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A Palindromic Half-Line Criterion for Absence of Eigenvalues and Applications to Substitution Hamiltonians

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Abstract. We prove a criterion for absence of decaying solutions on the half-line for one-dimensional discrete Schrödinger operators. As necessary inputs, we require infinitely many palindromic prefixes and upper and lower bounds for the traces of associated transfer matrices. We apply this criterion to Schrödinger operators with potentials generated by substitutions.

1 Introduction

The use of local symmetries in the study of spectral properties of one-dimensional Schrödinger operators has a long history dating back at least to the work of Gordon in 1976 [22]. Criteria in this spirit are particularly useful in the study of models centered around the Fibonacci Hamiltonian; see [13] for a review. The general idea is that local symmetries in the potential should be reflected in the solutions of the associated eigenvalue equation which prevents them from being square-summable. Quite often one can even prove the stronger property that the solutions do not decay at infinity. In one dimension, there are of course two types of local symmetries: repetition of blocks (i.e., powers) and reflection symmetry of blocks (i.e., palindromes). Moreover, the criteria can be classified as half-line methods and whole-line methods, according to whether they study properties of the potentials and the solutions on a half-line or the whole line. Half-line methods are usually slightly more involved, but they have the advantage that they provide stronger conclusions. Thus, there are in principle four types of criteria for absence of eigenvalues which employ local symmetries.

Based on Gordon's work, Delyon-Petritis [20] and Sütő [34] have found whole-line and half-line criteria, respectively, using the occurrence of local repetitions in the potential. These criteria have subsequently been applied to large classes of potentials generated by substitutions and circle maps; see, for example, [8, 9, 10, 11, 12, 15, 16, 17, 18, 20, 23, 34]. It is important to note that these criteria do not only exclude eigenvalues, they also establish explicit solution estimates which were crucial in the study of more refined spectral properties such as α-continuity; compare [10, 17].

Based on Jitomirskaya-Simon [25], Hof et al. [24] have found a whole-line criterion for absence of eigenvalues which is based on palindromes. This criterion is applicable to large classes of potentials but it has the slight drawback that it does not exclude decaying solutions. Moreover, its scope is somewhat limited; see [1, 2, 19] for results on non-applicability of palindromic criteria.

Let us summarize the current situation:

	powers	palindromes
whole-line	Delyon-Petritis [20] (based on [22])	Hof-Knill-Simon [24]
half-line	Sütő [34] (based on [22])	?

Our motivation for filling in the gap in this table is now twofold. On the one hand, it is interesting in its own right to find a palindromic half-line criterion. This is further motivated by the fact that, as mentioned above, half-line methods give stronger conclusions. On the other hand, we will show — in our application of such a criterion to Schrödinger operators with potentials generated by substitutions — how one obtains a more detailed understanding of both the solution behavior and the concrete realizations of the potentials one can treat for many examples (including, e.g., the prominent Thue-Morse case). We remark that the Thue-Morse case displays little power symmetries, and hence is quite impossible to study using Gordon-type criteria, whereas palindromic symmetries are abundant.

The organization of this article is as follows. In Section 2 we prove a half-line criterion for absence of decaying solutions for general potentials provided that one finds suitable palindromic structures and bounds for the traces of the transfer matrices associated to them. In Section 3 we apply this criterion to potentials generated by substitutions.

2 A Palindromic Criterion for Absence of Decaying Solutions on the Half-Line

In this section we show how one can exclude the presence of decaying solutions for a half-line eigenvalue problem with a potential having infinitely many palindromes as prefixes. The necessary input are upper and lower bounds on transfer matrix traces on (not necessarily related) subsequences of these palindromic prefixes. As explained in the introduction, this complements the whole-line palindrome method of Hof et al. and also the several variants of Gordon-type criteria.

Consider a discrete one-dimensional Schrödinger operator

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n) \tag{1}$$

in $\ell^2(\mathbb{Z})$ with potential $V : \mathbb{Z} \to \mathbb{R}$. We shall study the solutions to the difference equation

$$\phi(n+1) + \phi(n-1) + V(n)\phi(n) = E\phi(n) \tag{2}$$

for $E \in \mathbb{R}$. As usual, we introduce the transfer matrices

$$M_E(n) = T_E(V(n)) \times \cdots \times T_E(V(1))$$

where for $\zeta \in \mathbb{R}$,

$$T_E(\zeta) = \begin{pmatrix} E-\zeta & -1 \\ 1 & 0 \end{pmatrix}.$$

Then, any solution ϕ of (2) obeys

$$\Phi(n) = M_E(n)\Phi(0),$$

where $\Phi(i)$ denotes $(\phi(i+1), \phi(i))^T$. The main result of this section is the following:

Theorem 1 *Fix some $E \in \mathbb{R}$. Suppose that*

(i) *There exists a sequence of integers $n_k \to \infty$ such that for every k and every $1 \le i \le \lfloor n_k/2 \rfloor$, we have $V(i) = V(n_k - i + 1)$.*

(ii) *There exists a constant C_1 such that $|\mathrm{tr} M_E(n_k)| \le C_1$ for infinitely many n_k.*

(iii) *There exists a constant C_2 such that $|\mathrm{tr} M_E(n_k)| \ge C_2$ for infinitely many n_k.*

Then no solution ϕ of (2) tends to 0 at $+\infty$. In particular, no solution of (2) is square-summable and hence E is not an eigenvalue of H.

Remark In other words, if the potential restricted to the right half-line has infinitely many palindromic prefixes and the traces of the transfer matrices associated with these palindromes neither tend to 0 nor to ∞ for some energy E, then the solutions corresponding to this energy do not tend to 0 at $+\infty$ and hence E is not an eigenvalue of H.

Proof. We will first show that for every k, the transfer matrix $M_E(n_k)$ has the form

$$M_E(n_k) = \begin{pmatrix} a_k & -b_k \\ b_k & d_k \end{pmatrix} \tag{3}$$

for suitable numbers a_k, b_k, d_k. We consider first the case where n_k is even. Let

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and denote the matrix entries of $M_E(n_k/2)$ by

$$M_E(n_k/2) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix}.$$

Then, using assumption (i), we get

$$\begin{aligned}
M_E(n_k) &= M_E(n_k/2) \cdot T \cdot (M_E(n_k/2))^{-1} \cdot T \\
&= \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \cdot T \cdot \begin{pmatrix} \delta_k & -\beta_k \\ -\gamma_k & \alpha_k \end{pmatrix} \cdot T \\
&= \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \cdot \begin{pmatrix} \alpha_k & -\gamma_k \\ -\beta_k & \delta_k \end{pmatrix} \\
&= \begin{pmatrix} \alpha_k^2 - \beta_k^2 & -(\alpha_k\gamma_k - \beta_k\delta_k) \\ \alpha_k\gamma_k - \beta_k\delta_k & -\gamma_k^2 + \delta_k^2 \end{pmatrix} \\
&=: \begin{pmatrix} a_k & -b_k \\ b_k & d_k \end{pmatrix}.
\end{aligned}$$

This proves (3) for n_k even.

Let us now consider the case where n_k is odd. Writing

$$M_E(\lfloor n_k/2 \rfloor) = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix},$$

we can proceed with $\zeta = V(\lfloor n_k/2 \rfloor + 1)$ as follows:

$$\begin{aligned}
M_E(n_k) &= M_E(\lfloor n_k/2 \rfloor) \cdot T_E(\zeta) \cdot T \cdot (M_E(\lfloor n_k/2 \rfloor))^{-1} \cdot T \\
&= \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \cdot \begin{pmatrix} E-\zeta & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_k & -\gamma_k \\ -\beta_k & \delta_k \end{pmatrix} \\
&= \begin{pmatrix} (E-\zeta)\alpha_k^2 + 2\alpha_k\beta_k & -((E-\zeta)\alpha_k\gamma_k + 2\beta_k\gamma_k) \\ (E-\zeta)\alpha_k\gamma_k + 2\beta_k\gamma_k & -((E-\zeta)\gamma_k^2 + 2\gamma_k\delta_k) \end{pmatrix} \\
&=: \begin{pmatrix} a_k & -b_k \\ b_k & d_k \end{pmatrix}
\end{aligned}$$

and hence we have (3) also for n_k odd.

Now assume that there exists a decaying solution ϕ of (2). Then $\|M_E(n_k)\|$ and thus $\max\{|a_k|, |b_k|, |d_k|\}$ tends to infinity. By assumption (ii), on a subsequence n_{k_j} we have $|a_{k_j} + d_{k_j}| \le C_1$. Using this and the relation $\det M_E(n_k) = a_k d_k + b_k^2 = 1$, we get

$$\min\{|a_{k_j}|, |b_{k_j}|, |d_{k_j}|\} \to \infty. \tag{4}$$

Now, with $v = \Phi(0)$, we have $\|M_E(n_k)v\| \to 0$ as $k \to \infty$. We see that, as an element of $P(\mathbb{R}^2)$, $v \neq (1,0)^T$, for otherwise $a_k, b_k \to 0$ and hence $d_k \to \infty$, contradicting assumption (ii). Similarly, it follows that in $P(\mathbb{R}^2)$, $v \neq (0,1)^T$. Therefore $v = (1,\xi)^T$ in $P(\mathbb{R}^2)$ with some $\xi \neq 0$. We get

$$a_k - \xi b_k \to 0, \; b_k + \xi d_k \to 0.$$

Hence,

$$a_k + \xi^2 d_k \to 0.$$

In particular, on the subsequence n_{k_j} we have both $a_{k_j}+\xi^2 d_{k_j} \to 0$ and $|a_{k_j}+d_{k_j}| \leq C_1$, so that (4) implies $\xi^2 = 1$. But then $\mathrm{tr} M_E(n_k) = a_k + d_k \to 0$, contradicting assumption (iii). □

The above proof shows in fact the following:

Corollary 1 *Fix some energy $E \in \mathbb{R}$. Suppose as in Theorem 1 that the potential on the right half-line starts with infinitely many palindromes and suppose further that the traces of the associated transfer matrices are bounded on a subsequence. Then the existence of a decaying solution of (2) implies that these traces converge to zero and the initial condition of the decaying solution is equal to either $(1,1)^T$ or $(1,-1)^T$ in $P(\mathbb{R}^2)$.*

This result can be interpreted as a result for half-line operators on $\ell^2(\mathbb{N})$ where one has to impose a boundary condition at the origin to ensure self-adjointness. On the set of energies where the transfer matrix traces do not diverge to infinity, one therefore has absence of decaying (and thus ℓ^2-) solutions for all but two boundary conditions.

3 Absence of Eigenvalues for Substitution Schrödinger Operators

In this section, we want to apply our criterion to the particular case of Schrödinger operators with potentials generated by primitive substitutions. Some classes of such operators have already been shown to exhibit absence of eigenvalues [6, 9, 11, 12, 15, 16, 21, 23, 24, 34]. Here we will show how the application of our criterion allows one to treat an additional subclass.

Schrödinger operators with potentials generated by primitive substitutions have been studied mainly because of their relevance to quasicrystals and their exotic spectral properties. Since the discovery of quasicrystals by Shechtman et al. in 1984 [33], a lot of effort has been made to find appropriate structural models. The two most heavily studied models are generated by either a cut and project scheme or a substitution process. On the other hand, there has been intense research activity on Schrödinger operators with purely singular continuous spectra over the last decade and it turned out that as a rule, one-dimensional Schrödinger operators with potentials generated by primitive substitution apparently exhibit this "exotic" spectral type. While absence of absolutely continuous spectrum follows in full generality from Kotani [26] and Last and Simon [27], absence of point spectrum is not known in similar generality. However, no counterexample is known, and there is a huge number of positive partial results; see [13] for a review and [9] for the conjecture that the hypothesis sufficient to prove the singularity of the spectrum, that is, semi-primitivity of a reduced trace map and existence of a square in u (Theorem 2 below), are also sufficient to prove absence of eigenvalues . Apart from results on the spectral type, other interesting results have been obtained for these operators, such as zero-measure Cantor spectrum [6, 8, 9, 35], general gap-labeling (heuristic [28], K-theoretic [7] and constructive [30]), and results on the opening of gaps at low coupling (heuristic [28] and precise [5, 6]) and large coupling [30].

3.1 A class of models with empty point spectrum

Let us first recall some definitions concerning these operators.

A *substitution* [29] is a map S from a finite alphabet $\mathcal{A}$ to the set $\mathcal{A}^*$ of words on $\mathcal{A}$, which can be naturally extended to a map from $\mathcal{A}^*$ to $\mathcal{A}^*$ and also to a map from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{A}^{\mathbb{N}}$. S is said to be *primitive* if there exists $k \in \mathbb{N}$ such that for every pair $(\alpha, \beta) \in \mathcal{A}^2$, $S^k(\alpha)$ contains β. A *substitution sequence* u is a fixed point of S given by indefinite iteration of S on a letter $a \in \mathcal{A}$ such that $S(a)$ begins with a. The *hull* of u, Ω_u, is defined as the set of two-sided infinite sequences over $\mathcal{A}$ that have all their finite subwords occurring in u. Fix some function $f : \mathcal{A} \to \mathbb{R}$. One says that a Schrödinger operator of type (1) is *associated with S and f* if the sequence $(V(n))_{n\in\mathbf{Z}}$ has the form $V(n) = f(\omega_n)$ for some $\omega \in \Omega_u$. It follows from general principles (see, e.g., [13]) that primitivity of S implies the existence of a closed set $\Sigma \subseteq \mathbb{R}$ such that the spectrum of every associated operator H is equal to Σ. However, in general the spectral type of H need not be independent of ω.

We will always assume in the following that S and f are such that the potentials V are not periodic since the spectral theory of the periodic case is well established.

Let us recall some notions which are crucial to the approach of [9]. Define for any word $w = w_1 \dots w_m \in \mathcal{A}^*$ and every $E \in \mathbb{R}$,

$$M_E(w) = T_E(f(w_n)) \times \cdots \times T_E(f(w_1))$$

and

$$M_E^{(k)}(w) = M_E(S^k(w)).$$

The substitution rule naturally leads to recursive relations between the matrices $M_E^{(k)}(w)$. From these recursions one can obtain an even more useful system of recursive equations for the traces of these matrices. In general there exists a finite subset of words $\mathcal{B} \subset \mathcal{A}^*$ containing $\mathcal{A}$ for which these equations yield a closed set of recursive polynomial equations, which is called the *trace map*. It turns out that to each trace map one can associate a *reduced trace map* that is *monomial*. To this reduced trace map, one can then associate a substitution $\hat{S}$ on $\mathcal{B}$ whose properties are ultimately crucial for the spectral analysis. In short terms, the reduced trace map is obtained by keeping in the recursive equations the terms of highest degree which determine the behavior of the norms of the transfer matrices at large n, allowing one, under the hypothesis of the theorem below, to identify the spectrum with the set of energies with zero Lyapunov exponent and thus to apply Kotani's theorem [26] to prove the singularity of the spectrum; see [9] for details of the proof. We call such a substitution $\hat{S}$ *semi-primitive* if:

(i) There exists $\mathcal{C} \subset \mathcal{B}$ such that $\hat{S}|_{\mathcal{C}}$ is a primitive substitution from $\mathcal{C}$ to $\mathcal{C}^*$.

(ii) There exists k such that for all $\beta \in \mathcal{B}$, $\hat{S}^k(\beta)$ contains at least one letter from $\mathcal{C}$.

The following theorem was proven in [9]:

Theorem 2 *Assume that S is primitive and has a fixed point u, f is such that the associated potentials are aperiodic, and the following conditions are satisfied:*

(i) *u contains the square of a word in $\mathcal{B}$.*

(ii) *There exists a trace map whose associated substitution $\hat{S}$ is semi-primitive.*

Then for every $\omega \in \Omega_u$, the spectrum of the operator H in (1) *with $V(n) = f(\omega_n)$ is singular and supported on a set of zero Lebesgue measure.*

Of special interest for an application of our criterion from Section 2 to substitution models is the fact that semi-primitivity of the trace map implies the existence of non-divergent subsequences of trace map iterates for energies in the spectrum [9, Lemma 3.4]. It has been shown in [9] that semi-primitivity of the derived substitution $\hat{S}$ holds for many prominent substitutions including (in the case where $\mathcal{A} = \{a, b\}$) Fibonacci ($a \mapsto ab, b \mapsto a$), period doubling ($a \mapsto ab, b \mapsto aa$), binary non-Pisot ($a \mapsto ab, b \mapsto aaa$), and Thue-Morse ($a \mapsto ab, b \mapsto ba$, to be discussed in detail below).

Let $x_n(E) = \mathrm{tr}(M_E^{(n)}(a))$, where a is the first symbol of u. As a consequence we get that in the case of semi-primitive $\hat{S}$, for every energy E from the spectrum, $|x_n(E)|$ is bounded on a subsequence (which may depend on the energy).

For concrete models with a semi-primitive $\hat{S}$ and which display the required palindromic symmetries, we can therefore focus our attention on *lower* bounds for a subsequence of $|x_n(E)|$.

We have the following theorem:

Theorem 3 *Assume that S is primitive and has a fixed point u, f is such that the associated potentials are aperiodic, and the following conditions are satisfied:*

(i) *$S^n(a)$ is a palindrome for every n, where a is the first symbol of u.*

(ii) *There exists a trace map whose associated substitution $\hat{S}$ is semi-primitive.*

(iii) *For every $E \in \Sigma$, $x_n(E) \not\to 0$ as $n \to \infty$.*

Then there is $\omega \in \Omega_u$ such that u is the restriction of ω to $\mathbb{N}$ and the operator H in (1) with $V(n) = f(\omega_n)$ has no eigenvalues.

Proof. We only have to construct an element $\omega \in \Omega_u$ whose restriction to the right half-line coincides with u. The assertion is then a consequence of Theorem 1. The existence of such an element follows from the repetitivity of u (i.e., its finite subwords occur infinitely often) and compactness of Ω_u (since it is clearly a closed subset of the compact $\mathcal{A}^{\mathbb{Z}}$). Namely, using these two properties, one can construct a subsequence of elements of Ω which on the right half-line converges pointwise to u and then choose a converging subsequence by compactness. The limit ω of this subsequence then coincides with u on the right half-line. By construction, we have that assumptions (i) and (iii) imply conditions (i) and (iii) of Theorem 1, respectively, while, as mentioned above, assumption (ii) implies condition (ii) of Theorem 1 by [9, Lemma 3.4]. □

If u contains a square, which is the case, for example, if there is a palindrome of even length, the operator H above verifies all the assumptions of Theorem 2. We can therefore state the following corollary to Theorem 3.

Corollary 2 *Under the assumptions of Theorem 3, if u contains the square of a word, the spectrum of H is purely singular continuous and supported on a Cantor set of Lebesgue measure zero.*

Remark This corollary is a partial answer to the conjecture in [9] mentioned in the introduction of this section.

3.2 The Thue-Morse case

As a first example, we consider the Thue-Morse sequence; compare, in particular, [5]. Generic absence of eigenvalues was shown by Delyon-Peyrière [21], in [6] (in an implicit way that will be made explicit here), and by Hof et al. [24]. In fact, as was claimed in [6] (see the remark at the end of Section II), the palindromicity of some $S^n(a)$ is a crucial ingredient in the proof. However, our result additionally yields the absence of decaying solutions at $+\infty$ and, if one considers the symmetric extension of u (the reader may verify that this sequence belongs to Ω_u), we even have an explicit potential from the Thue-Morse hull for which every solution to (2) with E in the spectrum decays neither at $+\infty$ nor at $-\infty$.

Let us recall the definition of the Thue-Morse substitution: It is defined on the alphabet $\mathcal{A} = \{a, b\}$ by $S(a) = ab$ and $S(b) = ba$. It is clearly primitive and the sequence $u = \lim_{n\to\infty} S^n(a)$ is called the *Thue-Morse sequence*.

Since one also has $u = \lim_{n\to\infty} S^{2n}(a)$, S^2 generates the same hull and hence the same family of associated operators. It is therefore sufficient to verify all the assumptions of Theorem 3 for the substitution S^2.

Aperiodicity of the associated potentials holds for all non-constant functions f. Moreover, we have

(i) The iterates of S^2 on a are palindromes of even length since $S^2(a) = abba$ and $S^2(b) = baab$.

(ii) Define $y_n(E) = \mathrm{tr}(M_E^{(n)}(b)), z_n(E) = \mathrm{tr}(M_E^{(n)}(ab))$. The trace map for S^2 is as follows (we drop E for simplicity of notation):

$$\begin{aligned} x_{n+1} &= x_n y_n z_n - x_n^2 - y_n^2 + 2 \\ y_{n+1} &= x_n y_n z_n - x_n^2 - y_n^2 + 2 \\ z_{n+1} &= x_n y_n z_n^3 - x_3^2 z_n^2 - y_n^2 z_n^2 + x_n y_n z_n + 2 \end{aligned}$$

which gives rise to a semi-primitive reduced trace map:

$$x \mapsto xyz,\ y \mapsto xyz,\ z \mapsto xyz^3.$$

(iii) Since $z_n(E)$ corresponds to the evolution of the traces associated with the word $\gamma_0 = ab$ which contains a, Lemma 3.4 of [9] shows that for every $E \in \Sigma$, there is a subsequence of $(|z_n(E)|)_{n\in\mathbb{N}}$ which is bounded from above. Then the above recursions for $x_n = y_n$ show that the sequence $(x_n(E))_{n\in\mathbb{N}}$ cannot converge to 0.

Thus, by Corollary 2, there is $\omega \in \Omega_u$ such that u is the restriction of ω to $\mathbb{N}$ and for every non-constant f, the spectrum of H with $V(n) = f(\omega_n)$ is purely singular continuous and supported on a Cantor set of Lebesgue measure zero.

3.3 A family of examples

We can verify the assumptions of Corollary 2 for a class of two-letter substitutions, namely all those which are defined by $S(a) =$ *palindrome of even length beginning with* a, $S(b) = a^p$, $p \in \mathbb{N}$.

First, we remark that this class clearly verifies the hypothesis of semi-primitivity. Second, it is easy to see in this case that if for some E in the spectrum, the sequence $(x_n(E))_{n\in\mathbb{N}}$ converges to 0, then for $y_n(E), z_n(E)$ defined as in the previous subsection, the sequence $(|y_n(E)|)_{n\in\mathbb{N}}$ converges to 0 if p is odd and 2 if p is even. Since there is a subsequence of $(|z_n(E)|)_{n\in\mathbb{N}}$ which is bounded from above, this leads immediately to a contradiction because the specific form of $S(a)$ implies that the trace map expression for $x_n(E)$ contains a constant term of absolute value 2 and no term $y_n(E)$ or $-y_n(E)$.

A natural goal is now the generalization of this result to an arbitrary finite alphabet $\mathcal{A} = \{a_1, ..., a_m\}$, in the sense that $S(a_1) =$ *palindrome of even length beginning with* a_1 and all the other $S(a_k)$'s are powers of a_1, because in this case the convergence of $(x_n(E))_{n\in\mathbb{N}}$ to 0 implies the convergence to 0, 2, or -2 of all the other sequences of traces associated to one letter, while $x_n(E)$ contains a constant term of absolute value 2 and no term of the type $y_n(E)$ or $-y_n(E)$ associated to other letters.

3.4 Examples with an invariant

We have seen that for an attempt to apply Theorem 3, the hardest part is to establish condition (iii) since condition (i) can be verified easily and condition (ii) can be checked by a simple algorithmic procedure; compare [9]. We therefore want to point out that condition (iii) can sometimes be established by using some soft arguments involving a trace map invariant. For example, in the Fibonacci case we have that $x_n(E) = \mathrm{tr}(M_E^{(n)}(a))$ obeys

$$(x_{n+1}(E))^2 + (x_n(E))^2 + (x_{n-1}(E))^2 - x_{n+1}(E)x_n(E)x_{n-1}(E) = C_f$$

for every n and E, with $C_f \neq 0$ if f is not constant [34]. It is clear that this invariant prevents $x_n(E)$ from converging to zero! Models with invariant have been discussed in [3, 4, 14, 31, 32]. We remark that also in the period doubling case, an invariant plays an important role (this is implicit in [6]).

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Note added in proof During the publishing process of this article, we became aware of a preprint by Q-M. Lin, B. Tan, Z-W. Wen and J. Wu, where they claimed that, for any primitive substitution S, there exists a power of S and an associated trace map such that the corresponding induced substitution is semi-primitive.

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