $\beta \geq 0$. In the Fourier space, we have

$$\|Au\|_{\dot{H}^\beta} \lesssim \left\| |\xi|^\beta \int \tilde{A}(\xi - \eta)\tilde{u}(\eta)d\eta \right\|_{L^2_\xi}. \tag{3.10}$$

We split the η integral into those on the region $R_1 := \{|\xi| \lesssim |\eta|\}$ and $R_2 := \{|\eta| \ll |\xi| \sim |\xi - \eta|\}$. In the region R_1, we may replace the ξ weight by $|\eta|^\beta$ so that we can estimate the above norm by

$$\|\tilde{A}\|_{L^1} \||\xi|^\beta \tilde{u}\|_{L^2} \lesssim \|A\|_{B^0} \|u\|_{\dot{H}^\beta}. \tag{3.11}$$

In the region R_2, we may replace the ξ weight by $|\xi - \eta|^\beta$. Then, by the generalized Hölder and Young inequalities, we can bound the above norm by

$$\||\xi|^\beta \tilde{A}\|_{L^{n/(n-\beta),\infty}} \|\tilde{v}\|_{L^{2n/(n+2\beta),2}} \lesssim \|A\|_{B^0} \|v\|_{\dot{H}^\beta}. \tag{3.12}$$

Thus we obtain the desired result.

(iii) : This estimate immediately follows from explicit calculation of the Fourier transform.

(iv) : Let $\gamma := 2\theta - \beta - \sigma$. Then we have $\gamma < n/2$ and $2\theta = \beta + \gamma + \sigma$. By duality and the Plancherel identity, it suffices to show that

$$\begin{aligned} |\langle T^U(A)u, v\rangle_{L^2}| &\sim \left| \iint (e^{it(|\xi|^2 - |\eta|^2)/2} - 1)\tilde{A}(\xi - \eta)\tilde{u}(\eta)\overline{\tilde{v}(\xi)} d\eta d\xi \right| \\ &\lesssim |t|^\theta \|A\|_{B^\sigma} \|u\|_{\dot{H}^\beta} \|v\|_{\dot{H}^\gamma}. \end{aligned} \tag{3.13}$$

We split the double integral region into three regions: $R_1 := \{|\xi - \eta| \lesssim |\xi| \sim |\eta|\}$, $R_2 := \{|\xi| \lesssim |\xi - \eta| \sim |\eta|\}$ and $R_3 := \{|\eta| \lesssim |\xi| \sim |\xi - \eta|\}$. In the first region R_1, we have

$$|e^{it(|\xi|^2 - |\eta|^2)/2} - 1| \lesssim |t|^\theta |\xi - \eta|^\theta |\xi + \eta|^\theta \lesssim |t|^\theta |\xi - \eta|^\sigma |\eta|^\beta |\xi|^\gamma, \tag{3.14}$$

where we needed the assumption $\sigma \leq \theta$, so that the above integral can be estimated

$$\begin{aligned} \iint_{R_1} |\cdots| &\lesssim |t|^\theta \||\xi|^\sigma \tilde{A}\|_{L^1} \||\xi|^\beta \tilde{u}(\xi)\|_{L^2} \||\xi|^\gamma \tilde{v}(\xi)\|_{L^2} \\ &\lesssim |t|^\theta \|A\|_{B^\sigma} \|u\|_{\dot{H}^\beta} \|v\|_{\dot{H}^\gamma}, \end{aligned} \tag{3.15}$$

as desired. In the second region R_2 we have

$$|e^{it(|\xi|^2 - |\eta|^2)/2} - 1| \lesssim |t|^\theta |\xi - \eta|^{\sigma+\gamma} |\eta|^\beta, \tag{3.16}$$

then we obtain from the generalized Young and Hölder inequalities,

$$\begin{aligned} \iint_{R_2} |\cdots| &\lesssim |t|^\theta \||\xi|^{\sigma+\gamma} \tilde{A}\|_{L^{n/(n-\gamma),\infty}} \||\xi|^\beta \tilde{u}(\xi)\|_{L^2} \|\tilde{v}(\xi)\|_{L^{2n/(n+2\gamma),2}} \\ &\lesssim |t|^\theta \|A\|_{B^\sigma} \|u\|_{\dot{H}^\beta} \|v\|_{\dot{H}^\gamma}, \end{aligned} \tag{3.17}$$

if $\gamma \geq 0$. In the case $\gamma < 0$, we have (3.14), so the above argument in the region R_1 works also in this region. The remaining region R_3 is treated in the same way as R_2. Except the borderline case, we may replace the 1-Besov norm with the ∞-Besov by the bilinear real interpolation.

(v): In the Fourier space, we have

$$\begin{aligned} T^\psi(A; v, w) = c \iiint (e^{it(|\xi|^2-|\eta|^2)/2} - e^{it(|\zeta|^2-|\zeta-\xi+\eta|^2)/2}) \\ \times \tilde{A}(\xi-\eta)\tilde{v}(\eta)\overline{\tilde{\psi}(\zeta-\xi)\tilde{w}(\zeta)} d\xi d\eta d\zeta, \end{aligned} \tag{3.18}$$

and the phase factor can be rewritten as

$$e^{it(|\xi|^2-|\eta|^2)/2}(1 - e^{-it(\xi-\eta)(\xi-\zeta)}), \tag{3.19}$$

so that it can be bounded by

$$|t|^\theta|\xi-\eta|^\theta|\xi-\zeta|^\theta, \tag{3.20}$$

Let $\hat{A} := \mathcal{F}^{-1}|\tilde{A}|$, etc. By the above estimate on the phase, we have

$$\begin{aligned} |T^\psi(A; v, w)| &\lesssim |t|^\theta \langle |\xi|^\theta |\tilde{A}| * |\tilde{v}|, |\xi|^\theta |\tilde{\psi}| * |\tilde{w}| \rangle_{L^2} \\ &\lesssim |t|^\theta \langle (|\nabla|^\theta \hat{A})\hat{v}, (|\nabla|^\theta \hat{\psi})\hat{w} \rangle_{L^2} \\ &\lesssim |t|^\theta \left\| (|\nabla|^\theta \hat{A})\hat{v} \right\|_{\dot{H}^{\beta+\sigma-\theta}} \left\| (|\nabla|^\theta \hat{\psi})\hat{w} \right\|_{\dot{H}^{\gamma+\sigma'-\theta}} \\ &\lesssim |t|^\theta \|\hat{A}\|_{B^\sigma} \|\hat{v}\|_{\dot{H}^\beta} \|\hat{\psi}\|_{B^{n/2+\sigma'}_{2,\infty}} \|\hat{w}\|_{\dot{H}^\gamma} \\ &\lesssim |t|^\theta \|A\|_{B^\sigma} \|v\|_{\dot{H}^\beta} \|\psi\|_{B^{n/2+\sigma'}_{2,\infty}} \|w\|_{\dot{H}^\gamma}, \end{aligned} \tag{3.21}$$

as desired, where we used (i) and (ii) in the fourth inequality. □

4 Energy estimate

In this section, we derive an L^2 bound of difference of the modified field w. We consider only positive small time $0 < t \lesssim 1$. Assume that $w = w_k$ $(k = 0, 1)$ solves

$$2i\dot{w} + t^{\nu-2}T(A, \phi)w = 0, \tag{4.1}$$

with $A = A_k$ and $\phi = \phi_k$, where A_k is a real-valued space-time function and

$$\begin{aligned} T(A, \phi) &= e^{-i\Phi}\{U(-t)AU(t) - V(\phi)\}e^{i\Phi}, \\ \Phi(\phi) &= V(\phi)\frac{t^{\nu-1}}{2(\nu-1)}. \end{aligned} \tag{4.2}$$

We abbreviate $T_k := T(A_k, \phi_k)$. We decompose the operator T as

$$T = T^\Phi + T^U + T^V, \tag{4.3}$$

where

$$\begin{aligned} T^{\Phi} &:= e^{-i\Phi}U(-t)AU(t)e^{i\Phi} - U(-t)AU(t), \\ T^{U} &:= U(-t)AU(t) - A, \\ T^{V} &:= A - V(\phi). \end{aligned} \tag{4.4}$$

Then we have a general identity

$$\langle T^{\Phi}w, v\rangle_{L^2} = \langle A, S^0(\Phi; w, v)\rangle_{L^2}. \tag{4.5}$$

Now the difference of the solutions w_1 and w_0 satisfies

$$2i\delta_1\dot{w} + t^{\nu-2}\{T_0\delta_1 w + (\delta_1 T)w_1\} = 0, \tag{4.6}$$

so that by the energy identity we have

$$\partial_t\|\delta_1 w\|_{L^2}^2 = t^{\nu-2}\Re\langle i(\delta_1 T)w_1, \delta_1 w\rangle_{L^2}. \tag{4.7}$$

We can rewrite this by using (4.5) as

$$\begin{aligned} &\langle(\delta_1 T)w_1, \delta_1 w\rangle_{L^2} \\ &= \langle\delta_1 A, S^0(\Phi_0; w_1, \delta_1 w)\rangle_{L^2} + \langle A_1, S^{\delta}(\Phi_0, \Phi_1; w_1, \delta_1 w)\rangle_{L^2} \\ &\quad + \langle T^U(\delta_1 A)w_1, \delta_1 w\rangle_{L^2} + \langle(\delta_1 A - \delta_1 V(\phi_*))w_1, \delta_1 w\rangle_{L^2}. \end{aligned} \tag{4.8}$$

We apply Lemma 2.2 (i) to the first term with $\theta = \alpha - \kappa$, $\beta = s'$, $\gamma = 0$ and $\kappa > 0$ sufficiently small. The conditions required in the lemma can be satisfied if $n \geq 2$, $\alpha < 1/2$, $0 < s' < 1$ and $\kappa > 0$ is sufficiently small. Then we obtain

$$\begin{aligned} |\langle\delta_1 A, S^0(\Phi_0; w_1, \delta_1 w)\rangle_{L^2}| &\lesssim \|\delta_1 A\|_{B^{2\alpha-s'-2\kappa}}\|S^0(w_1, \delta_1 w)\|_{B_*^{s'-2\alpha+2\kappa}} \\ &\lesssim t^{\alpha-\kappa}d(\Phi_0)\|\delta_1 A\|_{B^{2\alpha-s'-2\kappa}}\|w_1\|_{\dot{H}^{s'}}\|\delta_1 w\|_{L^2}. \end{aligned} \tag{4.9}$$

For the second term, we take $\theta = \alpha - 2\kappa$, $\sigma = s' + \kappa$, $\beta = s'$ and $\gamma = 0$ in Lemma 2.2 (ii). Then we obtain

$$\begin{aligned} &|\langle A_1, S^{\delta}(\Phi_0, \Phi_1; w_1, \delta_1 w)\rangle_{L^2}| \\ &\lesssim \|A_1\|_{B^{2\alpha-3\kappa}}\|S^{\delta}(\Phi_0, \Phi_1; w_1, \delta_1 w)\|_{B_*^{-2\alpha+3\kappa}} \\ &\lesssim \|A_1\|_{B^{2\alpha-3\kappa}}t^{\alpha-2\kappa}d(\Phi_{0+})t^{\alpha}\|\delta_1\Phi\|_{B^{2\alpha-s'-\kappa}}\|w_1\|_{\dot{H}^{s'}}\|\delta_1 w\|_{L^2}. \end{aligned} \tag{4.10}$$

For the T^U part, we have

$$\begin{aligned} |\langle T^U(\delta_1 A)w_1, \delta_1 w\rangle_{L^2}| &\lesssim \|T^U(\delta_1 A)w_1\|_{L^2}\|\delta_1 w\|_{L^2} \\ &\lesssim t^{\alpha-\kappa}\|\delta_1 A\|_{B^{2\alpha-s'-2\kappa}}\|w_1\|_{\dot{H}^{s'}}\|\delta_1 w\|_{L^2}, \end{aligned} \tag{4.11}$$

where we used Lemma 3.1 (iv) in the second inequality, so we need $s' \geq \alpha - \kappa$, which is satisfied if $s' \geq 1/2$.

For the T^V part, we use Lemma 3.1 (ii) to have

$$\begin{aligned} |\langle(\delta_1 T^V)w_1, \delta_1 w\rangle_{L^2}| &\lesssim \|(\delta_1 A - \delta_1 V(\phi_*))w_1\|_{L^2}\|\delta_1 w\|_{L^2} \\ &\lesssim \|\delta_1 A - \delta_1 V(\phi_*)\|_{B^{-s'}}\|w_1\|_{\dot{H}^{s'}}\|\delta_1 w\|_{L^2}. \end{aligned} \tag{4.12}$$

Putting these estimates together, we obtain

Lemma 4.1 *Let $1/2 < \nu < 1$, $n \geq 3$, $1-\nu < \alpha = s - \nu/2 < 1/2$, and $1/2 \leq s' < 1$. Assume that $w = w_k$, $k = 0, 1$ satisfies (4.1) with $\phi = \phi_k$ and real-valued $A = A_k$. Let $\kappa > 0$ be sufficiently small depending on ν, α, s' and n. Then we have for $0 < t \lesssim 1$,*

$$\begin{aligned} &|\partial_t\|w_1 - w_0\|_{L^2}| \\ &\lesssim t^{\nu-2}\|w_1\|_{\dot{H}^{s'}}\Big\{\|A_1 - V(\phi_1) - A_0 + V(\phi_0)\|_{B^{-s'}} \\ &\quad + t^{\alpha-2\kappa}(D(\phi_0) + D(\phi_1)) \\ &\qquad \times (\|A_1 - A_0\|_{B^{2\alpha-s'-4\kappa}} + \|A_1\|_{B^{2\alpha-3\kappa}}t^{\alpha}\|\Phi_1 - \Phi_0\|_{B^{2\alpha-s'-\kappa}})\Big\}, \end{aligned} \tag{4.13}$$

where we denote

$$D(\phi) := \|\phi\|_{H^s}^m + 1, \tag{4.14}$$

with sufficiently large m depending on ν, α and s'.

We can bound $d(\Phi)$ by $D(\phi)$ since

$$\|V(\phi)\|_{B^{2\alpha}} \lesssim \|\phi\|_{\dot{H}^s}^2 \tag{4.15}$$

by (3.3), if $\alpha = s - \nu/2$. From now on, we fix $\alpha = s - \nu/2$.

By the same argument, we can prove the following.

Lemma 4.2 *Let $1/2 < \nu < 1$, $n \geq 3$, $1-\nu < \alpha = s - \nu/2 < 1/2$, and $1/2 \leq s' < 1$. Then we have for $0 < t \lesssim 1$,*

$$\begin{aligned} &\|T(A,\phi)w\|_{L^2} \\ &\lesssim \|w\|_{\dot{H}^{s'}}(t^{\alpha-2\kappa}D(\phi)\|A\|_{B^{2\alpha-s'-4\kappa}} + \|A - V(\phi)\|_{B^{-s'}}), \end{aligned} \tag{4.16}$$

where $\kappa > 0$ and $D(\phi)$ is as in the above lemma.

This implies that $\dot{w} \in L_t^1 L_x^2$.

5 Potential estimates

In this section, we derive a few estimates on the potential term $V(u^*) = V(U(t) e^{i\Phi}w)$. As in [3, 4, 12], we should derive a decay estimate (or convergence) at $t = +0$ for the potential by using the equation. Otherwise, if we would regard $V(u^*) - V(w(0))$ simply as a multiplication to dominate it by $w(t) - w(0)$, then we could get a closed estimate only for small data and only for $\nu = 1$. As was indicated by the estimates in the previous section, we need to estimate difference of the potentials also. In this section, we fix $\alpha = s - \nu/2$.

Let $A_k = V(U(t)e^{i\Phi_k} w_k)$ for $k = 0, 1$, where $\Phi_k = \Phi(\phi_k)$. First we derive a bound on the potential. Here we do not need the equation. We decompose $A = V(U(t)e^{i\Phi} w)$ as

$$V(U(t)e^{i\Phi} w) = K_V * S^0(\Phi; w, w) + K_V * S(w, w). \tag{5.1}$$

For the difference, we have

$$\begin{aligned} \delta_1 A =& K_V * \delta_1 S^0(\Phi_0; w_*, w_*) + K_V * \delta_1 S(w_*, w_*) \\ &+ K_V * S^\delta(\Phi_0, \Phi_1; w_1, w_1). \end{aligned} \tag{5.2}$$

For the S^0 part, we use Lemma 2.2 (i) with $\theta = 0$, $\beta = \gamma = s$ and $\alpha = s - \nu/2$. Then we obtain

$$\|K_V * S^0(w, w)\|_{B^{2s-\nu}} \sim \|S^0(w, w)\|_{B^{2s-n}} \lesssim d(\Phi)\|w\|^2_{\dot{H}^s}. \tag{5.3}$$

Similarly, choosing $\theta = 0$, $\beta = s$ and $\gamma = 0$, we obtain

$$\|K_V * \delta_1 S^0(\Phi_0; w_*, w_*)\|_{B^{2\alpha-s}} \lesssim d(\Phi_0)\|w_{0+}\|_{\dot{H}^s}\|\delta_1 w\|_{L^2}. \tag{5.4}$$

The S part can be easily estimated by using (3.3) as

$$\|K_V * S(w, w)\|_{B^{2\alpha}} \sim \|S(w, w)\|_{B^{2s-n}} \lesssim \|U(t)w\|^2_{\dot{H}^s} \sim \|w\|^2_{\dot{H}^s}, \tag{5.5}$$

and similarly,

$$\|K_V * \delta_1 S(w_*, w_*)\|_{B^{2\alpha-s}} \lesssim \|w_{0+}\|_{H^s}\|\delta_1 w\|_{L^2}. \tag{5.6}$$

A lower Besov norm is easily estimated by the Sobolev embedding as

$$\begin{aligned} \|V(U(t)e^{i\Phi} w)\|_{\dot{B}^{n/2-\nu}_{2,\infty}} &\lesssim \|V(U(t)e^{i\Phi} w)\|_{\dot{B}^{n-\nu}_{1,\infty}} \\ &\lesssim \left\||U(t)e^{i\Phi} w|^2\right\|_{L^1} \lesssim \|w\|^2_{L^2}. \end{aligned} \tag{5.7}$$

Similarly we have

$$\begin{aligned} \|K_V * \delta_1 S^0(\Phi_0; w_*, w_*) + K_V * \delta_1 S(w_*, w_*)\|_{\dot{B}^{n/2-\nu}_{2,\infty}} \\ \lesssim \|w_{0+}\|_{L^2}\|\delta_1 w\|_{L^2}. \end{aligned} \tag{5.8}$$

For the term of the phase change, we use Lemma 2.2 (ii) with $\theta = 0$, $\beta = \gamma = s$, $\alpha = s - \nu/2$, $\sigma = s$ to obtain

$$\begin{aligned} \|K_V * S^\delta(\Phi_0, \Phi_1; w_1, w_1)\|_{B^{2\alpha-s}} &\lesssim d(\Phi_{0+})t^\alpha \|\delta_1 \Phi\|_{B^{2\alpha-s}}\|w_1\|^2_{\dot{H}^s} \\ &\lesssim d(\Phi_{0+})\|\phi_{0+}\|_{H^s}\|\delta_1 \phi\|_{L^2}\|w_1\|^2_{\dot{H}^s}, \end{aligned} \tag{5.9}$$

where we used (3.3) to estimate $\delta_1\Phi$ as

$$\begin{aligned} t^{\alpha}\|\delta_1\Phi\|_{B^{2\alpha-s}} &\lesssim \|(\phi_0+\phi_1)\overline{\delta_1\phi}\|_{\dot{B}^{s-n/2}_{2,1}} \\ &\lesssim \|\phi_{0+}\|_{\dot{H}^s}\|\delta_1\phi\|_{L^2}. \end{aligned} \tag{5.10}$$

Repeating this argument with $\theta=0$, $\beta=\gamma=\nu/4+\kappa<s$ and $\sigma=s$, we obtain a lower Besov bound

$$\begin{aligned} &\|K_V{*}S^{\delta}(\Phi_0,\Phi_1;w_1,w_1)\|_{B^{-\alpha-\nu+2\kappa}} \\ &\quad\lesssim d(\Phi_{0+})\|\phi_{0+}\|_{H^s}\|\delta_1\phi\|_{L^2}\|w_1\|^2_{\dot{H}^{\nu/4+\kappa}}. \end{aligned} \tag{5.11}$$

If κ is sufficiently small, we have $-\alpha-\nu+2\kappa<-\nu$.

Gathering all estimates, we obtain the following.

Lemma 5.1 *Let $1/2<\nu<1$, $n\geq 3$ and $1-\nu/2<s<3/4$. Let $A_k=V(U(t)e^{i\Phi_k}w_k)$ with $\Phi_k=\Phi(\phi_k)$ for $k=0,1$. Then we have*

$$\begin{aligned} &\|A_k\|_{B^{-\nu}\cap B^{2s-\nu}} \lesssim D(\phi_k)\|w_k\|^2_{H^s}, \\ &\|A_1-A_0\|_{B^{-\nu}\cap B^{s-\nu}} \\ &\lesssim (D(\phi_0)+D(\Phi_1))(\|w_0\|_{H^s}+\|w_1\|_{H^s})(\|w_1-w_0\|_{L^2}+\|\phi_1-\phi_0\|_{L^2}), \end{aligned} \tag{5.12}$$

where D is as defined in (4.14).

Next we proceed to the decay estimate of the potential at $t=+0$. Now suppose that w_k solves

$$2i\dot{w}_k+t^{\nu-2}T(A'_k,\phi'_k)w_k=0, \tag{5.13}$$

and denote $\Phi'_k:=\Phi(\phi'_k)$. We decompose

$$V(U(t)e^{i\Phi}w)-V(w(0))=V^{\Phi}(\Phi,w)+V^U(w)+V^w(w), \tag{5.14}$$

where

$$\begin{aligned} V^{\Phi}(\Phi,w)&:=K_V*S^0(\Phi;w,w), \\ V^U(w)&:=V(U(t)w)-V(w), \\ V^w(w)&:=V(w)-V(w(0)). \end{aligned} \tag{5.15}$$

For the difference, we have

$$\delta_1 A-\delta_1 V(w_*(0))=\delta_1 V^{\Phi}(\Phi_*,w_*)+\delta_1 V^U(w_*)+\delta_1 V^w(w_*), \tag{5.16}$$

where the first term is further decomposed as

$$\delta_1 V^{\Phi}(\Phi_*,w_*)=K_V*S^{\delta}(\Phi_0,\Phi_1;w_1,w_1)+K_V*\delta_1 S^0(\Phi_0;w_*,w_*). \tag{5.17}$$

We use the equation of w only for the V^w part, since the other two parts already have decay properties. In fact, we can use Lemma 2.2 (i) with $\theta = \alpha - \kappa$ and $\beta = \gamma = s$, where $\kappa > 0$ is sufficiently small, to obtain

$$\|V^\Phi(w)\|_{B^{2\kappa}} \lesssim \|S^0(w,w)\|_{\dot{B}^{2\kappa+\nu-n/2}_{2,1}} \lesssim t^{\alpha-\kappa} d(\Phi)\|w\|^2_{\dot{H}^s}. \tag{5.18}$$

For $\delta_1 V^\Phi$, we can use Lemma 2.2 (ii) with $\theta = \alpha - \kappa$, $\beta = \gamma = s - \kappa$, $\sigma = s$ for the first term and (i) with $\beta = s - 2\kappa$, $\gamma = 0$ for the second term to obtain

$$\begin{aligned}\|\delta_1 V^\Phi(\Phi_*, w_*)\|_{B^{-s}} &\lesssim t^{\alpha-\kappa} d(\Phi_{0+}) t^\alpha \|\delta_1 \Phi\|_{B^{2\alpha-s}} \|w_1\|^2_{\dot{H}^{s-\kappa}} \\ &\quad + t^{\alpha-\kappa} d(\Phi_0)\|w_{0+}\|_{\dot{H}^{s-2\kappa}} \|\delta_1 w\|_{L^2} \\ &\lesssim t^{\alpha-\kappa}(d(\Phi_0) + d(\Phi_{0+})\|\phi_{0+}\|_{\dot{H}^s})\|w_{0+}\|^{[1,2]}_{H^{s-\kappa}} \\ &\qquad \times (\|\delta_1 w\|_{L^2} + \|\delta_1 \phi\|_{L^2}).\end{aligned} \tag{5.19}$$

The V^U part is treated by Lemma 3.1 as

$$\begin{aligned}\|V^U(w)\|_{B^{2\kappa}} &\lesssim \|(U(t)-I)w\overline{(U(t)+I)w}\|_{\dot{B}^{2\kappa+\nu-n/2}_{2,1}} \\ &\lesssim \|(U(t)-I)w\|_{\dot{H}^{2\kappa+\nu-s}} \|(U(t)+I)w\|_{\dot{H}^s} \\ &\lesssim t^{\alpha-\kappa}\|w\|^2_{\dot{H}^s}.\end{aligned} \tag{5.20}$$

We use the duality to estimate $\delta_1 V^U$, V^w and $\delta_1 V^w$. For any real-valued Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\langle V(U(t)w) - V(w), \psi\rangle_{L^2} = \langle T^U(K_V * \psi)w, w\rangle_{L^2}. \tag{5.21}$$

So we can estimate $\delta_1 V^U$ by using Lemma 3.1 as

$$\begin{aligned}|\langle \delta_1 V^U(w_*), \psi\rangle_{L^2}| &\lesssim \|T^U(K_V*\psi)\delta_1 w\|_{\dot{H}^{-s}} \|w_{0+}\|_{\dot{H}^s} \\ &\quad + \|T^U(K_V*\psi)w_{0+}\|_{L^2}\|\delta_1 w\|_{L^2} \\ &\lesssim t^\alpha \|K_V * \psi\|_{B^{n/2+s-\nu}_{2,\infty}} \|w_{0+}\|_{\dot{H}^s}\|\delta_1 w\|_{L^2} \\ &\lesssim t^\alpha \|\psi\|_{B^s_*}\|w_{0+}\|_{\dot{H}^s}\|\delta_1 w\|_{L^2},\end{aligned} \tag{5.22}$$

which implies by duality that

$$\|\delta_1 V^U(w_*)\|_{B^{-s}} \lesssim t^\alpha \|w_{0+}\|_{\dot{H}^s}\|\delta_1 w\|_{L^2}. \tag{5.23}$$

As for V^w, we have from the equation of w,

$$\partial_t |w|^2 = \Re\,(2i\dot{w}, iw)_{\mathbb{C}} = -t^{\nu-2}\Re\left(U(-t)A'U(t)e^{i\Phi'}w, ie^{i\Phi'}w\right)_{\mathbb{C}}. \tag{5.24}$$

Taking the L^2 coupling with the test function ψ, we obtain

$$\begin{aligned}\partial_t\langle |w|^2, \psi\rangle_{L^2} &= -t^{\nu-2}\Re\langle A'U(t)e^{i\Phi'}w, iU(t)(\psi e^{i\Phi'}w)\rangle_{L^2} \\ &= -t^{\nu-2}\Re\langle A', S^0(\Phi'; w, i\psi w) + S(w, i\psi w)\rangle_{L^2} \\ &= -\frac{t^{\nu-2}}{2}\Re\langle A', iS^\psi(\Phi'; w, w)\rangle_{L^2} - \frac{t^{\nu-2}}{2}\Re T^\psi(A'; w, iw).\end{aligned} \tag{5.25}$$

Taking the difference for $k=0,1$, we obtain

$$\begin{aligned}-2t^{2-\nu}\partial_t\langle\delta_1|w|^2,\psi\rangle_{L^2} =&\Re\langle\delta_1 A', iS^\psi(\Phi_1';w_1,w_1)\rangle_{L^2}\\&+\Re\langle A_0', iS^{\psi,\delta}(\Phi_0',\Phi_1';w_1,w_1)\rangle_{L^2}\\&+\Re\langle A_0', i\delta_1 S^\psi(\Phi_0';w_*,w_*)\rangle_{L^2}\\&+\Re T^\psi(\delta_1 A';w_1,iw_1)\\&+\Re\delta_1 T^\psi(A_0';w_*,iw_*)\\&=: I_1+\cdots+I_5.\end{aligned}\tag{5.26}$$

The S^ψ part in (5.25) is estimated as

$$\begin{aligned}|\langle A', iS^\psi(w,w)\rangle_{L^2}| &\lesssim \|A'\|_{B^{2s-\nu}}\|S^\psi(w,w)\|_{B_*^{\nu-2s}}\\&\lesssim t^{2\alpha-\kappa/2}d(\Phi')\|A'\|_{B^{2\alpha}}\|\psi\|_{\dot B^{n/2-\nu-\kappa}_{2,\infty}}\|w\|^2_{\dot H^s},\end{aligned}\tag{5.27}$$

where we applied Lemma 2.2 (iii) with $\theta=\alpha$, $\theta'=\alpha-\kappa/2$, $\sigma'=-\nu-\kappa$ and $\beta=\gamma=s$. As for the term involving T^ψ, we estimate it by using (3.7) as

$$|T^\psi(A';w,iw)|\lesssim t^{2\alpha-\kappa}\|A'\|_{B^{2\alpha-\kappa}}\|\psi\|_{\dot B^{n/2-\nu-\kappa}_{2,\infty}}\|w\|^2_{\dot H^s}.\tag{5.28}$$

For the S^* parts in $\delta_1|w|^2$, we apply Lemma 2.2 with $\theta=\theta'=\alpha-\kappa/2$, $\beta=s$ and $\sigma'=s-\nu$. We set $\gamma=s$ for I_1 and I_2, $\gamma=0$ for I_3, and $\sigma=s$ for I_2. Then we obtain

$$\begin{aligned}|I_1| &\lesssim t^{2\alpha-\kappa}d(\Phi_1')\|\delta_1 A'\|_{B^{s-\nu-2\kappa}}\|\psi\|_{\dot B^{n/2+s-\nu}_{2,\infty}}\|w_1\|^2_{\dot H^s},\\|I_2| &\lesssim t^{2\alpha-\kappa}d(\Phi_{0+}')\|\delta_1\phi'\|_{L^2}\|\phi_{0+}'\|_{\dot H^s}\|A_0'\|_{B^{2\alpha-2\kappa}}\|\psi\|_{\dot B^{n/2+s-\nu}_{2,\infty}}\|w_1\|^2_{\dot H^s},\\|I_3| &\lesssim t^{2\alpha-\kappa}d(\Phi_0')\|A_0'\|_{B^{2\alpha-2\kappa}}\|\psi\|_{\dot B^{n/2+s-\nu}_{2,\infty}}\|w_{0+}\|_{\dot H^s}\|\delta_1 w\|_{L^2}.\end{aligned}\tag{5.29}$$

where we used (5.10) for I_2. For the remaining two terms I_4 and I_5, we apply (3.7) with $\theta=2\alpha-\kappa$. Then we get

$$\begin{aligned}|I_4| &\lesssim t^{2\alpha-\kappa}\|\delta_1 A'\|_{B^{s-\nu-2\kappa}}\|\psi\|_{\dot B^{n/2+s-\nu}_{2,\infty}}\|w_k\|^2_{\dot H^s},\\|I_5| &\lesssim t^{2\alpha-\kappa}\|A_0'\|_{B^{2\alpha-2\kappa}}\|\psi\|_{\dot B^{n/2+s-\nu}_{2,\infty}}\|w_{0+}\|_{\dot H^s}\|\delta_1 w\|_{L^2}.\end{aligned}\tag{5.30}$$

By duality, we obtain

$$\|\partial_t V(w)\|_{B^\kappa}\lesssim t^{\alpha-\kappa-1}d(\Phi')\|A'\|_{B^{2\alpha}\cap B^{2\alpha-\kappa}}\|w\|^2_{\dot H^s},\tag{5.31}$$

$$\begin{aligned}\|\partial_t\delta_1 V(w)\|_{B^{-s}} \lesssim\ & t^{\alpha-\kappa-1}d(\Phi_{0+}')\|\phi_{0+}'\|^{[0,1]}_{\dot H^s}\|A_0'\|^{[0,1]}_{B^{2\alpha-2\kappa}}\|w_{0+}\|^{[1,2]}_{\dot H^s}\\&\times(\|\delta_1 A'\|_{B^{s-\nu-2\kappa}}+\|\delta_1\phi'\|_{L^2}+\|\delta_1 w\|_{L^2}).\end{aligned}\tag{5.32}$$

where we used that $1-\nu<\alpha$. In conclusion, we have the following decay estimate on the potential.

Lemma 5.2 *Let $1/2 < \nu < 1$, $n \geq 3$ and $1 - \nu/2 < s < 3/4$. Let $A_k = V(U(t)e^{i\Phi_k} w_k)$ with $\Phi_k = \Phi(\phi_k)$ for $k = 0, 1$. Assume that w_k solves*

$$2i\dot{w}_k + t^{\nu-2} T(A'_k, \phi'_k) w_k = 0, \tag{5.33}$$

with real-valued A'_k. Let $\kappa > 0$ be sufficiently small depending on ν and s. Let $\alpha = s - \nu/2$. Then we have

$$\begin{aligned} \|A_k - V(w_k(0))\|_{B^\kappa} \lesssim\ & t^{\alpha-\kappa}(D(\phi_k) + D(\phi'_k)) \\ & \times \sup_t \{ \|A'_k\|^{[0,1]}_{B^{2\alpha} \cap B^{2\alpha-\kappa}} \|w_k\|^2_{\dot{H}^s} \}, \end{aligned} \tag{5.34}$$

$$\begin{aligned} & \|A_1 - V(w_1(0)) - A_0 + V(w_0(0))\|_{B^{-s}} \\ & \lesssim t^{\alpha-\kappa}(D(\phi_0) + D(\phi_1) + D(\phi'_0) + D(\phi'_1)) \\ & \quad \times \sup_t \{ \|A'_0\|^{[0,1]}_{B^{2\alpha-2\kappa}} (\|w_0\|_{H^s} + \|w_1\|_{H^s})^{[1,2]} \\ & \quad \times (\|A'_1 - A'_0\|_{B^{2\alpha-s-2\kappa}} + \|\phi_1 - \phi_0\|_{L^2} + \|\phi'_1 - \phi'_0\|_{L^2} + \|w_1 - w_0\|_{L^2}) \} \end{aligned} \tag{5.35}$$

where D is as defined in (4.14), *and $\sup_t$ should be understood as the operator defined by*

$$(\sup_t f)(t) := \sup_{0<s<t} f(s). \tag{5.36}$$

6 Modified wave operators

In this section, we construct modified wave operators by solving the Cauchy problem for (1.10). The iteration scheme is the same as in [12]. We may concentrate only on very small positive time $0 < t \ll 1$, since the continuation for larger time is easy and well known. Let $n \geq 3$, $1/2 < \nu < 1$ and $1 - \nu/2 < s < 3/4$, $\alpha := s - \nu/2$. We first construct the modified wave operator W for $s < 3/4$, and then will show that it is also continuous in the topology of H^s, $3/4 \leq s < 1$.

Let $\phi \in H^s$. Let $\kappa > 0$ be so small depending on s, ν and n that all the above Lemmas can work and moreover we have $\alpha - 10\kappa > 1 - \nu$.

We want to solve the Cauchy problem for w:

$$2i\dot{w} + t^{\nu-2} e^{-i\Phi} \{ U(-t) V(U(t) e^{i\Phi} w) U(t) - V(\phi) \} e^{i\Phi} w = 0, \tag{6.1}$$

with $w(0) = \phi$, where $\Phi = \Phi(\phi)$ is defined by (1.9) as before. We solve this by iteration starting with $w_0 := \phi$ and

$$\begin{aligned} & A_k := V(U(t) e^{i\Phi} w_k), \\ & 2i\dot{w}_k + t^{\nu-2} T(A_{k-1}, \phi) w_k = 0, \\ & w_k(0) = \phi, \end{aligned} \tag{6.2}$$

where T is as defined in (4.2). In this paper, we do not use the equation for $u_k := U(t) e^{i\Phi} w_k$.

First of all, we have $D(\phi) < \infty$ by definition (4.14). In the following, we often regard such a quantity (depending only on ϕ) as a constant. From Lemma 5.1, we have

$$\|A_0\|_{B^{-\nu}\cap B^{2\alpha}} \lesssim \|\phi\|_{H^s}^2. \tag{6.3}$$

We may drop the term V^w when estimating the decay of A_0 since w_0 is independent of t. Then we obtain from the argument of Lemma 5.2,

$$\|A_0 - V(\phi)\|_{B^\kappa} \lesssim t^{\alpha-\kappa}\|\phi\|_{\dot{H}^s}^2. \tag{6.4}$$

Now we can start an induction to establish uniform bounds on A_k and w_k. Let $k \in \mathbb{N}$ and suppose that we have

$$\|w_j\|_{H^s} \le C_0,\ \|A_j\|_{B^{-\nu}\cap B^{2\alpha}} \le C_1,\ \|A_j(t) - V(\phi)\|_{B^\kappa} \le C_2 t^{\alpha-\kappa}, \tag{6.5}$$

for any $j < k$. First we presume that we can solve the equation for w_k such that $w_k \in C(H^s)$ with $w_k(0) = \phi$. Now we apply Lemma 4.1 with $w_1 := w_k(x+h)$ and $w_0 := w_k$. Then we obtain

$$\begin{aligned} &|\partial_t\|\delta^h w_k\|_{L^2}| \\ &\lesssim t^{\nu-2}\|w_k\|_{\dot{H}^{s'}}\big\{\|\delta^h(A_{k-1} - V(\phi))\|_{B^{-s'}} \\ &\quad + t^{\alpha-2\kappa}D(\phi)(\|\delta^h A_{k-1}\|_{B^{2\alpha-s'-4\kappa}} + \|A_{k-1}\|_{B^{2\alpha-3\kappa}} t^\alpha \|\delta^h\Phi\|_{B^{2\alpha-s'-\kappa}})\big\} \\ &\lesssim t^{\nu-2}|h|^{s'+\kappa}\|w_k\|_{\dot{H}^{s'}}\big\{\|A_{k-1} - V(\phi)\|_{B^\kappa} \\ &\quad + t^{\alpha-2\kappa}(\|A_{k-1}\|_{B^{2\alpha-3\kappa}} + \|A_{k-1}\|_{B^{2\alpha-3\kappa}}\|\phi\|_{H^s}^2)\big\} \\ &\lesssim |h|^{s'+\kappa}\|w_k\|_{\dot{H}^{s'}}\big\{t^{4\kappa-1}\|A_{k-1}\|_{B^{2\alpha-3\kappa}} + t^{\nu-2}\|A_{k-1} - V(\phi)\|_{B^\kappa}\big\}, \end{aligned} \tag{6.6}$$

where we used (5.10) in the second inequality after estimating the difference by the Besov space $B^{2\alpha}$. Let $s = s'$ and use the assumed bounds (6.5). Then we obtain

$$|\partial_t\|\delta^h w_k\|_{L^2}| \lesssim |h|^{s+\kappa} t^{4\kappa-1}(C_1 + C_2)\|w_k\|_{\dot{H}^s}. \tag{6.7}$$

By (2.13) this implies

$$|\partial_t\|w_k\|_{\dot{H}^s}| \le C_3 t^{4\kappa-1}\|w_k\|_{\dot{H}^s}, \tag{6.8}$$

where C_3 depends on C_0, C_1, C_2 and κ. On the other hand, by the L^2 conservation, we have

$$\|\delta^h w_k\|_{L^2} \le 2\|\phi\|_{L^2}. \tag{6.9}$$

Then, by the Gronwall inequality, we obtain

$$\|w_k\|_{H^s} \le M_0 e^{C_4 t^{4\kappa}}\|\phi\|_{H^s}, \tag{6.10}$$

where M_0 is an absolute constant and C_4 is a constant dependent on C_i $(i < 4)$ and κ. If we have chosen $C_0 > 2M_0\|\phi\|_{H^s}$, then for sufficiently small time (depending on C_i), we have

$$\|w_k\|_{H^s} < 2M_0\|\phi\|_{H^s} < C_0. \tag{6.11}$$

Actually, we have a better regularity for the perturbation. Plugging the above bound into (6.8) and using (2.13), we obtain

$$\|\varphi_J * w_k\|_{\dot{H}^s} \lesssim \|\varphi_J * \phi\|_{H^s} + J^{-\kappa} t^{4\kappa} C_0(C_1 + C_2). \tag{6.12}$$

Now we give a rigorous proof of that we can obtain w_k solving the equation (6.2) with $w_k(0) = \phi$ and belonging to $C(H^s)$. If we assume the initial data at positive time $t_0 > 0$ $w_k(t_0) = \phi$, then it is easy to solve the equation for $t > t_0$ by the standard L^2 estimate since we are away from the singularity at $t = 0$. Then we obtain the uniform bound (6.10) in H^s for this approximate solution. Moreover, Lemma 4.2 implies a uniform $L_t^{1+} L_x^2$ bound of $\dot{w}_k$, which yields uniform continuity of w_k in L^2. Thus, by letting $t_0 \to +0$, we obtain a solution w_k of (6.2) with $w_k(0) = \phi$ in $L^\infty(H^s)$ and $C(L^2)$. Then the frequency-localized bound (6.12) implies that w_k is continuous also in H^s.

Next we turn to the estimate on A_k. By Lemma 5.1 we have

$$\|A_k\|_{B^{-\nu} \cap B^{2\alpha}} \lesssim \|w_k\|_{H^s}^2, \tag{6.13}$$

and from Lemma 5.2 we obtain

$$\begin{aligned} \|A_k - V(\phi)\|_{B^\kappa} &\lesssim t^{\alpha-\kappa} \sup_t \|A_{k-1}\|_{B^{2\alpha} \cap B^{2\alpha-\kappa}}^{[0,1]} \|w_k\|_{\dot{H}^s}^2 \\ &\lesssim t^{\alpha-\kappa} (1 + C_1) \|w_k\|_{\dot{H}^s}^2. \end{aligned} \tag{6.14}$$

Thus, if we have chosen C_0, C_1 and C_2 sufficiently large compared with $M_0 \|\phi\|_{H^s}$, then we obtain the above bounds (6.5) uniformly for k and for small $t > 0$ by induction on k. In conclusion,

$$\|w_k\|_{H^s}, \quad \|A_k\|_{B^{-\nu} \cap B^{2\alpha}}, \quad t^{\kappa-\alpha} \|A_k - V(\phi)\|_{B^\kappa} \tag{6.15}$$

are bounded for small $t > 0$ and any $k \geq 0$.

Next we consider the convergence. We will regard those bounds on w_k and A_k obtained above just as constants. We apply Lemma 4.1 with $s = s'$, $w_1 := w_{k+1}$ and $w_0 := w_k$. Then we obtain

$$|\partial_t \|\delta_{k+1} w\|_{L^2}| \lesssim t^{4\kappa-1} \|\delta_k A\|_{B^{2\alpha-s-4\kappa}} + t^{\nu-2} \|\delta_k A\|_{B^{-s}}, \tag{6.16}$$

and then we need to estimate $\delta_k A$. Applying Lemma 5.1 with $A_1 := A_k$ and $A_0 := A_{k-1}$, we obtain

$$\|\delta_k A\|_{B^{-\nu} \cap B^{2\alpha-s}} \lesssim \|\delta_k w\|_{L^2}. \tag{6.17}$$

We also use Lemma 5.2 with $A_1 := A_k$ and $A_0 = A_{k-1}$. Then we obtain

$$\begin{aligned} \|\delta_k A\|_{B^{-s}} &\lesssim t^{\alpha-\kappa} (\|\delta_{k-1} A\|_{L_t^\infty(B^{2\alpha-s-2\kappa})} + \|\delta_k w\|_{L_t^\infty(L^2)}) \\ &\lesssim t^{\alpha-\kappa} \|\delta_{k-1+} w\|_{L_t^\infty(L^2)}. \end{aligned} \tag{6.18}$$

(Notice that $\delta_k V(w_*(0)) = 0$ in this case.)

Putting (6.16)–(6.18) together, we obtain

$$|\partial_t \|\delta_{k+1} w\|_{L^2}| \lesssim t^{4\kappa-1} \|\delta_{k-1+} w\|_{L_t^\infty(L^2)}, \tag{6.19}$$

By integration in time, we obtain

$$\|\delta_{k+1} w\|_{L^\infty(0,t;L^2)} \lesssim t^{4\kappa} \|\delta_{k-1+} w\|_{L^\infty(0,t;L^2)}, \tag{6.20}$$

which implies that w_k converges in $L^\infty(0,t;L^2)$ for sufficiently small $t > 0$. Since we have additional regularity for the nonlinear term (6.12), we can enhance the L^2 convergence to the H^s one. In fact, (6.12) implies that for any $\varepsilon > 0$, there exists some large N depending on ϕ and ε such that for small $t > 0$ and for any k we have

$$\|w_k|_{|\xi|\geq N}\|_{H^s} < \varepsilon, \tag{6.21}$$

where ξ denotes the Fourier variable. Then, by the Lebesgue dominant convergence theorem, we obtain H^s convergence of w_k as $k \to \infty$. The above contraction property (6.20) also implies the uniform continuity of w_k as $t \to +0$, first in L^2, and then by the same reasoning as above, in H^s. Thus we have the strong limit w_∞ of w_k in $C(H^s)$, and then by Lemma 5.1, A_k converges to $A_\infty = V(U(t)e^{i\Phi}w_\infty)$ in $B^{-\nu} \cap B^{2\alpha}$, which satisfies

$$\|A_\infty - V(\phi)\|_{B^\kappa} \lesssim t^{\alpha-\kappa}. \tag{6.22}$$

Using these convergence, it is easy to see that this limit function w_∞ solves the equation (6.1) with $w_\infty(0) = \phi$ as desired. The uniqueness of such a solution follows from the estimate for the difference of two solutions by using Lemma 4.1 as above. Then we obtain the well-defined modified wave operator W via the pseudo-conformal inversion.

Since the asymptotic behavior described in the theorem uniquely determines the limit of $V(u^*(t))$ as $t \to +0$, the injectivity of the modified wave operator W easily follows.

Notice that the regularity gain κ in (6.12) also implies that w_∞ is bounded in $H^{s+\kappa/2}$ if ϕ belongs to this space. Then we may apply the energy estimate with $s' = s + \kappa/2$, getting again certain amount of regularity. It is easy to check that the amount of regularity gain κ can be taken uniformly as long as s' is away from 1. Therefore, if $\phi \in H^{s'}$ with $s < s' < 1$, then w is bounded also in $H^{s'}$. Thus W maps $\mathcal{F}H^s$ into $\mathcal{F}H^s$, for any $s \in (1-\nu/2, 1)$. We also obtain (6.12) for s in this range.

Remark 6.1 *In principle, we may consider Sobolev norms higher than* 1 *by taking higher differences. If we want to go beyond* $H^{n/2}$*, then we also need to jack up the regularity of the potential. We do not pursue this problem in this paper.*

7 Continuity of Modified wave operators

In this section, we see the continuity of the modified wave operator in $\mathcal{F}H^s$. By the local wellposedness in $\mathcal{F}H^s$ of the original equation (1.1), it is equivalent to the continuity of the map $W^* : \phi \mapsto w(1)$ in H^s, where w is the solution of (6.1) with $w(0) = \phi$ obtained above. Notice that here the phase factor Φ also changes depending on ϕ, and it is the reason why the continuity is not trivial from the above iterative construction, where Φ was fixed. However, we have already derived the necessary estimates in Sections 4 and 5, and so we have just to check that they work.

We will show L^2 continuity of W^* in a bounded set of H^s. Once it is obtained, we can easily enhance it to the strong continuity in H^s via the frequency-localized uniform bound (6.12), which has been extended to $s \in (1-\nu/2, 1)$ by the argument at the end of the previous section.

Let $\phi_k \in H^s$ for $k = 0, 1$ and let w_k be the solutions to (6.1) with $w_k(0) = \phi_k$ and $\Phi = \Phi_k := \Phi(\phi_k)$. We assume that ϕ_k are bounded in H^s, so that we may regard those norms are dominated by a constant. We apply Lemma 4.1 with $s' = s$. Then we obtain

$$\begin{aligned} |\partial_t \|\delta_1 w\|_{L^2}| \lesssim t^{4\kappa-1}(\|\delta_1 A\|_{B^{2\alpha-s-4\kappa}} + \|\delta_1 \phi\|_{L^2}) \\ + t^{\nu-2}\|\delta_1 A - \delta_1 V(\phi_*)\|_{B^{-s}}, \end{aligned} \tag{7.1}$$

where we used (5.10) to estimate $\delta_1\Phi$. Then we need to estimate $\delta_1 A$. By Lemma 5.1, we have

$$\|\delta_1 A\|_{B^{-\nu}\cap B^{2\alpha-s}} \lesssim \|\delta_1 w\|_{L^2} + \|\delta_1 \phi\|_{L^2}. \tag{7.2}$$

Next we use Lemma 5.2 with $A'_k = A_k$ and $\phi'_k = \phi_k$. Then we obtain

$$\begin{aligned} \|\delta_1 A - \delta_1 V(\phi_*)\|_{B^{-s}} &\lesssim t^{\alpha-\kappa} \sup_t (\|\delta_1 A\|_{B^{2\alpha-s-\kappa}} + \|\delta_1 w\|_{L^2} + \|\delta_1 \phi\|_{L^2}) \\ &\lesssim t^{\alpha-\kappa} \sup_t (\|\delta_1 w\|_{L^2} + \|\delta_1 \phi\|_{L^2}), \end{aligned} \tag{7.3}$$

where we used the above obtained bound (7.2).

In conclusion, we obtain

$$|\partial_t \|\delta_1 w\|_{L^2}| \lesssim t^{4\kappa-1}(\|\delta_1 w\|_{L^\infty(L^2)} + \|\delta_1 \phi\|_{L^2}), \tag{7.4}$$

which, through integration in time, implies that for small $t > 0$,

$$\|\delta_1 w\|_{L^2} \lesssim \|\delta_1 \phi\|_{L^2}. \tag{7.5}$$

Thus we obtain L^2 continuity of W^* in any bounded set of H^s. It is easy to enhance this convergence into the strong one in H^s by (6.12), which implies the following. For any $\varepsilon > 0$, there exists a small ball B_ε of H^s around ϕ_0 and $N \in \mathbb{N}$ large such that for small $t > 0$ (independent of ε), we have

$$\left\| w_1(t)|_{|\xi|>N} \right\|_{H^s} < \varepsilon, \tag{7.6}$$

if $w_1(0) \in B_\varepsilon$. When combined with the L^2 continuity obtained above, this implies the strong continuity of W^* in H^s. Returning by the pseudo-conformal inversion, we obtain $\mathcal{F}H^s$ continuity of W.

8 Asymptotic completeness

In this section, we show the openness of the modified wave operator W, which implies the asymptotic completeness in a small ball of H^s around any solution having the asymptotic profile described by the modified wave operator. Here again we invert the problem by the pseudo-conformal transform. By the local wellposedness of the Hartree equation, it suffices to show the following: Let w_0 be a solution of (6.1) with $w_0(0) = \phi_0$ and $\Phi = \Phi_0 := \Phi(\phi_0)$. Then for some $t_0 > 0$ and any ψ' sufficiently close to $\psi_0 := w_0(t_0)$ in H^s, there exists a solution w' of (6.1) with $\Phi = \Phi' := \Phi(w'(0))$, satisfying $w'(t_0) = \psi'$. Moreover, when ψ' converges to ψ_0, $w'(0)$ also converges to $w_0(0)$ in H^s.

To find the solution w', we again use the iteration method. Let w_0 be given as above. Then we define A_k, Φ_k and w_k inductively by

$$\begin{aligned} &\Phi_k := \Phi(\phi_k), \quad \phi_k := w_k(0) \\ &A_k := V(U(t)e^{i\Phi_k} w_k), \\ &2i\dot{w}_k + t^{\nu-2} T(A_{k-1}, \phi_{k-1}) w_k = 0, \\ &w_k(t_0) = \psi'. \end{aligned} \tag{8.1}$$

We remark that for general data ψ' at a fixed $t_0 > 0$, this iteration can not possibly work, since we do not have the asymptotic completeness in the whole space in general. It is essential that we can choose t_0 as small as we need and also ψ' close to $w_0(t_0)$ (it suffices to be bounded).

First we derive uniform bounds for w_k and A_k. Assume that we have

$$\|w_j\|_{H^s} \le C_0, \ \|A_j\|_{B^{-\nu} \cap B^{2\alpha}} \le C_1, \ \|A_j - V(\phi_j)\|_{B^\kappa} \le C_2 t^{\alpha-\kappa}, \tag{8.2}$$

for $j < k$. It is clear by the result obtained above that we have (8.2) for $j = 0$. So it suffices to show (8.2) for $j = k$ to get the uniform bounds. Applying Lemma 4.1 with $w_1 := w_k(x+h)$ and $w_0 = w_k$, we obtain

$$\begin{aligned} |\partial_t \|\delta^h w_k\|_{L^2}| &\lesssim |h|^{s+\kappa} t^{\nu-2} \|w_k\|_{\dot{H}^s} \big\{ \|A_{k-1} - V(\phi_{k-1})\|_{B^\kappa} \\ &\quad + t^{\alpha-2\kappa} D(\phi_{k-1}) (\|A_{k-1}\|_{B^{2\alpha-3\kappa}} + \|A_{k-1}\|_{B^{2\alpha-3\kappa}} t^\alpha \|\Phi_{k-1}\|_{B^{2\alpha}}) \big\} \\ &\lesssim |h|^{s+\kappa} \|w_k\|_{\dot{H}^s} t^{4\kappa-1} (C_2 + C_0^{[0,m+2]} C_1). \end{aligned} \tag{8.3}$$

Taking the difference norm of $\dot{H}^s$, we obtain

$$|\partial_t \|w_k\|_{\dot{H}^s}| \le C_3 t^{4\kappa-1} \|w_k\|_{\dot{H}^s}, \tag{8.4}$$

where C_3 is a constant dependent on C_i $(i < 3)$ and κ. By the Gronwall inequality and the L^2 conservation, we obtain a uniform bound

$$\|w_k(t)\|_{H^s} \le e^{C_4|t^{4\kappa}-t_0^{4\kappa}|}\|\psi'\|_{H^s}, \tag{8.5}$$

where C_4 is determined by C_i $(i < 4)$ and κ. By Lemma 5.1, we have

$$\|A_k\|_{B^{-\nu}\cap B^{2\alpha}} \lesssim D(\phi_k)\|w_k\|_{H^s}^2, \tag{8.6}$$

and by Lemma 5.2,

$$\begin{aligned}\|A_k - V(\phi_k)\|_{B^\kappa} &\lesssim t^{\alpha-\kappa}D(\phi_{k-1+})\sup_t \|A_{k-1}\|_{B^{2\alpha}\cap B^{2\alpha-\kappa}}^{[0,1]}\|w_k\|_{H^s}^2 \\ &\lesssim t^{\alpha-\kappa}D(\phi_{k-1+})(1+C_1)\|w_k\|_{L_t^\infty H^s}^2.\end{aligned} \tag{8.7}$$

Now we choose C_0 sufficiently large compared with $\|w_0\|_{L_t^\infty H^s}$, and C_1 sufficiently large compared with $C_0^{[0,m]}C_0^2$, and then C_2 sufficiently large compared with $C_0^{[2,m+2]}(1 + C_1)$, and finally we choose t_0 sufficiently small. Then we can make those bounds in (8.5), (8.6) and (8.7) smaller than needed to proceed the induction for (8.2). Thus we obtain the uniform bounds (8.2) for any j.

Strictly speaking, we have to carry out this procedure first for $s < 3/4$ and then extend the bound to general $s < 1$ in the same way as in the construction of W (see the end of Section 6).

Next we show the convergence of w_k in L^2. We regard the bounds in (8.2) just as constants. By Lemma 4.1 with $w_1 := w_{k+1}$ and $w_0 := w_k$, we have

$$\begin{aligned}|\partial_t\|\delta_{k+1}w\|_{L^2}| \lesssim\; & t^{\nu-2}\|\delta_k A - \delta_k V(\phi_*)\|_{B^{-s}} \\ &+ t^{4\kappa-1}(\|\delta_k A\|_{B^{2\alpha-s-4\kappa}} + \|\delta_k\phi\|_{L^2}),\end{aligned} \tag{8.8}$$

where we used (5.10). Meanwhile, Lemma 5.1 with $A_1 := A_k$ and $A_0 := A_{k-1}$ implies that

$$\|\delta_k A\|_{B^{-\nu}\cap B^{2\alpha-s}} \lesssim \|\delta_k w\|_{L^2} + \|\delta_k\phi\|_{L^2}, \tag{8.9}$$

and Lemma 5.2 yields that

$$\begin{aligned}&\|\delta_k A - \delta_k V(\phi_*)\|_{B^{-s}} \\ &\quad\lesssim t^{\alpha-\kappa}\sup_t(\|\delta_{k-1}A\|_{B^{2\alpha-s-2\kappa}} + \|\delta_{k-1+}\phi\|_{L^2} + \|\delta_k w\|_{L^2}).\end{aligned} \tag{8.10}$$

Plugging these estimates into (8.8), we obtain

$$|\partial_t\|\delta_{k+1}w\|_{L^2}| \lesssim t^{4\kappa-1}\|\delta_{k-1+}w\|_{L_t^\infty L^2}. \tag{8.11}$$

By integration in time, we obtain

$$\|\delta_{k+1}w\|_{L_t^\infty L^2} \lesssim |t^{4\kappa} - t_0^{4\kappa}|\|\delta_{k-1+}w\|_{L_t^\infty L^2}. \tag{8.12}$$

In the special case $k = 1$, we have

$$\|\delta_2 w\|_{L_t^\infty L^2} \lesssim |t^{4\kappa} - t_0^{4\kappa}| \|\delta_1 w\|_{L_t^\infty L^2}, \tag{8.13}$$

since the δ_{k-1} parts in (8.10) disappear thanks to the equation for w_0. Therefore, if we choose t_0 sufficiently small, then w_k converges to some function w_∞ in L^2 for $0 < t < 2t_0$. By the same argument as for (6.12), we can derive from (8.3) that

$$\|\varphi_J * w_k(t)\|_{\dot{H}^s} \lesssim \|\varphi_J * \psi'\|_{\dot{H}^s} + J^{-\kappa}|t^{4\kappa} - t_0^{4\kappa}|C_3, \tag{8.14}$$

which implies that the above L^2 convergence is actually strong in H^s. Then $w' := w_\infty$ is the desired solution of (6.1) with $w'(t_0) = \psi'$ and $\Phi = \Phi(w'(0))$. Moreover, the above contraction property (8.12)–(8.13) implies that if ψ' converges to ψ_0 in L^2 (in a bounded set in H^s), then w' converges to w_0 in L^2 for $0 \le t \le 2t_0$. If ψ' converges in H^s, then the high frequency part of ψ' in (8.14) is uniformly bounded (more precisely, compact in ℓ^2 for the dyadic parameter J). Hence (8.14) together with the corresponding estimate for w_0 (see (6.12)) implies the convergence of w' to w_0 in H^s for $0 \le t \le 2t_0$. This finishes the proof of the openness of W, and so we have completed the proof of Theorem 1.1.

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