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Convergent Perturbative Solutions of the Schrödinger Equation for Two-Level Systems with Hamiltonians Depending Periodically on Time

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Abstract. We study the Schrödinger equation of a class of two-level systems under the action of a periodic time-dependent external field in the situation where the energy difference 2ϵ between the free energy levels is sufficiently small with respect to the strength of the external interaction. Under suitable conditions we show that this equation has a solution in terms of converging power series expansions in ϵ. In contrast to other expansion methods, like in the Dyson expansion, the method we present is not plagued by the presence of "secular terms". Due to this feature we were able to prove uniform convergence of the Fourier series involved in the computation of the wave functions and to prove absolute convergence of the ϵ-expansions leading to the "secular frequency" and to the coefficients of the Fourier expansion of the wave function.

I Introduction

This paper is dedicated to the mathematical study of a class of periodically time-depending two-level systems. It is well know that the usual perturbative approach, based, f.i., on the Dyson series, leads to difficulties involving secular terms and (for quasi-periodic interactions) small divisors. In [1] a new algorithm has been devised to overcome the secular terms in the general case of quasi-periodic interactions. Roughly speaking it involves an inductive "renormalization" of an effective field introduced via an exponential Ansatz (the function g to be introduced below). Here we apply that algorithm to the case of periodic interactions in the strong coupling regime, a situation of particular interest in several branches of physics (for references, see [2] or below). As we will show, our method not only recovers the Floquet form of the solution of the time-depending Schrödinger equation, but also allows the computation of the secular frequency and of the Fourier coefficients in terms of explicit *convergent* ϵ-expansions, what constitutes a feature of our algorithm, compared to other expansion methods.

Let us describe more precisely the systems we will study. Consider the following Hamiltonian for a two-level system under the action of an external time-dependent field

$$H_1(t) \ = \ H_0 + H_I(t) \ = \ \epsilon\sigma_3 - f(t)\sigma_1 \qquad \text{(I.1)}$$

and the corresponding Schrödinger equation[1]

$$i\partial_t \Psi(t) \;=\; H_1(t)\Psi(t), \tag{I.2}$$

with $\Psi : \mathbb{R} \to \mathbb{C}^2$. Here $f(t)$ is a function of time t and $\epsilon \in \mathbb{R}$ is a parameter representing half of the energy difference between the "free" (i.e., for $f \equiv 0$) energy levels. The symbols σ_1, σ_2 and σ_3 denote the Pauli matrices in their usual representations: $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The "interaction Hamiltonian" $H_I(t) := -f(t)\sigma_1$ represents a time-dependent external interaction coupled to the system inducing transitions between the two eigen-states of the free Hamiltonian $H_0 := \epsilon\sigma_3$.

Since the Schrödinger equation (I.2) can be read as

$$i\partial_\tau \Psi_0(\tau) \;=\; \left[\sigma_3 - \epsilon^{-1} f\left(\epsilon^{-1}\tau\right)\sigma_1\right]\Psi_0(\tau), \tag{I.3}$$

where $\tau \equiv \epsilon t$ and $\Psi_0(t) \equiv \Psi(\epsilon^{-1}t)$, the situation where ϵ is "small" characterizes the "strong coupling" and, for periodic f, "large frequency" regime [3, 4].

The system described above is certainly one of the simplest non-trivial time-depending quantum systems and the study of the solutions of (I.2) is of basic importance for many physical applications as, e.g., in quantum optics, in the theory of spin resonance or in problems of quantum tunneling.

Equation (I.2) has been analyzed by many authors in various approximations. In the wide literature on the subject of time-depending two-level systems we mention the pioneering works of Rabi [5], of Bloch and Siegert [6] and of Autler and Townes [7]. In [7] the authors studied the solutions of (I.2) for the case where, in our notation, $f(t) = -2\beta\cos(\omega t)$, $\beta \in \mathbb{R}$. Their work is exact but non-rigorous and involved a combination of the method of continued fractions, for relating the coefficients the Fourier decomposition of the wave functions, with numerical analysis. No proof has been exhibited that the continued fractions converge and further unjustified restrictions have been made in order to transform some transcendental equations into low order algebraic equations, which are then solved either exactly or, specially for strong fields, numerically.

For related treatments using different approaches and for related systems, see [8, 9, 10, 11, 12, 13, 14] and other references therein. For a recent review on the mathematical theory of quantum systems submitted to time-depending periodic and quasi-periodic perturbations see [3]. For an introduction to the subjects of "quantum chaos" and quantum stability, two subjects strongly linked to the problems considered here, see [15]. See also [4] for results on the spectral analysis of the quasi-energy operator for two-level atoms in the quasi-periodic case.

In [1] we studied the system described by (I.2) in the situation where f is a quasi-periodic function of time and a special perturbative expansion (power series expansion in ϵ) has been developed. Its main virtue is to be free of the so-called "secular terms", i.e., polynomials in t that appear order by order in perturbation

[1] For simplicity we shall adopt here a system of units with $\hbar = 1$.

theory and that spoil the analysis of convergence of the series and the proofs of quasi-periodicity of the perturbative terms. Although we have not been able to prove convergence of our power series expansion in the general case where f is quasi-periodic it has been established that the coefficients of the expansion are indeed quasi-periodic functions of time.

One of the obstacles found in the attempt to prove convergence of our expansion in the general case of quasi-periodic f is the presence of "small denominators". This typical feature of perturbative approximations for solutions of differential equations with quasi-periodic coefficients is well known as one of the main sources of problems in the mathematically precise treatment of such equations. On what concerns proofs of convergence it should, therefore, be expected that better results could be obtained if the function f were restricted to be periodic since, in this case, no problems with small denominators should afflict our expansions.

In the present paper we show how the difficulties analyzed in [1] can be circumvented in the case of periodic f and establish convergence of our perturbative expansion for that case.

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By a time-independent unitary transformation, representing a rotation of $\pi/2$ around the 2-axis, we may replace $H_1(t)$ by

$$H_2(t) \;:=\; \left(e^{-i\pi\sigma_2/4}\right) H_1(t) \left(e^{i\pi\sigma_2/4}\right) \;=\; \epsilon\sigma_1 + f(t)\sigma_3 \tag{I.4}$$

and the Schrödinger equation becomes

$$i\partial_t\Phi(t) \;=\; H_2(t)\Phi(t), \tag{I.5}$$

with

$$\Phi(t) \;:=\; e^{-i\pi\sigma_2/4}\Psi(t). \tag{I.6}$$

The theorem below, proven in [1], presents the solution of the Schrödinger equation (I.5) in terms of particular solutions of a generalized Riccati equation.

Theorem I.1 *Let $f : \mathbb{R} \to \mathbb{R}$, $f \in C^1(\mathbb{R})$ and $\epsilon \in \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{C}$, $g \in C^1(\mathbb{R})$, be a particular solution of the generalized Riccati equation*

$$G' - iG^2 - 2ifG + i\epsilon^2 = 0. \tag{I.7}$$

Then, the function $\Phi : \mathbb{R} \to \mathbb{C}^2$ given by

$$\Phi(t) \;=\; \begin{pmatrix} \phi_+(t) \\ \phi_-(t) \end{pmatrix} \;=\; U(t)\Phi(0) \;=\; U(t,\, 0)\Phi(0), \tag{I.8}$$

where

$$U(t) \;:=\; \begin{pmatrix} R(t)\left(1 + ig(0)S(t)\right) & -i\epsilon R(t)S(t) \\ & \\ -i\epsilon\overline{R(t)}\;\overline{S(t)} & \overline{R(t)}\left(1 - i\,\overline{g(0)}\;\overline{S(t)}\right) \end{pmatrix}, \tag{I.9}$$

with

$$R(t) \ := \ \exp\left(-i\int_0^t (f(\tau)+g(\tau))\, d\tau\right) \tag{I.10}$$

and

$$S(t) \ := \ \int_0^t R(\tau)^{-2}\, d\tau \tag{I.11}$$

is a solution of (I.5) with initial value $\Phi(0) = \begin{pmatrix} \phi_+(0) \\ \phi_-(0) \end{pmatrix} \in \mathbb{C}^2$.

For a proof of Theorem I.1, see [1]. Let us briefly describe some of the ideas leading to Theorem I.1 and to other results of [1]. As we saw in [1], the solutions of the Schrödinger equation (I.5) can be studied in terms of the solutions of a particular complex version of Hill's equation:

$$\phi''(t) + \left(if'(t) + \epsilon^2 + f(t)^2\right)\phi(t) = 0. \tag{I.12}$$

In fact, a simple computation (see [1]) shows that the components $\phi_\pm$ of $\Phi(t)$ satisfy precisely

$$\begin{array}{rcl} \phi_+'' + \left(+if' + \epsilon^2 + f^2\right)\phi_+ & = & 0 \\ \phi_-'' + \left(-if' + \epsilon^2 + f^2\right)\phi_- & = & 0 \end{array} . \tag{I.13}$$

As a side remark we note that equations (I.13) are simpler and more convenient than the equations obtained by separating ψ_+ and ψ_- from (I.2):

$$\begin{array}{rcl} f\psi_+'' - f'\psi_+' + \left(\epsilon^2 f + f^3 - i\epsilon f'\right)\psi_+ & = & 0 \\ f\psi_-'' - f'\psi_-' + \left(\epsilon^2 f + f^3 + i\epsilon f'\right)\psi_- & = & 0 \end{array} . \tag{I.14}$$

These equations, mentioned (but not used) in [7], are mathematically less convenient because they may be non-regular, since f may have zeros in typical cases, like the simple monochromatic case $f(t) = -2\beta\cos(\omega t)$, analyzed in [7].

In [1] we attempted to solve (I.12) using the Ansatz

$$\phi(t) = \exp\left(-i\int_0^t (f(\tau)+g(\tau))d\tau\right). \tag{I.15}$$

It follows that g has to satisfy the generalized Riccati equation (I.7) and we tried to find solutions for g in terms of a power expansion in ϵ like

$$g(t) \ = \ q(t)\sum_{n=1}^{\infty} \epsilon^n\, c_n(t), \tag{I.16}$$

where

$$q(t) \ := \ \exp\left(i\int_0^t f(\tau)d\tau\right). \tag{I.17}$$

The heuristic idea behind the Ansätze (I.15) and (I.16) is the following. For $\epsilon \equiv 0$ a solution for (I.12) is given by $\exp\left(-i\int_0^t f(\tau)d\tau\right)$. Thus, in (I.15) and (I.16) we are searching for solutions in terms of an "effective external field" of the form $f+g$, with g vanishing for $\epsilon = 0$.

Note that a solution of the form (I.15) leads to only one of the two independent solutions of the second order Hill's equation (I.12). The complete solution of the Schrödinger equation (I.5) in terms of solutions of the generalized Riccati equation (I.7) is that described in Theorem I.1.

As mentioned above, perturbative solutions of quasi-periodically time-dependent systems are usually plagued by small denominators and by the presence of the so-called "secular terms". In [1] we discovered a particular way to eliminate completely the secular terms from the perturbative expansion of g (see Appendix A for a brief description of the strategy developed in [1]) and we were able to show, under some special assumptions, that the coefficients $c_n(t)$ are all quasi-periodic functions. In [1] we proved convergence of our perturbative solution in the somewhat trivial case where $f(t)$ is a non-zero constant function. Unfortunately no conclusion could be drawn about the convergence of the perturbative expansion for g in the general case of quasi-periodic f. We conjectured, however, that our expansion is uniformly convergent in the situation where $f(t)$ has small fluctuations about its mean value.

The technically central result of the present paper is the proof that, under suitable assumptions, the series (I.16) converges uniformly on $\mathbb{R}$ as a function of time for $|\epsilon|$ small enough and f periodic. This is the content of Theorem III.1. Moreover, we show that the functions c_n and, hence, g, have uniformly converging Fourier series representations. We use this fact together with the solution (I.9) to find the Floquet representation of the components $\phi_\pm$ of the wave function in terms of uniformly converging Fourier series representations. This is the content of Theorem I.2. Absolutely converging power series in ϵ for the Fourier coefficients and for the secular frequency are also presented.

We believe that the methods employed in this paper are also of importance for the general theory of Hill's equation. It would be of great interest to know whether the ideas described in [1] and here can be generalized and applied to a larger class of Hill's equations than those we studied so far.

I.1 The Main Result

On what concerns the solutions of the Schrödinger equation (I.5) the next theorem summarizes our main results.

Theorem I.2 *Let f be a real T_ω-periodic function of time ($T_\omega := 2\pi/\omega$) whose Fourier decomposition*

$$f(t) \;=\; \sum_{n\in\mathbb{Z}} F_n e^{in\omega t}, \tag{I.18}$$

with $\omega > 0$, contains only a finite number of terms, i.e., the set of integers $\{n \in \mathbb{Z} |\ F_n \neq 0\}$ is a finite set. We also assume that either $F_0 = 0$ or $2F_0 \in \mathbb{R} \backslash \{k\omega,\ k \in \mathbb{Z}\}$.

Consider the two following mutually exclusive conditions on f:
I) $M(q^2) \neq 0$.
II) $M(q^2) = 0$ but $M(\mathcal{Q}_1) \neq 0$, where

$$\mathcal{Q}_1(t) \ := \ q(t)^2 \int_0^t q^{-2}(\tau) d\tau. \tag{I.19}$$

Then, for each f as above, satisfying condition I or II, there exists a constant $K > 0$ (depending on the Fourier coefficients $\{F_n,\ n \in \mathbb{Z}\ , n \neq 0\}$ and on $\omega > 0$) such that, for each ϵ with $|\epsilon| < K$, there exist $\Omega \in \mathbb{R}$ and T_ω-periodic functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ such that the propagator $U(t)$ of (I.8) can be written as

$$U(t) \ = \ \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix} \ = \ \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ -\overline{U_{12}(t)} & \overline{U_{11}(t)} \end{pmatrix}, \tag{I.20}$$

with

$$U_{11}(t) \ = \ e^{-i\Omega t}\, u_{11}^{-}(t) + e^{i\Omega t}\, u_{11}^{+}(t), \tag{I.21}$$
$$U_{12}(t) \ = \ e^{-i\Omega t}\, u_{12}^{-}(t) + e^{i\Omega t}\, u_{12}^{+}(t). \tag{I.22}$$

The functions $u_{11}^{\pm}$ and $u_{12}^{\pm}$ have absolutely and uniformly converging Fourier expansions

$$u_{11}^{\pm}(t) \ = \ \sum_{n \in \mathbb{Z}} \mathcal{U}_{11}^{\pm}(n) e^{in\omega t},$$
$$u_{12}^{\pm}(t) \ = \ \sum_{n \in \mathbb{Z}} \mathcal{U}_{12}^{\pm}(n) e^{in\omega t}.$$

Moreover, under the same assumptions, Ω and the Fourier coefficients $\mathcal{U}_{11}^{\pm}(n)$ and $\mathcal{U}_{12}^{\pm}(n)$ can be expressed in terms of absolutely converging power series on ϵ.

Remarks on Theorem I.2

1. Expressions (I.21) and (I.22) represent the so-called "Floquet form" of the matrix elements $U_{11}(t)$ and $U_{12}(t)$. The frequency Ω is sometimes called the "secular frequency". The existence of the Floquet form is, of course, guaranteed by the well known Floquet's theorem. Hence, our algorithm not only recovers the Floquet form but also allows the explicit computation of the secular frequency and the Fourier coefficients in terms of *convergent* ϵ-expansions.

2. For a discussion of some physical implications of the solution described in the last theorem, see [2].

3. The physically realistic condition that the Fourier decomposition of f contains only a finite number of terms can be weakened. The only condition we use is the fast decay for $|m| \to \infty$ of the Fourier coefficients Q_m of the function $q(t)$ (defined in (I.17)), as found in Proposition II.2.

4. The second equality in (I.20) is due to (I.9).

5. It is important to stress that conditions I and II are restrictions on the function f and not on the parameter ϵ.

6. Possibly there are other conditions beyond *I* and *II* which could be considered, but they have not been explored so far. They are relevant in some cases. Theorem I.2 still does not provide a complete solution of (I.5) for all possible periodic functions f, but examples and some qualitative arguments show that the remaining cases are rather exceptional. For instance, for the monochromatic case where $f(t) = \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t)$ condition I covers all pairs $(\varphi_1, \varphi_2) \in \mathbb{R}^2$, except the countable family of circles centered at the origin with radius $x_a \omega/2$, $a = 1, 2, \ldots$, where x_a if the a-th zero of J_0 in $\mathbb{R}_+$ (J_0 is the Bessel function of order zero). However, in these circles condition II is nowhere fulfilled. See the discussion in Section VI.

7. From the computational point of view the solution given by our method can be easily implemented in numerical programs and has been successfully tested, providing ways to study our two-level system for large times with controllable errors (due to the uniform convergence).

8. Unitarity of $U(t)$ for all $t \in \mathbb{R}$ is a well known consequence of Dyson's expansion (see f.i. [18]).

9. Conditions I and II define, in principle, distinct solutions of the generalized Riccati equation (I.7) and, hence, of the Schrödinger equation (I.5). To fix a name we will call these solutions "classes" of solutions.

10. As we will discuss, condition I is mostly important for the case $F_0 = 0$, while condition II is mostly important for the case $F_0 \neq 0$. There are, however, particular cases where condition I holds for $F_0 \neq 0$ and condition II for $F_0 = 0$, but examples indicate that such situations are rather exceptional. See Section VI.1.

For the proof of Theorem I.2 we have to consider two distinct cases, the case where $F_0 = 0$ and the case where $F_0 \neq 0$. The former will be considered in Section III and the later in Section IV.

I.2 Some Definitions and Some Remarks on the Notation

Let us make some remarks on the notation we use here and recall the notation used in [1]. Given the Fourier representation[2]

$$f(t) \;=\; \sum_{\underset{\sim}{m}\in\mathbb{Z}^B} F_{\underset{\sim}{m}}\, e^{i\underset{\sim}{m}\cdot\underset{\sim}{\omega}_f t} \tag{I.23}$$

of a quasi-periodic function f, we denote (as in [1]) by $\underline{\omega}$ the vector of frequencies defined by

$$\underline{\omega} \;:=\; \begin{cases} \underset{\sim}{\omega}_f \in \mathbb{R}^B, & \text{if } F_{\underset{\sim}{0}} = 0 \\ \\ (\underset{\sim}{\omega}_f,\ F_{\underset{\sim}{0}}) \in \mathbb{R}^{B+1}, & \text{if } F_{\underset{\sim}{0}} \neq 0, \end{cases} \quad . \tag{I.24}$$

Since we assume that $\underset{\sim}{\omega}_f \in \mathbb{R}_+^B$, the definition above says that all components of $\underline{\omega}$ are always non-zero. Moreover, we denote

$$A \;:=\; \begin{cases} B, & \text{if } F_{\underset{\sim}{0}} = 0 \\ \\ B+1, & \text{if } F_{\underset{\sim}{0}} \neq 0 \end{cases} \quad . \tag{I.25}$$

We will frequently use $F_0 \equiv F_{\underset{\sim}{0}}$.

We will denote vectors in $\mathbb{Z}^B$ (or $\mathbb{R}^B$) by $\underset{\sim}{v}$ and vectors in $\mathbb{Z}^A$ (or $\mathbb{R}^A$) by $\underline{v}$. The symbol $|\underline{n}|$ denotes the $l^1(\mathbb{Z}^A)$ norm of a vector $\underline{n} = (n_1, \ldots, n_A) \in \mathbb{Z}^A$: $|\underline{n}| := |n_1| + \cdots + |n_A|$.

We denote by $\lfloor x \rfloor$ the largest integer lower or equal to $x \in \mathbb{R}$ and by $\lceil x \rceil$ the smallest integer larger or equal to $x \in \mathbb{R}$

For $m \in \mathbb{Z}$ we denote by $\ll\! m\! \gg$ the following function:

$$\ll\! m\! \gg \;:=\; \begin{cases} |m|, & \text{for } m \neq 0 \\ 1, & \text{for } m = 0 \end{cases} \quad . \tag{I.26}$$

In the case where f is a quasi-periodic function as in (I.23) we will denote by $\mathbf{Q}_{\underline{m}}$ the Fourier coefficients of the function q, defined in (I.17):

$$q(t) \;=\; \sum_{\underline{m}\in\mathbb{Z}^A} \mathbf{Q}_{\underline{m}} e^{i\underline{m}\cdot\underline{\omega}t}, \tag{I.27}$$

and by $\mathbf{Q}^{(2)}_{\underline{m}}$ the Fourier coefficients of the function q^2:

$$q(t)^2 \;=\; \sum_{m\in\mathbb{Z}^A} \mathbf{Q}^{(2)}_{\underline{m}} e^{i\underline{m}\cdot\underline{\omega}t}. \tag{I.28}$$

[2] For convenience we adopt here a different notation of that found in [1], where the Fourier decomposition of f was written as $f(t) \;=\; \sum_{\underset{\sim}{m}\in\mathbb{Z}^B} f_{\underset{\sim}{m}}\, e^{i\underset{\sim}{m}\cdot\underset{\sim}{\omega}_f t}$.

Finally, for an almost periodic function h we denote by $M(h)$ its "mean value", defined as

$$M(h) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} h(t)\, dt.$$

See, e.g. [16, 17]. The mean value $M(h)$ equals the constant term in the Fourier expansion of h. One has, for instance, $M(q^2) = \mathbf{Q}^{(2)}_{\underline{0}}$.

II Some Previous Results

In [1] some results could be proven about the nature of some particular solutions of (I.7) for the case where f is a quasi-periodic function subjected to some additional restrictions. These results are described in Theorem II.1.

Theorem II.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be quasi-periodic with*

$$f(t) \;=\; \sum_{\underset{\sim}{n}\in\mathbb{Z}^B} F_{\underset{\sim}{n}}\, e^{i\underset{\sim}{\omega}_f\cdot\underset{\sim}{n}t},$$

and such that the sum above contains only a finite number of terms. Assume that the vector $\underline{\omega}$ (defined in (I.24)) satisfies Diophantine conditions, i.e., assume the existence of constants $\Delta > 0$ and $\sigma > 0$ such that, for all $\underline{n} \in \mathbb{Z}^A$, $\underline{n} \neq \underline{0}$,

$$|\underline{n}\cdot\underline{\omega}| \geq \Delta^{-1}|\underline{n}|^{-\sigma}.$$

I. *Assume that f satisfies the condition $M(q^2) \neq 0$. Then, there exists a formal power series*

$$g(t) \;=\; q(t)\sum_{n=1}^{\infty} c_n(t)\epsilon^n, \tag{II.1}$$

representing a particular solution of the generalized Riccati equation (I.7) such that all coefficients c_n can be chosen to be quasi-periodic and can be represented as

$$c_n(t) \;=\; \sum_{\underline{m}\in\mathbb{Z}^A} \mathbf{C}^{(n)}_{\underline{m}}\, e^{i\underline{m}\cdot\underline{\omega}t}, \tag{II.2}$$

where, for the Fourier coefficients $\mathbf{C}^{(n)}_{\underline{m}}$, we have

$$|\mathbf{C}^{(n)}_{\underline{m}}| \;\leq\; \mathcal{K}_n e^{-\chi_0|\underline{m}|},$$

where $\chi_0 > 0$ is a constant and $\mathcal{K}_n \geq 0$.

II. *Assume that f satisfies the conditions*

$$M(q^2) = 0 \qquad \textit{and} \qquad M(\mathcal{Q}_1) \neq 0,$$

where $\mathcal{Q}_1$ is defined in (I.19). Then, there exists a formal power series

$$g(t) \ = \ q(t) \sum_{n=1}^{\infty} e_n(t) \epsilon^{2n}, \tag{II.3}$$

representing a particular solution of the generalized Riccati equation (I.7) such that all coefficients e_n can be chosen to be quasi-periodic and can be represented as

$$e_n(t) \ = \ \sum_{\underline{m} \in \mathbb{Z}^A} \mathbf{E}^{(n)}_{\underline{m}} \, e^{i\underline{m} \cdot \underline{\omega} t}, \tag{II.4}$$

where, for the Fourier coefficients $\mathbf{E}^{(n)}_{\underline{m}}$, we have

$$|\mathbf{E}^{(n)}_{\underline{m}}| \ \leq \ \mathcal{L}_n e^{-\chi_0 |\underline{m}|},$$

where $\chi_0 > 0$ is a constant and $\mathcal{L}_n \geq 0$.

There are other conditions beyond *I* and *II* which could be considered, but they have not been explored so far. See the discussion in Section VI.

The statements of this last theorem are not sufficient for proving convergence of the power series expansions in ϵ for g in the general case of quasi-periodic f. Unfortunately, as discussed in [1], the behavior for large n of the constants $\mathcal{K}_n$ and $\mathcal{L}_n$ is too bad to guarantee absolute convergence of the formal power series above.

For the restricted case were f is periodic we will in the present paper prove stronger results (Theorem III.1 below) than that implied by Theorem II.1. As we will see, these stronger results, in contrast, imply convergence of the ϵ-power series for g (Theorem III.3 below).

Some of the more technical results of [1] have been obtained through the analysis of the Fourier coefficients of the functions c_n and e_n defined in Theorem II.1 above. Specially important for us are the recursion relations found in [1] for the Fourier coefficients $\mathbf{C}^{(n)}_{\underline{m}}$ and $\mathbf{E}^{(n)}_{\underline{m}}$ defined in (II.2) and (II.4), respectively. Those recursion relations follow by imposing the generalized Riccati equation (I.7) to the power expansions (II.1) and (II.3). In Appendix A we reproduce some of the main ideas of [1] leading to a power series expansion for g free of secular terms and leading to the recursion relations below.

It is important for our present purposes to reproduce those recursive relations here, what we shall do now.

As in (I.27)–(I.28), we denote by $\mathbf{Q}_{\underline{m}}$ the Fourier coefficients of the function q and by $\mathbf{Q}^{(2)}_{\underline{m}}$ the Fourier coefficients of the function q^2. For the Fourier coefficients of the functions c_n we have the following relations:

$$\mathbf{C}^{(1)}_{\underline{m}} = \alpha_1 \mathbf{Q}_{\underline{m}}, \tag{II.5}$$

$$\mathbf{C}^{(2)}_{\underline{m}} = \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{\left(\alpha_1^2\mathbf{Q}^{(2)}_{\underline{n}} - \overline{\mathbf{Q}^{(2)}_{-\underline{n}}}\right)}{\underline{n}\cdot\underline{\omega}} \left[\mathbf{Q}_{\underline{m}-\underline{n}} - \frac{\mathbf{Q}_{\underline{m}}\mathbf{Q}^{(2)}_{-\underline{n}}}{\mathbf{Q}^{(2)}_{\underline{0}}}\right], \tag{II.6}$$

$$\begin{aligned}\mathbf{C}^{(n)}_{\underline{m}} = & \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \frac{1}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} \left[\mathbf{Q}_{\underline{m}-(\underline{n}_1+\underline{n}_2)} - \frac{\mathbf{Q}_{\underline{m}}\mathbf{Q}^{(2)}_{-\underline{n}_1-\underline{n}_2}}{\mathbf{Q}^{(2)}_{\underline{0}}}\right] \sum_{p=1}^{n-1} \mathbf{C}^{(p)}_{\underline{n}_1}\mathbf{C}^{(n-p)}_{\underline{n}_2} \\ & - \frac{\mathbf{Q}_{\underline{m}}}{2\alpha_1\mathbf{Q}^{(2)}_{\underline{0}}} \sum_{\underline{n}\in\mathbb{Z}^A}\sum_{p=2}^{n-1} \mathbf{C}^{(p)}_{\underline{n}}\mathbf{C}^{(n+1-p)}_{-\underline{n}}, \qquad \text{for } n\geq 3. \end{aligned} \tag{II.7}$$

Above $\underline{m}\in\mathbb{Z}^A$, $\alpha_1^2 = \dfrac{\overline{M(q^2)}}{M(q^2)}$. For the Fourier coefficients of the functions e_n we have the following relations.

$$\mathbf{E}^{(1)}_{\underline{m}} = \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{\mathbf{Q}_{\underline{m}+\underline{n}}\overline{\mathbf{Q}^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}} + \frac{\mathbf{Q}_{\underline{m}}}{2iM(\mathcal{Q}_1)} \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1\neq\underline{0},\ \underline{n}_2\neq\underline{0}}} \frac{\mathbf{Q}^{(2)}_{\underline{n}_1+\underline{n}_2}\,\overline{\mathbf{Q}^{(2)}_{\underline{n}_1}}\,\overline{\mathbf{Q}^{(2)}_{\underline{n}_2}}}{(\underline{n}_1\cdot\underline{\omega})(\underline{n}_2\cdot\underline{\omega})}, \tag{II.8}$$

$$\begin{aligned}\mathbf{E}^{(n)}_{\underline{m}} = & \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1+\underline{n}_2\neq\underline{0}}} \left[\mathbf{Q}_{\underline{m}-\underline{n}_1-\underline{n}_2} + \frac{\mathbf{Q}_{\underline{m}}}{iM(\mathcal{Q}_1)}\left(\mathbf{Q}^{(2)}_{-\underline{n}_1-\underline{n}_2}\mathcal{R} + \sum_{\substack{\underline{n}\in\mathbb{Z}^A\\ \underline{n}\neq\underline{0}}} \frac{\mathbf{Q}^{(2)}_{\underline{n}-\underline{n}_1-\underline{n}_2}\overline{\mathbf{Q}^{(2)}_{\underline{n}}}}{\underline{n}\cdot\underline{\omega}}\right)\right] \\ & \times \frac{\sum_{p=1}^{n-1}\mathbf{E}^{(p)}_{\underline{n}_1}\mathbf{E}^{(n-p)}_{\underline{n}_2}}{(\underline{n}_1+\underline{n}_2)\cdot\underline{\omega}} + \frac{\mathbf{Q}_{\underline{m}}}{2iM(\mathcal{Q}_1)} \sum_{\underline{n}\in\mathbb{Z}^A}\sum_{p=2}^{n-1} \mathbf{E}^{(p)}_{\underline{n}}\mathbf{E}^{(n+1-p)}_{-\underline{n}}, \quad n\geq 2. \end{aligned} \tag{II.9}$$

Above $\underline{m}\in\mathbb{Z}^A$, $\mathcal{Q}_1$ is defined in (I.19) and

$$\mathcal{R} := \frac{1}{2iM(\mathcal{Q}_1)} \sum_{\substack{\underline{n}_1,\,\underline{n}_2\in\mathbb{Z}^A\\ \underline{n}_1\neq\underline{0},\ \underline{n}_2\neq\underline{0}}} \frac{\mathbf{Q}^{(2)}_{\underline{n}_1+\underline{n}_2}\,\overline{\mathbf{Q}^{(2)}_{\underline{n}_1}}\,\overline{\mathbf{Q}^{(2)}_{\underline{n}_2}}}{(\underline{n}_1\cdot\underline{\omega})(\underline{n}_2\cdot\underline{\omega})}. \tag{II.10}$$

The above expressions for the Fourier coefficients are somewhat complex but two important features can be distinguished. The first is the inevitable presence of "small denominators", represented by the various factors of the form $(\underline{n}\cdot\underline{\omega})^{-1}$ (with $\underline{n}\neq\underline{0}$) appearing above. The second is the presence of convolution products (a consequence, lately, of the quadratic character of the generalized Riccati equation). The presence of the later is the additional source of complications mentioned before, for they also, together with the small denominators, contribute to spoil the decay of the Fourier coefficients needed to prove convergence of the ϵ-expansions.

II.1 The Fourier Coefficients $\mathbf{Q}_{\underline{m}}$ and $\mathbf{Q}^{(2)}_{\underline{m}}$

For future purposes, it is important now to look more closely at the Fourier coefficients $\mathbf{Q}_{\underline{m}}$ and $\mathbf{Q}^{(2)}_{\underline{m}}$.

By assumption, the set $\{\underset{\sim}{n} \in \mathbb{Z}^B,\ \underset{\sim}{n} \neq \underset{\sim}{0} |\ F_{\underset{\sim}{n}} \neq \underset{\sim}{0}\}$ is a finite set and, by the condition that f is real, it contains an even number of elements, say $2J$ with $J \geq 1$. Let us write this set as $\{\underset{\sim}{n}_1, \ldots, \underset{\sim}{n}_{2J}\}$ with the convention $\underset{\sim}{n}_a = -\underset{\sim}{n}_{2J-a+1} \neq \underset{\sim}{0}$, $1 \leq a \leq J$, and let us write f in the form

$$f(t) \ = \ F_0 + \sum_{a=1}^{2J} f_a e^{i\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f t}, \tag{II.11}$$

with $f_a \equiv F_{\underset{\sim}{n}_a}$. Clearly $\overline{f_a} = f_{2J-a+1}$, $1 \leq a \leq J$, since f is real.

A simple computation [1] shows that

$$q(t) \ = \ e^{i\gamma_f} \sum_{p_1,\, \ldots,\, p_{2J}=0}^{\infty} \exp\left(i \left(F_0 + \underset{\sim}{\omega}_f \cdot \sum_{b=1}^{2J} p_b \, \underset{\sim}{n}_b \right) t \right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right)^{p_a} \right], \tag{II.12}$$

with

$$\gamma_f \ := \ i \sum_{a=1}^{2J} \frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f}. \tag{II.13}$$

One sees that $\gamma_f \in \mathbb{R}$. The function q^2 is obtained by the replacement $f \to 2f$:

$$q(t)^2 \ = \ e^{i2\gamma_f} \sum_{p_1,\, \ldots,\, p_{2J}=0}^{\infty} \exp\left(i \left(2F_0 + \underset{\sim}{\omega}_f \cdot \sum_{b=1}^{2J} p_b \, \underset{\sim}{n}_b \right) t \right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{2f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right)^{p_a} \right].$$

From (II.12) we conclude that, if F_0 is not of the form $F_0 = \underset{\sim}{\omega}_f \cdot \underset{\sim}{k}$, for some vector of integers $\underset{\sim}{k}$, one has

$$q(t) = \sum_{\underline{m} \in \mathbb{Z}^A} \mathbf{Q}_{\underline{m}} e^{i\underline{m}\cdot\underline{\omega}t}$$

with $\underline{\omega}$ defined in (I.24) and

$$\mathbf{Q}_{\underline{m}} \ = \ e^{i\gamma_f} \sum_{p_1,\, \ldots,\, p_{2J}=0}^{\infty} \delta\left(\underline{P},\ \underline{m}\right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f} \right)^{p_a} \right], \tag{II.14}$$

where

$$\underline{P} \ \equiv \ \underline{P}(p_1, \ldots, p_{2J}, \underset{\sim}{n}_1, \ldots, \underset{\sim}{n}_{2J}) \ := \ \begin{cases} \displaystyle\sum_{b=1}^{2J} p_b \underset{\sim}{n}_b \in \mathbb{Z}^B, & \text{if } F_0 = 0, \\[2ex] \left(\displaystyle\sum_{b=1}^{2J} p_b \underset{\sim}{n}_b,\ 1 \right) \in \mathbb{Z}^{B+1}, & \text{if } F_0 \neq 0. \end{cases} \tag{II.15}$$

and where

$$\delta\left(\underline{P},\ \underline{m}\right) := \begin{cases} 1, & \text{if } \underline{P} = \underline{m}, \\ 0, & \text{else.} \end{cases} \tag{II.16}$$

For q^2, and if F_0 is not of the form $2F_0 = \underset{\sim}{\omega}_f \cdot \underset{\sim}{k}$, for some vector of integers $\underset{\sim}{k}$, we have

$$q^2(t) = \sum_{\underline{m} \in \mathbb{Z}^A} \mathbf{Q}^{(2)}_{\underline{m}} e^{i\underline{m}\cdot\underline{\omega} t},$$

where

$$\mathbf{Q}^{(2)}_{\underline{m}} = e^{i2\gamma_f} \sum_{p_1,\ldots,p_{2J}=0}^{\infty} \delta\left(\underline{P}^{(2)},\ \underline{m}\right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!}\left(\frac{2f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f}\right)^{p_a}\right], \tag{II.17}$$

with

$$\underline{P}^{(2)} \equiv \underline{P}^{(2)}(p_1,\ldots,p_{2J},\underset{\sim}{n}_1,\ldots,\underset{\sim}{n}_{2J}) := \begin{cases} \displaystyle\sum_{b=1}^{2J} p_b \underset{\sim}{n}_b \in \mathbb{Z}^B, & \text{if } F_0 = 0, \\ \left(\displaystyle\sum_{b=1}^{2J} p_b \underset{\sim}{n}_b,\ 2\right) \in \mathbb{Z}^{B+1}, & \text{if } F_0 \neq 0. \end{cases}$$

Let us now study the condition $M(q^2) = \mathbf{Q}^{(2)}_{\underline{0}} = 0$ for $F_0 \neq 0$, F_0 not of the form $2F_0 = \underset{\sim}{\omega}_f \cdot \underset{\sim}{k}$, for some vector of integers $\underset{\sim}{k}$. We have from (II.17)

$$M(q^2) = e^{i2\gamma_f} \sum_{p_1,\ldots,p_{2J}=0}^{\infty} \delta\left(\underline{P}^{(2)},\ \underline{0}\right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!}\left(\frac{2f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f}\right)^{p_a}\right]. \tag{II.18}$$

Since the last component of $\underline{P}^{(2)}$ equals 2 for $F_0 \neq 0$, we always have $\delta(\underline{P}^{(2)},\ \underline{0}) = 0$ in the sum above and, hence, $M(q^2) = 0$. This means that, for $F_0 \neq 0$ condition I never happens, except perhaps for the case where $2F_0 = \underset{\sim}{\omega}_f \cdot \underset{\sim}{k}$, $\underset{\sim}{k} \in \mathbb{Z}^B$, much in contrast to the case $F_0 = 0$, where condition I holds almost everywhere in the space of the functions f (see Section VI.1).

From (II.14) and (II.17) it is clear that for $F_0 \neq 0$, and $2F_0 \neq \underset{\sim}{\omega}_f \cdot \underset{\sim}{k}$, with $\underset{\sim}{k} \in \mathbb{Z}^B$, one has, writing $\underline{m} = (\underset{\sim}{m},\ m_A)$,

$$\mathbf{Q}_{\underline{m}} = Q_{\underset{\sim}{m}}\, \delta_{m_A,\,1} \quad \text{and} \quad \mathbf{Q}^{(2)}_{\underline{m}} = Q^{(2)}_{\underset{\sim}{m}}\, \delta_{m_A,\,2}, \tag{II.19}$$

where δ is the usual Krönecker delta and where

$$Q_{\underset{\sim}{m}} := e^{i\gamma_f} \sum_{p_1,\ldots,p_{2J}=0}^{\infty} \delta\left(\underset{\sim}{P},\ \underset{\sim}{m}\right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!}\left(\frac{f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f}\right)^{p_a}\right], \tag{II.20}$$

and

$$Q^{(2)}_{\underset{\sim}{m}} := e^{2i\gamma_f} \sum_{p_1,\ldots,p_{2J}=0}^{\infty} \delta\left(\underset{\sim}{P},\ \underset{\sim}{m}\right) \prod_{a=1}^{2J} \left[\frac{1}{p_a!}\left(\frac{2f_a}{\underset{\sim}{n}_a \cdot \underset{\sim}{\omega}_f}\right)^{p_a}\right], \tag{II.21}$$

with

$$\underset{\sim}{P} \; := \; \sum_{b=1}^{2J} p_b \underset{\sim}{n}_b \in \mathbb{Z}^B . \tag{II.22}$$

The observation taken from (II.19) that $\mathbf{Q}_{\underline{m}}$ and $\mathbf{Q}^{(2)}_{\underline{m}}$ are zero except if $m_A = 1$, respectively, if $m_A = 2$, will be of crucial importance for the analysis of the case $F_0 \neq 0$, given in Section IV. This is because these restrictions propagate in a specific way to the Fourier coefficients $\mathbf{E}^{(n)}_{\underline{m}}$.

Below we will make use of the following proposition on the decay of the coefficients Q_m and $Q^{(2)}_m$:

Proposition II.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic and be represented by a finite Fourier series as in (I.18). Then, for any $\chi > 0$ there is a positive constant $\mathcal{Q} \equiv \mathcal{Q}(\chi)$ such that*

$$|Q_m| \; \leq \; \mathcal{Q} \frac{e^{-\chi |m|}}{\ll m \gg^2} \qquad \textit{and} \qquad |Q^{(2)}_m| \; \leq \; \mathcal{Q} \frac{e^{-\chi |m|}}{\ll m \gg^2} \tag{II.23}$$

for all $m \in \mathbb{Z}$, where the symbol $\ll m \gg$ is defined in (I.26).

The proof is found in Appendix B. Finally, we mention the following important lemma, whose proof is given in Appendix C.

Lemma II.3 *For $\chi > 0$ and $m \in \mathbb{Z}$ define*

$$\mathcal{B}(m) \equiv \mathcal{B}(m, \, \chi) \; := \; \sum_{n \in \mathbb{Z}} \frac{e^{-\chi(|m-n|+|n|)}}{\ll m - n \gg^2 \, \ll n \gg^2} . \tag{II.24}$$

Then one has

$$\mathcal{B}(m) \; \leq \; B_0 \frac{e^{-\chi |m|}}{\ll m \gg^2} , \tag{II.25}$$

for some constant $B_0 \equiv B_0(\chi) > 0$ and for all $m \in \mathbb{Z}$.

We are ready now to start the analysis of the recursion relations (II.5)–(II.7) and (II.8)–(II.9) for the periodic case. As already mentioned, we have to consider two separated cases: the case where $F_0 = 0$, we will deal with now, and the case $F_0 \neq 0$, which will be treated in Section IV.

III The Periodic Case With $F_0 = 0$

In [1] the recursion relations (II.5)–(II.7) and (II.8)–(II.9) have been used to prove inductively exponential bounds for the Fourier coefficients. As mentioned before two main difficulties have to be faced in this enterprise: the presence of "small denominators" and of convolution products in the recursion relations. Both are responsible for reducing the rate of decay of the Fourier coefficients at each induction step.

Let us consider the origin of the "small denominators problem" in our expansions. It comes from the many factors of the form $(\underline{n}\cdot\underline{\omega})^{-1}$ (with $\underline{n}\neq\underline{0}$) appearing in the recursion relations. In the case where f is a periodic function with frequency ω with $F_0\neq 0$, we have $A=2$, $\underline{n}=(n_1,\ n_2)\in\mathbb{Z}^2$ and $\underline{n}\cdot\underline{\omega}=n_1\omega+n_2F_0$. On the other hand, in the case where f is a periodic function with frequency ω and with $F_0=0$, we have $A=1$, $\underline{n}=n\in\mathbb{Z}$ and $\underline{n}\cdot\underline{\omega}=n\omega$. To avoid the quasi-resonant situation where $n_1\omega+n_2F_0$ is small we will first consider the case where $F_0=0$.

III.1 The Recursive Relations in the Periodic Case for $F_0=0$

Under the hypothesis, the recursive relations for the Fourier coefficients of the functions c_n become

$$C_m^{(1)} = \alpha_1 Q_m, \tag{III.1}$$

$$C_m^{(2)} = \sum_{\substack{n_1\in\mathbb{Z}\\ n_1\neq 0}} \frac{\left(\alpha_1^2 Q_{n_1}^{(2)} - \overline{Q_{-n_1}^{(2)}}\right)}{n_1\omega}\left[Q_{m-n_1} - \frac{Q_m Q_{-n_1}^{(2)}}{Q_0^{(2)}}\right], \tag{III.2}$$

$$\begin{aligned} C_m^{(n)} &= \sum_{\substack{n_1,\,n_2\in\mathbb{Z}\\ n_1+n_2\neq 0}} \frac{1}{(n_1+n_2)\cdot\omega}\left[Q_{m-(n_1+n_2)} - \frac{Q_m Q_{-n_1-n_2}^{(2)}}{Q_0^{(2)}}\right]\sum_{p=1}^{n-1} C_{n_1}^{(p)} C_{n_2}^{(n-p)} \\ &\quad - \frac{Q_m}{2\alpha_1 Q_0^{(2)}}\sum_{n_1\in\mathbb{Z}}\sum_{p=2}^{n-1} C_{n_1}^{(p)} C_{-n_1}^{(n+1-p)}, \qquad \text{for } n\geq 3. \end{aligned} \tag{III.3}$$

Above $m\in\mathbb{Z}$ and $\alpha_1^2 = \dfrac{\overline{Q_0^{(2)}}}{Q_0^{(2)}}$.

For the Fourier coefficients of the functions e_n we have:

$$E_m^{(1)} = \sum_{\substack{n_1\in\mathbb{Z}\\ n_1\neq 0}} \frac{Q_{m+n_1}\overline{Q_{n_1}^{(2)}}}{n_1\omega} + \frac{Q_m}{2iM(\mathcal{Q}_1)}\sum_{\substack{n_1,\,n_2\in\mathbb{Z}\\ n_1\neq 0,\ n_2\neq 0}} \frac{Q_{n_1+n_2}^{(2)}\,\overline{Q_{n_1}^{(2)}}\,\overline{Q_{n_2}^{(2)}}}{(n_1\omega)(n_2\omega)} \tag{III.4}$$

$$\begin{aligned} E_m^{(n)} &= \sum_{\substack{n_1,\,n_2\in\mathbb{Z}\\ n_1+n_2\neq 0}}\left[Q_{m-n_1-n_2} + \frac{Q_m}{iM(\mathcal{Q}_1)}\left(Q_{-n_1-n_2}^{(2)}\mathcal{R} + \sum_{\substack{n_3\in\mathbb{Z}\\ n_3\neq 0}}\frac{Q_{n_3-n_1-n_2}^{(2)}\overline{Q_{n_3}^{(2)}}}{n_3\omega}\right)\right] \\ &\quad\times \frac{\sum_{p=1}^{n-1} E_{n_1}^{(p)} E_{n_2}^{(n-p)}}{(n_1+n_2)\omega} + \frac{Q_m}{2iM(\mathcal{Q}_1)}\sum_{p=2}^{n-1}\sum_{n_1\in\mathbb{Z}} E_{n_1}^{(p)} E_{-n_1}^{(n+1-p)}, \quad n\geq 2. \end{aligned} \tag{III.5}$$

It is clear here that no "small denominators" appear in this case, since now $|(\underline{n}\cdot\underline{\omega})^{-1}|\leq\omega^{-1}$ for $\underline{n}\neq\underline{0}$. Hence, the convolution products are the only remaining factors eventually forcing the reduction of the decay rate of the Fourier coefficients at the successive induction steps.

In the Section III.2 we will show how the effect of the convolution products can be taken under control. The result is expressed in the following three theorems.

Theorem III.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic with a finite Fourier decomposition as in (I.18) and with $F_0 = 0$.*

Case I. *Consider the Fourier coefficients $C_m^{(n)}$ satisfying the recursion relations (III.1), (III.2) and (III.3). Under the hypothesis that $M(q^2) \neq 0$ we have*

$$|C_m^{(n)}| \;\leq\; K_n \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{III.6}$$

for all $n \in \mathbb{N}$, and all $m \in \mathbb{Z}$, where $\chi > 0$ is a constant and $\ll m \gg$ is defined in (I.26). Above, the coefficients K_n do not depend on m and satisfy the recursion relation

$$K_n \;=\; \mathcal{C}_2 \left[\left(\sum_{p=1}^{n-1} K_p K_{n-p} \right) + \left(\sum_{p=2}^{n-1} K_p K_{n+1-p} \right) \right] , \tag{III.7}$$

with $K_1 = K_2 = \mathcal{C}_1$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are positive constants which can be chosen larger than or equal to 1.

Case II. *Consider the Fourier coefficients $E_m^{(n)}$ satisfying the recursion relations (III.4) and (III.5). Under the hypothesis that $M(q^2) = 0$ and $M(\mathcal{Q}_1) \neq 0$ we have*

$$|E_m^{(n)}| \;\leq\; K'_n \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{III.8}$$

for all $n \in \mathbb{N}$, and all $m \in \mathbb{Z}$, where $\chi > 0$ is a constant. Above, the coefficients K'_n do not depend on m and satisfy the recursion relation

$$K'_n \;=\; \mathcal{E}_2 \left[\left(\sum_{p=1}^{n-1} K'_p K'_{n-p} \right) + \left(\sum_{p=2}^{n-1} K'_p K'_{n+1-p} \right) \right] , \tag{III.9}$$

with $K'_1 = K'_2 = \mathcal{E}_1$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are positive constants which can be chosen larger than or equal to 1.

Theorem III.1 will be proven in Section III.2. The importance of the recursive definition of the constants K_n given in (III.7) or (III.9) is expressed in the following crucial theorem, which says that the constants K_n grow at most exponentially with n.

Theorem III.2 *Let the constants K_n be defined through the recurrence relations (III.7) or (III.9). Then there exist constants $K > 0$ and $K_0 > 0$ (depending eventually on f) such that $K_n \leq K_0 K^n$ for all $n \in \mathbb{N}$.*

The proof of Theorem III.2 is found in Appendix D and makes interesting use of properties of the Catalan sequence. Theorems III.1 and III.2 have the following immediate corollary:

Theorem III.3 *The power series expansions in (II.1) and (II.3) are absolutely convergent for all $\epsilon \in \mathbb{C}$ with $|\epsilon| < K^{-1}$ for all $t \in \mathbb{R}$ and, hence, (II.1) and (II.3) define particular solutions of the generalized Riccati equation (I.7) in cases I and II, respectively, of Theorem III.1. The function g can be expressed in terms of an absolutely and uniformly converging Fourier series whose coefficients can be expressed in terms of absolutely converging power series in ϵ for all $\epsilon \in \mathbb{C}$ with $|\epsilon| < K^{-1}$.*

Proof of Theorem III.3. We prove the statement for case I. Case II is analogous. The first step is to determine the Fourier expansion of the function g, as given in (I.16), and to study some of their properties. One clearly has

$$g(t) \;=\; \sum_{m\in\mathbb{Z}} G_m e^{im\omega t}, \tag{III.10}$$

with

$$G_m \;\equiv G_m(\epsilon) \;\;=\; \sum_{n=1}^{\infty} \epsilon^n G_m^{(n)}, \tag{III.11}$$

where

$$G_m^{(n)} \;:=\; \sum_{l\in\mathbb{Z}} Q_{m-l} C_l^{(n)}. \tag{III.12}$$

We have the following proposition:

Proposition III.4 *For all $\chi > 0$ there exists a constant $\mathcal{C}_g \equiv \mathcal{C}_g(\chi) > 0$ such that*

$$|G_m^{(n)}| \;\leq\; \mathcal{C}_g K_n \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{III.13}$$

for all $m \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Consequently, for $|\epsilon| < K$ one has

$$|G_m| \;\leq\; \mathcal{C}'_g \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{III.14}$$

for some constant $\mathcal{C}'_g(\chi, \epsilon) > 0$ and for all $m \in \mathbb{Z}$.

Proof of Proposition III.4. Inserting (II.23) and (III.6) into (III.12) we have, for any $\chi > 0$,

$$\left| G_m^{(n)} \right| \;\leq\; \mathcal{Q} K_n \, \mathcal{B}(m, \chi), \tag{III.15}$$

where $\mathcal{B}(m, \chi)$ is defined in (II.24). Relation (III.13) follows now from Lemma II.3. $\square$

From this, the proof of Theorem III.3 follows immediately. $\square$

The solutions for the generalized Riccati equation (I.7) mentioned in Theorem III.3 are, through (I.9), the main ingredient for the solution of the Schrödinger equation (I.5). This will be further discussed in Section V. Now we have to prove Theorem III.1.

III.2 Inductive Bounds for the Fourier Coefficients

In this section we will prove Theorem III.1 in cases I and II. We will make use of Proposition II.2 on the decay of the Fourier coefficients Q_m and $Q_m^{(2)}$ of the functions q and q^2, respectively.

III.2.1 Case I

In this section we will prove Theorem III.1 in case I. Making use of Proposition II.2 and of relations (III.1)–(III.3) we easily derive the following estimates:

$$|C_m^{(1)}| \leq \mathcal{Q} \frac{e^{-\chi|m|}}{\ll m \gg^2}, \tag{III.16}$$

$$|C_m^{(2)}| \leq \frac{2\mathcal{Q}}{\omega} \sum_{n_1 \in \mathbb{Z}} \frac{e^{-\chi|n_1|}}{\ll n_1 \gg^2} \left[\frac{e^{-\chi|m-n_1|}}{\ll m-n_1 \gg^2} + \frac{\mathcal{Q}}{|Q_0^{(2)}|} \frac{e^{-\chi(|m|+|n_1|)}}{\ll m \gg^2 \ll n_1 \gg^2} \right], \tag{III.17}$$

$$\begin{aligned} |C_m^{(n)}| \leq & \frac{\mathcal{Q}}{\omega} \sum_{n_1,\, n_2 \in \mathbb{Z}} \left(\sum_{p=1}^{n-1} |C_{n_1}^{(p)}|\, |C_{n_2}^{(n-p)}| \right) \times \\ & \times \left[\frac{e^{-\chi|m-(n_1+n_2)|}}{\ll m-(n_1+n_2) \gg^2} + \frac{\mathcal{Q}}{|Q_0^{(2)}|} \frac{e^{-\chi(|m|+|n_1+n_2|)}}{\ll m \gg^2 \ll n_1+n_2 \gg^2} \right] \\ & + \frac{\mathcal{Q}}{2|Q_0^{(2)}|} \frac{e^{-\chi|m|}}{\ll m \gg^2} \sum_{n_1 \in \mathbb{Z}} \sum_{p=2}^{n-1} |C_{n_1}^{(p)}|\, |C_{-n_1}^{(n+1-p)}|, \quad \text{for } n \geq 3. \end{aligned} \tag{III.18}$$

It follows from (III.17), from the definition of $\mathcal{B}(m)$ in (II.24) and from Lemma II.3 that

$$|C_m^{(2)}| \leq 2\omega^{-1}\mathcal{Q} \left(\mathcal{B}(m) + \frac{\mathcal{Q}}{|Q_0^{(2)}|} \frac{e^{-\chi|m|}}{\ll m \gg^2} \sum_{n_1 \in \mathbb{Z}} \frac{e^{-2\chi|n_1|}}{\ll n_1 \gg^4} \right) \leq K_2 \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{III.19}$$

for some convenient choice of the constant K_2.

Now, we will use an induction argument to establish (III.6) for all $n \geq 3$. Let us assume that, for a given $n \in \mathbb{N}$, $n \geq 3$, one has

$$|C_m^{(p)}| \; \leq \; K_p \frac{e^{-\chi|m|}}{\ll m \gg^2}, \qquad \forall m \in \mathbb{Z}, \tag{III.20}$$

for all p such that $1 \leq p \leq n-1$, for some convenient constants K_p. We will establish that this implies the same sort of bound for $p = n$. Note, by taking

$K_1 \geq \mathcal{Q}$, that relation (III.16) guarantees (III.20) for $p = 1$ and that relation (III.19) guarantees the case $p = 2$.

From (III.18) and from the induction hypothesis,

$$\begin{aligned} |C_m^{(n)}| &\leq \omega^{-1}\mathcal{Q}\left(\sum_{p=1}^{n-1} K_p K_{n-p}\right)\left[\sum_{n_1,\, n_2\in\mathbb{Z}} \frac{e^{-\chi(|m-(n_1+n_2)|+|n_1|+|n_2|)}}{\ll m-(n_1+n_2)\gg^2 \ll n_1\gg^2 \ll n_2\gg^2}\right. \\ &+ \left.\frac{\mathcal{Q}}{|Q_0^{(2)}|}\frac{e^{-\chi|m|}}{\ll m\gg^2}\sum_{n_1,\, n_2\in\mathbb{Z}} \frac{e^{-\chi(|n_1+n_2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2 \ll n_1\gg^2 \ll n_2\gg^2}\right] \\ &+ \frac{\mathcal{Q}}{2|Q_0^{(2)}|}\frac{e^{-\chi|m|}}{\ll m\gg^2}\left(\sum_{p=2}^{n-1} K_p K_{n+1-p}\right)\sum_{n_1\in\mathbb{Z}} \frac{e^{-2\chi|n_1|}}{\ll n_1\gg^4}. \end{aligned} \tag{III.21}$$

Now,

$$\sum_{n_1,\, n_2\in\mathbb{Z}} \frac{e^{-\chi(|n_1+n_2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2 \ll n_1\gg^2 \ll n_2\gg^2}$$

and

$$\sum_{n_1\in\mathbb{Z}} \frac{e^{-2\chi|n_1|}}{\ll n_1\gg^4}$$

are just finite constants and

$$\begin{aligned} \sum_{n_1,\, n_2\in\mathbb{Z}} \frac{e^{-\chi(|m-(n_1+n_2)|+|n_1|+|n_2|)}}{\ll m-(n_1+n_2)\gg^2 \ll n_1\gg^2 \ll n_2\gg^2} &= \sum_{n_1\in\mathbb{Z}} \frac{e^{-\chi|n_1|}}{\ll n_1\gg^2}\mathcal{B}(m-n_1) \\ &\leq B_0 \sum_{n_1\in\mathbb{Z}} \frac{e^{-\chi(|n_1|+|m-n_1|)}}{\ll n_1\gg^2 \ll m-n_1\gg^2} \\ &= B_0\mathcal{B}(m) \\ &\leq (B_0)^2\frac{e^{-\chi|m|}}{\ll m\gg^2}, \end{aligned} \tag{III.22}$$

where we again used Lemma II.3. Therefore, we conclude

$$|C_m^{(n)}| \leq \left[\mathcal{C}_a\left(\sum_{p=1}^{n-1} K_p K_{n-p}\right) + \mathcal{C}_b\left(\sum_{p=2}^{n-1} K_p K_{n+1-p}\right)\right]\frac{e^{-\chi|m|}}{\ll m\gg^2}, \tag{III.23}$$

for two positive constants $\mathcal{C}_a$ and $\mathcal{C}_b$. Taking $\mathcal{C}_2 := \max\{\mathcal{C}_a,\ \mathcal{C}_b,\ 1\}$ relation (III.7) is proven with $\mathcal{C}_2 \geq 1$.

Note that, without loss, we are allowed to choose $K_1 = K_2 \geq 1$ by choosing both equal to $\max\{K_1,\ K_2,\ 1\}$. □

III.2.2 Case II

In this section we will prove Theorem III.1 in case II. From (III.4)–(III.5), from Proposition II.2 and from the assumption (III.8) we have

$$\begin{aligned}
\left|E_m^{(1)}\right| &\leq \frac{\mathcal{Q}^2}{\omega}\sum_{n_1\in\mathbb{Z}}\frac{e^{-\chi(|m+n_1|+|n_1|)}}{\ll m+n_1\gg^2\ll n_1\gg^2} \\
&+ \frac{\mathcal{Q}^4 e^{-\chi|m|}}{2\ll m\gg^2\,\omega^2|M(\mathcal{Q}_1)|}\sum_{n_1,\,n_2\in\mathbb{Z}}\frac{e^{-\chi(|n_1+n2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}\quad ,\\
\left|E_m^{(n)}\right| &\leq \frac{1}{\omega}\sum_{n_1,\,n_2\in\mathbb{Z}}\Bigg[\mathcal{Q}\frac{e^{-\chi(|m-n_1-n_2|+|n_1|+|n_2|)}}{\ll m-n_1-n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}\\
&\quad+\frac{\mathcal{Q}^2 e^{-\chi|m|}}{|M(\mathcal{Q}_1)|\ll m\gg^2}\Bigg(\frac{e^{-\chi(|n_1+n_2|+|n_1|+|n_2|)}|\mathcal{R}|}{\ll n_1+n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}\\
&\quad+\frac{\mathcal{Q}}{\omega}\sum_{n_3\in\mathbb{Z}}\frac{e^{-\chi(|n_1+n_2+n_3|+|n_1|+|n_2|+|n_3|)}}{\ll n_1+n_2+n_3\gg^2\ll n_1\gg^2\ll n_2\gg^2\ll n_3\gg^2}\Bigg)\Bigg]\sum_{p=1}^{n-1}K'_pK'_{n-p}\\
&\quad+\frac{\mathcal{Q}e^{-\chi|m|}}{2|M(\mathcal{Q}_1)|\ll m\gg^2}\left(\sum_{n_1\in\mathbb{Z}}\frac{e^{-2\chi|n_1|}}{\ll n_1\gg^4}\right)\left(\sum_{p=2}^{n-1}K'_pK'_{n+1-p}\right),\quad n\geq 2.
\end{aligned}$$

Since sums like

$$\sum_{n_1,\,n_2\in\mathbb{Z}}\frac{e^{-\chi(|n_1+n2|+|n_1|+|n_2|)}}{\ll n_1+n_2\gg^2\ll n_1\gg^2\ll n_2\gg^2}$$

and

$$\sum_{n_1,\,n_2,\,n_3\in\mathbb{Z}}\frac{e^{-\chi(|n_1+n_2+n_3|+|n_1|+|n_2|+|n_3|)}}{\ll n_1+n_2+n_3\gg^2\ll n_1\gg^2\ll n_2\gg^2\ll n_3\gg^2}$$

are just finite constants, and by applying Lemma II.3 we get

$$\begin{aligned}
|E_m^{(1)}| &\leq \mathcal{E}_a\frac{e^{-\chi|m|}}{\ll m\gg^2},\\
|E_m^{(n)}| &\leq \frac{e^{-\chi|m|}}{\ll m\gg^2}\left[\mathcal{E}_b\left(\sum_{p=1}^{n-1}K'_pK'_{n-p}\right)+\mathcal{E}_c\left(\sum_{p=2}^{n-1}K'_pK'_{n+1-p}\right)\right],\qquad n\geq 2,
\end{aligned}$$

where $\mathcal{E}_a$, $\mathcal{E}_b$ and $\mathcal{E}_c$ are constants. The rest of the proof follows the same steps of the proof of Theorem III.1 in case I. □

IV The Periodic Case With $F_0\neq 0$

Now we will consider the case where f is periodic with $F_0\neq 0$, for which we have $A=2$. The denominators $\underline{n}\cdot\underline{\omega}$ are of the form $n_1\omega+n_2F_0$, with $n_1,\ n_2\in\mathbb{Z}$, and one has to fear the presence of small denominators in the recursion relations if both n_1 and n_2 can be arbitrarily large. Due to (II.19), we will see, however, that the range of values of n_2 is limited one single value. Hence, no small divisors appear and we are back to a situation analogous to the case $F_0 = 0$.

IV.1 The Structure of the Coefficients $\mathbf{E}_{\underline{m}}^{(n)}$

Let us now return to the periodic case with $B = 1$, $F_0 \neq 0$ and $2F_0 \neq k\omega$ for any $k \in \mathbb{Z}$. Recalling relations (II.19) let us first prove the following theorem:

Theorem IV.1 *For periodic f with a finite Fourier decomposition as above and with $F_0 \neq 0$ and $2F_0 \neq k\omega$, $k \in \mathbb{Z}$, the Fourier coefficients $\mathbf{E}_{\underline{m}}^{(n)}$, $n \geq 1$, are given by*

$$\mathbf{E}_{\underline{m}}^{(n)} = E_{m_1}^{(n)} \delta_{m_2,\,-1}, \tag{IV.1}$$

for all $\underline{m} = (m_1,\ m_2) \in \mathbb{Z}^2$, where, for $m \in \mathbb{Z}$,

$$E_m^{(1)} := \sum_{a_1 \in \mathbb{Z}} \frac{Q_{m+a_1} \overline{Q_{a_1}^{(2)}}}{a_1\omega + 2F_0}, \tag{IV.2}$$

and

$$E_m^{(n)} := \sum_{p=1}^{n-1} \sum_{a_1,\, b_1 \in \mathbb{Z}} \frac{Q_{m-a_1-b_1}\, E_{a_1}^{(p)}\, E_{b_1}^{(n-p)}}{(a_1+b_1)\omega - 2F_0}, \qquad n \geq 2. \tag{IV.3}$$

Proof. Let us first consider the case $n = 1$. The other cases will follow by induction. From (II.8), using (II.19) and writing $\underline{a} = (a_1,\ a_2)$, $\underline{b} = (b_1,\ b_2)$ and $\underline{c} = (c_1,\ c_2) \in \mathbb{Z}^2$, we get

$$\begin{aligned}
\mathbf{E}_{\underline{m}}^{(1)} &= \sum_{\substack{\underline{a} \in \mathbb{Z}^2 \\ \underline{a} \neq \underline{0}}} \frac{Q_{m_1+a_1} \overline{Q_{a_1}^{(2)}}}{\underline{a} \cdot \underline{\omega}} \left(\delta_{m_2+a_2,\,1}\, \delta_{a_2,\,2}\right) \\
&\quad + \frac{Q_{m_1} \delta_{m_2,\,1}}{2iM(\mathcal{Q}_1)} \sum_{\substack{\underline{b},\, \underline{c} \in \mathbb{Z}^2 \\ \underline{b} \neq \underline{0},\ \underline{c} \neq \underline{0}}} \frac{Q_{b_1+c_1}^{(2)}\, \overline{Q_{b_1}^{(2)}}\, \overline{Q_{c_1}^{(2)}}}{(\underline{b} \cdot \underline{\omega})(\underline{c} \cdot \underline{\omega})} \left(\delta_{b_2+c_2,\,2}\, \delta_{b_2,\,2}\, \delta_{c_2,\,2}\right) \\
&= \left(\sum_{a_1 \in \mathbb{Z}} \frac{Q_{m_1+a_1} \overline{Q_{a_1}^{(2)}}}{a_1\omega + 2F_0} \right) \delta_{m_2,\,-1},
\end{aligned} \tag{IV.4}$$

since $\delta_{b_2+c_2,\,2}\, \delta_{b_2,\,2}\, \delta_{c_2,\,2} = \delta_{4,\,2}\, \delta_{b_2,\,2}\, \delta_{c_2,\,2} = 0$. This proves Theorem IV.1 for $n = 1$.

For any $n \geq 2$ relation (II.9) is very much simplified with the observation that, for F_0 as above, one has $\mathcal{R} = 0$. To see this, write $\mathcal{R}$ according to the definition (II.10) and use (II.19) to get

$$\mathcal{R} \;=\; \frac{1}{2iM(\mathcal{Q}_1)} \sum_{\substack{\underline{a},\, \underline{b} \in \mathbb{Z}^A \\ \underline{a} \neq \underline{0},\, \underline{b} \neq \underline{0}}} \frac{Q^{(2)}_{a_1+b_1} \, \overline{Q^{(2)}_{a_1}} \, \overline{Q^{(2)}_{b_1}}}{(\underline{a} \cdot \underline{\omega})(\underline{b} \cdot \underline{\omega})} \left(\delta_{a_2+b_2,\, 2}\, \delta_{a_2,\, 2}\, \delta_{b_2,\, 2}\right) \;=\; 0, \tag{IV.5}$$

since $\delta_{a_2+b_2,\, 2}\, \delta_{a_2,\, 2}\, \delta_{b_2,\, 2} = \delta_{4,\, 2}\, \delta_{a_2,\, 2}\, \delta_{b_2,\, 2} = 0$.

The proof is now done by induction. Let $n \geq 2$ and assume that for all p with $1 \leq p \leq n-1$ one has

$$\mathbf{E}^{(p)}_{\underline{m}} \;=\; E^{(p)}_{m_1}\, \delta_{m_2,\, -1} \tag{IV.6}$$

for all $\underline{m} = (m_1,\ m_2) \in \mathbb{Z}^2$. According to (II.9) we have

$$\mathbf{E}^{(n)}_{\underline{m}} \;=\; \sum_{p=1}^{n-1} \left(\mathcal{A}^{(n,\, p)}_{\underline{m}} + \frac{\mathbf{Q}_{\underline{m}}}{iM(\mathcal{Q}_1)} \mathcal{B}^{(n,\, p)} \right) + \frac{\mathbf{Q}_{\underline{m}}}{2iM(\mathcal{Q}_1)} \sum_{p=2}^{n-1} \mathcal{C}^{(n,\, p)}, \tag{IV.7}$$

where

$$\mathcal{A}^{(n,\, p)}_{\underline{m}} \;:=\; \sum_{\substack{\underline{a},\, \underline{b} \in \mathbb{Z}^2 \\ \underline{a}+\underline{b} \neq \underline{0}}} \mathbf{Q}_{\underline{m}-\underline{a}-\underline{b}} \, \frac{\mathbf{E}^{(p)}_{\underline{a}} \mathbf{E}^{(n-p)}_{\underline{b}}}{(\underline{a}+\underline{b}) \cdot \underline{\omega}}, \tag{IV.8}$$

$$\mathcal{B}^{(n,\, p)} \;:=\; \sum_{\substack{\underline{a} \in \mathbb{Z}^2 \\ \underline{a} \neq \underline{0}}} \sum_{\substack{\underline{b},\, \underline{c} \in \mathbb{Z}^2 \\ \underline{b}+\underline{c} \neq \underline{0}}} \frac{\mathbf{Q}^{(2)}_{\underline{a}-\underline{b}-\underline{c}} \overline{\mathbf{Q}^{(2)}_{\underline{a}}} \mathbf{E}^{(p)}_{\underline{b}} \mathbf{E}^{(n-p)}_{\underline{c}}}{(\underline{a} \cdot \underline{\omega})((\underline{b}+\underline{c}) \cdot \underline{\omega})}, \tag{IV.9}$$

and

$$\mathcal{C}^{(n,\, p)} \;:=\; \sum_{\underline{a} \in \mathbb{Z}^2} \mathbf{E}^{(p)}_{\underline{a}} \mathbf{E}^{(n+1-p)}_{-\underline{a}}. \tag{IV.10}$$

By (II.19) and by the induction hypothesis,

$$\begin{aligned} \mathcal{A}^{(n,\, p)}_{\underline{m}} \;&=\; \sum_{\substack{\underline{a},\, \underline{b} \in \mathbb{Z}^2 \\ \underline{a}+\underline{b} \neq \underline{0}}} Q_{m_1-a_1-b_1} \, \frac{E^{(p)}_{a_1} E^{(n-p)}_{b_1} \left[\delta_{m_2-a_2-b_2,\, 1}\, \delta_{a_2,\, -1}\, \delta_{b_2,\, -1}\right]}{(a_1+b_1)\omega + (a_2+b_2)F_0} \\ &=\; \left(\sum_{a_1,\, b_1 \in \mathbb{Z}} \frac{Q_{m_1-a_1-b_1}\, E^{(p)}_{a_1}\, E^{(n-p)}_{b_1}}{(a_1+b_1)\omega - 2F_0} \right) \delta_{m_2,\, -1}. \end{aligned} \tag{IV.11}$$

Moreover,

$$\mathcal{B}^{(n,\,p)} \;=\; \sum_{\substack{\underline{a}\in\mathbb{Z}^2\\ \underline{a}\neq\underline{0}}} \sum_{\substack{\underline{b},\,\underline{c}\in\mathbb{Z}^2\\ \underline{b}+\underline{c}\neq\underline{0}}} \frac{Q^{(2)}_{a_1-b_1-c_1}\overline{Q^{(2)}_{a_1}}E^{(p)}_{b_1}E^{(n-p)}_{c_1}\left[\delta_{a_2-b_2-c_2,\,2}\,\delta_{a_2,\,2}\,\delta_{b_2,\,-1}\,\delta_{c_2,\,-1}\right]}{(a_1\omega+a_2F_0)((b_1+c_1)\omega+(b_2+c_2)F_0)}$$

equals to zero, since $\delta_{a_2-b_2-c_2,\,2}\,\delta_{a_2,\,2}\,\delta_{b_2,\,-1}\,\delta_{c_2,\,-1} = \delta_{4,\,2}\,\delta_{a_2,\,2}\,\delta_{b_2,\,-1}\,\delta_{c_2,\,-1} = 0$. Finally,

$$\mathcal{C}^{(n,\,p)} \;=\; \sum_{\underline{a}\in\mathbb{Z}^2} E^{(p)}_{a_1}E^{(n+1-p)}_{-a_1}\left(\delta_{a_2,\,-1}\,\delta_{-a_2,\,-1}\right) \;=\; 0. \tag{IV.12}$$

Hence, for $n\geq 2$,

$$\mathbf{E}^{(n)}_{\underline{m}} \;=\; \sum_{p=1}^{n-1}\mathcal{A}^{(n,\,p)}_{\underline{m}} \;=\; \left(\sum_{p=1}^{n-1}\sum_{a_1,\,b_1\in\mathbb{Z}}\frac{Q_{m_1-a_1-b_1}\,E^{(p)}_{a_1}\,E^{(n-p)}_{b_1}}{(a_1+b_1)\omega-2F_0}\right)\delta_{m_2,\,-1}, \tag{IV.13}$$

completing the proof of Theorem IV.1. □

IV.2 Inductive Upper Bounds and Convergence

Theorem IV.1 is of crucial importance, since it shows that actually no problems with small denominators are present in the recursion relations defining the Fourier coefficients $\mathbf{E}^{(n)}_{\underline{m}}$. This allows to find upper bounds for the absolute values of the coefficients $\mathbf{E}^{(n)}_{\underline{m}}$ in essentially the same way as performed in for the case $F_0 = 0$. This is what we do now.

As we already mentioned, the coefficients Q_m and $Q^{(2)}_m$ can be bounded as in Proposition II.2. Moreover, we have

$$|a_1\omega+2F_0| \geq \min_{a\in\mathbb{Z}}\big|\,|a|\omega-2|F_0|\,\big| =: \eta > 0. \tag{IV.14}$$

Note that $\eta = 2|F_0|$ for $|F_0|\leq\omega/2$ and, hence, $\eta\to 0$ when $F_0\to 0$. This remark will be relevant in Section VI.3. Using Proposition II.2 and Lemma II.3,

$$\begin{aligned}\left|\mathbf{E}^{(1)}_{\underline{m}}\right| \;&=\; \left|\sum_{a_1\in\mathbb{Z}}\frac{Q_{m_1+a_1}\overline{Q^{(2)}_{a_1}}}{a_1\omega+2F_0}\right|\delta_{m_2,\,-1}\\ &\leq\; \frac{\mathcal{Q}^2}{\eta}\,\mathcal{B}(m_1)\,\delta_{m_2,\,-1} \;\leq\; \left(\frac{\mathcal{Q}^2B_0}{\eta}\right)\frac{e^{-\chi|m_1|}}{\ll m_1\gg^2}\,\delta_{m_2,\,-1},\end{aligned} \tag{IV.15}$$

where $\mathcal{B}(m)$ is defined in (II.24). Defining $K''_1 := \mathcal{Q}^2B_0/\eta$, taking $n\geq 2$ and assuming the induction hypothesis

$$\left|\mathbf{E}^{(p)}_{\underline{m}}\right| \;\leq\; K''_p\,\frac{e^{-\chi|m_1|}}{\ll m_1\gg^2}\,\delta_{m_2,\,-1}, \tag{IV.16}$$

for all p with $1 \leq p \leq n-1$, where K''_p are constants independent of $\underline{m}$, we have from (IV.13),

$$\begin{aligned}
\left|\mathbf{E}^{(n)}_{\underline{m}}\right| &\leq \frac{1}{\eta}\left(\sum_{p=1}^{n-1}\sum_{a_1,\, b_1\in\mathbb{Z}} |Q_{m_1-a_1-b_1}|\, \left|E^{(p)}_{a_1}\right| \, \left|E^{(n-p)}_{b_1}\right|\right) \delta_{m_2,\,-1} \\
&\leq \frac{\mathcal{Q}}{\eta}\left[\sum_{a_1,\, b_1\in\mathbb{Z}} \frac{e^{-\chi(|m_1-a_1-b_1|+|a_1|+|b_1|)}}{\ll m_1-a_1-b_1\gg^2 \ll a_1\gg^2 \ll b_1\gg^2}\right]\sum_{p=1}^{n-1} K''_p K''_{n-p}\, \delta_{m_2,\,-1} \\
&\leq \frac{\mathcal{Q}B_0^2}{\eta}\left(\sum_{p=1}^{n-1} K''_p K''_{n-p}\right) \frac{e^{-\chi|m_1|}}{\ll m_1\gg^2}\ \delta_{m_2,\,-1}, \qquad \text{(IV.17)}
\end{aligned}$$

where, above, we used Lemma II.3. Defining inductively

$$K''_n := \frac{\mathcal{Q}B_0^2}{\eta}\left(\sum_{p=1}^{n-1} K''_p K''_{n-p}\right) \qquad \text{(IV.18)}$$

we have proven that

$$\left|\mathbf{E}^{(n)}_{\underline{m}}\right| \leq K''_n \frac{e^{-\chi|m_1|}}{\ll m_1\gg^2}\, \delta_{m_2,\,-1}, \qquad \text{(IV.19)}$$

for all $n \in \mathbb{N}$ and all $\underline{m} = (m_1,\ m_2) \in \mathbb{Z}^2$. With the same methods employed Appendix D, we can show that $K''_n \leq K''_0 (K'')^n$ for all $n \in \mathbb{N}$, where K''_0 and K'' are positive constants.

From all this, it follows that, for all n,

$$|e_n(t)| \leq K''_0 (K'')^n \sum_{m_1\in\mathbb{Z}} \frac{e^{-\chi|m_1|}}{\ll m_1\gg^2} = K'''_0 (K'')^n \qquad \text{(IV.20)}$$

where K'''_0 is a constant and

$$|g(t)| \leq K'''_0 \sum_{n=1}^{\infty} |\epsilon^2|^n (K'')^n. \qquad \text{(IV.21)}$$

We have thus established that the Fourier series of the functions e_n converge absolutely and uniformly and that, for $|\epsilon|^2 < (K'')^{-1}$, the power series (II.3), which defines the solution g, is absolutely convergent. The Fourier expansion for g is also absolutely and uniformly convergent.

We conclude from the lines above that the true radius of convergence R_ϵ of the ϵ-expansion of g is bounded from below by $(K'')^{-1/2}$. Note that K'' is proportional to η^{-1} and, hence, $(K'')^{-1/2}$ shrinks to zero when $F_0 \to 0$ (see the definition of η in equation (IV.14)). As we will remark in Section VI.3, there are indications that R_ϵ also shrinks o zero when $F_0 \to 0$.

Let us finish this section with a closer look at the Fourier expansion of g. Theorem IV.1 says that the functions e_n have the following Fourier decomposition:

$$e_n(t) \;=\; e^{-iF_0 t} \sum_{m\in\mathbb{Z}} E_m^{(n)} e^{im\omega t}, \tag{IV.22}$$

while for $q(t)$ we have

$$q(t) \;=\; e^{iF_0 t} \sum_{m\in\mathbb{Z}} Q_m e^{im\omega t}. \tag{IV.23}$$

Thus,

$$g(t) \;=\; \sum_{m\in\mathbb{Z}} G_m e^{im\omega t} \tag{IV.24}$$

where

$$G_m \;\equiv\; G_m(\epsilon) \;=\; \sum_{n=1}^{\infty} \lambda^n G_m^{(n)} \tag{IV.25}$$

with $\lambda = \epsilon^2$ and

$$G_m^{(n)} \;=\; \sum_{l\in\mathbb{Z}} Q_{m-l} E_l^{(n)}. \tag{IV.26}$$

Note by (IV.24) that F_0 is present in g only in the Fourier coefficients G_m and not in the frequencies.

For the coefficients $G_m^{(n)}$ we have the following expressions, which will need when we discuss the ϵ-expansion of Ω in Section VI.3:

$$G_m^{(1)} \;=\; \sum_{a_1\in\mathbb{Z}} \frac{Q_{m+a_1}^{(2)}\,\overline{Q_{a_1}^{(2)}}}{a_1\omega + 2F_0} \tag{IV.27}$$

and

$$G_m^{(n)} \;=\; \sum_{p=1}^{n-1} \sum_{a_1,\, b_1\in\mathbb{Z}} \frac{Q_{m-a_1-b_1}^{(2)}\; E_{a_1}^{(p)}\; E_{b_1}^{(n-p)}}{(a_1+b_1)\omega - 2F_0}, \qquad n \geq 2. \tag{IV.28}$$

V The Fourier Expansion for the Wave Function

Now we return to the discussion of the solution (I.9) of the Schrödinger equation (I.5). Our intention is to find the Fourier expansion of the wave function $\Phi(t)$.

V.1 The Floquet Form of the Wave Function. The Fourier Decomposition and the Secular Frequency

As explained in [1] and in Section I, the components $\phi_\pm$ of the wave function $\Phi(t)$ are solutions of Hill's equation (I.13). For periodic f the classical theorem of Floquet (see e.g. [21] and [22]) claims that there are particular solutions of equations like (I.13) with the general form $e^{i\Omega t}u(t)$, where $u(t)$ is periodic with the same period of f. In order to preserve unitarity we must have $\Omega \in \mathbb{R}$. This form of the particular solutions is called the "Floquet form" and the frequencies Ω are called "secular frequencies".

In this section we will recover the Floquet form of the wave function in terms of Fourier expansions and we will find out expansions for the secular frequencies as converging power series expansions in ϵ.

According to the solution expressed in relation (I.8) and (I.9), we have first to find out the Fourier expansion for the functions R and S defined in (I.10) and (I.11), respectively.

We start with the function R. The Fourier expansion of the function $f+g$ is

$$f(t)+g(t) \;=\; \Omega + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} (F_n + G_n(\epsilon))\, e^{in\omega t}, \tag{V.1}$$

where

$$\Omega \;\equiv\; \Omega(\epsilon) \;:=\; F_0 + G_0(\epsilon). \tag{V.2}$$

One has,

$$R(t) \;=\; e^{-i\gamma_f(\epsilon)}\, e^{-i\Omega t}\, \exp\left(-\sum_{n\in\mathbb{Z}} H_n e^{in\omega t} \right) \tag{V.3}$$

with

$$H_n \;\equiv\; H_n(\epsilon) \;:=\; \begin{cases} \dfrac{F_n + G_n(\epsilon)}{n\omega}, & \text{for } n \neq 0 \\[2ex] 0, & \text{for } n = 0 \end{cases}\;, \tag{V.4}$$

and

$$\gamma_f(\epsilon) \;:=\; i \sum_{m\in\mathbb{Z}} H_m. \tag{V.5}$$

Note that $\gamma_f(0) = \gamma_f$, where γ_f is defined in (B.4).

Since we are assuming that there are only finitely many non-vanishing coefficients F_n, we have the following proposition as an obvious corollary of Proposition III.4:

Proposition V.1 *For all $\chi > 0$ and $|\epsilon|$ small enough, there exists a constant $\mathcal{C}_H \equiv \mathcal{C}_H(\chi,\, \epsilon) > 0$ such that*

$$|H_m| \;\leq\; \mathcal{C}_H \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{V.6}$$

for all $m \in \mathbb{Z}$.

Writing now the Fourier expansion of $R(t)$ in the form

$$R(t) \;=\; e^{-i\Omega t} \sum_{n\in\mathbb{Z}} R_n e^{in\omega t} \tag{V.7}$$

we find from (V.3)

$$R_0 = e^{-i\gamma_f(\epsilon)} \left(1 + \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{(p+1)!} \sum_{n_1, \ldots, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} H_{-N_p} \right), \qquad \text{(V.8)}$$

$$R_n = e^{-i\gamma_f(\epsilon)} \left(-H_n + \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{(p+1)!} \sum_{n_1, \ldots, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} H_{n-N_p} \right), \qquad \text{(V.9)}$$

for $n \neq 0$, with

$$N_p := \sum_{a=1}^{p} n_a, \qquad \text{(V.10)}$$

for $p \geq 1$.

In order to compute the Fourier expansion of S we have to compute first the Fourier expansion of R^{-2}. This is now an easy task, since the replacement $R(t) \to R(t)^{-2}$ corresponds to the replacement $(f+g) \to -2(f+g)$ and, hence, to $H_n \to -2H_n$. We get

$$R(t)^{-2} = e^{2i\Omega t} \sum_{n \in \mathbb{Z}} R_n^{(-2)} e^{in\omega t}, \qquad \text{(V.11)}$$

with

$$R_0^{(-2)} = e^{2i\gamma_f(\epsilon)} \left(1 + \sum_{p=1}^{\infty} \frac{2^{p+1}}{(p+1)!} \sum_{n_1, \ldots, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} H_{-N_p} \right),$$

$$R_n^{(-2)} = e^{2i\gamma_f(\epsilon)} \left(2H_n + \sum_{p=1}^{\infty} \frac{2^{p+1}}{(p+1)!} \sum_{n_1, \ldots, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} H_{n-N_p} \right),$$

for $n \neq 0$.

The following proposition will be used below.

Proposition V.2 *For all $\chi > 0$ and $|\epsilon|$ small enough, there exist constants $\mathcal{C}_R \equiv \mathcal{C}_R(\chi, \epsilon) > 0$ and $\mathcal{C}_{R^{(-2)}} \equiv \mathcal{C}_{R^{(-2)}}(\chi, \epsilon) > 0$ such that*

$$|R_m| \leq \mathcal{C}_R \frac{e^{-\chi|m|}}{\ll m \gg^2} \qquad \text{(V.12)}$$

$$|R_m^{(-2)}| \leq \mathcal{C}_{R^{(-2)}} \frac{e^{-\chi|m|}}{\ll m \gg^2} \qquad \text{(V.13)}$$

for all $m \in \mathbb{Z}$.

Proof of Proposition V.2. Using Proposition V.1 we have, for any $p \geq 1$,

$$\left| \sum_{n_1, \ldots, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} H_{n-N_p} \right| \leq$$
$$(\mathcal{C}_H)^{p+1} \sum_{n_1, \ldots, n_p \in \mathbb{Z}} \frac{\exp\left(-\chi(|n_1| + \cdots + |n_p| + |n - n_1 - \cdots - n_p|)\right)}{\left(\ll n_1 \gg \cdots \ll n_p \gg \ll n - n_1 - \cdots - n_p \gg\right)^2}.$$

Making repeated use of Lemma II.3, we get

$$\left| \sum_{n_1, \ldots, n_p \in \mathbb{Z}} H_{n_1} \cdots H_{n_p} H_{n-N_p} \right| \leq \frac{(\mathcal{C}_H B_0)^{p+1}}{B_0} \frac{e^{-\chi|n|}}{\ll n \gg^2}. \tag{V.14}$$

Inserting this into (V.8)–(V.9) gives (since $B_0 > 1$)

$$|R_n| \leq \left(\frac{e^{|\mathrm{Im}(\gamma_f(\epsilon))| + \mathcal{C}_H B_0}}{B_0} \right) \frac{e^{-\chi|n|}}{\ll n \gg^2} \tag{V.15}$$

for all $n \in \mathbb{Z}$, as desired. The proof for $R_n^{(-2)}$ is analogous. □

Assuming for a while

$$n\omega + 2\Omega \neq 0 \qquad \text{for all } n \in \mathbb{Z}, \tag{V.16}$$

we have[3]

$$S(t) = \sigma_0 + e^{2i\Omega t} \sum_{n \in \mathbb{Z}} S_n e^{in\omega t} \tag{V.17}$$

with

$$S_n := -i \frac{R_n^{(-2)}}{n\omega + 2\Omega} \qquad \text{and} \qquad \sigma_0 := -\sum_{n \in \mathbb{Z}} S_n. \tag{V.18}$$

Assumption (V.16) is actually a consequence of unitarity, as will be discussed in Section V.2.

The following proposition is an elementary corollary of Proposition V.2:

Proposition V.3 *For all $\chi > 0$ and $|\epsilon|$ small enough, there exists a constant $\mathcal{C}_S \equiv \mathcal{C}_S(\chi, \epsilon) > 0$ such that*

$$|S_m| \leq \mathcal{C}_S \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{V.19}$$

for all $m \in \mathbb{Z}$.

[3]For the case $n = 0$, (V.16) says that $\Omega \neq 0$. This must hold except for $\epsilon = 0$ when $\Omega = 0$.

Writing

$$U(t) = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{pmatrix} = \begin{pmatrix} U_{11}(t) & U_{12}(t) \\ -\overline{U_{12}(t)} & \overline{U_{11}(t)} \end{pmatrix}, \tag{V.20}$$

we have for U_{11} and U_{12}:

$$U_{11}(t) = e^{-i\Omega t}\, u_{11}^{-}(t) + e^{i\Omega t}\, u_{11}^{+}(t) \tag{V.21}$$

$$U_{12}(t) = e^{-i\Omega t}\, u_{12}^{-}(t) + e^{i\Omega t}\, u_{12}^{+}(t) \tag{V.22}$$

with

$$\begin{aligned} u_{11}^{-}(t) &:= (1 + ig(0)\sigma_0)\, r(t), & u_{11}^{+}(t) &:= ig(0)\, v(t), \\ u_{12}^{-}(t) &:= -i\epsilon\sigma_0\, r(t), & u_{12}^{+}(t) &:= -i\epsilon\, v(t), \end{aligned} \tag{V.23}$$

for

$$r(t) := \sum_{n\in\mathbb{Z}} R_n\, e^{in\omega t} \qquad \text{and} \qquad v(t) := \sum_{n\in\mathbb{Z}} V_n\, e^{in\omega t}, \tag{V.24}$$

with

$$V_n := \sum_{m\in\mathbb{Z}} S_{n-m} R_m. \tag{V.25}$$

This provides the desired Floquet form for the components of the wave function $\Phi(t)$. We note from the expressions above that the secular frequencies are $\pm\Omega$. For Ω we have the ϵ-expansion

$$\Omega = \sum_{n=1}^{\infty} \epsilon^n G_0^{(n)}, \tag{V.26}$$

for $F_0 = 0$ or

$$\Omega = F_0 + \sum_{n=1}^{\infty} \epsilon^{2n} G_0^{(n)}, \tag{V.27}$$

for $F_0 \neq 0$, where the coefficients $G_0^{(n)}$ are given by (III.12) or (IV.26), according to the case. Analogously, we have for $g(0)$

$$g(0) = \sum_{m\in\mathbb{Z}} G_m = \sum_{n=1}^{\infty} \epsilon^n \sum_{m\in\mathbb{Z}} G_m^{(n)}, \tag{V.28}$$

for $F_0 = 0$ or

$$g(0) = \sum_{m\in\mathbb{Z}} G_m = \sum_{n=1}^{\infty} \epsilon^{2n} \sum_{m\in\mathbb{Z}} G_m^{(n)}, \tag{V.29}$$

for $F_0 \neq 0$. All these series converge absolutely for $|\epsilon|$ small enough.

As before, we have the following corollary of Propositions V.2, V.3 and Lemma II.3:

Proposition V.4 *For all $\chi > 0$ and $|\epsilon|$ small enough, there exists a constant $\mathcal{C}_V \equiv \mathcal{C}_V(\chi, \epsilon) > 0$ such that*

$$|V_m| \leq \mathcal{C}_V \frac{e^{-\chi|m|}}{\ll m \gg^2} \tag{V.30}$$

for all $m \in \mathbb{Z}$.

This last proposition closed the proof of Theorem I.2.

V.2 Remarks on the Unitarity of the Propagator. Crossings

The unitarity of the propagator $U(t)$ means $U(t)^* U(t) = \mathbb{1}$. After (V.20), this means

$$|U_{11}(t)|^2 + |U_{12}(t)|^2 = 1. \tag{V.31}$$

Looking at relations (V.21) and (V.22) two conclusions can be drawn from (V.31). The first is the following proposition:

Proposition V.5 *For $\epsilon \in \mathbb{R}$ and under the hypothesis leading to (V.21) and (V.22) one has $\Omega \in \mathbb{R}$.*

The proof follows from the obvious observation that (V.31) would be violated for $|t|$ large enough if Ω had a non-vanishing imaginary part. Unfortunately a proof of this fact using directly the ϵ-expansions of Ω, (V.26) or (V.27), is difficult and has not been found yet.

The second conclusion is that (V.16) must indeed hold. For, without this assumption there would be a term linear in t in (V.17), violating (V.31) for large $|t|$.

As in the case of Proposition V.5, no direct proof of this fact out of the ϵ-expansions for Ω, (V.26) or (V.27), has been found yet. The proof will probably follow the fact that $|\Omega|$ had to be smaller than 2ω in the region of convergence.

Finally, note that on results say that the spectrum of the quasi-energy operator is a subset of $\{\pm\Omega + k\omega | \ k \in \mathbb{Z}\}$. Hence, the condition (V.16) $2\Omega \neq k\omega$, $k \in \mathbb{Z}$, implies the absence of crossings in the spectrum of the quasi-energy operator when ϵ varies within the convergence region. This is, of course, relevant for the adiabatic limit of systems where ϵ is a slowly varying function of time.

VI Discussion on the Classes of Solutions

Let us now discuss some aspects of conditions I and II of Theorem I.2 for the case $F_0 = 0$.

As in (II.11) or (B.1), let us write the Fourier decomposition of f as

$$f(t) = \sum_{a=1}^{2J} f_a e^{in_a \omega t}, \tag{VI.1}$$

with $n_a = -n_{2J-a+1}$ and $\overline{f_a} = f_{2J-a+1}$ for all a with $1 \le a \le J$. Comparing with (I.18) one has $f_a \equiv F_{n_a}$, $1 \le a \le J$.

Hence, for $F_0 = 0$ and for fixed J and ω, there are J independent complex coefficients f_a and we can identify the parameter space $\mathbb{R}^{2J}$ with the set $\mathfrak{F}_{J,\,\omega}$ of all possible functions f with a given J and ω.

Condition $M(q^2) = 0$ determines a $(2J-1)$ or $(2J-2)$-dimensional subset of $\mathfrak{F}_{J,\,\omega}$ and there condition II applies. It is also on this subset that the more restrictive condition $M(q^2) = M(\mathcal{Q}_1) = 0$ should hold, restricting the parameter space of f to a $(2J-2)$, $(2J-3)$ or $(2J-4)$-dimensional subset. Hence, successive conditions like I and II would eventually exhaust completely the parameter space $\mathfrak{F}_{J,\,\omega}$.

Conditions beyond I and II have not been yet analyzed and many questions concerning the classes of solutions are still open. For instance, will further conditions like I and II really exhaust the parameter space of the functions f? Will the subtraction method of [1] and the convergence proofs of the present paper also work under these further conditions? What are the physically qualitative distinctions between the classes? Are these classes of solutions in some sense analytic continuations of each other? In Section VI.3 we give indications that the answer to the last question is no.

A distinction between class I and II may be pointed out with the observation that in class I we have power expansions in ϵ while in II we have power expansions in ϵ^2. Compare relations (II.1) and (II.3) of Theorem II.1. See also Section VI.3.

VI.1 An Explicit Example

In order to illustrate these ideas and point to some problems let us consider the important example where f represents a monochromatic interaction given by

$$f(t) \;=\; \varphi_1 \cos(\omega t) + \varphi_2 \sin(\omega t), \tag{VI.2}$$

$\varphi_1,\, \varphi_2 \in \mathbb{R}$. We have $f(t) = f_1 e^{-i\omega t} + f_2 e^{i\omega t}$ with $f_1 = (\varphi_1 + i\varphi_2)/2$, $f_2 = \overline{f_1}$, $J = 1$, $n_1 = -1$, $n_2 = 1$. Applying now (II.17) for this case with $m = 0$ we get

$$M(q^2) \;=\; Q_0^{(2)} \;=\; e^{2i\gamma_f} \sum_{p=0}^{\infty} \frac{(-1)^p}{(p!)^2} \left(\frac{4|f_1|}{2\omega}\right)^{2p} \;=\; e^{2i\gamma_f} J_0\left(\frac{2\varphi_0}{\omega}\right), \tag{VI.3}$$

where $\varphi_0 := \sqrt{\varphi_1^2 + \varphi_2^2}$ and where J_0 is the Bessel function of first kind and order zero. In this case $\gamma_f = \varphi_2/\omega$.

Relation (VI.3) shows that condition I is not empty and that the locus in the $(\varphi_1,\ \varphi_2)$-space of the condition $M(q^2) = 0$ (necessary for condition II) is the countable family of circles centered at the origin with radius $x_a\omega/2$, $a = 1, 2, \ldots$, where x_a if the a-th zero of J_0 in $\mathbb{R}_+$.

One shows analogously that

$$Q_m \;=\; e^{i\gamma_f} \left(\frac{\overline{f_1}}{|f_1|}\right)^m J_m\left(\frac{2|f_1|}{\omega}\right) \tag{VI.4}$$

and

$$Q_m^{(2)} \;=\; e^{2i\gamma_f}\left(\frac{\overline{f_1}}{|f_1|}\right)^m J_m\left(\frac{4|f_1|}{\omega}\right), \tag{VI.5}$$

for all $m \in \mathbb{Z}$, where J_m is the Bessel function of first kind and order m.

For $Q_0^{(2)} = 0$ the function $\mathcal{Q}_1$ is periodic and we have in general

$$M(\mathcal{Q}_1) \;=\; \frac{i}{\omega}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\left|Q_m^{(2)}\right|^2}{m} \;=\; \frac{i}{\omega}\sum_{m=1}^{\infty}\left(\frac{\left|Q_m^{(2)}\right|^2 - \left|Q_{-m}^{(2)}\right|^2}{m}\right). \tag{VI.6}$$

Since $|J_m(x)| = |J_{-m}(x)|$ for all $x \in \mathbb{R}$, $\forall m \in \mathbb{Z}$, it follows that $|Q_m^{(2)}| = |Q_{-m}^{(2)}|$, $\forall m \in \mathbb{Z}$. Hence, for functions f like (VI.2)

$$M(\mathcal{Q}_1) \;=\; 0. \tag{VI.7}$$

Therefore, condition II is nowhere fulfilled. For a complete solution of the problem for functions like (VI.2), including the circles mentioned above, higher restrictions than that implied by condition II are necessary.

VI.2 A Second Example

For functions f with $J > 1$ the situation leading to (VI.7) is not expected in general and condition II, and eventually others, may hold in non-empty regions of the parameter space of f. This can be seen in the following example with $J = 2$. Let us take

$$f(t) \;=\; f_1(t) + f_2(t)$$

with

$$\begin{aligned} f_1(t) &= f_1 e^{-i\omega t} + \overline{f_1} e^{i\omega t} \\ f_2(t) &= f_2 e^{-i2\omega t} + \overline{f_2} e^{i2\omega t} \end{aligned}$$

$f_i \in \mathbb{C}$, $i = 1, 2$. We have $q(t) = q_1(t)q_2(t)$, where

$$\begin{aligned} q_1(t) &:= e^{i\gamma_{f_1}} \sum_{n\in\mathbb{Z}} e^{in\zeta_1} J_n\left(\frac{2|f_1|}{\omega}\right) e^{in\omega t}, \\ q_2(t) &:= e^{i\gamma_{f_2}} \sum_{n\in\mathbb{Z}} e^{in\zeta_2} J_n\left(\frac{|f_2|}{\omega}\right) e^{in2\omega t}, \end{aligned}$$

with

$$e^{i\zeta_i} = \frac{\overline{f_i}}{|f_i|}, \qquad i = 1, 2.$$

It follows that

$$
\begin{aligned}
Q_m &= e^{i(\gamma_{f_1}+\gamma_{f_2})} \sum_{k\in\mathbb{Z}} e^{i((m-2k)\zeta_1+k\zeta_2)} J_{m-2k}\left(\frac{2|f_1|}{\omega}\right) J_k\left(\frac{|f_2|}{\omega}\right), \\
Q_m^{(2)} &= e^{2i(\gamma_{f_1}+\gamma_{f_2})} \sum_{k\in\mathbb{Z}} e^{i((m-2k)\zeta_1+k\zeta_2)} J_{m-2k}\left(\frac{4|f_1|}{\omega}\right) J_k\left(\frac{2|f_2|}{\omega}\right).
\end{aligned}
$$

From this we see (using $J_{-n}(x) = (-1)^n J_n(x)$) that

$$
\begin{aligned}
\overline{Q_{-m}^{(2)}} &= (-1)^m e^{-4i(\gamma_{f_1}+\gamma_{f_2})} \\
&\quad \times \left\{ e^{2i(\gamma_{f_1}+\gamma_{f_2})} \sum_{k\in\mathbb{Z}} (-1)^k e^{i((m-2k)\zeta_1+k\zeta_2)} J_{m-2k}\left(\frac{4|f_1|}{\omega}\right) J_k\left(\frac{2|f_2|}{\omega}\right) \right\}.
\end{aligned}
$$

The factor between brackets differs from $Q_m^{(2)}$ due to the presence of the factor $(-1)^k$ in the sum over $k \in \mathbb{Z}$. Hence, we should rather expect $|Q_m^{(2)}| \neq |Q_{-m}^{(2)}|$ in this case, what most likely implies $M(\mathcal{Q}_1) \neq 0$ for $M(q^2) = 0$, leading to a non-empty condition II.

VI.3 The Secular Frequency

For $F_0 = 0$, case I, relation (V.26) says that

$$
\Omega \;=\; \epsilon \left| Q_0^{(2)} \right| + \epsilon^2 G_0^{(2)} + \sum_{n=3}^{\infty} \epsilon^n G_0^{(n)}. \tag{VI.8}
$$

Because of condition I, the first order contribution in ϵ is non-vanishing. However, as one easily checks, $G_0^{(2)} = 0$ and, hence, the second order contribution to Ω is always zero. As we will see, this no longer happens in the case $F_0 \neq 0$.

For $F_0 \neq 0$ we have from (V.27), (IV.27) and (IV.28)

$$
\begin{aligned}
\Omega &= F_0 + \sum_{n=1}^{\infty} \epsilon^{2n} G_0^{(n)} \\
&= F_0 + \epsilon^2 \left(\sum_{a_1\in\mathbb{Z}} \frac{\left| Q_{a_1}^{(2)} \right|}{a_1\omega + 2F_0} \right) \\
&\quad + \sum_{n=2}^{\infty} \epsilon^{2n} \left(\sum_{p=1}^{n-1} \sum_{a_1,\, b_1\in\mathbb{Z}} \frac{Q_{-a_1-b_1}^{(2)}\, E_{a_1}^{(p)}\, E_{b_1}^{(n-p)}}{(a_1+b_1)\omega - 2F_0} \right).
\end{aligned} \tag{VI.9}
$$

It is interesting to study the limit $F_0 \to 0$ of Ω given in (VI.9). If $Q_0^{(2)} \neq 0$ the limit $F_0 \to 0$ of Ω given in (VI.9) is termwise singular, in contrast to the expression for Ω obtained <u>under</u> the condition $F_0 = 0$.

For $Q_0^{(2)} = 0$ the situation is analogous, as we discuss briefly now. For $E_m^{(1)}$ we have

$$E_m^{(1)} := \sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \neq 0}} \frac{Q_{m+a_1} \overline{Q_{a_1}^{(2)}}}{a_1 \omega + 2F_0} \quad \Longrightarrow \quad \lim_{F_0 \to 0} E_m^{(1)} = \mathcal{E}_m^{(1)} := \sum_{\substack{a_1 \in \mathbb{Z} \\ a_1 \neq 0}} \frac{Q_{m+a_1} \overline{Q_{a_1}^{(2)}}}{a_1 \omega}, \tag{VI.10}$$

and hence $\lim_{F_0 \to 0} E_m^{(1)}$ exists and is well defined for all $m \in \mathbb{Z}$. However, for $E_m^{(2)}$, we have

$$E_m^{(2)} = \sum_{a_1,\, b_1 \in \mathbb{Z}} \frac{Q_{m-a_1-b_1}}{(a_1+b_1)\omega - 2F_0} E_{a_1}^{(1)} E_{b_1}^{(1)} = S_0 + S_1 \tag{VI.11}$$

with

$$S_0 := -\frac{Q_m}{2F_0} \sum_{a_1 \in \mathbb{Z}} E_{a_1}^{(1)} E_{-a_1}^{(1)}, \qquad S_1 := \sum_{\substack{a_1,\, b_1 \in \mathbb{Z} \\ a_1 + b_1 \neq 0}} \frac{Q_{m-a_1-b_1}}{(a_1+b_1)\omega - 2F_0} E_{a_1}^{(1)} E_{b_1}^{(1)}. \tag{VI.12}$$

The limit $F_0 \to 0$ exists for S_1, but not for S_0.

One easily sees that

$$\lim_{F_0 \to 0} G_0^{(1)} = \sum_{a_1 \in \mathbb{Z}} \frac{|Q_{a_1}^{(2)}|}{a_1 \omega} \tag{VI.13}$$

and that

$$\lim_{F_0 \to 0} G_0^{(2)} = \sum_{\substack{a_1,\, b_1 \in \mathbb{Z} \\ a_1 + b_1 \neq 0}} \frac{Q_{-a_1-b_1}^{(2)}\ \mathcal{E}_{a_1}^{(1)}\ \mathcal{E}_{b_1}^{(1)}}{(a_1+b_1)\omega}, \tag{VI.14}$$

where $\mathcal{E}_m^{(1)}$ is defined in (VI.10). However,

$$G_0^{(3)} = \sum_{a_1,\, b_1 \in \mathbb{Z}} \frac{Q_{-a_1-b_1}^{(2)}\ E_{a_1}^{(1)}\ E_{b_1}^{(2)}}{(a_1+b_1)\omega - 2F_0} \tag{VI.15}$$

and the limit $F_0 \to 0$ of the right hand side does not exist, since it does not exist for $E_{b_1}^{(2)}$. The same must hold for $G_0^{(n)}$ with $n > 3$. The conclusion is, thus, the same as in the case $Q_0^{(2)} \neq 0$.

The remarks above indicate that the limit $F_0 \to 0$ of the solution of (I.5) obtained here is singular and does not converge to the solution corresponding to the case $F_0 = 0$. All this strongly suggests that the radius of convergence of the ϵ-expansions for the case $F_0 \neq 0$ shrinks to zero when the limit $F_0 \to 0$ is performed. An indication to this was already discussed in the paragraphs following equation (IV.19). More generally, the same must happen when $2F_0$ approaches an integer multiple of ω.

All this should not be surprising since there is no reason to expect analyticity or even continuity of, for instance, the secular frequency Ω as a function of the parameters defining f. Recall that, generically, we have $Q_0^{(2)} \neq 0$ for $F_0 = 0$ but, generically, $\mathbf{Q}_{\underline{0}}^{(2)} = 0$ for $F_0 \neq 0$ and, hence, both expansions can be rather different.

Appendices

A Short Description of the Strategy Followed in [1]

For convenience of the reader we reproduce the main steps of the strategy developed in [1] for finding a power series solution of the generalized Riccati equation (I.7) without secular terms.

As discussed in Section I, a natural proposal is to express g, a particular solution of (I.7), as a formal power expansion on ϵ which vanishes at $\epsilon = 0$. For convenience, we write this expansion as in (I.16) where $q(t)$ is defined in (I.17). This would give the desired solution, provided the infinite sum converges. Inserting (I.16) into (I.7) leads to

$$\sum_{n=1}^{\infty}\left((qc_n)' - i\sum_{p=1}^{n-1} q^2 c_p c_{n-p} - 2ifqc_n\right)\epsilon^n + i\epsilon^2 = 0. \tag{A.1}$$

Assuming that the coefficients vanish order by order we conclude

$$(qc_1)' - 2ifqc_1 = 0, \tag{A.2}$$

$$(qc_2)' - iq^2c_1^2 - 2ifqc_2 + i = 0, \tag{A.3}$$

$$(qc_n)' - i\sum_{p=1}^{n-1} q^2 c_p c_{n-p} - 2ifqc_n = 0, \quad n \geq 3. \tag{A.4}$$

The solutions of (A.2)–(A.3) are

$$c_1(t) \;=\; \alpha_1\, q(t), \tag{A.5}$$

$$c_2(t) \;=\; q(t)\left[i\int_0^t \left(\alpha_1^2 q(t')^2 - q(t')^{-2}\right) dt' + \alpha_2\right], \tag{A.6}$$

$$c_n(t) \;=\; q(t)\left[i\left(\sum_{p=1}^{n-1}\int_0^t c_p(t')c_{n-p}(t')\, dt'\right) + \alpha_n\right], \quad \text{for } n \geq 3, \tag{A.7}$$

where the α_n's above, $n = 1, 2, \ldots$, are arbitrary integration constants.

The key idea is to fix the integration constants α_i in such a way as to eliminate the constant terms from the integrands in (A.6) and (A.7). The remaining terms involve sums of exponentials like $e^{in\omega t}$, $n \neq 0$, which do not develop secular terms when integrated, in contrast to the constant terms. For instance, fixing α_1 such that $M(\alpha_1^2 q^2 - q^{-2}) = 0$, that means, $\alpha_1^2 = M(q^{-2})/M(q^2)$, prevents secular terms in (A.6).

As shown in [1] this procedure can be implemented in all orders, fixing all constants α_i and preventing secular terms in all functions $c_n(t)$. In case I, relations (II.5)–(II.7) represent precisely relations (A.5)–(A.7) in Fourier space with the integration constants fixed as explained above. Case II is analogous.

B The Decay of the Fourier Coefficients of q and q^2

To prove our main results on the Fourier coefficients of the functions c_n and e_n we have to establish some results on the decay of the Fourier coefficients of q and q^2.

For periodic f we write the Fourier series (I.18) in the form

$$f(t) \; = \; F_0 + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} F_n e^{in\omega t},$$

with $\overline{F_n} = F_{-n}$, since f is real. In order to simplify our analysis we will consider here the case where the sum above is a finite sum. This situation is physically more realistic anyway.

By assumption, the set of integers $\{n \in \mathbb{Z},\ n \neq 0 |\ F_n \neq 0\}$ is a finite set and, by the condition that f is real and $F_0 = 0$, it contains an even number of elements, say $2J$ with $J \geq 1$. Let us write this set of integers as $\{n_1, \ldots, n_{2J}\}$ and write

$$f(t) \; = \; F_0 + \sum_{a=1}^{2J} f_a e^{in_a\omega t}, \tag{B.1}$$

with the convention that $n_a = -n_{2J-a+1}$, for all $1 \leq a \leq J$, with $f_a \equiv F_{n_a}$. Clearly $\overline{f_a} = f_{2J-a+1}$, $1 \leq a \leq J$. Relation (II.20) becomes

$$Q_m \; = \; e^{i\gamma_f} \sum_{p_1,\, \ldots,\, p_{2J}=0}^{\infty} \delta\,(P,\ m) \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{f_a}{n_a\omega} \right)^{p_a} \right], \tag{B.2}$$

where

$$P \; \equiv \; P(p_1, \ldots, p_{2J}, n_1, \ldots, n_{2J}) \; := \; \sum_{b=1}^{2J} p_b n_b \in \mathbb{Z}, \tag{B.3}$$

and where

$$\gamma_f \; := \; i \sum_{a=1}^{2J} \frac{f_a}{n_a\omega}. \tag{B.4}$$

As one easily sees, $\gamma_f \in \mathbb{R}$. Above $\delta\,(P,\ m)$ is the Krönecker delta:

$$\delta\,(P,\ m) \; := \; \begin{cases} 1, & \text{if } P = m, \\ 0, & \text{else.} \end{cases}$$

Relation (II.21) becomes

$$Q_m^{(2)} = e^{2i\gamma_f} \sum_{p_1, \ldots, p_{2J}=0}^{\infty} \delta(P, m) \prod_{a=1}^{2J} \left[\frac{1}{p_a!} \left(\frac{2f_a}{n_a \omega} \right)^{p_a} \right]. \tag{B.5}$$

The coefficients Q_m and $Q_m^{(2)}$ can also be expressed in terms of Bessel functions of the first kind and integer order. See Section VI for some examples.

As in [1], define

$$\varphi := \max_{1 \le a \le 2J} \left| \frac{f_a}{n_a \omega} \right| \qquad \text{and} \qquad \mathcal{N} := \sum_{b=1}^{2J} |n_b|.$$

Note that, since the n_b's are fixed by the choice of f, $\mathcal{N}$ is non-zero.

The following important bounds have been proven in [1], Appendix D:

$$|Q_m| \le \left(2Je^{(2J-1)\varphi}\right) \frac{\varphi^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil!} \left(1 - \frac{\varphi}{\lceil \mathcal{N}^{-1}|m| \rceil + 1}\right)^{-1}, \tag{B.6}$$

and

$$|Q_m^{(2)}| \le \left(2Je^{(2J-1)2\varphi}\right) \frac{(2\varphi)^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil!} \left(1 - \frac{2\varphi}{\lceil \mathcal{N}^{-1}|m| \rceil + 1}\right)^{-1}, \tag{B.7}$$

for all m with $\lceil \mathcal{N}^{-1}|m| \rceil + 1 > 2\varphi$. Above $\lceil x \rceil$ is the lowest integer larger than or equal to x.

In [1] we derived from (B.6) a simple exponential bound for $|Q_m|$, namely,

$$|Q_m| \le \mathcal{Q}\, e^{-\chi |m|}, \tag{B.8}$$

where $\mathcal{Q}$ and χ are some positive constants. For the purposes of this paper a sharper bound than (B.8) is needed and we have to study relation (B.6) more carefully. The result is expressed in Proposition II.2 whose proof we present now.

Proof of Proposition II.2. Let us consider first the coefficients Q_m. Due to the dominating factor $\lceil \mathcal{N}^{-1}|m| \rceil!$, one has

$$\lim_{|m| \to \infty} \frac{\ll m \gg^2}{e^{-\chi|m|}} \frac{\varphi^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil!} = 0.$$

for any constant $\chi > 0$. Hence, one can choose a constant $M_1 > 0$ depending on χ such that

$$\frac{\varphi^{\lceil \mathcal{N}^{-1}|m| \rceil}}{\lceil \mathcal{N}^{-1}|m| \rceil!} \le M_1 \frac{e^{-\chi|m|}}{\ll m \gg^2}$$

for all $m \in \mathbb{Z}$. Therefore, there exists a positive constant $\mathcal{Q}_1 > 0$ (depending on χ) such that $|Q_m| \le \mathcal{Q}_1 \ll m \gg^{-2} e^{-\chi|m|}$ for all $m \in \mathbb{Z}$. For $Q_m^{(2)}$ we proceed in the same way and get the bound $|Q_m^{(2)}| \le \mathcal{Q}_2 \ll m \gg^{-2} e^{-\chi|m|}$ for all $m \in \mathbb{Z}$. In (II.23) we adopt $\mathcal{Q} = \max\{\mathcal{Q}_1,\ \mathcal{Q}_2\}$. $\square$

C Bounds on Convolutions

Here we will prove Lemma II.3. Consider for $\chi > 0$ and $m \in \mathbb{Z}$

$$\mathcal{B}(m) \equiv \mathcal{B}(m,\,\chi) \;:=\; \sum_{n\in\mathbb{Z}} \frac{e^{-\chi(|m-n|+|n|)}}{\ll m-n\gg^2 \ll n\gg^2}. \tag{C.1}$$

First, note that $\mathcal{B}(m) = \mathcal{B}(-m)$ for all $m \in \mathbb{Z}$. Choosing B_0 to be such that

$$B_0 \geq \sum_{n\in\mathbb{Z}} \frac{e^{-2\chi|n|}}{\ll n\gg^4}$$

the statement of the lemma becomes trivially correct for $m = 0$. Hence, it is enough to consider the case where $m > 0$.

In (C.1), the sum over all $n \in \mathbb{N}$ can be split into three sums:

$$\begin{aligned}\mathcal{B}(m) \;&=\; e^{-\chi m}\sum_{n=-\infty}^{-1} \frac{e^{2\chi n}}{(m-n)^2 n^2} \;+\; e^{-\chi m}\sum_{n=0}^{m} \frac{1}{\ll m-n\gg^2 \ll n\gg^2} \\ &\quad + e^{\chi m}\sum_{n=m+1}^{\infty} \frac{e^{-2\chi n}}{(m-n)^2 n^2}\;.\end{aligned} \tag{C.2}$$

In the first sum above we perform the change of variables $n \to -n$ and in the third sum we perform the change of variables $n \to n+m$. The result is

$$\mathcal{B}(m) \;=\; e^{-\chi m}\left(2\sum_{n=1}^{\infty} \frac{e^{-2\chi n}}{(m+n)^2 n^2} + \sum_{n=0}^{m} \frac{1}{\ll m-n\gg^2 \ll n\gg^2}\right) \tag{C.3}$$

Now we will study separately each of the sums in (C.3). Since for $n \geq 1$ one has $m + n \geq \ll m\gg$ one has for the first sum

$$\sum_{n=1}^{\infty} \frac{e^{-2\chi n}}{(m+n)^2 n^2} \leq \frac{B_1}{\ll m\gg^2} \tag{C.4}$$

where $B_1 := \displaystyle\sum_{n=1}^{\infty} \frac{e^{-2\chi n}}{n^2}$.

The second sum in (C.3) is a little more involving. We have

$$\begin{aligned}&\sum_{n=0}^{m} \frac{1}{\ll m-n\gg^2 \ll n\gg^2} \quad = \\ &\sum_{n=0}^{\lfloor m/2\rfloor} \frac{1}{\ll m-n\gg^2 \ll n\gg^2} \;+\; \sum_{n=\lfloor m/2\rfloor+1}^{m} \frac{1}{\ll m-n\gg^2 \ll n\gg^2}.\end{aligned} \tag{C.5}$$

For the first sum in the right hand side of (C.5) we have $\ll m - n \gg \, \geq m - n \geq m - \lfloor m/2 \rfloor \geq m/2$. For the second sum in the right hand side of (C.5) we have $n \geq \lfloor m/2 \rfloor + 1 \geq m/2$. Hence, for $m > 0$,

$$\begin{aligned}
\sum_{n=0}^{m} \frac{1}{\ll m-n \gg^2 \ll n \gg^2} &\leq \left(\frac{2}{m}\right)^2 \left[\sum_{n=0}^{\lfloor m/2 \rfloor} \frac{1}{\ll n \gg^2} + \sum_{n=\lfloor m/2 \rfloor + 1}^{m} \frac{1}{\ll m-n \gg^2}\right] \\
&\leq 2 \left(\frac{2}{\ll m \gg}\right)^2 \sum_{n=0}^{\infty} \frac{1}{\ll n \gg^2} \ . \qquad \text{(C.6)}
\end{aligned}$$

Therefore, choosing

$$B_0 = 2B_1 + 8 \sum_{n=0}^{\infty} \frac{1}{\ll n \gg^2} \qquad \text{(C.7)}$$

the lemma is proven. □

D Catalan Numbers. Bounds on the Constants K_n

Here we will prove the crucial Theorem III.2. Let us start recalling that we have chosen $K_1 = K_2 = \mathcal{C}_1$ for some constant $\mathcal{C}_1$ which, in turn, can be chosen without loss to be larger than or equal to 1. The proof of Theorem III.2 will be presented on four steps.

Step 1. In this step we show that the sequence K_n, defined in (III.7), is an increasing sequence.

First, note that $K_3 = \mathcal{C}_2(2K_1K_2 + (K_2)^2) = 3\mathcal{C}_2(K_2)^2$. Since $K_1 = K_2 \geq 1$ and $\mathcal{C}_2 \geq 1$, we have $K_1 = K_2 < K_3$.

Let us now suppose that

$$K_1 = K_2 < K_3 < \cdots < K_n \qquad \text{(D.1)}$$

for some $n \geq 3$. We will show that $K_{n+1} > K_n$. We have

$$\begin{aligned}
&K_{n+1} - K_n = \\
&\mathcal{C}_2 \left[\sum_{p=1}^{n} K_pK_{n-p+1} + \sum_{p=2}^{n} K_pK_{n-p+2} - \sum_{p=1}^{n-1} K_pK_{n-p} - \sum_{p=2}^{n-1} K_pK_{n-p+1}\right] = \\
&\mathcal{C}_2 \left[2K_1K_n + \sum_{p=2}^{n} K_pK_{n-p+2} - \sum_{p=1}^{n-1} K_pK_{n-p}\right] = \\
&\mathcal{C}_2 \left[2K_1K_n + (K_n - K_{n-2})K_1 + (K_3 - K_1)K_{n-1} + \cdots + (K_n - K_{n-2})K_2\right],
\end{aligned}$$

where in the last equality we used $K_1 = K_2$. Now, from hypothesis (D.1) we conclude that $K_{n+1} > K_n$, thus proving that K_n is an increasing sequence.

Step 2. Here we show that the sequence K_n defined in (III.7) satisfies

$$K_n \ \leq \ 3\mathcal{C}_2 \sum_{p=2}^{n-1} K_p K_{n-p+1} \tag{D.2}$$

for all $n \geq 3$.

We have already shown that $K_3 = 3\mathcal{C}_2(K_2)^2$. Hence, (D.2) is obeyed for $n = 3$.

Assume now that (D.2) is satisfied for all K_p with $p \in \{1, \ldots, n-1\}$, for some $n \geq 4$. We will show that it is also satisfied for K_n. In fact, we have from (III.7)

$$K_n \ = \ \mathcal{C}_2 \left[K_1 K_{n-1} + \sum_{a=2}^{n-1} K_a (K_{n-a} + K_{n-a+1}) \right] . \tag{D.3}$$

From this and from the fact proven in step 1 that the sequence K_n is increasing, it follows that

$$K_n \ \leq \ \mathcal{C}_2 \left[K_1 K_{n-1} + 2 \sum_{a=2}^{n-1} K_a K_{n-a+1} \right] \ . \tag{D.4}$$

Now, using the obvious relation

$$K_1 K_{n-1} = K_2 K_{n-1} \leq \sum_{a=2}^{n-1} K_a K_{n-a+1}$$

we get finally from (D.4)

$$K_n \ \leq \ 3\mathcal{C}_2 \sum_{p=2}^{n-1} K_p K_{n-p+1}, \tag{D.5}$$

thus proving (D.2).

Step 3. Here we will prove the following statement. Let L_n be defined as the sequence such that $L_1 = L_2 = K_1 = K_2 = \mathcal{C}_1$ and

$$L_n \ = \ 3\mathcal{C}_2 \sum_{p=2}^{n-1} L_p L_{n-p+1}. \tag{D.6}$$

Then, one has

$$K_n \leq L_n, \qquad \forall n \in \mathbb{N}. \tag{D.7}$$

First, note that $K_3 = 3\mathcal{C}_2(K_1)^2 = 3\mathcal{C}_2(L_1)^2 = L_3$. Hence, (D.7) is valid for $n \in \{1,\ 2,\ 3\}$. Now suppose $K_p \leq L_p$ for all $p \in \{1, \ldots, n-1\}$ for some $n \geq 4$. One has from (D.2)

$$K_n \ \leq \ 3\mathcal{C}_2 \sum_{p=2}^{n-1} K_p K_{n-p+1} \ \leq \ 3\mathcal{C}_2 \sum_{p=2}^{n-1} L_p L_{n-p+1} \ = \ L_n, \tag{D.8}$$

thus proving (D.7).

Step 4. Consider the sequence $\mathbf{c}_n$ defined as follows: $\mathbf{c}_1 = \mathbf{c}_2 = 1$ and

$$\mathbf{c}_n = \sum_{p=2}^{n-1} \mathbf{c}_p \mathbf{c}_{n-p+1} \tag{D.9}$$

for $n \geq 3$. The so defined numbers $\mathbf{c}_n$ are called "Catalan numbers", after the mathematician Eugène C. Catalan. The Catalan numbers arise in several combinatorial problems (for a historical account with proofs, see [19]) and can be expressed in a closed form as

$$\mathbf{c}_n = \frac{(2n-4)!}{(n-1)!(n-2)!}, \qquad n \geq 2. \tag{D.10}$$

(see, f.i, [19] or [20]). Using Stirling's formula we get the following asymptotic behaviour for the Catalan numbers:

$$\mathbf{c}_n \approx \frac{1}{16\sqrt{\pi}} \frac{4^n}{n^{3/2}}, \qquad n \text{ large}. \tag{D.11}$$

The existence of a connection between the Catalan numbers and the sequence L_n defined above is evident. Two distinctions are the factor $3\mathcal{C}_2$ appearing in (D.6) and the fact that $L_1 = L_2 = \mathcal{C}_1$ is not necessarily equal to 1. Nevertheless, using the definition of the Catalan numbers in (D.9), it is easy to prove the following closed expression for the numbers L_n:

$$L_n = (\mathcal{C}_1)^{n-1} (3\mathcal{C}_2)^{n-2} \frac{(2n-4)!}{(n-1)!(n-2)!}, \qquad n \geq 2. \tag{D.12}$$

We omit the proof here. Hence, the following asymptotic behaviour can be established:

$$L_n \approx \frac{1}{144\mathcal{C}_1\mathcal{C}_2^2\sqrt{\pi}} \frac{(12\mathcal{C}_1\mathcal{C}_2)^n}{n^{3/2}}, \qquad n \text{ large}. \tag{D.13}$$

From the inequality $K_n \leq L_n$, proven in step 3, it follows that $K_n \leq K_0(12\mathcal{C}_1\mathcal{C}_2)^n$ for some constant $K_0 > 0$, for all $n \in \mathbb{N}$. Theorem III.2 is now proven. $\square$

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