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On Birkhoff Coordinates for KdV

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Abstract. We prove that on the Sobolev spaces H_0^N $(N \geq 0)$ of 1-periodic functions in $H_{loc}^N(\mathbb{R})$ with average 0, the Korteweg-deVries equation (KdV) admits global Birkhoff coordinates.

0 Introduction

Consider the Korteweg-deVries equation (KdV) on $[0,1]$ with periodic boundary conditions,

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u \quad (t \in \mathbb{R}, x \in \mathbb{R}).$$

This equation can be viewed as a Hamiltonian system on the phase space $H^N(N \geq 0)$ with Poisson structure given by ∂_x,

$$\partial_t u = \partial_x \frac{\partial \mathcal{H}}{\partial q(x)}(u).$$

Here $\mathcal{H}$ is the KdV-Hamiltonian $\mathcal{H}(q) := \int_0^1 \left(\frac{1}{2}(\partial_x q)^2 + q^3\right) dx$, $\frac{\partial \mathcal{H}}{\partial q(x)}$ denotes the L_2-gradient of $\mathcal{H}$, and H^N is the Sobolev space

$$H^N := \{q(x) = \sum_k \hat{q}(k) e^{2\pi i k x} \mid ||q||_N < \infty\}$$

where $\hat{q}(k)$ $(k \in \mathbb{Z})$ are the Fourier coefficients of q,

$$\hat{q}(k) = \int_0^1 q(x) e^{-2\pi i k x} dx$$

and

$$||q||_N^2 = \sum_k |\hat{q}(k)|^2 (1 + |k|)^{2N}.$$

The Poisson structure ∂_x is degenerate: the average $[q] := \int_0^1 q(x)dx$ is a Casimir and the symplectic leaves of the induced foliation on H^N are given by the affine spaces $H_c^N := \{q \in H^N \mid [q] = c\}$. It has been proved in a series of papers [**Ka**], [**BBGK**], and [**BKM1**] that for $N \in \mathbb{Z}_{\geq 0}$, each symplectic leaf admits Birkhoff coordinates, i.e. that the corresponding symplectic polar coordinates are action-angle variables.

Let us formulate this result in the case $c = 0$ more precisely: For $r \geq 0$, denote by $h^r := h^r(\mathbb{N};\mathbb{R}^2)$ the model space $\{z = (x_j, y_j)_{j\geq 1} \mid ||z||_r^2 = \sum_{j\geq 1}^{\infty} j^{2r}(x_j^2 + y_j^2) < \infty\}$ endowed with the Poisson bracket defined by $\{x_k, y_n\} = \delta_{n,k}$, $\{x_k, x_n\} = 0$, $\{y_k, y_n\} = 0$. As usual denote by $L^2[0,1]$ the space of real valued L^2-integrable functions on $[0,1]$ and let $L_c^2 \equiv L_c^2[0,1] = \{q \in L^2[0,1] \mid [q] = c\}$.

Theorem 1 *There exists a symplectomorphism*

$$\Omega : L_0^2 \to h^{1/2}(\mathbb{N};\mathbb{R}^2), \quad q \mapsto (x_n(q), y_n(q))_{n\geq 1}$$

with the following properties:

(1) $(x_n, y_n)_{n\geq 1}$ *are Birkhoff coordinates for KdV, i.e. the symplectic polar coordinates* $(I_n, \theta_n)_{n\geq 1}$ *associated to* $(x_n, y_n)_{n\geq 1}, I_n := (x_n^2 + y_n^2)/2$ *and* $\theta_n := \operatorname{arctg}\left(\frac{y_n}{x_n}\right)$, *are action-angle variables for KdV.*

(2) *For any* $N \in \mathbb{Z}_{\geq 0}$, *the restriction* $\Omega^{(N)}$ *of* Ω *to* H_0^N *is a real analytic diffeomorphism,* $\Omega^{(N)} : H_0^N \to h^{N+\frac{1}{2}}$.

A similar result has been proved for action-angle variables with respect to the second bracket of KdV (cf. [**KaMa**]).

Let us mention, among many others, the following two applications of Theorem 1:
(A) The KdV-Hamiltonian $\mathcal{H}$ can be brought into a convergent Birkhoff normal form: when expressed in the new coordinates, $\mathcal{H}$ admits a convergent power series expansion in the action variables $I_1, I_2, \ldots$.
(B) The image $\mathcal{I} := \left\{(I_n(q))_{n\geq 1} \mid q \in L_0^2\right\}$ is all of the positive quadrant of the weighted ℓ_1-sequence space, $\ell_1^1(\mathbb{N};\mathbb{R}_{\geq 0})$. It is a (non-compact) infinite dimensional convex polytope which is the image of the momentum map $(I_n(q))_{n\geq 1}$. This map arises from the action of an infinite dimensional torus on the function space L_0^2. This suggests that the theory of the convexity of the image of momentum map developed in the finite dimensional case (cf [**At**], [**GS**]) extends to an infinite dimensional setting.

In this paper we present a new proof of Theorem 1 which is considerably shorter than the one given in the series of papers [**Ka**], [**BBGK**], and [**BKM1**]. First we introduce action and angle variables, $(I_n)_{n\geq 1}$ and $(\theta_n)_{n\geq 1}$. Heuristically, the formulas for $(I_n)_{n\geq 1}$ and $(\theta_n)_{n\geq 1}$ can be derived as in classical mechanics (cf sections 2 and 3). Following computations for the defocusing nonlinear Schrödinger equation (NLS) due to McKean and Vaninsky [**MV**], we show that $(\theta_n)_{n\geq 1}$ and $(I_n)_{n\geq 1}$ satisfy canonical relations. We then use these variables to construct the map Ω as follows: for q with $I_n(q) \neq 0$, define $\Omega_n(q) = (x_n(q),\ y_n(q))$ by $x_n = \sqrt{2I_n}\cos\theta_n$, $y_n = \sqrt{2I_n}\sin\theta_n$. We prove that $\Omega(q)$ admits an analytic continuation to a complex neighborhood of L_0^2. One of the main new features of the proof of Theorem 1 is to use some of these canonical relations to show that Ω is a local diffeomorphism.

The paper is organized as follows:

In section 1, for the convenience of the reader, we review regularity properties and asymptotic estimates of the action variables I_n $(n \geq 1)$ obtained in [**BBGK**].
In section 2, we introduce the angle variables θ_n $(n \geq 1)$ given by the Abel map, the latter being defined with the help of certain holomorphic differentials studied in [**BKM2**], prove regularity properties, and provide asymptotic estimates of θ_n.
In section 3, we define the map $\Omega : L_0^2 \to h^{1/2}$ using the action-angle variables $(I_n, \theta_n)_{n\geq 1}$ and prove that Ω is real analytic.
A natural way to prove that Ω is a symplectomorphism would be to verify the canonical relations for actions and angles. These relations imply that Ω is a local diffeomorphism. To show that Ω is $1-1$ and onto it is to establish that Ω is proper and $\Omega^{-1}\{0\} = \{0\}$.
However, due to the fact that the Poisson structure ∂_x is a first order differential operator, additional regularity for the L_2-gradients of the action-angle variables are needed to justify the computations used to establish the canonical relations for them. As a consequence, we modify the plan of proof proposed above as follows: It is easy to see that the gradients of the actions have the additional regularity needed to verify all the canonical relations involving the actions (section 4). These canonical relations are used to conclude that Ω is a local diffeomorphism (section 5).
In section 6, we show that Ω is bijective and in section 7 we study the restriction of Ω to the Sobolev space H_0^N.
The property of Ω being a local diffeomorphism allows to consider the push forward $\Omega_*\omega$ of the Gardner symplectic structure ω and to verify that $\Omega_*\omega$ is the standard symplectic form (section 8).
In section 9 we establish, among other things, regularity properties for the Birkhoff coordinates which will be used in subsequent work.
For the convenience of the reader we present several auxilary results in four appendices. Notation is standard, except the one for denoting error terms: For $1 \leq p \leq \infty$, $O_p(n^\alpha)$ respectively $o_p(n^\alpha)$, denotes a sequence of functions $(f_n)_{n\geq 1}$ in L^p such that $n^{-\alpha}||f_n||_{L^p} \leq C$ respectively $\lim_{n\to\infty} n^{-\alpha}||f_n||_{L^p} = 0$.

1 Action variables

In this section we recall the formulas for the actions $(I_n)_{n\geq 1}$, found by Flaschka-McLaughlin [**FM**], and state regularity properties and asymptotic estimates presented in [**BBGK**] and [**BKM1**].

For $q \in L^2_{0,\mathbb{C}} \equiv L^2_0([0,1];\mathbb{C})$ consider the Schrödinger equation

$$-y'' + qy = \lambda y. \tag{1.1}$$

Denote by $y_1(x, \lambda, q)$ and $y_2(x, \lambda, q)$ the fundamental solutions of (1.1) (which are elements in $H^2_{loc}(\mathbb{R};\mathbb{C})$) and by $\Delta(\lambda, q)$ the discriminant,

$$\Delta(\lambda, q) := y_1(1, \lambda, q) + y_2'(1, \lambda, q)$$

and write $\dot{\Delta}(\lambda)$ for $\frac{d}{d\lambda}\Delta(\lambda, q)$. Further denote by *spec*(q) the spectrum $(\lambda_n(q))_{n\geq 0}$ of the operator $-\frac{d^2}{dx^2} + q$ when considered with periodic boundary conditions on the interval [0,2] where $(\lambda_n(q))_{n\geq 0}$ are ordered in such a way that

$$\mathtt{Re}\,\lambda_n < \mathtt{Re}\,\lambda_{n+1} \qquad \text{or} \qquad \mathtt{Re}\,\lambda_n = \mathtt{Re}\,\lambda_{n+1} \text{ and } \mathtt{Im}\,\lambda_n \leq \mathtt{Im}\,\lambda_{n+1}.$$

We point out that $\lambda_n(q)$ are not continuous with respect to q due to this choice of the ordering and the assumption that q is complex valued. In the sequel, we will always assume that $\mathtt{Im}\,q$ of an element $q \in L^2_{0,\mathbb{C}}$ is sufficiently small so that, for any $n \geq 1, \{\lambda_{2n-1}, \lambda_{2n}\}$ is an isolated pair of eigenvalues.

For such a potential q, according to Flaschka and McLaughlin [**FM**], the action variables of KdV, with respect to the Poisson structure ∂_x, are given by

$$I_n(q) := \frac{1}{\pi}\int_{\Gamma_n} \mu \frac{\dot{\Delta}(\mu)}{\sqrt{\Delta(\mu)^2 - 4}} d\mu. \tag{1.2}$$

Here $\sqrt{\Delta(\mu)^2 - 4}$ denotes the branch on the complex plane slit open along $(-\infty, \lambda_0), (\lambda_{2n-1}, \lambda_{2n})$ $(n \geq 1)$ with the sign of the radical chosen so that for q real , $i\sqrt{\Delta(\mu)^2 - 4} > 0$ for $\lambda_0 < \mu < \lambda_1$ and Γ_n $(n \geq 1)$ is a circuit around the interval $(\lambda_{2n-1}, \lambda_{2n})$ with counterclockwise orientation. Flaschka and McLaughlin have obtained formula (1.2) by applying a well known procedure due to Arnold in the case of finite dimensional integrable systems: they defined the action variable I_n by $I_n := \frac{1}{2\pi}\int_{c_n} \alpha$ where α is a 1-form satisfying $\omega = d\alpha$ and $(c_n)_n$ is a (appropriately chosen) basis of cycles of an invariant torus. Expressing $\frac{1}{2\pi}\int_{c_n} \alpha$ in conveniently chosen canonical coordinates they obtain the integral in (1.2) .
Denote by $(\gamma_n)_{n\geq 1}$ the sequence of gap lengths, $\gamma_n := \lambda_{2n} - \lambda_{2n-1}$.

Proposition 1 *Let $q_0 \in L^2_0$. Then there exist a neighborhood U_{q_0} of q_0 in $L^2_{0,\mathbb{C}}$ and a constant $C \geq 1$ so that, for any $n \geq 1$, I_n is analytic on U_{q_0} and*

$$2I_n = \frac{1}{n\pi}\left(\frac{\gamma_n}{2}\right)^2 (1 + r_n)$$

where the error r_n is analytic on U_{q_0}, satisfies $\frac{1}{C} \leq |1 + r_n| \leq C$ and $\frac{1}{C} \leq$ $\mathtt{Re}(1 + r_n) \leq C$ as well as the asymptotic estimate $r_n = O\left(\frac{\log n}{n}\right)$.

As a consequence,

$$\xi_n(q) := \left(\frac{2I_n}{(\gamma_n/2)^2}\right)^{1/2} \tag{1.3}$$

is analytic and does not vanish on U_{q_0} (with $z^{1/2}$ denoting the branch of the square root which equals 1 at $z = 1$) and satisfies the asymptotic estimate $(q \in U_{q_0})$

$$|\xi_n - \frac{1}{\sqrt{n\pi}}| \leq C' \frac{\log n}{n}$$

where $C' \geq 1$ is independent of q.

Proof. (in [**BBGK**], section 2) □

Integrating (1.2) by parts, the L^2-gradient $\frac{\partial I_n}{\partial q(x)}$ can be computed

$$\frac{\partial I_n}{\partial q(x)} = -\frac{1}{\pi}\int_{\Gamma_n} \frac{\frac{\partial \Delta(\mu)}{\partial q(x)}}{\sqrt{\Delta^2(\mu)-4}}\, d\mu.$$

2 Angle variables

To define the angle variables, introduce the holomorphic differentials investigated in [**BKM2**] (cf also [**MT2**]).

Proposition 2 *There exists an open neighborhood $U = U_{L^2_0}$ in $L^2_{0,\mathbb{C}}$ so that for any q in U, one can find a sequence of entire functions $\psi_j(\lambda) \equiv \psi_j(\lambda, q)$ $(j \geq 1)$ satisfying*

$$\frac{1}{2\pi}\int_{\Gamma_n} \frac{\psi_j(\lambda, q)\ d\lambda}{\sqrt{\Delta(\lambda, q)^2 - 4}} = \delta_{j,n} \tag{2.1}$$

The functions ψ_j depend analytically on λ and q and admit a product representation

$$\psi_j(\lambda) = \frac{c_j}{j^2\pi^2}\prod_{k\neq j} \frac{\mu_k^{(j)} - \lambda}{k^2\pi^2} \tag{2.2}$$

with $\mu_k^{(j)} = \mu_k^{(j)}(q)$ and $c_j = c_j(q)$ depending analytically on $q \in U$ and satisfying

$$|\mu_k^{(j)} - \tau_k| \leq C\frac{1}{k}\ |\gamma_k|^2 \quad (k \neq j); \quad \tau_k = \frac{1}{2}(\lambda_{2k-1} + \lambda_{2k}) \tag{2.3}$$

$$|c_j - 2\pi j| \leq C\frac{1}{j} \tag{2.4}$$

where $C > 0$ can be chosen locally uniformly with respect to q and independently of $j \geq 1$.

Proof. cf Theorem A.5 (in Appendix A.2), Lemma 3.2, and Lemma 3.3 in [**BKM2**]. □

It is convenient to introduce the following

Definition *An open set U in $L^2_{0,\mathbb{C}}$ is said to be a G-neighborhood if U satisfies the properties stated in Proposition 2.*

In the sequel, let U_{q_0} always denote a bounded G-neighborhood of $q_0 \in L^2_0$.

To define the angle variables, introduce the hyperelliptic surface Σ_q, $y = \sqrt{\Delta^2(\lambda) - 4}$, associated with *spec*$(q)$.

For q in $U_{q_0} \setminus D_n$ with

$$D_n := \{q \mid \lambda_{2n} = \lambda_{2n-1}\}$$

the angle variable $\theta_n(q)$ is defined formally - to be the n'th component of the Abel map associated to Σ_q, evaluated at $(\mu_k^*)_{k\geq 1}$ with $\mu_k^* := (\mu_k, \sqrt{\Delta^2(\mu_k) - 4}) \in \Sigma_q$. Here $\mu_k = \mu_k(q)$ ($k \geq 1$) denote the Dirichlet eigenvalues of the operator $-\frac{d^2}{dx^2} + q$ considered on $[0, 1]$.

More precisely, we define for q in $U_{q_0} \setminus D_n$,

$$\theta_n(q) := \sum_{k\geq 1} \int_{\lambda_{2k}(q)}^{\mu_k^*(q)} \frac{\psi_n(\lambda, q)}{\sqrt{\Delta^2(\lambda, q) - 4}} d\lambda \tag{2.5}$$

where for each $k \geq 1$ the path in the integral

$$\eta_{n,k}(q) := \int_{\lambda_{2k}(q)}^{\mu_k^*(q)} \frac{\psi_n(\lambda, q)}{\sqrt{\Delta^2(\lambda, q) - 4}} d\lambda \tag{2.6}$$

is near λ_{2k}, but otherwise arbitrary.

Formula (2.5)for the variables $(\theta_n)_n$ conjugate to the actions can be obtained - at least formally - by taking the derivative of $\alpha = \sum_n I_n d\theta_n$ with respect to I_n, $\frac{\partial\alpha}{\partial I_n} = d\theta_n$ and integrating on an invariant torus with $I_n \neq 0$, $\theta_n = \int_{q_0}^q \frac{\partial\alpha}{\partial I_n}$ where q_0 is a base point of the invariant torus under consideration. By then expressing $\frac{\partial\alpha}{\partial I_n}$ in conveniently chosen canonical coordinates one obtains formula (2.5) under the assumption that α coincides with the 1-form introduced in [FM].

In the remainder of this section we show that the $\eta_{n,k}$ are well defined analytic functions on $U_{q_0} \setminus D_n$, multivalued in the case $k = n$, and that they satisfy estimates to make the infinite sum in (2.5) convergent and $\theta_n(q)$ analytic.

Lemma 3 (i) *For $k \neq n$, $\eta_{n,k}$ is a well defined function defined on U_{q_0}. In particular, the integral in (2.6) is independent of the path chosen (as long as the latter stays near λ_{2k}).*

(ii) *$\eta_{n,n}$ is well defined as a multivalued function on $U_{q_0} \setminus D_n$ with values differing by multiples of 2π.*

Proof. (i) First notice that $\eta_{n,k}$ is well defined for q with $\gamma_k(q) = 0$. In such a case $\mu_k^{(n)} = \lambda_{2k}$. Therefore $\psi_n(\lambda)$ and $\sqrt{\Delta^2(\lambda) - 4}$ both contain the factor $(\lambda_{2k} - \lambda)$ and $\frac{\psi_n(\lambda)}{\sqrt{\Delta^2(\lambda)-4}}$ is analytic near λ_{2k}. Thus by Cauchy's theorem, $\eta_{n,k}$ is well defined in this case.

The independence of $\eta_{n,k}$ of the path of integration in the case $\gamma_k \neq 0$ follows from the normalization (2.1)

$$\int_{\lambda_{2k}}^{\lambda_{2k-1}} \frac{\psi_n(\lambda) d\lambda}{\sqrt{\Delta^2(\lambda, q) - 4}} = \pi\delta_{n,k} \mod 2\pi. \tag{2.7}$$

(ii) First we notice that as $\gamma_n(q) \neq 0$, the integral in (2.6) is well defined. Due to the normalization condition (2.7), we have

$$\int_{\lambda_{2n}}^{\lambda_{2n-1}} \frac{\psi_n(\lambda) d\lambda}{\sqrt{\Delta^2(\lambda, q) - 4}} = \pi \mod 2\pi. \tag{2.8}$$

By Cauchy's theorem, $\eta_{n,n}$ is thus well defined $\mod 2\pi$. □

To prove the boundedness result below, it is convenient to consider the model for Σ_q, obtained by glueing two copies of the complex plane, slit open along $(-\infty, \lambda_0), (\lambda_{2n-1}, \lambda_{2n})$ $(n \geq 1)$. These copies are refered to as the sheets of Σ_q.

Lemma 4 *Let U_{q_0} be a bounded G-neighborhood of $q_0 \in L_0^2$. Then there exists $C > 0$ so that for any $n \geq 1$ the following holds:*

(i) *for all $k \neq n$ and $q \in U_{q_0}$,*

$$|\eta_{n,k}(q)| \leq \frac{Cn}{|k^2 - n^2|} \frac{1}{k} (|\mu_k - \tau_k| + |\gamma_k|);$$

(ii) *for $q \in U_{q_0} \setminus D_n$*

$$|\eta_{n,n}(q) \mod 2\pi| \leq C \log\left(2 + \left|\frac{\mu_n - \tau_n}{\gamma_n}\right|\right);$$

(iii) *for all $q \in U_{q_0}$,*

$$\sum_{k \neq n} |\eta_{n,k}(q)| \leq \frac{C}{n} \left(\left(\sum_{k \geq 1} |\mu_k - \tau_k|^2 \right)^{1/2} + \left(\sum_{k \geq 1} |\gamma_k|^2 \right)^{1/2} \right).$$

Proof. is provided in Appendix A. □

To prove regularity properties of $\eta_{n,k}$, introduce

$$\begin{aligned} S_k &:= \{q \in U_{q_0} \mid \gamma_k(q) = 0\} \\ W_k &:= \{q \in U_{q_0} \mid \mu_k \in \{\lambda_{2k-1}, \lambda_{2k}\}\}. \end{aligned}$$

Notice that S_k and W_k are analytic subvarieties as $S_k = \{q \in U_{q_0} \mid \Delta(\dot{\lambda}_k) = (-1)^k 2, \dot{\Delta}(\dot{\lambda}_k) = 0\}$ (where $\dot{\lambda}_k$ is the root of $\dot{\Delta}(\lambda) = 0$ near λ_{2k}) and $W_k = \{q \in U_{q_0} \mid y_1(1, \mu_k) = (-1)^k\} \equiv \{q \in U_{q_0} \mid y_1(1, \mu_k) - y_2'(1, \mu_k) = 0\}$ where for the characterization of W_k we used that the Wronskian identity $[y_1(x, \lambda), y_2(x, \lambda)] = 1$, evaluated at $(x, \lambda) = (1, \mu_k)$, is given by $y_1(1, \mu_k) y_2'(1, \mu_k) = 1$.

Lemma 5 *Let U_{q_0} be a G-neighborhood of $q_0 \in L_0^2$. Then:*

(i) *for $k \neq n$, $\eta_{n,k}$ is analytic on U_{q_0};*

(ii) *$\eta_{n,n}$ is an analytic, multivalued function on $U_{q_0} \setminus D_n$ whose values can be identified modulo π;*

(iii) *when restricted to real potentials, $\eta_{n,n}$ is a continuous, multivalued function whose values can be identified modulo 2π.*

Proof. (i) Notice that for $q \in U_{q_0} \setminus S_k$ and a small q-neighborhood $V \subseteq U_{q_0} \setminus S_k$, there exist analytic functions λ_k^+, λ_k^- on V with $\{\lambda_k^+, \lambda_k^-\} = \{\lambda_{2k}, \lambda_{2k-1}\}$. In view of (2.7) $\eta_{n,k}(q) := \int_{\lambda_k^+(q)}^{\mu_k^*(q)} \frac{\psi_n(\lambda,q)}{\sqrt{\Delta^2(\lambda,q)-4}} d\lambda$. From this deduce that $\eta_{n,k}$ is analytic on $V \setminus (S_k \cup W_k)$ and as a consequence, analytic on $U_{q_0} \setminus (S_k \cup W_k)$.

It remains to prove the analyticity of $\eta_{n,k}$ for $q \in S_k \cup W_k$. By [[**PT**], Appendix A] this amounts to prove that $\eta_{n,k}$ is locally bounded and weakly analytic. By Lemma 4, $\eta_{n,k}$ is bounded on U_{q_0}. For $\eta_{n,k}$ to be weakly analytic it is to show that for any given $q \in S_k \cup W_k$ and any $p \in L^2_{0,\mathbb{C}}$, $\eta_{n,k}(q + zp)$ is analytic for $z \in \mathbb{C}$ near $z = 0$. Introduce $D_\epsilon := \{q + zp \mid z \in \mathbb{C},\ |z| < \epsilon\}$ and chose ϵ sufficiently small so that $D_\epsilon \subseteq U_{q_0}$. Due to the fact that S_k and W_k are analytic submanifolds of U_{q_0} it follows that, for ϵ sufficiently small, the following two cases occur:

$$\text{case } 1_S \ : \quad S_k \cap D_\epsilon \subseteq \{q\}; \qquad \text{case } 2_S \ : \quad S_k \cap D_\epsilon = D_\epsilon$$

and, similarly,

$$\text{case } 1_W \ : \quad W_k \cap D_\epsilon \subseteq \{q\}; \qquad \text{case } 2_W \ : \quad W_k \cap D_\epsilon = D_\epsilon .$$

Combining them, we obtain four different cases, (i_S, j_W) $(1 \leq i, j \leq 2)$ which are treated separately. First we notice that the cases $(i_S, 2_W)$ $(i = 1, 2)$ are particularly easy as $\eta_{n,k} = 0$ on D_ϵ. In the case $(2_S, 1_W)$ we have $\lambda_{2k} = \lambda_{2k-1} = \tau_k$ on D_ϵ and as τ_k is analytic it follows that $\eta_{n,k}$ is continuous on D_ϵ. As, by considerations above, $\eta_{n,k}$ is analytic on $D_\epsilon \setminus \{q\}$ it follows that $\eta_{n,k}$ is analytic on D_ϵ (removable singularity). It remains to treat the case $(1_S, 1_W)$. Again by the considerations above, $\eta_{n,k}$ is analytic on $D_\epsilon \setminus \{q\}$. As $\lim_{\substack{r \to q \\ r \in D_\epsilon}} \lambda_j(r) = \lambda_{2k}(q)$ for $j = 2k, 2k-1$, $\eta_{n,k}|_{D_\epsilon}$ is continuous at q. It follows that $\eta_{n,k}$ is analytic on D_ϵ in case $(1_S, 1_W)$.
(ii) By Lemma 3, $\eta_{n,n}$ is a multivalued function whose values coincide modulo 2π. For $q \in U_{q_0} \setminus D_n$, there exist a neighborhood $V \subseteq U_{q_0} \setminus D_n$ and analytic functions λ_n^+, λ_n^- on V so that $\{\lambda_n^+, \lambda_n^-\} = \{\lambda_{2n}, \lambda_{2n-1}\}$. As

$$\int_{\lambda_{2n}}^{\lambda_{2n-1}} \frac{\psi_n(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda = \pi \qquad \mod 2\pi$$

and $\int_{\lambda_n^+}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta^2(\lambda)-4}} d\lambda$ is continuous on V, we conclude that $\eta_{n,k}$ is continuous on V when viewed as a multivalued function whose values coincide modulo π.

Arguing as in (i), we conclude that $\eta_{n,n}$ is analytic on V, and therefore on $U_{q_0} \setminus D_n$ as well, when considered as a multivalued function.

(iii) As λ_{2n} and λ_{2n-1} are real for q real valued, they are continous in q. This implies that $\eta_{n,n}$ is continuous on $U_{q_0} \setminus D_n \cap L^2_0$ when viewed as a multivalued function whose values coincide modulo 2π. □

We summarize our results in the following

Proposition 6 *There exists a G-neighborhood $U = U_{L^2_0}$ of L^2_0 in $L^2_{0,\mathbb{C}}$ so that, for any $n \geq 1$, the following statements hold:*

(i) $\tilde{\theta}_n := \sum_{k \neq n} \eta_{n,k}$ *converges absolutely, is analytic on U, and satisfies $\tilde{\theta}_n = O\left(\frac{1}{n}\right)$ locally uniformly in q (cf Lemma 4);*

(ii) θ_n *is an analytic, multivalued function on $U \setminus D_n$ with values equal modulo π;*

(iii) *when restricted to real valued potentials in $U \setminus D_n$, θ_n is a continuous multivalued function with values equal modulo 2π.*

3 Ω : Definition and regularity properties

In this section we define a real analytic map $\Omega = (\Omega_n)_{n \geq 1} : L^2_0 \to h^{1/2}(\mathbb{N}; \mathbb{R}^2)$ which satisfies - as will be proved in the subsequent sections - all the properties listed in Theorem 1.

We begin by defining the $n'th$ component of Ω, $\Omega_n(q) := (x_n(q), y_n(q))$. Let $U \equiv U_{L^2_0}$ be a G-neighborhood of L^2_0 in $L^2_{0,\mathbb{C}}$.

Definition *For $q \in U \setminus D_n$, set*

$$\Omega_n(q) := (x_n(q), y_n(q)) := \xi_n(q) \frac{\gamma_n(q)}{2} (\cos\ \theta_n(q), \sin\ \theta_n(q)),$$

where $\xi_n(q)$ has been introduced in section 1, $\theta_n(q)$ in section 2, and where $\gamma_n(q) := \lambda_{2n}(q) - \lambda_{2n-1}(q)$, is related to the actions $I_n(q)$ by $2I_n(q) = \left(\xi_n(q) \frac{\gamma_n(q)}{2}\right)^2$.

Recall that $\gamma_n(q)$ is not continuous on $U \setminus D_n$ due to the choice of the ordering of the eigenvalues. Further recall that

$$\theta_n = \eta_{n,n} \ + \ \tilde{\theta}_n$$

where $\tilde{\theta}_n := \sum_{k \neq n} \eta_{n,k}$ is analytic on U whereas

$$\eta_{n,n}(q) = \int_{\lambda_{2n}}^{\mu_n^*} \frac{\varepsilon_n \psi_n}{\sqrt{\Delta^2 - 4}} d\lambda$$

is analytic on $U \setminus D_n$ when viewed as a multivalued function whose values coincide mod π (cf Lemma 5).

Lemma 7 *On $U \setminus D_n$, $x_n(q)$ and $y_n(q)$ are analytic.*

Proof. Let $p \in U \setminus D_n$. Then there exist a neighborhood $V \subseteq U \setminus D_n$ and analytic functions $\lambda_n^{\pm}$ on V with $\{\lambda_n^-(q), \lambda_n^+(q)\} = \{\lambda_{2n-1}(q), \lambda_{2n}(q)\}$.

It follows from the proof of Lemma 5 that $\eta_{n,n}^+(q) := \int_{\lambda_n^+}^{\mu_n^*} \frac{\psi_n}{\sqrt{\Delta^2-4}} d\lambda$ is analytic on V when viewed as a multivalued function (mod 2π). Introduce on V the following functions

$$\gamma_n^+ := \lambda_n^+ - \lambda_n^-; \qquad \theta_n^+ := \eta_{n,n}^+ + \tilde{\theta}_n;$$
$$x_n^+ := \xi_n \frac{\gamma_n^+}{2} \cos\theta_n^+; \qquad y_n^+ := \xi_n \frac{\gamma_n^+}{2} \sin\ \theta_n^+.$$

Then $\gamma_n^+, \theta_n^+, x_n^+, y_n^+$ are analytic on V. Thus the claimed statement follows if

$$x_n = x_n^+ \text{ and } y_n = y_n^+.$$

Take q in V. If $\lambda_n^+(q) = \lambda_{2n}(q)$ then, according to the definition of γ_n and θ_n, and Lemma 3

$$\gamma_n^+(q) = \gamma_n(q), \qquad \theta_n^+(q) \equiv \theta_n(q) \mod 2\pi$$

whereas in the case $\lambda_n^+(q) = \lambda_{2n-1}(q)$, in view of (2.7),

$$\gamma_n^+(q) = -\gamma_n(q), \qquad \theta_n^+(q) \equiv (\theta_n(q) + \pi) \mod 2\pi.$$

Thus in both cases we conclude that $x_n(q) = x_n^+(q)$ and $y_n(q) = y_n^+(q)$. □

The next result shows that Ω_n can be extended:

Proposition 8 *There exists a G-neighborhood $U = U_{L_0^2}$ of L_0^2 in $L_{0,\mathbb{C}}^2$ so that for any $n \geq 1$, $\Omega_n = (x_n, y_n)$ admits an analytic continuation on U.*

Let us outline our proof of Proposition 8. First we show that, for any $n \geq 1$, Ω_n admits a continuous extension on U (Corollary 11) and has a bound of the form

$$|\Omega_n(q)| \leq \frac{C}{n^{1/2}}(|\gamma_n| + |\mu_n - \tau_n|)$$

where $C > 0$ can be chosen independently of q for q in a bounded G-neighborhood of q_0 (Corollary 11). Using Lemma 7, Proposition 8 then follows by showing that Ω_n is weakly analytic.

We begin by establishing an auxilary result. For $q \in U_{q_0}$, U_{q_0} a G- neighborhood of $q_0 \in L_0^2$, and $n \geq 1$ introduce the functions

$$\zeta_n \equiv \zeta_n(\lambda, q) = \frac{\psi_n(\lambda, q)}{v_n(\lambda, q)} \tag{3.1}$$

defined for $\lambda \in \mathbb{C}$ near $\{\lambda_{2n}(q_0), \lambda_{2n-1}(q_0)\}$ where

$$v_n(\lambda, q) := (-1)^{n-1} \frac{2}{n\pi} \frac{(\lambda - \lambda_0)^{1/2}}{n\pi} \prod_{k \neq n} \frac{((\lambda_{2k} - \lambda)(\lambda_{2k-1} - \lambda))^{1/2}}{k^2\pi^2} \tag{3.2}$$

and $z^{1/2}$ denotes the branch defined on $\mathbb{C} \setminus \mathbb{R}_-$ with $1^{1/2} = 1$. Then, for $\left(\lambda, \sqrt{\Delta(\lambda)^2 - 4}\right) \in \Sigma_q$ near the branch points $\{\lambda_{2n}, \lambda_{2n-1}\}$, $\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}$ is defined by

$$\frac{\zeta_n(\lambda)}{\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} = \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}}. \tag{3.3}$$

Lemma 9 *Given a bounded G-neighborhood U_{q_0} of $q_0 \in L^2_0$, there exists a constant $C > 0$ so that, for q in U_{q_0} and $n \geq 1$,*

$$|\zeta_n(\tau_n) - 1| \leq C|\gamma_n|.$$

Proof. For $q \in U_{q_0} \setminus D_n$ real valued, by formula (2.1),

$$\frac{1}{\pi} \int_{\lambda_{2n}}^{\lambda_{2n-1}} \zeta_n(\lambda, q) \frac{1}{\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda = 1. \tag{3.4}$$

Choose $\lambda(t) := \tau_n - t\frac{\gamma_n}{2}$ $(-1 \leq t \leq 1)$ as path of integration. As q is realvalued

$$\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})} = -\frac{\gamma_n}{2} \left(1 - t^2\right)^{1/2}. \tag{3.5}$$

Substituting (3.5) into (3.4) yields

$$1 = \frac{1}{\pi} \int_{-1}^{1} \zeta_n(\lambda(t)) \frac{dt}{\left(1 - t^2\right)^{1/2}} = \frac{1}{\pi} \int_0^1 \left(\zeta_n(\lambda(t)) + \zeta_n(\lambda(-t))\right) \frac{dt}{\left(1 - t^2\right)^{1/2}}. \tag{3.6}$$

Notice that $\zeta_n(\lambda(t)) + \zeta_n(\lambda(-t))$ is even in $t\gamma_n$. Further, $\zeta_n(\lambda)$ as well as γ_n^2 are analytic in q, hence (3.6) remains valid on all of $U_{q_0} \setminus D_n$. The integral in (3.6) is split up into two parts, $F_I(q) + F_{II}(q)$, with

$$F_I(q) := \zeta_n(\tau_n) \frac{1}{\pi} \int_{-1}^{1} \frac{dt}{\left(1 - t^2\right)^{1/2}} = \zeta_n(\tau_n).$$

Then (3.6) leads to

$$|\zeta_n(\tau_n) - 1| \leq |F_{II}(q)|. \tag{3.7}$$

To estimate

$$F_{II}(q) := \frac{1}{\pi} \int_{-1}^{1} \left(\zeta_n(\lambda) - \zeta_n(\tau_n)\right) \frac{dt}{\left(1 - t^2\right)^{1/2}},$$

notice that, as $\lambda(t) - \tau_n = -t\frac{\gamma_n}{2}$,

$$\begin{aligned} \zeta_n(\lambda) - \zeta_n(\tau_n) &= \int_0^1 \frac{\partial \zeta_n}{\partial \lambda}(\tau_n + s(\lambda - \tau_n))(\lambda - \tau_n) ds \\ &= -t\frac{\gamma_n}{2} \int_0^1 \frac{\partial \zeta_n}{\partial \lambda}(\tau_n + st\frac{\gamma_n}{2}) ds. \end{aligned}$$

This leads to

$$F_{II}(q) = -\frac{\gamma_n}{2}\frac{1}{\pi}\int_{-1}^{1}\int_{0}^{1}\frac{t}{(1-t^2)^{1/2}}\frac{\partial\zeta_n}{\partial\lambda}(\tau_n + st\frac{\gamma_n}{2})dtds.$$

Choose $C > 0$ so that

$$\sup_{\substack{0\leq s\leq 1\\ 0\leq |t|\leq 1}}\left|\frac{\partial\zeta_n}{\partial\lambda}(\tau_n + st\frac{\gamma_n}{2})\right| \leq C \quad \forall q \in U_{q_0}.$$

Thus, for $q \in U_{q_0} \setminus D_n$,

$$|\zeta_n(\tau_n) - 1| \leq C|\gamma_n|. \tag{3.8}$$

As $\zeta_n(\tau_n)$ and $|\gamma_n|$ are continuous and $U_{q_0} \setminus D_n$ is dense in U_{q_0}, (3.8) holds on the whole neighborhood U_{q_0}. □

Recall that in section 2, we have introduced the real analytic submanifolds

$$\begin{aligned} W_n &:= \{q \in U_{q_0} \mid \mu_n \in \{\lambda_{2n}, \lambda_{2n-1}\}\}, \\ S_n &:= \{q \in U_{q_0} \mid \lambda_{2n} = \lambda_{2n-1}\} \end{aligned}$$

where U_{q_0} is a bounded G-neighborhood of $q_0 \in L_0^2$. To formulate our next result, introduce, for $q \in U_{q_0}$,

$$p_n(q) := (\mu_n - \tau_n)\int_0^1\int_0^1 \frac{\partial\zeta_n}{\partial\lambda}(\tau_n + st(\mu_n - \tau_n))dsdt. \tag{3.9}$$

Use the model for Σ_q near λ_{2n} obtained by glueing two copies of the complex plane, slit open along the interval $\mathcal{G}_n = \{(1-t)\lambda_{2n-1} + t\lambda_{2n} \mid 0 \leq t \leq 1\}$. For $\lambda^* = (\lambda, \sqrt{\Delta(\lambda)^2 - 4}) \in \Sigma_q$ with $\lambda \notin \mathcal{G}_n$ and near λ_{2n}, define $\epsilon_n \equiv \epsilon_n(\lambda^*) = \pm 1$ by

$$\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})} = i\epsilon_n \cdot (\lambda - \tau_n)\left(1 - \left(\frac{\gamma_n/2}{\lambda - \tau_n}\right)^2\right)^{1/2} \tag{3.10}$$

where $(1-z^2)^{1/2}$ denotes the square root on $\mathbb{C} \setminus (-\infty, -1) \cup (1, \infty)$ with $1^{1/2} = 1$. Formula (3.10) then leads to

$$\sqrt{\Delta(\lambda)^2 - 4} = \zeta_n(\lambda) i\epsilon_n \cdot (\lambda - \tau_n)\left(1 - \left(\frac{\gamma_n/2}{\lambda - \tau_n}\right)^2\right)^{1/2}. \tag{3.11}$$

Define $\Omega_n \equiv (x_n, y_n)$ on S_n as follows

$$(x_n, y_n) := (0,0) \quad \text{on} \quad S_n \cap W_n \tag{3.12}$$

$$(x_n, y_n) := (\mu_n - \tau_n)\xi_n e^{i\epsilon_n\tilde{\theta}_n + p_n}(1, -i\epsilon_n) \quad \text{on} \quad S_n \setminus W_n \tag{3.13}$$

with $\epsilon_n = \epsilon_n(\mu_n^*)$, $\mu_n^* = (\mu_n,\ y_1(1,\mu_n) - y_2'(1,\mu_n))$ and $\tilde{\theta}_n := \sum_{k\neq n}\eta_{n,k}$. Notice that $\Omega_n|_{S_n}$ is continuous on S_n.

Lemma 10 *For $q_1 \in S_n \setminus W_n$,*

$$\lim_{\substack{q \to q_1 \\ q \notin S_n \cup W_n}} \Omega_n(q) = \Omega_n(q_1).$$

Proof. We first evaluate the limits of $x_n(q) \pm iy_n(q) = \xi_n \frac{\gamma_n}{2} e^{\pm i\theta_n}$ for $q \to q_1$ with $q \in U_{q_0} \setminus (S_n \cup W_n)$. By Proposition 6, $\lim_{q\to q_1} e^{\pm i\tilde{\theta}_n(q)} = e^{\pm i\tilde{\theta}_n(q_1)}$ and by Proposition 1, $\lim_{q\to q_1} \xi_n(q) = \xi_n(q_1)$. Thus it remains to find the limit of $\frac{\gamma_n}{2} e^{\pm i\eta_{n,n}(q)}$ as $q \to q_1$. For $q \in U_{q_0} \setminus (S_n \cup W_n)$,

$$\eta_{n,n}(q) = \int_{\lambda_{2n}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda = \int_{\lambda_{2n}}^{\mu_n^*} \frac{\zeta_n(\lambda)}{\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}} d\lambda \tag{3.14}$$

where $\zeta_n(\lambda)$ is given by (3.1) and the square root $\sqrt{(\lambda_{2n} - \lambda)(\lambda - \lambda_{2n-1})}$ is defined on Σ_q for λ near λ_{2n} by (3.10). For $q \in U_{q_0} \setminus (S_n \cup W_n)$ with $|\mu_n - \tau_n| \leq 4|\gamma_n|$, by Lemma 4,

$$|\eta_{n,n}(q)| \leq C \quad (\text{ for } q \text{ with } |\mu_n - \tau_n| \leq 4|\gamma_n| \text{ }). \tag{3.15}$$

To evaluate $\int_{\lambda_{2n}}^{\mu_n^*} \frac{\zeta_n(\lambda)}{\sqrt{(\lambda_{2n}-\lambda)(\lambda-\lambda_{2n-1})}} d\lambda$ for $q \in U_{q_0} \setminus (S_n \cup W_n)$ with $|\mu_n - \tau_n| > 4|\gamma_n|$ we consider two cases:

$$\text{case 1 : } \quad \mathtt{Re}\, w_n \geq 0; \qquad \text{case 2 : } \quad \mathtt{Re}\, w_n < 0$$

where $w_n = \frac{\mu_n - \tau_n}{\gamma_n/2}$.

Let us first consider case 1. Choose as path of integration

$$\lambda(t) = \lambda_{2n} + t(\mu_n - \lambda_{2n}) = \tau_n + \frac{\gamma_n}{2} w(t)$$

where

$$w(t) = 1 - t + tw_n \quad (0 \leq t \leq 1).$$

Then

$$\begin{aligned} (\lambda_{2n} - \lambda(t))(\lambda(t) - \lambda_{2n-1}) &= \left(\frac{\gamma_n}{2}\right)^2 (1 - w(t))(1 + w(t)) \\ &= -\left(\frac{\gamma_n}{2}\right)^2 w(t)^2 \left(1 - \frac{1}{w(t)^2}\right). \end{aligned}$$

Notice that $\mathtt{Re}\, w(t) = 1 - t + t\,\mathtt{Re}\, w_n \geq 0$ (case 1). Moreover, for $0 \leq t \leq 1$, (cf (3.10))

$$\sqrt{(\lambda_{2n} - \lambda(t))(\lambda(t) - \lambda_{2n-1})} = i\epsilon_n \frac{\gamma_n}{2} w(t) \left(1 - \frac{1}{w(t)^2}\right)^{1/2}. \tag{3.16}$$

Substituting (3.16) into the integral in (3.14) we get

$$\begin{aligned}\eta_{n,n}(q) &= \int_0^1 \frac{\zeta_n(\lambda(t))(\mu_n-\lambda_{2n})dt}{i\epsilon_n\frac{\gamma_n}{2}w(t)\left(1-\frac{1}{w(t)^2}\right)^{1/2}} \\ &= \frac{\epsilon_n}{i}\int_0^1 \frac{\zeta_n(\lambda(t))}{w(t)\left(1-\frac{1}{w(t)^2}\right)^{1/2}}(w_n-1)dt \\ &= \frac{\epsilon_n}{i}\int_1^{w_n} \frac{\zeta_n(\tau_n+\frac{\gamma_n}{2}w)}{w\left(1-\frac{1}{w^2}\right)^{1/2}}dw \qquad \text{mod } 2\pi.\end{aligned} \tag{3.17}$$

Using the Taylor expansion

$$\zeta_n\left(\tau_n+\frac{\gamma_n}{2}w\right)=\zeta_n(\tau_n)+\frac{\gamma_n}{2}w\int_0^1\frac{\partial\zeta_n}{\partial\lambda}(\tau_n+s\frac{\gamma_n}{2}w)ds,$$

the last integral in (3.17) can be split into two parts, $\eta_{n,n}(q)=I(q)+II(q)$ where

$$I(q):=\frac{\epsilon_n}{i}\zeta_n(\tau_n)\int_1^{w_n}\frac{1}{w\left(1-\frac{1}{w^2}\right)^{1/2}}dw \tag{3.18}$$

and

$$II(q):=\frac{\epsilon_n}{i}\int_1^{w_n}\int_0^1\frac{\frac{\partial\zeta_n}{\partial\lambda}(\tau_n+s\frac{\gamma_n}{2}w)}{\left(1-\frac{1}{w^2}\right)^{1/2}}\frac{\gamma_n}{2}dwds. \tag{3.19}$$

Then, as $\texttt{Re}\, w(t)>0$ for $0\le t<1$, and $w(0)=1$

$$I(q)=\frac{\epsilon_n}{i}\zeta_n(\tau_n)\log\left(w+w(1-\frac{1}{w^2})^{1/2}\right)\Big|_{w=w_n} \qquad (\mod 2\pi) \tag{3.20}$$

and with $\frac{\gamma_n}{2}dw=\frac{\gamma_n}{2}(w_n-1)dt=(\mu_n-\lambda_{2n})dt$

$$II(q)=(\mu_n-\lambda_{2n})\frac{\epsilon_n}{i}\int_0^1\int_0^1\frac{\partial\zeta_n}{\partial\lambda}(\tau_n+s(\frac{\gamma_n}{2}+t(\mu_n-\lambda_{2n})))\frac{dtds}{(1-\frac{1}{w(t)^2})^{1/2}}. \tag{3.21}$$

Notice that, for $0<t\le 1$,

$$\left|\frac{1}{(1-\frac{1}{w(t)^2})^{1/2}}\right|=\left|\frac{1}{(1-\frac{1}{w(t)})^{1/2}(1+\frac{1}{w(t)})^{1/2}}\right|\le\frac{2}{t^{1/2}} \tag{3.22}$$

where using that $|w_n|\ge 4$,

$$\left|1-\frac{1}{w(t)}\right|^{-1/2}=\frac{1}{t^{1/2}}\left|\frac{1+t(w_n-1)}{w_n-1}\right|^{1/2}\le\frac{2}{t^{1/2}} \tag{3.23}$$

and, using that $\operatorname{Re} w(t) = 1 + t \operatorname{Re} w_n \geq 1$

$$\left|1 + \frac{1}{w(t)}\right|^{-1/2} = \left|\frac{1 + t(w_n - 1)}{2 + t(w_n - 1)}\right|^{1/2} \leq 1. \tag{3.24}$$

Before continuing our argument for case 1 let us first consider the case 2: $\operatorname{Re} w_n < 0$. Then

$$\eta_{n,n}(q) = \int_{\lambda_{2n}}^{\mu_n^*} \frac{\psi_n(\lambda) d\lambda}{\sqrt{\Delta(\lambda)^2 - 4}} = \pi + \int_{\lambda_{2n-1}}^{\mu_n^*} \frac{\psi_n(\lambda) d\lambda}{\sqrt{\Delta(\lambda)^2 - 4}} \mod 2\pi \tag{3.25}$$

where we used (2.7). For the last integral in (3.25), choose as path of integration $\lambda(t) = \lambda_{2n-1} + t(\mu_n - \lambda_{2n-1})$ and argue as in case 1. It leads to the following formula,

$$\eta_{n,n} = I(q) + II(q) + III(q)$$

where $I(q)$ is defined as in (3.20) but

$$II(q) := (\mu_n - \lambda_{2n-1}) \frac{\epsilon_n}{i} \int_0^1 \int_0^1 \frac{\partial \zeta_n}{\partial \lambda}(\tau(s,t)) \frac{dtds}{\left(1 - \frac{1}{w(t)^2}\right)^{1/2}} \mod 2\pi \tag{3.26}$$

where $\tau(s,t) := \tau_n + s(-\frac{\gamma_n}{2} + t(\mu_n - \lambda_{2n-1}))$ and

$$III(q) := (\epsilon_n \zeta_n(\tau_n) + 1)\pi \mod 2\pi. \tag{3.27}$$

The estimates (3.23), (3.24) allow to take the limit under the integral in (3.21) and (3.26) to obtain

$$\lim_{q \to q_1} II(q) = (\mu_n - \lambda_{2n}) \frac{\epsilon_n}{i} \int_0^1 \int_0^1 \frac{\partial \zeta_n}{\partial \lambda}(\tau(s,t)) \frac{dtds}{(1 - \frac{1}{w(t)^2})^{1/2}} \Bigg|_{q=q_1} = p_n(q_1) \tag{3.28}$$

where we used that $\lim_{q \to q_1} \gamma_n(q) = 0$ and $\lim_{q \to q_1} \lambda_{2n}(q) = \tau_n(q_1)$.

Now let us continue with the proof of case 1 and case 2 simultaneously. From (3.20) we obtain

$$\begin{aligned} \lim_{q \to q_1} \frac{\gamma_n}{2} e^{\pm iI(q)} &= \lim_{q \to q_1} \frac{\gamma_n}{2} \left(w + w \left(1 - \frac{1}{w^2}\right)^{1/2} \right)^{\pm \epsilon_n \zeta_n(\tau_n)} \Bigg|_{w = w_n} \\ &= (\mu_n - \tau_n)(\pm \epsilon_n(q_1) + 1) \end{aligned} \tag{3.29}$$

where we used $|\zeta_n(\tau_n) - 1| \leq C|\gamma_n|$ (Lemma 9) and thus

$$\lim_{q \to q_1} \frac{\gamma_n}{2} \left(\frac{1}{\gamma_n/2}\right)^{\zeta_n(\tau_n)} = 1. \tag{3.30}$$

Notice that $III(q)$ (cf 3.27) is continuous in q and

$$\lim_{q\to q_1} e^{\pm iIII(q)} = \lim_{q\to q_1} \exp\left(\pm i(\epsilon_n\zeta_n(\tau_n)+1)\pi\right) = 1. \tag{3.31}$$

Combining (3.28), (3.29), and (3.31) we conclude that $\lim_{q\to q_1} \frac{\gamma_n}{2} e^{\pm\eta_{n,n}}$ exists.

For $q_1 \in S_n \setminus W_n$ we then obtain ($q \in U_{q_0} \setminus (S_n \cup W_n)$)

$$\begin{aligned}\lim_{q\to q_1}(x_n+iy_n) &= \xi_n e^{i\tilde\theta_n} \lim_{q\to q_1} \frac{\gamma_n}{2} e^{i\eta_{n,n}} \\ &= \xi_n e^{i\tilde\theta_n} \lim_{q\to q_1} \left(\frac{\gamma_n}{2}(2w_n)^{\epsilon_n\zeta_n(\tau_n)} e^{\epsilon_n p_n}\right) \\ &= (1+\epsilon_n)\xi_n e^{i\tilde\theta_n}(\mu_n-\tau_n)e^{p_n}\end{aligned}$$

where $p_n \equiv p_n(q_1)$ (cf (3.9)). Similarly,

$$\begin{aligned}\lim_{q\to q_1}(x_n-iy_n) &= \xi_n e^{-i\tilde\theta_n} \lim_{q\to q_1} \frac{\gamma_n}{2} e^{i\eta_{n,n}} \\ &= \xi_n e^{-i\tilde\theta_n} \lim_{q\to q_1} \left(\frac{\gamma_n}{2}(2w_n)^{-\epsilon_n\zeta_n(\tau_n)} e^{-\epsilon_n p_n}\right) \\ &= (1-\epsilon_n)\xi_n e^{-i\tilde\theta_n}(\mu_n-\tau_n)e^{p_n}.\end{aligned}$$

Thus

$$\lim_{q\to q_1} x_n = \xi_n e^{\epsilon_n i\tilde\theta_n}(\mu_n-\tau_n)e^{p_n}$$

and

$$\lim_{q\to q_1} y_n = -i\epsilon_n\xi_n e^{\epsilon_n i\tilde\theta_n}(\mu_n-\tau_n)e^{p_n} = -i\epsilon_n x_n(q_1).$$

□

Corollary 11 (i) *Ω_n is continuous on U_{q_0}.*
(ii) *There exists $C>0$ so that for $q\in U_{q_0}$ and $n\geq 1$,*

$$|x_n|+|y_n| \leq \frac{C}{n^{1/2}}(|\mu_n-\tau_n|+|\gamma_n|).$$

Proof. (i) Follows from Lemma 7, Lemma 10 and the definitions (3.12), (3.13).
(ii) On U_{q_0}, $\left(e^{\pm i\tilde\theta_n(q)}\right)_{n\geq 1}$ (cf Proposition 6) and $(\sqrt{n}\xi_n)_{n\geq 1}$ (cf Proposition 1) are bounded. It remains to bound $\frac{\gamma_n}{2} e^{\pm i\eta_{n,n}}$ by $C(|\mu_n-\tau_n|+|\gamma_n|)$. This follows from (3.15), the boundedness of $e^{\pm iII(q)}$ (cf (3.21) and (3.26)), the boundedness of $e^{\pm iIII(q)}$ (cf (3.27), Lemma 9), and the boundedness of $\frac{\gamma_n}{2}e^{\pm iI(q)}$ (cf (3.20), Lemma 9). □

Proof. (of Proposition 8). The claimed statement follows if for any $q_0 \in L_0^2$, there exists a G-neighborhood U_{q_0} of q_0 in $L^2_{0,\mathbb{C}}$ so that x_n, y_n are bounded on U_{q_0} and weakly analytic (cf [**PT**]). By Corollary 11, x_n, y_n are bounded on U_{q_0}. From

Lemma 7 and Corollary 11 one concludes, similarly as in the proof of Lemma 5, that $x_n(q), y_n(q)$ are weakly analytic. □

The results of this section lead to

Theorem 2 $\Omega := (\Omega_n)_{n\geq 1} : L_0^2 \to h^{1/2}(\mathbb{N}; \mathbb{R}^2)$ *is real analytic.*

Proof. Let $q_0 \in L_0^2$. By Corollary 11 there exist $C > 0$ and a G-neighborhood U_{q_0} of q_0 in $L_{0,\mathbb{C}}^2$ so that for any $n \geq 1$ Ω_n is analytic on U_{q_0} and, for q in U_{q_0},

$$|x_n|^2 + |y_n|^2 \leq \frac{C}{n} \left(|\gamma_n(q)|^2 + |\mu_n(q) - \tau_n(q)|^2 \right).$$

By Proposition 28, U_{q_0} and $C > 0$ can be chosen so that, for $q \in U_{q_0}$,

$$\sum_{n\geq 1} \left(|\gamma_n(q)|^2 + |\mu_n(q) - \tau_n(q)|^2 \right) \leq C.$$

Thus $\Omega(q) \in h^{1/2}(\mathbb{N}; \mathbb{R}^2)$ and Ω is bounded on U_{q_0}. Together with the analyticity of Ω_n on U_{q_0} $(n \geq 1)$, this implies that Ω is analytic on U_{q_0}. □

4 Canonical relations: part 1

In this section we prove a first set of canonical relations for the variables I_n, θ_n $(n \geq 1)$ introduced in sections 1 and 2 respectivly. These relations will be used in the next section to prove that the map Ω, defined in section 3, is a local diffeomorphism. Let $\mathcal{O}(q)$ be the set of open gaps,

$$\mathcal{O} \equiv \mathcal{O}(q) := \{n \in \mathbb{N} \mid \gamma_n(q) \neq 0\}.$$

Proposition 12 (i) *For $q \in L_0^2$ and $m, n \geq 1$,*

$$\{I_n, I_m\} = 0.$$

(ii) *For $q \in L_0^2$, $m \in \mathcal{O}(q)$, and $n \geq 1$,*

$$\{\theta_m, I_n\}(q) = -\delta_{n,m}.$$

(iii) *For $q \in L_0^2$ and $m, n \notin \mathcal{O}(q)$,*

$$\begin{aligned} \{x_n, x_m\} &= \{y_n, y_m\} = 0; \\ \{x_n, y_m\} &= 0 \quad (m \neq n); \quad \{x_n, y_n\} \neq 0. \end{aligned}$$

We prove parts (i), (ii), and (iii) of Proposition 12 separately.

Proof of Proposition 12(i) Recall that

$$\frac{\partial I_k}{\partial q(x)} = -\frac{2}{\pi}\int_{\lambda_{2k-1}}^{\lambda_{2k}} \frac{1}{\sqrt{\Delta^2(\lambda)-4}}\frac{\partial \Delta(\lambda)}{\partial q(x)} d\lambda \tag{4.1}$$

where the path of integration is given by $\lambda = \lambda_{2k-1} + t\gamma_k - i0$ with $0 \leq t \leq 1$. For $a, b \in \mathbb{R}$, we have (cf (B.3) in Appendix B)

$$\{\Delta(a,q), \Delta(b,q)\} = 0.$$

Therefore $\{I_n, I_m\} = 0$. □

The proof of Proposition 12(ii) requires several auxiliary results which we present first.

For $q \in L_0^2$, let $Iso(q)$ denote the set of isospectral potentials. As $Iso(q)$ is compact and generically not contained in a finite dimensional space, $Iso(q)$ generically is not a manifold. Nevertheless its normal space $N_q Iso(q)$ and its tangent space $T_q Iso(q)$ at q are well defined (cf [**MT1**]) : $T_q Iso(q)$ is the L_2-closure of the span of $\frac{d}{dx}(f_{2n}^2 - f_{2n-1}^2)$ with $n \in \mathcal{O} \equiv \mathcal{O}(q)$ where $(f_n)_{n\geq 0}$ denotes an orthonormal set of eigenfunctions of the Schrödinger operator $-\frac{d^2}{dx^2} + q$ on $[0,2]$, considered with periodic boundary conditions. The normal space $N_q Iso(q)$ is the orthogonal complement of $T_q Iso(q)$ in L_0^2.

Lemma 13 *For $n \geq 1$ and $q \in L_0^2$, $\frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \in T_q Iso(q)$.*

Proof. It suffices to consider $n \in \mathcal{O}$ as, for $n \in \mathbb{N} \setminus \mathcal{O}$, $\frac{\partial I_n}{\partial q(x)} = 0$. Similarly as in the proof of Proposition 12(i) one shows that, for any $\lambda \in \mathbb{R}$,

$$\{\Delta(\lambda), I_n\} = 0.$$

Therefore $\Delta(\cdot, q)$ remains unchanged along the flow generated by $\frac{d}{dx}\frac{\partial I_n}{\partial q(x)}$. As $\Delta(\cdot, q)$ determines the spectrum of q, $\{\lambda_n(q)\}_{n=0}^{\infty} = \{\lambda \mid \Delta(\lambda, q) = \pm 2\}$, we conclude that $\frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \in T_q Iso(q)$. □

Denote by $m_{ij} = m_{ij}(\lambda, q)$ $(1 \leq i, j \leq 2)$ the entries of the Floquet matrix $m_{ij} := \partial_x^{i-1} y_j(1, \lambda, q)$.

Lemma 14 *For any $k \geq 1$, $q \in L_0^2$, and $\lambda \neq \mu_k(q)$,*

$$\{\mu_k(\cdot), \Delta(\lambda, \cdot)\}(q) = \frac{1}{2}\frac{m_{11}(\mu_k(q), q) - m_{22}(\mu_k(q), q)}{\dot{m}_{12}(\mu_k(q), q)}\frac{m_{12}(\lambda, q)}{\lambda - \mu_k(q)}.$$

Proof. By the definition of the Poisson bracket,

$$\{\mu_k, \Delta(\lambda)\}(q) = -\int_0^1 \frac{\partial \Delta(\lambda, q)}{\partial q(x)}\frac{d}{dx}\frac{\partial \mu_k(q)}{\partial q(x)} dx. \tag{4.2}$$

Using that (cf. [PT]) $\frac{\partial \mu_k}{\partial q(x)} = \frac{y_2^2(x,\mu_k,q)}{\dot{m}_{12}(\mu_k) m_{22}(\mu_k)}$ we obtain (cf. (B.4) in Appendix B)

$$\begin{aligned} 2(\lambda - \mu_k)\{\mu_k, \Delta(\lambda)\} &= \frac{m_{12}(\lambda)}{\dot{m}_{12}(\mu_k)} \left(\frac{1}{m_{22}(\mu_k)} - m_{22}(\mu_k) \right) \\ &= \frac{m_{12}(\lambda)}{\dot{m}_{12}(\mu_k)} \left(m_{11}(\mu_k) - m_{22}(\mu_k) \right). \end{aligned}$$

□

Corollary 15 *For any $k, n \geq 1$ and $q \in L_0^2$,*

$$\{\mu_k(\cdot), I_n(\cdot)\} = -\frac{1}{\pi} \frac{m_{11}(\mu_k) - m_{22}(\mu_k)}{\dot{m}_{12}(\mu_k)} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{m_{12}(\lambda)}{\lambda - \mu_k} \frac{d\lambda}{\sqrt{\Delta^2(\lambda) - 4}}$$

where we have omitted q from the list of parameters.

Proof. The claimed formula follows from Lemma 14 and

$$\frac{\partial I_n}{\partial q(x)} = -\frac{2}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{1}{\sqrt{\Delta^2(\lambda) - 4}} \frac{\partial \Delta(\lambda)}{\partial q(x)} d\lambda.$$

□

As $\frac{d}{dx} \frac{\partial I_n(x)}{\partial q(x)} \in T_q Iso(q)$, only the projection of $\frac{\partial \theta_m(x)}{\partial q(x)}$ onto $T_q Iso(q)$ will matter for the computation of $\{\theta_m, I_n\}(q)$. As $\theta_m = \sum_{k \geq 1} \eta_{m,k}$ we introduce, for $k \in \mathcal{O}$ and $m \geq 1$,

$$h_{m,k}(x,q) := \begin{cases} -\frac{\psi_m(\mu_k)}{\dot{\Delta}(\mu_k)} y_1(x,\mu_k) y_2(x,\mu_k) & \text{if } \mu_k \in \{\lambda_{2k-1}, \lambda_{2k}\} \\ \frac{\psi_m(\mu_k)}{\sqrt{\Delta^2(\mu_k) - 4}} \frac{\partial \mu_k}{\partial q(x)} & \text{if } \lambda_{2k-1} < \mu_k < \lambda_{2k} \end{cases}$$

where $\psi_m(\lambda)$ $(m \geq 1)$ is given in Proposition 2.

Lemma 16 *For $q \in L_0^2$, $k \in \mathcal{O}$, and $m, n \geq 1$,*
(i)

$$\left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx} \frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = \left\langle h_{m,k}, \frac{d}{dx} \frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2};$$

(ii)

$$\left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx} \frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = -\frac{\psi_m(\mu_k)}{\dot{m}_{12}(\mu_k)} \frac{1}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{m_{12}(\lambda)}{\lambda - \mu_k} \frac{d\lambda}{\sqrt{\Delta^2(\lambda) - 4}}.$$

Proof. (i) Consider the case $\lambda_{2k-1} < \mu_k < \lambda_{2k}$. To prove the statement we use C.3 in Appendix C. As $\lambda_{2k}(\cdot)$ is a spectral invariant, $\frac{\partial \lambda_{2k}}{\partial q(x)} \in N_q Iso(q)$.

By Lemma 13, $\left\langle \frac{\partial \lambda_{2k}}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = 0$. Similarly,

$$\left\langle \frac{\partial}{\partial q(x)} \left(\frac{\psi_m(y+\lambda_{2k})}{\sqrt{-G(y+\lambda_{2k})}} \right), \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = 0$$

where $G(\lambda, q) := \frac{\Delta(\lambda)^2-4}{\lambda_{2k}-\lambda}$. Therefore in this case we obtain (i). In the case $\mu_k = \lambda_{2k}$, we use Lemma 42 in Appendix C. By Corollary 40 in Appendix B, $\left\langle y_2^2(x,\mu_k), \frac{d}{dx}\frac{\partial \Delta(\lambda)}{\partial q(x)} \right\rangle_{L^2} = 0$, as $\lambda_{2k} = \mu_k$. Therefore $\left\langle y_2^2(x,\mu_k), \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = 0$ and, by Lemma 42, we obtain (i). The case $\mu_k = \lambda_{2k-1}$ is treated similarly.
(ii) For $q \in L_0^2$ with $\mu_k \neq \lambda_{2k}$, the statement follows from (i) and Corollary 15 (recall that $\sqrt{\Delta^2(\mu_k)-4} = m_{11}(\mu_k) - m_{22}(\mu_k)$). By continuity, (ii) holds for $m \neq k$, or $m = k$ and $m \in \mathcal{O}$. □

Denote by $Gap^0_{\leq K}$ the set of K-gap potentials

$$Gap^0_{\leq K} := \{q \in L_0^2 \mid \gamma_k = 0 \text{ iff } k > K\}. \tag{4.3}$$

Proof of Proposition 12(ii) Fix $m, n \geq 1$. By Proposition 41, for $K \geq \max\{m,n\}$ and $q \in Gap^0_{\leq K}$,

$$\{\theta_m, I_n\}(q) = \sum_{k=1}^{K} \left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} + \sum_{k=K+1}^{\infty} \left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2}.$$

Using Corollary 44 together with (B.4) (cf Appendix B), we obtain, for $k > K$ and $\lambda \neq \mu_k$, (using that for $\lambda_{2k} = \lambda_{2k-1}$, $m_{22}^2(\mu_k) = 1$ and $m_{21}(\mu_k) = 0$)

$$\left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx}\frac{\partial \Delta(\lambda, q)}{\partial q(x)} \right\rangle_{L^2} = 0.$$

Thus, for $k > K$,

$$\left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = -\frac{2}{\pi}\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{1}{\sqrt{\Delta^2(\lambda)-4}} \left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx}\frac{\partial \Delta(\lambda)}{\partial q(x)} \right\rangle_{L^2} d\lambda = 0.$$

Hence, for $q \in Gap^0_{\leq K}$, (cf Lemma 16 and Lemma 47 in Appendix D)

$$\begin{aligned}
\{\theta_m, I_n\}(q) &= \sum_{k=1}^{K} \left\langle \frac{\partial \eta_{m,k}}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} \\
&= -\frac{1}{\pi}\int_{\lambda_{2n-1}}^{\lambda_{2n}} \sum_{k=1}^{K} \frac{\psi_m(\mu_k)}{\dot{m}_{12}(\mu_k)} \frac{m_{12}(\lambda)}{\lambda - \mu_k} \frac{d\lambda}{\sqrt{\Delta^2(\lambda)-4}} \\
&= -\frac{1}{\pi}\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\psi_m(\lambda)}{\sqrt{\Delta^2(\lambda)-4}} d\lambda = -\delta_{nm}.
\end{aligned}$$

As $\frac{d}{dx}\frac{\partial I_n}{\partial q(x)}$ and $\frac{\partial \theta_m}{\partial q(x)}$ depend continuously on q, and the set $\cup_{k\geq K} Gap^0_{\leq k}$ is dense in L^2_0, we conclude that $\{\theta_m, I_n\} = -\delta_{n,m}$ for $q \in U \setminus D_m$. □

Corollary 17 *For $k, n \geq 1$,*

$$\{x_k, I_n\} = \delta_{k,n} y_k; \qquad \{y_k, I_n\} = -\delta_{k,n} x_k.$$

Proof. Assume that $q \in U \setminus D_k$. Then

$$\begin{aligned} \{x_k, I_n\} &= \left\langle \frac{1}{\sqrt{2I_k}} \cos\theta_k \frac{\partial I_k}{\partial q(x)} - \sqrt{2I_k} \sin\theta_k \frac{\partial \theta_k}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} \quad (4.4)\\ &= \delta_{k,n}\sqrt{2I_k}\sin\theta_k = \delta_{k,n} y_k. \end{aligned}$$

As x_k, y_k, and $\frac{d}{dx}\frac{\partial I_n}{\partial q(x)}$ are analytic, we conclude that (4.4) holds for $q \in L^2_0$. The other identity in the statement is obtained in a similar fashion. □

To prove Proposition 12(iii) we need the following two Lemmas. Recall that $\tilde{\theta}_n = \sum_{k\neq n} \eta_{n,k}(q)$ and introduce, for $q \in L^2_0$ with $\lambda_{2n-1} = \lambda_{2n}$, an $L_2[0,1]$-orthonormal basis $\tilde{f}_{2n-1}, \tilde{f}_{2n}$ of *span* $\langle y_1(\cdot,\lambda_{2n}), y_2(\cdot,\lambda_{2n})\rangle$ with $\tilde{f}_{2n} := \frac{y_2}{||y_2||}$ and $\tilde{f}_{2n-1}(0) > 0$. Then $\tilde{f}_{2n-1}$ is of the form ($y_j \equiv y_j(\cdot,\lambda_{2n})$, $j = 1,2$)

$$\tilde{f}_{2n-1} = \frac{y_1 + b_n y_2}{||y_1 + b_n y_2||}; \qquad b_n := -\frac{\langle y_1, y_2\rangle_{L^2}}{\langle y_2, y_2\rangle_{L^2}}.$$

Lemma 18 *Let $q \in L^2_0$ with $\lambda_{2n-1}(q) = \lambda_{2n}(q)$. Then*

$$\frac{\partial x_n}{\partial q(x)} = \xi_n \left(\cos\tilde{\theta}_n \frac{\tilde{f}^2_{2n} - \tilde{f}^2_{2n-1}}{2} - \kappa_n \sin\tilde{\theta}_n \; \tilde{f}_{2n}\tilde{f}_{2n-1} \right) \quad (4.5)$$

$$\frac{\partial y_n}{\partial q(x)} = \xi_n \left(\sin\tilde{\theta}_n \frac{\tilde{f}^2_{2n} - \tilde{f}^2_{2n-1}}{2} + \kappa_n \cos\tilde{\theta}_n \; \tilde{f}_{2n}\tilde{f}_{2n-1} \right) \quad (4.6)$$

where $\kappa_n \equiv \kappa_n(q)$ satisfies $\kappa_n \neq 0$. If q is a finite gap potential one has for $n \to \infty$

$$\kappa_n = -1 + O\left(\frac{\log n}{n}\right).$$

Proof. is given in Appendix C. □

Lemma 19 *Let $q \in L^2_0$ with $\lambda_{2m-1}(q) = \lambda_{2m}(q)$ and $\lambda_{2n-1}(q) = \lambda_{2n}(q)$. Then, with $\tilde{f}_j$ defined as above*

$$\left\langle \tilde{f}^2_{2n} - \tilde{f}^2_{2n-1}, \frac{d}{dx}\left(\tilde{f}^2_{2m} - \tilde{f}^2_{2m-1}\right) \right\rangle_{L^2} = 0 \quad (4.7)$$

$$\left\langle \tilde{f}_{2n}\tilde{f}_{2n-1}, \frac{d}{dx}\tilde{f}_{2m}\tilde{f}_{2m-1} \right\rangle_{L^2} = 0 \quad (4.8)$$

$$\left\langle \tilde{f}^2_{2n} - \tilde{f}^2_{2n-1}, \frac{d}{dx}\tilde{f}_{2m}\tilde{f}_{2m-1} \right\rangle_{L^2} = -\delta_{n,m}\, ||y_2||\, ||y_1 + b_n y_2||. \quad (4.9)$$

Proof. Assume that $q \in H_0^1$. The identities (4.7) and (4.8) clearly hold if $m = n$. If $m \neq n$, then, as $\tilde{f}_{2k-1}^2$, $\tilde{f}_{2k}^2$, and $\tilde{f}_{2k}\tilde{f}_{2k-1}$ with $k \in \{m, n\}$ are in H^3, we obtain by Lemma 39 in Appendix B that (4.7)-(4.9) hold.

It remains to verify (4.9) for $m = n$. Notice that

$$y_1(x, \lambda_{2n})y_2(x, \lambda_{2n}) = \alpha \tilde{f}_{2n-1}\tilde{f}_{2n} - b_n ||y_2||^2 \tilde{f}_{2n}^2$$

where, in view of $\tilde{f}_{2n-1} = \frac{y_1 + b_n y_2}{||y_1 + b_n y_2||}$, $\alpha = ||y_1 + b_n y_2||\, ||y_2||$. Let $W[f, g] := f'g - fg'$. By a straightforward computation,

$$\begin{aligned} \left\langle \tilde{f}_{2n}^2, \frac{d}{dx}\tilde{f}_{2n}\tilde{f}_{2n-1} \right\rangle_{L^2} &= \frac{1}{2} W[\tilde{f}_{2n-1}, \tilde{f}_{2n}](0); \\ \left\langle \tilde{f}_{2n-1}^2, \frac{d}{dx}\tilde{f}_{2n}\tilde{f}_{2n-1} \right\rangle_{L^2} &= -\frac{1}{2} W[\tilde{f}_{2n-1}, \tilde{f}_{2n}](0). \end{aligned}$$

Combining the two identities above leads to

$$\left\langle \tilde{f}_{2n}^2 - \tilde{f}_{2n-1}^2, \frac{d}{dx}\tilde{f}_{2n-1}\tilde{f}_{2n} \right\rangle_{L^2} = W[\tilde{f}_{2n-1}, \tilde{f}_{2n}](0) = -\frac{1}{\alpha}$$

and (4.9) holds for $n = m$.

Finally one can argue by continuity to conclude that (4.7)-(4.9) hold for $q \in L_0^2$. □

Proof of Proposition 12(iii) The claimed identities follow from Lemma 18 and Lemma 19. □

5 $d_q\Omega$ a local diffeomorphism

In this section we prove

Proposition 20 *For $q \in L_0^2$, the map $d_q\Omega : L_0^2 \to h^{\frac{1}{2}}(\mathbb{N}; \mathbb{R}^2)$ is invertible.*

Remark The derivative $d_q\Omega$ at $q = 0$ can be explicitly computed. It is given by ($p \in L_0^2$)

$$d_0\Omega(p) = \left(\frac{-1}{\sqrt{n\pi}} (p_{2n}, p_{2n-1}) \right)_{n \geq 1}$$

where $(p_n)_{n \geq 1}$ are the Fourier coefficents of p,

$$p_{2n} = \int_0^1 p(x) \cos(2\pi n x) dx; \quad p_{2n-1} = \int_0^1 p(x) \sin(2\pi n x) dx.$$

To prove Proposition 20 we show in a first step that $d_q\Omega$ is Fredholm (cf Lemma 23 below). For this we need the following

Lemma 21 *For $K \geq 0$ and $q \in Gap^0_{\leq K}$ (cf 4.3), we have:*

(i)
$$\sqrt{2n\pi}\frac{\partial x_n}{\partial q(x)} = -\sqrt{2}\cos 2\pi n x + O_\infty\left(\frac{\log n}{n}\right) \quad (n \to \infty)$$
$$\sqrt{2n\pi}\frac{\partial y_n}{\partial q(x)} = -\sqrt{2}\sin 2\pi n x + O_\infty\left(\frac{\log n}{n}\right) \quad (n \to \infty);$$

(ii)
$$\frac{1}{\sqrt{2n\pi}}\frac{d}{dx}\frac{\partial x_n}{\partial q(x)} = \sqrt{2}\sin 2\pi n x + O_\infty\left(\frac{\log n}{n}\right) \quad (n \to \infty)$$
$$\frac{1}{\sqrt{2n\pi}}\frac{d}{dx}\frac{\partial y_n}{\partial q(x)} = -\sqrt{2}\cos 2\pi n x + O_\infty\left(\frac{\log n}{n}\right) \quad (n \to \infty).$$

Proof. The estimate for $\frac{\partial y_n}{\partial q(x)}$ is obtained similarly as the estimate for $\frac{\partial x_n}{\partial q(x)}$, so we concentrate on $\frac{\partial x_n}{\partial q(x)}$.

(i) Fix $K \geq 0$ and $q \in Gap^0_{\leq K}$ and let $n > K$ be arbitrary. As $\lambda_{2n-1}(q) = \lambda_{2n}(q)$, by Lemma 18,

$$\frac{\partial x_n}{\partial q(x)} = \xi_n(q)\left(\cos\tilde{\theta}_n \frac{\tilde{f}^2_{2n} - \tilde{f}^2_{2n-1}}{2} - \kappa_n \sin\tilde{\theta}_n\ \tilde{f}_{2n}\tilde{f}_{2n-1}\right). \tag{5.1}$$

Recall that $\tilde{\theta}_n = \sum_{k\neq n}\eta_{n,k}$. As, for $k > K$, $\mu_k = \lambda_{2k}$, we get, for $k > K$, $\eta_{n,k} = 0$. Therefore $\tilde{\theta}_n = \sum_{k=1}^K \eta_{n,k}$. By Lemma 4

$$\tilde{\theta}_n = O\left(\frac{1}{n}\right). \tag{5.2}$$

Recall that $\xi_n = \frac{1}{\sqrt{n\pi}}\left(1 + O\left(\frac{\log n}{n}\right)\right)$, $\kappa_n = -1 + O\left(\frac{\log n}{n}\right)$. Further, as $y_1 = \cos n\pi x + O_\infty\left(\frac{1}{n}\right)$ and $y_2 = \frac{\sin n\pi x}{n\pi} + O_\infty\left(\frac{1}{n^2}\right)$ we have $\langle y_1, y_2\rangle_{L^2} = O\left(\frac{1}{n^2}\right)$ and $\langle y_2, y_2\rangle_{L^2} = O\left(\frac{1}{n^2}\right)$. Hence $b_n = -\frac{\langle y_1,y_2\rangle_{L^2}}{\langle y_2,y_2\rangle_{L^2}} = O(1)$ and $y_1 + b_n y_2 = \cos n\pi x + O_\infty\left(\frac{1}{n}\right)$. One thus obtains

$$\tilde{f}_{2n} = \frac{y_2(x,\lambda_{2n})}{||y_2(\cdot,\lambda_{2n})||} = \sqrt{2}\sin n\pi x + O_\infty\left(\frac{1}{n}\right). \tag{5.3}$$

and

$$\tilde{f}_{2n-1} = \frac{y_1 + b_n y_2}{||y_1 + b_n y_2||} = \sqrt{2}\cos n\pi x + O_\infty\left(\frac{1}{n}\right). \tag{5.4}$$

Therefore

$$\tilde{f}_{2n}\tilde{f}_{2n-1} = \sin 2n\pi x + O_\infty\left(\frac{1}{n}\right), \tag{5.5}$$

$$\tilde{f}^2_{2n} - \tilde{f}^2_{2n-1} = -2\cos 2n\pi x + O_\infty\left(\frac{1}{n}\right). \tag{5.6}$$

Substituting the above estimates in (5.1), one obtains the claimed asymptotic.

(ii) The proof for (ii) is similar, using the asymptotics of the derivatives of the fundamental solutions $y_1'(x,\lambda_{2n})$ and $y_2'(x,\lambda_{2n})$ stated in (C.9). □

Introduce ($n \geq 1$)

$$B_n \equiv B_n(q) := \sqrt{2n\pi}\frac{\partial x_n}{\partial q(x)}; \quad B_{-n} \equiv B_{-n}(q) := \sqrt{2n\pi}\frac{\partial y_n}{\partial q(x)};$$
$$T_n \equiv T_n(q) := -\sqrt{2}\cos 2\pi nx; \quad T_{-n} \equiv T_{-n}(q) := -\sqrt{2}\sin 2\pi nx.$$

From Lemma 21 we obtain, with

$$Gap^0_{finite} = \cup_{k\geq 1} Gap^0_{\leq k},$$

Corollary 22 *For $q \in Gap^0_{finite}$, the system $(B_m)_{m\neq 0}$ is quadratically close to $(T_m)_{m\neq 0}$, i.e.*

$$\sum_{m\neq 0} ||B_m - T_m||^2 < \infty.$$

The linear operator $d_q\Omega : L^2_0 \to h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$ is given by

$$d_q\Omega(h) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \langle h, B_m\rangle_{L^2}\, e_m \tag{5.7}$$

where $e_m = (2m\pi)^{-1/2}(\delta_{n,m},0)_{n\geq 1}$ and $e_{-m} = (2m\pi)^{-1/2}(0,\delta_{n,m})_{n\geq 1}$. Denote by $(e^*_m)_m$ the basis dual to $(e_m)_m$, i.e. $e^*_m = (2m\pi)^{1/2}(\delta_{n,m},0)_{n\geq 1}$ and $e^*_{-m} = (2m\pi)^{1/2}(0,\delta_{n,m})_{n\geq 1}$.

Lemma 23 *Let $q \in L^2_0$.*
(i) *The operator $d_q\Omega$ is a Fredholm operator with index* 0.
(ii) $B_m = T_m + o_2(1)$, *($\pm m \to \infty$).*

Proof. Introduce the operators $\mathcal{D} : L^2_0 \to h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$, and $A_q : L^2_0 \to h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$, given by

$$\mathcal{D}(h) := \sum_{m\in\mathbb{Z}\setminus\{0\}} \langle h, T_m\rangle_{L^2}\, e_m;$$
$$A_q := d_q\Omega - \mathcal{D}; \qquad A_q(h) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \langle h, B_m - T_m\rangle_{L^2}\, e_m.$$

(i) First we prove that, for $q \in Gap^0_{finite}$, the operator A_q is compact. It follows from Corollary 22 that, for any $q \in Gap^0_{finite}$ and $\epsilon > 0$, there exist $a > 0$ and $M > 0$ such that $\forall h \in L^2_0$ with $||h|| \leq 1$, the following inequalities hold

$$||A_q h|| \leq a; \qquad \sum_{|m|>M} \langle h, B_m - T_m\rangle^2_{L^2} < \epsilon.$$

Thus A_q is compact.

As $A_q = d_q\Omega - \mathcal{D}$ depends continuously on q and Gap^0_{finite} is dense in L^2_0, we conclude that A_q is compact for $q \in L^2_0$. As $\mathcal{D}$ is invertible, $d_q\Omega$ is a Fredholm operator of index 0.
(ii) Notice that, for $m \neq 0$, $(d_q\Omega)^* (e^*_m) = B_m$, where $(d_q\Omega)^* : h^{-\frac{1}{2}}(\mathbb{N};\mathbb{R}^2) \to L^2_0$ and $(e^*_m)_m$ denotes the basis dual to $(e_m)_m$ introduced above. Indeed, for $h \in L^2_0$,

$$\left\langle (d_q\Omega)^* (e^*_m), h \right\rangle_{L^2} = \langle e^*_m, d_q\Omega(h) \rangle = \langle h, B_m \rangle_{L^2}$$

where we used (5.7). By (i), $B_m = \mathcal{D}^*(e^*_m) + A^*_q(e^*_m)$. Notice that $\mathcal{D}^*(e^*_m) = T_m$. Further $A^*_q(e^*_m) = o_2(1)$ as $A^*_q : h^{-\frac{1}{2}}(\mathbb{N};\mathbb{R}^2) \to L^2_0$ is compact. □

As a second ingredient of the proof of Proposition 20, we show that $d_q\Omega$ is $1-1$. First we need to establish some auxilary results. Following [**GK**], we say that a sequence $(F_n)_{n\in\mathcal{J}}$ in L^2_0 ($\mathcal{J} \subset \mathbb{Z}$) is almost normalized if

$$0 < \inf_n \|F_n\| \text{ and } \sup_n \|F_n\| < \infty.$$

An almost normalized sequence $(F_n)_{n\in\mathcal{J}}$ is said to be ω-linearly independent in L^2_0 (cf [**GK**] p. 316) if for any sequence $(\alpha_n)_{n\in\mathcal{J}}$ with $\sum_{n\in\mathcal{J}} \alpha_n^2 < \infty$ and $\sum_{n\in\mathcal{J}} \alpha_n F_n = 0$, $\alpha_n = 0$ for all $n \in \mathcal{J}$.

Notice that, by Lemma 23, B_m is almost normalized.

Lemma 24 *Let $q \in L^2_0$. Then $d_q\Omega$ is invertible iff $(B_m)_{m\neq 0}$ is ω-linearly independent in L^2_0.*

Proof. By Lemma 23, $(d_q\Omega)^* : h^{-\frac{1}{2}}(\mathbb{N};\mathbb{R}^2) \to L^2_0$ is a Fredholm operator of index 0. Further, for $m \neq 0$, $(d_q\Omega)^* (e^*_m) = B_m$. Therefore, $Null\ (d_q\Omega)^* = \{0\}$ iff $(B_m)_{m\neq 0}$ is ω-linearly independent in L^2_0. □

For $n \in \mathcal{O}$, $\frac{\sqrt{2\pi n'}}{\sqrt{2I_n}} \frac{\partial I_n}{\partial q(x)} = \cos\theta_n B_n + \sin\theta_n B_{-n}$. Hence, by Lemma 21, the sequence $\left(\frac{\sqrt{2n\pi}}{\sqrt{2I_n}} \frac{\partial I_n}{\partial q(x)} \right)_{n\in\mathcal{O}}$ is almost normalized.

Lemma 25 *The system $(B_m)_{m\neq 0}$ is ω-linearly independent in L^2_0 iff the system $\left(\frac{\sqrt{2n\pi}}{\sqrt{2I_n}} \frac{\partial I_n}{\partial q(x)} \right)_{n\in\mathcal{O}}$ is ω-linearly independent in L^2_0.*

Proof. Assume that, for a sequence $(\alpha_m)_{m\neq 0}$ with $\sum_{m\in\mathbb{Z}\setminus\{0\}} \alpha_m^2 < \infty$,

$$f := \sum_{m\in\mathbb{Z}\setminus\{0\}} \alpha_m B_m = \sum_{n\geq 1} \sqrt{2n\pi} \left(\alpha_n \frac{\partial x_n}{\partial q(x)} + \alpha_{-n} \frac{\partial y_n}{\partial q(x)} \right) = 0.$$

Then, by Corollary 17, for $k \in \mathcal{O}$,

$$0 = \left\langle f, \frac{d}{dx} \frac{\partial I_k}{\partial q(x)} \right\rangle_{L^2} = \sqrt{2k\pi}(\alpha_k y_k - \alpha_{-k} x_k).$$

Thus, for $k \in \mathcal{O}$, $(\alpha_k, \alpha_{-k}) = \pm\sqrt{\alpha_k^2 + \alpha_{-k}^2}(\cos\theta_k, \sin\theta_k)$ and

$$\alpha_k \frac{\partial x_k}{\partial q(x)} + \alpha_{-k}\frac{\partial y_k}{\partial q(x)} = \pm\sqrt{\alpha_k^2 + \alpha_{-k}^2}\frac{1}{\sqrt{2I_k}}\frac{\partial I_k}{\partial q(x)}.$$

By Proposition 12(iii) and Corollary 17, for $k \notin \mathcal{O}$,

$$\begin{aligned} 0 &= \left\langle f, \frac{d}{dx}\frac{\partial x_k}{\partial q(x)}\right\rangle_{L^2} = \sqrt{2k\pi}\alpha_{-k}\left\langle \frac{\partial y_k}{\partial q(x)}, \frac{d}{dx}\frac{\partial x_k}{\partial q(x)}\right\rangle_{L^2} \\ 0 &= \left\langle f, \frac{d}{dx}\frac{\partial y_k}{\partial q(x)}\right\rangle_{L^2} = \sqrt{2k\pi}\alpha_{k}\left\langle \frac{\partial x_k}{\partial q(x)}, \frac{d}{dx}\frac{\partial y_k}{\partial q(x)}\right\rangle_{L^2}. \end{aligned}$$

Hence, by Proposition 12(iii), for $k \notin \mathcal{O}$, $\alpha_{\pm k} = 0$ and

$$0 = \sum_{m\in\mathbb{Z}\setminus\{0\}} \alpha_m B_m = \sum_{n\in\mathcal{O}}\left(\pm\sqrt{\alpha_n^2 + \alpha_{-n}^2}\right)\frac{\sqrt{2n\pi}}{\sqrt{2I_n}}\frac{\partial I_n}{\partial q(x)}.$$

From these considerations the claimed statement follows. □

Lemma 26 *The system* $\left(\frac{\sqrt{2n\pi}}{\sqrt{2I_n}}\frac{\partial I_n}{\partial q(x)}\right)_{n\in\mathcal{O}}$ *is ω-linearly independent in L_0^2.*

Proof. It is to show that for any $(\alpha_n)_{n\in\mathcal{O}}$ with $\sum_{n\in\mathcal{O}}\alpha_n^2 < \infty$ and

$$\sum_{n\in\mathcal{O}} \alpha_n \frac{\sqrt{2n\pi}}{\sqrt{2I_n}}\frac{\partial I_n}{\partial q(x)} = 0 \tag{5.8}$$

one has $\alpha_n = 0$ for any $n \in \mathcal{O}$.

Recall that, for $k \in \mathcal{O}$ and $m \geq 1$, we have introduced

$$h_{m,k}(x,q) := \begin{cases} -\frac{\psi_m(\mu_k)}{\dot\Delta(\mu_k)}y_1(x,\mu_k)y_2(x,\mu_k) & \mu_k \in \{\lambda_{2k-1}, \lambda_{2k}\} \\ \frac{\psi_m(\mu_k)}{\sqrt{\Delta^2(\mu_k)-4}}\frac{\partial\mu_k}{\partial q(x)} & \lambda_{2k-1} < \mu_k < \lambda_{2k} \end{cases}$$

and proved (cf Lemma 16)

$$\left\langle \frac{\partial I_n}{\partial q(x)}, \frac{d}{dx}h_{m,k}\right\rangle_{L^2} = \frac{\psi_m(\mu_k)}{\dot m_{12}(\mu_k)}\frac{1}{\pi}\int_{\lambda_{2n-1}}^{\lambda_{2n}}\frac{m_{12}(\lambda)}{\lambda-\mu_k}\frac{d\lambda}{\sqrt{\Delta^2(\lambda)-4}}.$$

For any $m \in \mathcal{O}$ given, we want to conclude from (5.8) that $\alpha_m = 0$. Indeed,

$$\begin{aligned} 0 &= \left\langle \sum_{n\in\mathcal{O}}\alpha_n\frac{\sqrt{2n\pi}}{\sqrt{2I_n}}\frac{\partial I_n}{\partial q(x)}, \frac{d}{dx}h_{m,k}\right\rangle_{L^2} \\ &= \sum_{n\in\mathcal{O}}\alpha_n\frac{\sqrt{2n\pi}}{\sqrt{2I_n}}\frac{\psi_m(\mu_k)}{\dot m_{12}(\mu_k)}\frac{1}{\pi}\int_{\lambda_{2n-1}}^{\lambda_{2n}}\frac{m_{12}(\lambda)}{\lambda-\mu_k}\frac{d\lambda}{\sqrt{\Delta^2(\lambda)-4}}. \end{aligned}$$

With the change of variable of integration $\lambda = \zeta_n(t) := \tau_n + t\frac{\gamma_n}{2}$ $(-1 \le t \le 1)$,

$$\int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{m_{12}(\lambda)}{\lambda - \mu_k} \frac{d\lambda}{\sqrt{\Delta^2(\lambda) - 4}} = \int_{-1}^{1} \frac{m_{12}(\zeta_n(t))}{\zeta_n(t) - \mu_k} \frac{\sqrt{1-t^2}\gamma_n/2}{\sqrt{\Delta^2(\zeta_n(t)) - 4}} \frac{dt}{\sqrt{1-t^2}}$$

and standard asymptotic estimates for $\sqrt{2I_n} = \xi_n\gamma_n/2$, $\psi_m(\lambda)$, and $\dot{m}_{12}(\lambda)$ one concludes that (for $n, k \ne m$)

$$\left| \alpha_n \frac{\sqrt{2n\pi}}{\sqrt{2I_n}} \frac{\psi_m(\mu_k)}{\dot{m}_{12}(\mu_k)} \frac{m_{12}(\zeta_n(t))}{\zeta_n(t) - \mu_k} \frac{\sqrt{1-t^2}\gamma_n/2}{\sqrt{\Delta^2(\zeta_n(t)) - 4}} \right| \le C \frac{m}{|k^2 - m^2|} \frac{|\alpha_n|}{n}.$$

Therefore

$$0 = \sum_{n \in \mathcal{O}} \alpha_n \frac{\sqrt{2n\pi}}{\sqrt{2I_n}} \frac{1}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \sum_{k \in \mathcal{O}} \frac{\psi_m(\mu_k)}{\dot{m}_{12}(\mu_k)} \frac{m_{12}(\lambda)}{\lambda - \mu_k} \frac{d\lambda}{\sqrt{\Delta^2(\lambda) - 4}}. \tag{5.9}$$

For, $k \notin \mathcal{O}$, $\psi_m(\mu_k) = 0$. Thus, by the sampling formula (cf Proposition 46 Appendix D),

$$\sum_{k \in \mathcal{O}} \frac{\psi_m(\mu_k)}{\dot{m}_{12}(\mu_k)} \frac{m_{12}(\lambda)}{\lambda - \mu_k} = \sum_{k \ge 1} \frac{\psi_m(\mu_k)}{\dot{m}_{12}(\mu_k)} \frac{m_{12}(\lambda)}{\lambda - \mu_k} = \psi_m(\lambda).$$

We now can rewrite (5.9) as

$$0 = \sum_{n \in \mathcal{O}} \alpha_n \frac{\sqrt{2n\pi}}{\sqrt{2I_n}} \frac{1}{\pi} \int_{\lambda_{2n-1}}^{\lambda_{2n}} \frac{\psi_m(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda = \sum_{n \in \mathcal{O}} \alpha_n \frac{\sqrt{2n\pi}}{\sqrt{2I_n}} \delta_{n,m}$$

and hence $\alpha_m = 0$. □

6 Ω a diffeomorphism

The main result of this section is the following

Theorem 3 *The map $\Omega : L_0^2 \to h^{\frac{1}{2}}(\mathbb{N}; \mathbb{R}^2)$ as well as its inverse is a real analytic diffeomorphism.*

First we need to prove

Proposition 27 *The map $\Omega : L_0^2 \to h^{\frac{1}{2}}(\mathbb{N}; \mathbb{R}^2)$ is proper.*

Proof. Given a compact subset $K \subset h^{\frac{1}{2}}(\mathbb{N}; \mathbb{R}^2)$, there exists $M \ge 1$ and, for any $\varepsilon > 0, n_\varepsilon \ge 1$ so that, for all $q \in Q := \Omega^{-1}(K) \subseteq L_0^2$,

$$\sum_{n \ge 1} n|I_n(q)| \le M; \tag{6.1}$$

$$\sum_{n \geq n_\varepsilon} n|I_n(q)| \leq \varepsilon. \tag{6.2}$$

It is proved in [BBGK, Lemma 2.2] that

$$I_n \geq \frac{1}{(8\pi)^2} \min\{(1/n)\gamma_n^2, n\gamma_n\}.$$

Thus the set $\{\gamma_n(q)_{n\geq 1} \mid q \in \Omega^{-1}(K)\}$ is compact in ℓ^2. Therefore $\Omega^{-1}(K)$ is compact in L_0^2 (cf [**GT**]). □

Proof of Theorem 3 We have established that $\Omega : L_0^2 \to h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$ is a real analytic map and a local diffeomorphism. It remains to show that Ω is 1-1 and onto. Consider the set $\mathcal{V} := \{z \in h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2) \mid \sharp\Omega^{-1}(z) = 1\}$. Then $\mathcal{V}$ is open and closed in $h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$ as Ω is proper and a local diffeomorphism. In order to prove that $\mathcal{V} = h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$ it suffices therefore to show that $\mathcal{V} \neq \emptyset$. Take $w = 0 \in h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$. Then, for any $q \in \Omega^{-1}(0)$ and $n \geq 1$, $\gamma_n(q) = 0$ and therefore $q \equiv 0$. □

7 Restriction of Ω to H_0^N $(N \geq 1)$

In this section we want to improve on Theorem 3. For any $N \geq 0$, denote by $\Omega^{(N)}$ the restriction of $\Omega \equiv \Omega^{(0)}$ to H_0^N. It turns out that the range of $\Omega^{(N)}$ is contained in $h^{N+1/2}(\mathbb{N};\mathbb{R}^2)$ (cf Lemma 29), hence $\Omega^{(N)}$ can be viewed as a map

$$\Omega^{(N)} : H_0^N \to h^{N+1/2}(\mathbb{N};\mathbb{R}^2).$$

Theorem 4 *For any $N \geq 0$,*

(i) $\Omega^{(N)}$ *is a diffeomorphism;*

(ii) $\Omega^{(N)}$ *is real analytic.*

The proof of Theorem 4 follows from the results stated in the remainder of this section.

Recall the following result from [**KM**] (cf also [**ST**]) and [**Ma**].

Proposition 28 *(i) For $q_0 \in H_0^N$, there exists a complex neighborhood $U_{q_0} \subseteq H_{0,\mathbb{C}}^N$ so that, for $q \in U_{q_0}$, $(\gamma_n(q))_{n\geq 1}$ and $(\mu_n(q) - \lambda_{2n}(q))_{n\geq 1}$ are uniformly bounded in $h^N(\mathbb{N};\mathbb{C})$.*
(ii) For any real valued $q \in L_0^2$ one has

$$q \in H_0^N \text{ iff } (\gamma_n(q))_{n\geq 1} \in h^N(\mathbb{N};\mathbb{R}).$$

As a consequence we obtain the following

Lemma 29 *Let $N \geq 0$.*

(i) *For* $q_0 \in H_0^N$ *there exists a complex neighborhood* U_{q_0} *of* q_0 *in* $H_{0,\mathbb{C}}^N$ *so that* $\Omega(U_{q_0})$ *is bounded in* $h^{N+1/2}(\mathbb{N};\mathbb{C}^2)$.

(ii) *For real valued potentials, the following characterization holds:*

$$q \in H_0^N \text{ iff } (x_n(q), y_n(q))_{n\geq 1} \in h^{N+1/2}(\mathbb{N};\mathbb{R}^2).$$

Proof. (i) By Proposition 28(i), there exists a complex neighborhood V_{q_0} of q_0 in $H_{0,\mathbb{C}}^N$ so that $(\gamma_n(q))_{n\geq 1}$ and $(\mu_n(q) - \lambda_{2n}(q))_{n\geq 1}$ are uniformly bounded in $h^N(\mathbb{N};\mathbb{C})$. By Corollary 11, there exists a complex neighborhood W_{q_0} of q_0 so that $|x_n|+|y_n| \leq \frac{C}{n^{1/2}}(|\mu_n - \tau_n|+|\gamma_n|)$ $(\forall n \geq 1)$. Hence $\Omega\left(V_{q_0} \cap W_{q_0}\right) \subseteq h^{N+1/2}(\mathbb{N};\mathbb{C}^2)$.
(ii) In view of (i) it remains to prove that for any element $(x_n, y_n)_{n\geq 1}$ $\in h^{N+1/2}(\mathbb{N};\mathbb{R}^2)$, $\Omega^{-1}\left((x_n, y_n)_{n\geq 1}\right) \in H_0^N$. By Theorem 3,

$$q := \Omega^{-1}\left((x_n, y_n)_{n\geq 1}\right) \in L_0^2.$$

As q is real valued

$$|x_n|^2 + |y_n|^2 = 2I_n.$$

By Proposition 1, $2I_n = O\left(\frac{1}{n}\right)\left(\frac{\gamma_n}{2}\right)^2$. As q is real valued and $(x_n, y_n)_{n\geq 1} \in h^{N+1/2}(\mathbb{N};\mathbb{R}^2)$ it then follows from Proposition 28(ii) that $q \in H_0^N$. □

As a conseqence of Lemma 29 one gets

Corollary 30 *For any* $N \geq 0$,

$$\Omega^{(N)} : H_0^N \to h^{N+1/2}(\mathbb{N};\mathbb{R}^2)$$

is real analytic and bijective.

Proof. To see that $\Omega^{(N)}$ is real analytic it suffices to show that $\Omega^{(N)}$ is weakly analytic and locally bounded. As Ω is real analytic, $\Omega^{(N)}$ is weakly analytic. By Lemma 29(i), $\Omega^{(N)}$ is locally bounded.

From the fact that $\Omega : L_0^2 \to h^{1/2}$ is bijective it follows that $\Omega^{(N)} : H_0^N \to h^{N+1/2}$ is 1-1 and by Lemma 29(ii), we have that $\Omega^{(N)}$ is onto. □

Let us now analyze the derivative $d_q\Omega$ in more detail. Clearly, for $q \in H_0^N$,

$$d_q\Omega_n^{(N)} = d_q\Omega_n|_{H_0^N}.$$

Using an inductive procedure, we obtain the following improvement of Lemma 21.

Lemma 31 *Let $q \in Gap^0_{\leq K}$ with $K \geq 0$ and $N \geq 0$. Then for any $p \in H_0^N$, the following statements hold:*

$$\begin{aligned} \left| \sqrt{2n\pi} \left\langle \frac{\partial x_n}{\partial q(x)}, p \right\rangle_{L^2} + \left\langle \sqrt{2} \cos 2n\pi x, p \right\rangle_{L^2} \right| &\leq C_n ||p||_{H^N}; \\ \left| \sqrt{2n\pi} \left\langle \frac{\partial y_n}{\partial q(x)}, p \right\rangle_{L^2} + \left\langle \sqrt{2} \sin 2n\pi x, p \right\rangle_{L^2} \right| &\leq C_n ||p||_{H^N} \end{aligned}$$

where the bounds C_n are independent of p and satisfy $C_n = O\left(\frac{\log n}{n^{N+1}}\right)$.

Proof. Both estimates are proved similarly, so we concentrate on the first one. The proof consists in verifying the statement for $N = 0, 1$ and in proving an inductive step. Let us start with the latter one. Assume that the statement has already been proved for $N \geq 0$. We want to show that the statement holds for $N + 2$. Let $p \in H_0^{N+2}$. According to Lemma 18 and as $q \in Gap^0_{\leq K}$, $\frac{\partial x_n}{\partial q(x)}$ is, for $n \geq K + 1$, a linear combination of the products $y_i(x, \lambda_{2n}, q) y_j(x, \lambda_{2n}, q) \in C^\infty$ $(1 \leq i, j \leq 2)$. Hence (straightforward verification)

$$L_q \frac{\partial x_n}{\partial q(x)} = 2\lambda_{2n} \frac{d}{dx} \frac{\partial x_n}{\partial q(x)} \tag{7.1}$$

where L_q is a skew symmetric differential operator of order 3, given by

$$L_q = -\frac{1}{2} \frac{d^3}{dx^3} + \frac{d}{dx} q + q \frac{d}{dx}.$$

Denote by $\left(\frac{d}{dx}\right)^{-1} : L_0^2 \to H_0^1$ the inverse of the restriction of $\frac{d}{dx}$ to H_0^1. It follows from (7.1) that

$$\frac{\partial x_n}{\partial q(x)} = \frac{1}{2\lambda_{2n}} \left(\frac{d}{dx}\right)^{-1} L_q \frac{\partial x_n}{\partial q(x)}. \tag{7.2}$$

Substitute (7.2) into $\left\langle \frac{\partial x_n}{\partial q(x)}, p \right\rangle_{L^2}$ and integrate by parts to get

$$\left\langle \frac{\partial x_n}{\partial q(x)}, p \right\rangle_{L^2} = \frac{1}{2\lambda_{2n}} \left\langle \frac{\partial x_n}{\partial q(x)}, \tilde{p} \right\rangle_{L^2} \tag{7.3}$$

where

$$\tilde{p} := L_q \left(\frac{d}{dx}\right)^{-1} p = -\frac{1}{2} p'' + 2qp + q' \left(\frac{d}{dx}\right)^{-1} p \quad \in H_0^N. \tag{7.4}$$

By the induction hypothesis

$$\left| \sqrt{2n\pi} \left\langle \frac{\partial x_n}{\partial q(x)}, \tilde{p} \right\rangle_{L^2} + \left\langle \sqrt{2} \cos 2n\pi x, \tilde{p} \right\rangle_{L^2} \right| \leq O\left(\frac{\log n}{n^{N+1}}\right) ||\tilde{p}||_{H^N}. \tag{7.5}$$

By (7.4), we have

$$||\tilde{p}||_{H^N} \leq C ||p||_{H^{N+2}}. \tag{7.6}$$

Further,

$$\left\langle \sqrt{2}\cos 2n\pi x, \tilde{p} \right\rangle_{L^2} = -\frac{1}{2}\left\langle \sqrt{2}\cos 2n\pi x, p'' \right\rangle_{L^2} + \left\langle \sqrt{2}\cos 2n\pi x, 2qp + q'\left(\frac{d}{dx}\right)^{-1} p \right\rangle_{L^2} \tag{7.7}$$

where

$$\left\langle \sqrt{2}\cos 2n\pi x, p'' \right\rangle_{L^2} = -(2n\pi)^2 \left\langle \sqrt{2}\cos 2n\pi x, p \right\rangle_{L^2}, \tag{7.8}$$

and

$$\left| \left\langle \sqrt{2}\cos 2n\pi x,\ 2qp + q'\left(\frac{d}{dx}\right)^{-1} p \right\rangle_{L^2} \right| \le O\left(\frac{1}{n^{N+2}}\right) ||p||_{H^{N+2}}. \tag{7.9}$$

Substituting (7.8) and (7.9) into (7.7) and using (7.6), (7.5) leads to the following estimate

$$\left| \sqrt{2n\pi}\left\langle \frac{\partial x_n}{\partial q(x)}, \tilde{p} \right\rangle_{L^2} + 2n^2\pi^2 \left\langle \sqrt{2}\cos 2n\pi x, p \right\rangle_{L^2} \right| \le O\left(\frac{\log n}{n^{N+1}}\right) ||p||_{H^{N+2}}. \tag{7.10}$$

Using (7.3) , (7.10) and the asymptotics $\lambda_{2n} = n^2\pi^2 + O(1)$, we obtain

$$\begin{aligned} &\left| \sqrt{2n\pi}\left\langle \frac{\partial x_n}{\partial q(x)}, p \right\rangle_{L^2} + \left\langle \sqrt{2}\cos 2n\pi x, p \right\rangle_{L^2} \right| \\ &\le \left| \frac{\sqrt{2n\pi}}{2\lambda_{2n}}\left\langle \frac{\partial x_n}{\partial q(x)}, \tilde{p} \right\rangle_{L^2} + \frac{2n^2\pi^2}{2\lambda_{2n}}\left\langle \sqrt{2}\cos 2n\pi x, p \right\rangle_{L^2} \right| \\ &+ \left| -\frac{2n^2\pi^2}{2ß\lambda_{2n}}\left\langle \sqrt{2}\cos 2n\pi x, p \right\rangle_{L^2} + \left\langle \sqrt{2}\cos 2n\pi x, p \right\rangle_{L^2} \right| \\ &\le O\left(\frac{\log n}{n^{N+3}}\right) ||p||_{H^{N+2}}. \end{aligned}$$

This proves the induction step.
It remains to verify the statements for $N = 0$ and $N = 1$. The case $N = 0$ is contained in Lemma 21(i). The case $N = 1$ is proved in similar fashion as the induction step using the operator $\left(\frac{d}{dx}\right)^{-1} L_q \left(\frac{d}{dx}\right)^{-1}$ instead of $L_q \left(\frac{d}{dx}\right)^{-1}$ together with Lemma 21(ii). □

Lemma 32 *For $q \in H_0^N$, $d_q\Omega^{(N)} : H_0^N \to h^{N+1/2}$ is bijective.*

Proof. By Theorem 3, $d_q\Omega : L_0^2 \to h^{1/2}$ is bijective, hence $d_q\Omega^{(N)} = d_q\Omega|_{H_0^N}$ is 1-1. To see that $d_q\Omega^{(N)}$ is onto it then suffices to prove that $d_q\Omega^{(N)}$ is a Fredholm operator of index 0. Using Lemma 31, this is verified in a similar way as in the proof of Lemma 23. □

8 Ω a symplectomorphism

The symplectic structure ω associated to the Poisson bracket $\{F,G\} = \left\langle \frac{\partial F}{\partial q(x)}, \frac{d}{dx}\frac{\partial G}{\partial q(x)} \right\rangle_{L^2}$ is given by $\omega(f,g) := \left\langle f, \left(\frac{d}{dx}\right)^{-1} g \right\rangle_{L^2}$ $(f,g \in L_0^2)$. Denote by ω_{can} the canonical symplectic structure $\omega_{\text{can}} = \sum_{k=1}^{\infty} dy_k \wedge dx_k$ on $h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$. In this section we prove

Theorem 5 *The map* $\Omega : \left(L_0^2, \omega\right) \to \left(h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2), \omega_{\text{can}}\right)$ *is a symplectomorphism.*

To establish Theorem 5, it remains to prove that $\Omega_* \omega = \omega_{can}$. We will establish this identity for finite gap potentials and then argue by continuity. First let us introduce some more notation. Recall that $D_m = \{q \mid \gamma_m(q) = 0\}$ and define, for any given $K \geq 0$, the map

$$\begin{aligned}\Lambda_K : \cap_{m\leq K}\left(L_0^2 \setminus D_m\right) &\to \left(\mathbb{R}_{>0} \times S^1\right)^K \times h^{\frac{1}{2}}\left(\mathbb{N}_{>K};\mathbb{R}^2\right)\\ q &\mapsto (I_n(q), \theta_n(q))_{1\leq n\leq K}, (x_n(q), y_n(q))_{n>K}.\end{aligned}$$

By Proposition 20, Λ_K is a local diffeomorphism. Further $d_q\Lambda_K : L_0^2 \to h^{\frac{1}{2}}(\mathbb{N};\mathbb{R}^2)$ is given by

$$\begin{aligned} d_q\Lambda_K(h) \quad = \quad & \sum_{n=1}^{K}\left(\left\langle \frac{\partial I_n}{\partial q(x)}, h\right\rangle_{L^2} e_n + \left\langle \frac{\partial \theta_n}{\partial q(x)}, h\right\rangle_{L^2} e_{-n}\right) + \\ & \sum_{n=K+1}^{\infty}\left(\left\langle \frac{\partial x_n}{\partial q(x)}, h\right\rangle_{L^2} e_n + \left\langle \frac{\partial y_n}{\partial q(x)}, h\right\rangle_{L^2} e_{-n}\right). \end{aligned}$$

Introduce $v_{\pm n} \equiv v_{\pm n}(q) := \left(d_q\Lambda_K\right)^{-1}(e_{\pm n})$ and let ω_K be the restriction of the symplectic form ω to $Gap^0_{\leq K}$ which we now analyze.

Lemma 33 *Let* $q \in Gap^0_{\leq K}$ *and* $1 \leq n,m \leq K$. *Then*
(i) $v_{\pm n}(q) \in T_q Gap^0_{\leq K}$.
(ii) $v_{-n}(q) = -\frac{d}{dx}\frac{\partial I_n}{\partial q(x)}$.
(iii) $\omega_K(v_{-m}, v_{-n}) = 0$; $\quad \omega_K(v_m, v_{-n}) = -\delta_{n,m}$.

Proof. Notice that the system $(\frac{\partial I_n}{\partial q(x)}, \frac{\partial \theta_n}{\partial q(x)})_{1\leq n\leq K}, (\frac{\partial x_n}{\partial q(x)}, \frac{\partial y_n}{\partial q(x)})_{n>K}$ is biorthogonal to $(v_n, v_{-n})_{n\geq 1}$, i.e. for $1 \leq n \leq K$ and $m \geq 1$,

$$\left\langle \frac{\partial I_n}{\partial q(x)}, v_m\right\rangle_{L^2} = \delta_{n,m}; \quad \left\langle \frac{\partial \theta_n}{\partial q(x)}, v_{-m}\right\rangle_{L^2} = \delta_{n,m}; \tag{8.1}$$

$$\left\langle \frac{\partial I_n}{\partial q(x)}, v_{-m}\right\rangle_{L^2} = 0; \quad \left\langle \frac{\partial \theta_n}{\partial q(x)}, v_m\right\rangle_{L^2} = 0 \tag{8.2}$$

and, for $n > K$, $m \geq 1$,

$$\left\langle \frac{\partial x_n}{\partial q(x)}, v_m \right\rangle_{L^2} = \delta_{n,m}; \quad \left\langle \frac{\partial y_n}{\partial q(x)}, v_{-m} \right\rangle_{L^2} = \delta_{n,m}; \tag{8.3}$$

$$\left\langle \frac{\partial x_n}{\partial q(x)}, v_{-m} \right\rangle_{L^2} = 0; \quad \left\langle \frac{\partial y_n}{\partial q(x)}, v_m \right\rangle_{L^2} = 0. \tag{8.4}$$

(i) As $Gap^0_{\leq K} = \{q \in L^2_0 \mid x_n(q) = y_n(q) = 0 \text{ iff } n > K\}$, it follows from (8.3) and (8.4) that, for $1 \leq m \leq K$, $v_{\pm m} \in T_q Gap^0_{\leq K}$.
(ii) By Lemma 13, $\frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \in T_q Iso(q) \subset T_q Gap^0_{\leq K}$. By Proposition 12(ii), for $1 \leq n, m \leq K$,

$$\left\langle \frac{\partial \theta_m}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = -\delta_{n,m}.$$

By Proposition 12(i) and Corollary 17, for $l > K$, $m \geq 1$, and $1 \leq n \leq K$, we have $\left\langle \frac{\partial I_m}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = 0$ and

$$\left\langle \frac{\partial x_l}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = 0; \quad \left\langle \frac{\partial y_l}{\partial q(x)}, \frac{d}{dx}\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = 0.$$

The conditions (8.1)-(8.4) determine $(v_n, v_{-n})_{n\geq 1}$ uniquely. Thus, for $1 \leq n \leq K$, $v_{-n}(q) = -\frac{d}{dx}\frac{\partial I_n}{\partial q(x)}$.
(iii) As, for $1 \leq l \leq K$, $v_{\pm l}(q) \in T_q Gap^0_{\leq K}$, we obtain, for $1 \leq n, m \leq K$, using (ii) and (8.1)

$$\begin{aligned} \omega_K(v_{-n}, v_{-m}) &= \omega(v_{-n}, v_{-m}) = \left\langle \frac{d}{dx}\frac{\partial I_n}{\partial q(x)}, \frac{\partial I_m}{\partial q(x)} \right\rangle_{L^2} = 0; \\ \omega_K(v_m, v_{-n}) &= \omega(v_m, v_{-n}) = \left\langle v_m, -\frac{\partial I_n}{\partial q(x)} \right\rangle_{L^2} = -\delta_{n,m}. \end{aligned}$$

□

When expressed in the coordinates $(I_n, \theta_n)_{1\leq n\leq K}$ on $Gap^0_{\leq K}$ the 2-form ω_K takes, in view of Lemma 33, the form

$$\omega_K = \sum_{n=1}^{K} d\theta_n \wedge dI_n + \sum_{1\leq i<j\leq K} c_{ij} dI_i \wedge dI_j \tag{8.5}$$

where c_{ij} are functions of $(I_n, \theta_n)_{1\leq n\leq K}$, ($1 \leq i, j \leq K$). As ω is closed, ω_K is closed as well. Therefore the coefficients c_{ij} depend only on $I_1, \ldots, I_K$. We want to show that c_{ij} vanish. To this end we prove that $c_{ij} = 0$ when evaluated at a potential $q \in Gap^0_{\leq K}$ with $\theta_1 = \cdots = \theta_K = 0$. Introduce, for $A \subseteq L^2$, the subset of normalized potentials in A

$$Nor A := \{q \in A \mid \mu_k(q) = \lambda_{2k}(q) \ \forall k \geq 1\}.$$

Notice that on $NorGap^0_{\le K}$, $\theta_1 = \cdots = \theta_K = 0$. In Appendix C, we derive an explicit formula for the gradient $\frac{\partial\theta_n}{\partial q(x)}$ on $NorL^2_0 \setminus D_n$ which turns out to be in H^2 (cf Proposition 41). Hence, on $\left(L^2_0 \setminus D_n\right) \cap \left(L^2_0 \setminus D_m\right) \cap NorL^2_0$, $\{\theta_m, \theta_n\}$ is well defined. Further in Appendix C, Lemma 45, the gradients $\frac{\partial x_l}{\partial q(x)}$ and $\frac{\partial y_l}{\partial q(x)}$ for potentials $q \in L^2_0$ with $\gamma_l(q) = 0$ are given which also turn out to be in H^2. Hence, for $q \in L^2_0$ with $\gamma_n \neq 0$ and $\gamma_l = 0$, $\{\theta_n, x_l\}(q)$ and $\{\theta_n, y_l\}(q)$ are both well defined.

Lemma 34 (i) *For $m, n \geq 1$ and $q \in \left(L^2_0 \setminus D_m\right) \cap \left(L^2_0 \setminus D_n\right) \cap NorL^2_0$,*

$$\{\theta_m, \theta_n\}(q) = 0.$$

(ii) *For $l, n \geq 1$ and $q \in NorL^2_0$ with $\gamma_l(q) = 0$ and $\gamma_n \neq 0$*

$$\{\theta_n, x_l\} = \{\theta_n, y_l\} = 0.$$

Proof. (i) For $k \geq 1$, introduce

$$a_k(x,q) := y_1(x, \mu_k(q), q) y_2(x, \mu_k(q), q); \quad g_k(x,q) := \frac{y_2(x, \mu_k(q), q)}{||y_2(\cdot, \mu_k(q), q)||_{L^2}}.$$

Then (cf [PT]), for $i, j \geq 1$,

$$\left\langle g_i^2, \frac{d}{dx} g_j^2 \right\rangle_{L^2} = 0; \quad \left\langle a_i, \frac{d}{dx} a_j \right\rangle_{L^2} = 0; \quad \left\langle a_j, \frac{d}{dx} g_i^2 \right\rangle_{L^2} = \frac{1}{2}\delta_{i,j}.$$

The claimed statement then follows from Proposition 41.
(ii) For $q \in \left(L^2_0 \setminus D_n\right) \cap \left(L^2_0 \setminus D_l\right) \cap NorL^2_0$, we conclude from (i) and Proposition 12 that the claimed statement holds. In view of Proposition 41, the general case is then obtained by a limiting argument. □

Lemma 35 *Let $q \in NorGap^0_{\le K}$ and $1 \leq n, m \leq K$. Then*
(i) $v_n(q) = \frac{d}{dx}\frac{\partial\theta_n}{\partial q(x)}$.
(ii) $\omega_K(v_n, v_m) = 0$.

Proof. By Lemma 34, for $1 \leq n \leq K$, $l > K$, and $q \in NorGap^0_{\le K}$

$$\{\theta_n, x_l\}(q) = \{\theta_n, y_l\}(q) = 0$$

and, for $1 \leq l \leq K$,

$$\{\theta_n, I_l\}(q) = -\delta_{n,l}; \quad \{\theta_n, \theta_l\}(q) = 0.$$

Thus it follows from (8.1)-(8.4) that, for $1 \leq n \leq K$, $v_n = \frac{d}{dx}\frac{\partial\theta_n}{\partial q(x)}$.
(ii) Follows from (i) and (8.2). □

Proposition 36 *When expressed in the coordinates $(I_n, \theta_n)_{1 \leq n \leq K}$ on $Gap^0_{\leq K}$ the 2-form ω_K is canonical, i.e.*

$$\omega_K = \sum_{n=1}^{K} d\theta_n \wedge dI_n.$$

Proof. By (8.5)

$$\omega_K = \sum_{n=1}^{K} d\theta_n \wedge dI_n + \sum_{1 \leq i < j \leq K} c_{ij} dI_i \wedge dI_j,$$

where the coefficients c_{ij} depend only on $I_1, \ldots, I_K$. By Lemma 35, $c_{ij} = 0$ if $\theta_1 = \cdots = \theta_K = 0$. Thus $c_{ij} \equiv 0$ on $Gap^0_{\leq K}$, for $1 \leq i < j \leq K$. □

Proof of Theorem 5 Introduce, for $q \in L^2_0$ and $n \geq 1$,

$$u_{\pm n} \equiv u_{\pm n}(q) := (d_q \Omega)^{-1} (e_{\pm n}). \tag{8.6}$$

We have to prove that, for any $m, n \geq 1$ and any $q \in L^2_0$,

$$\omega(u_m, u_n) = \omega(u_{-m}, u_{-n}) = 0; \quad \omega(u_m, u_{-n}) = -\delta_{m,n}. \tag{8.7}$$

Fix $m, n \geq 1$. For any $K \geq \max\{m, n\}$ and $q \in Gap^0_{\leq K}$ we have, by Proposition 36,

$$\omega(v_m, v_n) = \omega(v_{-m}, v_{-n}) = 0; \quad \omega(v_m, v_{-n}) = -\delta_{m,n}.$$

For $1 \leq k \leq K$,

$$\begin{aligned} u_k &= \sqrt{2I_k} v_k \cos\theta_k - \frac{1}{\sqrt{2I_k}} v_{-k} \sin\theta_k \\ u_{-k} &= \sqrt{2I_k} v_k \sin\theta_k + \frac{1}{\sqrt{2I_k}} v_{-k} \cos\theta_k. \end{aligned}$$

Therefore, by Proposition 36, we obtain (8.7), for $q \in Gap^0_{\leq K}$. The set $\cup_{K \geq \max\{m,n\}} Gap^0_{\leq K}$ is dense in L^2_0 and, as Ω is analytic, $u_{\pm m}(q), u_{\pm n}(q)$ depend continuously on q. Therefore (8.7) holds for any $q \in L^2_0$. □

9 Canonical relations: part 2

In this section we establish regularity properties of the L_2-gradients of θ_n, x_n, and y_n (cf Proposition 37 below) and apply them to prove the remaining cannonical relations.

Proposition 37 *For $n \geq 1$ and $N \geq 0$, the maps*

$$\begin{aligned}
\nabla\theta_n &: H_0^N \setminus D_n \to H_0^{N+1}; \quad \nabla\theta_n : q \mapsto \frac{\partial\theta_n}{\partial q(x)} \\
\nabla x_n &: H_0^N \to H_0^{N+1}; \quad \nabla x_n : q \mapsto \frac{\partial x_n}{\partial q(x)} \\
\nabla y_n &: H_0^N \to H_0^{N+1}; \quad \nabla y_n : q \mapsto \frac{\partial y_n}{\partial q(x)}
\end{aligned}$$

are real analytic.

Proof. We prove the statement for $N = 0$, as for $N > 0$ the proof is similar. Let $q \in L_0^2$ and $z := \Omega(q)$. As $\Omega^{-1} : h^{\frac{1}{2}}(\mathbb{N}; \mathbb{R}^2) \to L_0^2$ is analytic, $d_z\Omega^{-1}$ depends analytically on z. Thus, for $n \geq 1$, the maps $u_{\pm n}(\cdot) : L_0^2 \to L_0^2$, $q \mapsto u_{\pm n}(q)$ (cf (8.6)) are analytic.

Notice that the system $\left(\frac{\partial x_n}{\partial q(x)}, \frac{\partial y_n}{\partial q(x)}\right)_{n\geq 1}$ is biorthogonal to the basis $(u_n, u_{-n})_{n\geq 1}$. On the other hand, it follows from (8.7) that

$$\left\langle u_m, \left(\frac{d}{dx}\right)^{-1} u_n \right\rangle_{L^2} = \left\langle u_{-m}, \left(\frac{d}{dx}\right)^{-1} u_{-n} \right\rangle_{L^2} = 0; \tag{9.1}$$

$$\left\langle u_m, \left(\frac{d}{dx}\right)^{-1} u_{-n} \right\rangle_{L^2} = -\delta_{m,n}. \tag{9.2}$$

Thus $\left(-\left(\frac{d}{dx}\right)^{-1} u_{-n}, \left(\frac{d}{dx}\right)^{-1} u_n\right)_{n\geq 1}$ is a system, biorthogonal to $(u_n, u_{-n})_{n\geq 1}$. As a basis admits exactly one biorthogonal system, we conclude that, for $n \geq 1$,

$$\frac{\partial x_n}{\partial q(x)} = -\left(\frac{d}{dx}\right)^{-1} u_{-n}; \quad \frac{\partial y_n}{\partial q(x)} = \left(\frac{d}{dx}\right)^{-1} u_n. \tag{9.3}$$

In particular, for $q \in L_0^2$, $\frac{\partial x_n}{\partial q(x)}, \frac{\partial y_n}{\partial q(x)} \in H_0^1$ and $\nabla x_n : q \mapsto \frac{\partial x_n}{\partial q(x)}$ and $\nabla y_n : q \mapsto \frac{\partial y_n}{\partial q(x)}$, viewed as maps from L_0^2 to H_0^1, are analytic. As, for $q \in L_0^2 \setminus D_n$,

$$\begin{aligned}
\frac{\partial x_n}{\partial q(x)} &= \frac{1}{\sqrt{2I_n}} \cos\theta_n \frac{\partial I_n}{\partial q(x)} - \sqrt{2I_n} \sin\theta_n \frac{\partial\theta_n}{\partial q(x)}; \\
\frac{\partial y_n}{\partial q(x)} &= \frac{1}{\sqrt{2I_n}} \sin\theta_n \frac{\partial I_n}{\partial q(x)} + \sqrt{2I_n} \cos\theta_n \frac{\partial\theta_n}{\partial q(x)}
\end{aligned}$$

and the map $\nabla I_n : L_0^2 \to H_0^2$ is analytic, we conclude that $\nabla\theta_n : L_0^2 \setminus D_n \to H_0^1$ is a real analytic map. □

Theorem 6 (i) *For $q \in L_0^2$ and $m, n \geq 1$,*

$$\{x_m, x_n\} = 0; \quad \{y_m, y_n\} = 0; \quad \{x_n, y_m\} = \delta_{n,m}.$$

(ii) *For $m, n \geq 1$ and $q \in \left(L_0^2 \setminus D_m\right) \cap \left(L_0^2 \setminus D_n\right)$,*

$$\{\theta_m, \theta_n\} = 0.$$

Proof. (i) By Proposition 37, any bracket in the statement is well defined. The statement follows from Theorem 5 (cf 8.7) and (9.3).
(ii) For $q \in \left(L_0^2 \setminus D_n\right) \cap \left(L_0^2 \setminus D_m\right)$, $\{\theta_n, \theta_m\}$ is well defined by Proposition 37. By (i) we have

$$0 = \{x_n, x_m\} = \{\sqrt{2I_n}\cos\theta_n, \sqrt{2I_m}\cos\theta_m\}. \tag{9.4}$$

Using that $\{I_n, I_m\} = 0$ and $\{\theta_n, I_m\} = -\delta_{n,m}$ one verifies

$$\{\sqrt{2I_n}\cos\theta_n, \sqrt{2I_m}\cos\theta_m\} = \sin\theta_n \sin\theta_m \sqrt{2I_n}\sqrt{2I_m}\{\theta_n, \theta_m\}. \tag{9.5}$$

Combining (9.4) and (9.5) yields

$$\sin\theta_n \sin\theta_m \{\theta_n, \theta_m\} = 0$$

and thus, for $\theta_n,\ \theta_m \notin \{0, \pi\} \mod 2\pi$,

$$\{\theta_n, \theta_m\} = 0.$$

By continuity, $\{\theta_n, \theta_m\} = 0$ on $\left(L_0^2 \setminus D_n\right) \cap \left(L_0^2 \setminus D_m\right)$. □

A Appendix

In this appendix, we prove Lemma 4 stated in section 2:

Lemma 38 *Let U_{q_0} be a bounded G-neighborhood of $q_0 \in L_0^2$. Then there exists $C > 0$ so that for any $n \geq 1$ the following holds:*

(i) *for all $k \neq n$ and $q \in U_{q_0}$,*

$$|\eta_{n,k}(q)| \leq \frac{Cn}{|k^2 - n^2|}\frac{1}{k}(|\mu_k - \tau_k| + |\gamma_k|);$$

(ii) *for $q \in U_{q_0} \setminus D_n$*

$$|\eta_{n,n}(q) \mod 2\pi| \leq C \log\left(2 + \left|\frac{\mu_n - \tau_n}{\gamma_n}\right|\right);$$

(iii) *for all* $q \in U_{q_0}$,

$$\sum_{k \neq n} |\eta_{n,k}(q)| \leq \frac{C}{n} \left(\left(\sum_{k \geq 1} |\mu_k - \tau_k|^2 \right)^{1/2} + \left(\sum_{k \geq 1} |\gamma_k|^2 \right)^{1/2} \right).$$

Proof. (i) As $n \neq k$, one has by (2.7)

$$\eta_{n,k} = \int_{\lambda_{2k-1}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda = \int_{\lambda_{2k}}^{\mu_k^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda.$$

The following argument is not affected if one interchanges the roles of λ_{2k-1} and λ_{2k}. Therefore we may assume in the following that $|\mu_k - \lambda_{2k-1}| \leq |\mu_k - \lambda_{2k}|$. For λ near $G_k := \{t\lambda_{2k} + (1-t)\lambda_{2k-1} \mid 0 \leq t \leq 1\}$ we have

$$\frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} = \pm \frac{\mu_k^{(n)} - \lambda}{\sqrt{(\lambda_{2k} - \lambda)(\lambda - \lambda_{2k-1})}} \zeta_{n,k}(\lambda)$$

where, with $\mu_n^{(n)} = \tau_n$

$$\zeta_{n,k} := \pm \frac{c_n}{\tau_n - \lambda} \left(\prod_{j \neq k} \frac{\mu_j^{(n)} - \lambda}{j^2 \pi^2} \right) \frac{1}{k\pi} \left(4 \frac{\lambda - \lambda_0}{k^2 \pi^2} \prod_{j \neq k} \frac{(\lambda_{2j} - \lambda)(\lambda_{2j-1} - \lambda)}{(j^2 \pi^2)^2} \right)^{-1/2}.$$

Using that $c_n = O(n)$ (Proposition 2) we then conclude (cf [**PT**], Appendix E), that for λ near G_k, and any n, k with $n \neq k$

$$|\zeta_{n,k}(\lambda)| \leq C \frac{n}{k|n^2 - k^2|} \tag{A.1}$$

uniformly for $q \in U_{q_0}$. Moreover, if we integrate along a straight line l from λ_{2k-1} to μ_k on the sheet of Σ_q determined by μ_k^*, then we have

$$\sqrt{\frac{\mu_k^{(n)} - \lambda}{\lambda_{2k} - \lambda}} = O(1)$$

since $|\mu_k - \lambda_{2k-1}| \leq |\mu_k - \lambda_{2k}|$ and $\mu_k^{(n)} = \tau_k + O\left(\gamma_k^2\right)$. Thus it remains to show that

$$\int_{\lambda_{2k-1}}^{\mu_k^*} \sqrt{\frac{\lambda - \mu_k^{(n)}}{\lambda - \lambda_{2k-1}}} d\lambda = O\left(|\gamma_k| + |\mu_k - \tau_k|\right)$$

when integrating along the straight line l. But this follows with the substitution $\lambda = \lambda_{2k-1} + t(\mu_k - \lambda_{2k-1})$. Setting $\epsilon = |\mu_k^{(n)} - \lambda_{2k-1}|$ and $\delta = |\mu_k - \lambda_{2k-1}|$ we obtain the bound

$$\int_0^1 \frac{\sqrt{\epsilon + \delta}}{\sqrt{\delta}\sqrt{t}} \, \delta \, dt = 2\sqrt{\epsilon + \delta}\sqrt{\delta} \leq \epsilon + 2\delta.$$

As $\epsilon = O(|\gamma_k|)$ and $\delta = O(|\gamma_k| + |\mu_k - \tau_k|)$, the claim follows.
(ii) Arguing as in (i), we may assume, in view of (2.7) that $\mu_n \neq \lambda_{2n-1}, \lambda_{2n}$. In the case where μ_n satisfies $0 < |\mu_n - \lambda_n^+| \leq 2|\gamma_n|$, one obtains as in (i),

$$\left| \int_{\lambda_{2n}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} d\lambda \right| \leq \pi + C \int_0^1 \frac{1}{t^{1/2}|\mu_n - \lambda_n^+|^{1/2}|\gamma_n/2|^{1/2}} |\mu_n - \lambda_n^+| dt, \quad \text{(A.2)}$$

which establishes the claimed estimate in this case.

If $|\mu_n - \lambda_n^+| > 2|\gamma_n|$, the integral is split into two parts,

$$\left| \int_{\lambda_{2n}}^{\mu_n^*} \frac{\psi_n(\lambda)}{\sqrt{\Delta(\lambda)^2 - 4}} \, d\lambda \right| \leq \pi + \left| \int_{\lambda_n^+}^{z} \frac{\psi_n(\lambda) d\lambda}{\sqrt{\Delta(\lambda)^2 - 4}} \right| + \left| \int_z^{\mu_n} \frac{\psi_n(\lambda) d\lambda}{\sqrt{\Delta(\lambda)^2 - 4}} \right| \quad \text{(A.3)}$$

where $z = \tau_n + |\gamma_n| \, \frac{\mu_n - \tau_n}{|\mu_n - \tau_n|}$. The first integral on the right side of (A.3) is estimated as in (A.2). Arguing as in (i), the second integral can be estimated

$$\begin{aligned} \left| \int_z^{\mu_n} \frac{\psi_n(\lambda) d\lambda}{\sqrt{\Delta(\lambda)^2 - 4}} \right| &\leq C \int_2^{2|\frac{\mu_n - \tau_n}{\gamma_n}|} \frac{1}{|\frac{\gamma_n}{2}|(t^2 - 1)^{1/2}} \left| \frac{\gamma_n}{2} \right| dt \qquad \text{(A.4)} \\ &\leq C \operatorname{arccosh} \left(\left| \frac{\mu_n - \tau_n}{\gamma_n/2} \right| \right) \leq C \, \log \left(2 \left| \frac{\mu_n - \tau_n}{\gamma_n/2} \right| \right). \end{aligned}$$

Combining (A.3) and (A.4) leads to the claimed estimate.
(iii) We split the sum $\sum_{k \neq n} |\eta_{n,k}(q)|$ into two parts $\sum_{|k-n| \leq n/2} |\eta_{n,k}(q)|$ and $\sum_{|k-n| > n/2} |\eta_{n,k}(q)|$. The two parts are estimated separately,

$$\begin{aligned} \sum_{|k-n| \leq n/2} |\eta_{n,k}(q)| &\leq C \sum_{|k-n| \leq n/2} \frac{n}{n+k} \frac{1}{k} \frac{1}{|k-n|} (|\mu_k - \tau_k| + |\gamma_k|) \\ &\leq C \frac{2}{n} \sum_{k \neq n} \frac{1}{|k-n|} (|\mu_k - \tau_k| + |\gamma_k|) \\ &\leq C \frac{2}{n} \left(\sum_{k \neq n} \frac{1}{|k-n|^2} \right)^{1/2} \left(\left(\sum_{k \geq 1} |\mu_k - \tau_k|^2 \right)^{1/2} + \left(\sum_{k \geq 1} |\gamma_k|^2 \right)^{1/2} \right) \end{aligned}$$

where for the last inequality we have used the Cauchy-Schwartz inequality.
The sum $\sum_{|k-n| > n/2} |\eta_{n,k}(q)|$ is treated similarly. □

B Appendix

In this appendix, we prove various orthogonality relations.

For $\lambda \in \mathbb{R}$ and $q \in L^2$, introduce

$$F(x,\lambda,q) := \sum_{1\le i,j\le 2} a_{ij}(q) y_i(x,\lambda,q) y_j(x,\lambda,q)$$
$$G(x,\lambda,q) := \sum_{1\le i,j\le 2} b_{ij}(q) y_i(x,\lambda,q) y_j(x,\lambda,q)$$

with $a_{ij}(\cdot), b_{ij}(\cdot) \in C(L^2;\mathbb{R})$. Notice that for $q \in H^1$, F and G are in $H^3_{loc}(\mathbb{R})$, but not necessarily periodic.

Lemma 39 *Assume that $\alpha \neq \beta$, and $q \in H^1$. Then, with $F \equiv F(x,\alpha,q)$ and $G \equiv G(x,\beta,q)$,*

$$\left\langle F, \frac{d}{dx}G \right\rangle_{L^2} = \frac{1}{2(\beta-\alpha)}\left[-\frac{1}{2}\left(F''G - F'G' + FG''\right)\Big|_0^1 + 2\left(F(q-\alpha)G\right)\Big|_0^1\right]. \tag{B.1}$$

Moreover, if the right side of (B.1) is well defined and continuous for $q \in L^2$, (B.1) holds for $q \in L^2$.

Proof. For $a \in \mathbb{R}$, introduce

$$L_{q;a} := -\frac{1}{2}\left(\frac{d}{dx}\right)^3 + q\frac{d}{dx} + \frac{d}{dx}q - 2a\frac{d}{dx}.$$

One verifies that

$$L_{q;\alpha}F(x,\alpha,q) = L_{q;\beta}G(x,\beta,q) = 0. \tag{B.2}$$

As $\frac{d}{dx} = \frac{1}{2(\beta-\alpha)}(L_{q;\alpha} - L_{q;\beta})$, we obtain using (B.2)

$$\left\langle F, \frac{d}{dx}G \right\rangle_{L^2} = \frac{1}{2(\beta-\alpha)}\left\langle F, (L_{q;\alpha} - L_{q;\beta})G\right\rangle_{L^2} = \frac{1}{2(\beta-\alpha)}\left\langle F, L_{q;\alpha}G\right\rangle_{L^2}.$$

Integrating by parts, we obtain

$$\left\langle F, L_{q;\alpha}G\right\rangle_{L^2} = -\frac{1}{2}\left(F''G - F'G' + FG''\right)\Big|_0^1 + 2\left(F(q-\alpha)G\right)\Big|_0^1 - \left\langle L_{q;\alpha}F, G\right\rangle.$$

Using (B.2) once again we obtain (B.1). □

Corollary 40 (i) *Assume that $\alpha \neq \beta$ and, for $q \in H^1$, $F \equiv F(\cdot,\alpha,q)$, $G \equiv G(\cdot,\beta,q) \in H^3$. Then, for $q \in L^2$,*

$$\left\langle F, \frac{d}{dx}G \right\rangle_{L^2} = 0.$$

(ii) *For $\lambda,\ \beta$ arbitrary and $q \in L^2$,*

$$\left\langle \frac{\partial\Delta(\lambda,q)}{\partial q(x)}, \frac{d}{dx}\frac{\partial\Delta(\beta,q)}{\partial q(x)} \right\rangle_{L^2} = 0. \tag{B.3}$$

(iii) *For* $\lambda, a, b \in \mathbb{R}$, $k \geq 1$, *and* $q \in L^2$

$$\left\langle \frac{\partial \Delta(\lambda, q)}{\partial q(x)}, \frac{d}{dx}\left(ay_1(x,\mu_k,q)y_2(x,\mu_k,q) + by_2^2(x,\mu_k,q)\right)\right\rangle_{L^2} = \frac{m_{12}(\lambda)}{2(\lambda-\mu_k)}\left(am_{21}(\mu_k)m_{22}(\mu_k) + b(m_{22}^2(\mu_k)-1)\right). \tag{B.4}$$

Proof. (i) It follows from the assumption $F, G \in H^3$ that $\frac{d^j}{dx^j}F\Big|_0^1 = 0$ and $\frac{d^j}{dx^j}G\Big|_0^1 = 0$ for $0 \leq j \leq 2$. Hence the claimed statement is a direct consequence of Lemma 39.
(ii) For $q \in H^1$ and $\lambda \in \mathbb{R}$, $\frac{\partial\Delta(\lambda,q)}{\partial q(x)} \in H^3$ and (i) can be applied.
(iii) Assume that $q \in H^1$. Let $F := \frac{\partial\Delta(\lambda,q)}{\partial q(x)}$ and $G := ay_1(x,\mu_k,q)y_2(x,\mu_k,q) + by_2^2(x,\mu_k,q)$. Then $F \in H^3$ and $G \in H^3_{loc}(\mathbb{R})$. One verifies that $F(0) = F(1) = m_{12}(\lambda)$, $F'(0) = F'(1) = m_{22}(\lambda) - m_{11}(\lambda)$, $G(0) = G(1) = 0$, $G'(0) = a$, $G'(1) = am_{11}(\mu_k)m_{22}(\mu_k) = a$, $G''(0) = 2b$, $G(1)'' = 2(am_{21}(\mu_k)m_{22}(\mu_k) + bm_{22}^2(\mu_k))$. Therefore (B.4) holds for $q \in H^1$. As the right hand side of (B.4) is defined and continuous on L^2, we conclude from Lemma 39 that the identity (B.4) remains valid for $q \in L^2$. □

C Appendix

The purpose of this appendix is to derive an explicit formula for the gradient of the angle variables $\frac{\partial\theta_n}{\partial q(x)}$ for certain potentials. This formula is similar to the one obtained in [MV] for the nonlinear Schrödinger equation (NLS). In addition, we present formulas for $\frac{\partial x_n}{\partial q(x)}$ and $\frac{\partial y_n}{\partial q(x)}$ for $q \in L_0^2$ with $\lambda_{2n-1} = \lambda_{2n}$.

Recall that $D_n := \{q \mid \gamma_n(q) = 0\}$. For $k, n \geq 1$ and $q \in L_0^2 \setminus D_n$ introduce

$$c_{n,k} \equiv c_{n,k}(q) := -\left.\frac{\psi_n}{\dot\Delta}\right|_{\lambda_{2k},q}; \qquad d_k \equiv d_k(q) := (-1)^{k+1}\left.\frac{\dot m_{11}m_{21}}{\dot\Delta}\right|_{\lambda_{2k},q}.$$

Recall that $\psi_n(\lambda, q)$ is an entire function introduced in section 2 and $m_{ij} = m_{ij}(\lambda, q)$ $(1 \leq i, j \leq 2)$ denote the entries of the Floquet matrix $m_{ij} := \partial_x^{i-1} y_j(1, \lambda, q)$.

Proposition 41 *Let* $K, n \geq 1$ *and* $q \in L_0^2 \setminus D_n$ *with* $\mu_k(q) = \lambda_{2k}(q)$ *for* $k \geq K$. *Then*

$$\frac{\partial\theta_n}{\partial q(x)} = \sum_{k=1}^{K-1}\frac{\partial\eta_{n,k}}{\partial q(x)} + \sum_{k=K}^{\infty} c_{n,k}(q)\Big(y_1(x,\lambda_{2k},q)y_2(x,\lambda_{2k},q) + d_k(q)y_2^2(x,\lambda_{2k},q)\Big) \tag{C.1}$$

where the series converges in H^2.

To prove Proposition 41 we first study the gradient of $\eta_{n,k}$. Notice that

$$\eta_{n,k}(q) = \int_0^{\mu_k(q)-\lambda_{2k}(q)} \frac{\psi_n(y+\lambda_{2k}(q),q)}{\sqrt{yG(y+\lambda_{2k}(q),q)}}dy \tag{C.2}$$

where $G(\lambda,q) := \frac{\Delta^2(\lambda)-4}{\lambda_{2k}-\lambda}$. For $q \in L_0^2$ with $\lambda_{2k-1}(q) < \mu_k(q) < \lambda_{2k}(q)$, we can use (C.2) to write

$$\begin{aligned} \frac{\partial \eta_{n,k}}{\partial q(x)} &= \frac{\psi_n(\mu_k(q),q)}{\sqrt{\Delta^2(\mu_k(q),q)-4}} \left(\frac{\partial \mu_k}{\partial q(x)}(q) - \frac{\partial \lambda_{2k}}{\partial q(x)}(q) \right) \\ &+ \int_0^{\mu_k(q)-\lambda_{2k}(q)} \frac{1}{\sqrt{-y}} \frac{\partial}{\partial q(x)} \left(\frac{\psi_n(y+\lambda_{2k}(q),q)}{\sqrt{-G(y+\lambda_{2k}(q),q)}} \right) dy. \end{aligned} \tag{C.3}$$

Lemma 42 *For $p \in L_0^2$ with $\lambda_{2k-1}(p) < \mu_k(p) = \lambda_{2k}(p)$,*

$$\begin{aligned} \left.\frac{\partial \eta_{n,k}}{\partial q(x)}\right|_{q=p} &= \frac{(-1)^k \psi_n}{\dot\Delta^2} \left(\dot m_{22} \frac{\partial m_{11}}{\partial q(x)} - \dot m_{11} \frac{\partial m_{22}}{\partial q(x)} \right)\Big|_{\lambda_{2k},p} \\ &= c_{n,k} \left(y_1(x) y_2(x) + d_k y_2^2(x) \right)\big|_{\lambda_{2k},p} \end{aligned} \tag{C.4}$$

where $\dot{}$ denotes the derivative with respect to λ.

Proof. Introduce the open sets ($k \geq 1$)

$$V_k := \{ q \in L_0^2 \mid \lambda_{2k-1}(q) < \mu_k(q) < \lambda_{2k}(q) \}.$$

It follows from (C.3) and the analyticity of $\eta_{n,k}$ that

$$\lim_{\substack{q\in V_k \\ q\to p}} \frac{\partial \eta_{n,k}}{\partial q(x)} = \lim_{\substack{q\in V_k \\ q\to p}} \frac{\psi_n(\mu_k(q),q)}{\sqrt{\Delta^2(\mu_k(q),q)-4}} \left(\frac{\partial \mu_k}{\partial q(x)}(q) - \frac{\partial \lambda_{2k}}{\partial q(x)}(q) \right).$$

As $\Delta(\lambda_{2k}(q),q) = (-1)^k 2$ and $m_{12}(\mu_k(q),q) = 0$, we get, by implicit differentiation,

$$\frac{\partial \lambda_{2k}}{\partial q(x)}(q) = -\frac{\frac{\partial \Delta}{\partial q(x)}(\lambda_{2k}(q),q)}{\dot\Delta(\lambda_{2k}(q),q)}; \quad \frac{\partial \mu_k}{\partial q(x)}(q) = -\frac{\frac{\partial m_{12}}{\partial q(x)}(\mu_k,q)}{\dot m_{12}(\mu_k,q)}.$$

Differentiating the Wronskian identity, $m_{11}m_{22} - m_{12}m_{21} = 1$, with respect to λ at $\lambda = \mu_k(q)$, we get, using that $2m_{11} = \Delta + \sqrt{\Delta^2-4}$ and $2m_{22} = \Delta - \sqrt{\Delta^2-4}$ at $\lambda = \mu_k$,

$$2\dot m_{12} m_{21} = 2\dot m_{11} m_{22} + 2 m_{11} \dot m_{22} = \Delta(\dot m_{11} + \dot m_{22}) - \sqrt{\Delta^2-4}(\dot m_{11} - \dot m_{22}).$$

Similarly, differentiating the Wronskian identity with respect to q and evaluating the result at $\lambda = \mu_k(q)$ we get

$$\frac{\frac{\partial m_{12}}{\partial q(x)}}{\dot m_{12}} = \frac{\Delta \frac{\partial (m_{11}+m_{22})}{\partial q(x)} - \sqrt{\Delta^2-4}\frac{\partial(m_{11}-m_{22})}{\partial q(x)}}{\Delta(\dot m_{11}+\dot m_{22}) - \sqrt{\Delta^2-4}(\dot m_{11}-\dot m_{22})}.$$

Thus

$$\frac{\psi_n(\mu_k,q)}{\sqrt{\Delta^2(\mu_k,q)-4}}\left(\frac{\partial\mu_k}{\partial q(x)}(q)-\frac{\partial\lambda_{2k}}{\partial q(x)}(q)\right) \tag{C.5}$$
$$=\frac{\psi_n(\mu_k,q)}{\sqrt{\Delta^2(\mu_k,q)-4}}\left[\left.\frac{\frac{\partial\Delta}{\partial q(x)}}{\dot\Delta}\right|_{\lambda_{2k},q}-\left.\frac{\Delta\frac{\partial\Delta}{\partial q(x)}-\sqrt{\Delta^2-4}\frac{\partial}{\partial q(x)}(m_{11}-m_{22})}{\Delta\dot\Delta-\sqrt{\Delta^2-4}(\dot m_{11}-\dot m_{22})}\right|_{\mu_k,q}\right].$$

Taking the limit $q\to p$, (C.5) yields

$$(-1)^k\ \psi_n\dot\Delta^{-2}\left.\left(\dot m_{22}\frac{\partial m_{11}}{\partial q(x)}-\dot m_{11}\frac{\partial m_{22}}{\partial q(x)}\right)\right|_{\lambda_{2k},p}.$$

To finish the derivation, notice that, as $\mu_k(p)=\lambda_{2k}(p)$, $m_{12}(\lambda_{2k},p)=0$ and $m_{11}(\lambda_{2k},p)=m_{22}(\lambda_{2k},p)=(-1)^k$. Using that (cf [PT])

$$\begin{aligned}\frac{\partial m_{11}}{\partial q(x)} &= m_{12}y_1^2(x)-m_{11}y_1(x)y_2(x)\\ \frac{\partial m_{22}}{\partial q(x)} &= m_{22}y_1(x)y_2(x)-m_{21}y_2^2(x)\end{aligned}$$

we obtain at $(\lambda_{2k}(p),p)$

$$\dot m_{22}\frac{\partial m_{11}}{\partial q(x)}-\dot m_{11}\frac{\partial m_{22}}{\partial q(x)}=(-1)^{k+1}\dot\Delta y_1(x)y_2(x)+\dot m_{11}m_{21}y_2^2(x). \qquad \square$$

Lemma 43 (i) *Let $n\geq 1$ be fixed. $c_{n,k}(q)$ with $k\neq n$ and $d_k(q)$ with $k\geq 1$ can be extended continuously on L_0^2 and satisfy the asymptotics*

$$c_{n,k}=O\left(\frac{1}{k^2}\right);\quad d_k(q)=O\left(1\right).$$

(ii) *For $n\geq 1$, $\gamma_n c_{n,n}$ can be extended continuously on L_0^2 and satisfies the asymptotics*

$$\tilde c_{n,n}:=\gamma_n c_{n,n}=-4n\pi\left(1+O\left(\frac{\log n}{n}\right)\right)\neq 0.$$

Proof. (i) Recall that $\psi_n(\lambda,q)$ and $\dot\Delta(\lambda,q)$ have the following product representations

$$\psi_n(\lambda,q)=\frac{c_n(q)}{n^2\pi^2}\prod_{m\neq n}\frac{\mu_m^{(n)}-\lambda}{m^2\pi^2};\quad \dot\Delta(\lambda,q)=-\prod_{m\geq 1}\frac{\dot\lambda_m-\lambda}{m^2\pi^2}.$$

Thus $c_{n,k}(q) = -\frac{\psi_n(\lambda_{2k})}{\Delta(\lambda_{2k})}$ can be written as a product of three quotients

$$c_{n,k}(q) = \frac{c_n(q)}{\dot\lambda_n - \lambda_{2k}} \frac{f(\lambda_{2k})}{g(\lambda_{2k})} \frac{\mu_k^{(n)} - \lambda_{2k}}{\dot\lambda_k - \lambda_{2k}} \tag{C.6}$$

where $f(\lambda) := \prod_{\substack{m\geq 1\\ m\neq k,n}} \frac{\mu_m^{(n)}-\lambda}{m^2\pi^2}$ and $g(\lambda) := \prod_{\substack{m\geq 1\\ m\neq k,n}} \frac{\dot\lambda_m-\lambda}{m^2\pi^2}$. As, by assumption, $n \neq k$, the first two quotients on the right hand side of (C.6) are continuous on L_0^2. As $\lambda_{2k} = k^2\pi^2 + O(1)$, $\frac{c_n(q)}{\dot\lambda_n - \lambda_{2k}} = O\left(\frac{1}{k^2}\right)$ whereas $\frac{f(\lambda_{2k})}{g(\lambda_{2k})} = \left(1 + O\left(\frac{\log k}{k}\right)\right)$ (cf [PT] Appendix E). To estimate the third quotient, recall that ([BKM1, Theorem2.1] and [BKM2 Lemma 2.4])

$$|\mu_k^{(n)}(p) - \tau_k(p)| = \gamma_k^2(p) O\left(\frac{1}{k}\right); \quad |\dot\lambda_k(p) - \tau_k(p)| = \gamma_k^2(p) O\left(\frac{\log k}{k}\right) \tag{C.7}$$

uniformly in $\{(n,k) \in \mathbb{N}\times\mathbb{N} \mid k \neq n\}$ and p in a sufficiently small neighborhood of q. This leads to

$$\left|\frac{\mu_k^{(n)} - \lambda_{2k}}{\dot\lambda_k - \lambda_{2k}}\right| = \left|\frac{\mu_k^{(n)} - \tau_k - \gamma_k/2}{\dot\lambda_k - \tau_k - \gamma_k/2}\right| = \frac{1/2 + \gamma_k O(1/k)}{1/2 + \gamma_k O(\log k/k)}.$$

Thus the last quotient on the right hand side of (C.6) can be extended continuously on L_0^2 and is $O(1)$. The estimates for d_k are obtained in a similar way.
(ii) Notice that

$$\gamma_n c_{n,n} = \gamma_n \frac{c_n}{\dot\lambda_n - \lambda_{2n}} \frac{\prod_{m\neq n} \frac{\mu_m^{(n)}-\lambda_{2n}}{m^2\pi^2}}{\prod_{m\neq n} \frac{\dot\lambda_m-\lambda_{2n}}{m^2\pi^2}} \neq 0.$$

Similarly as in (i) one obtains

$$\frac{1}{2}\gamma_n c_{n,n} = -c_n \frac{\gamma_n/2}{\gamma_n/2 - (\dot\lambda_n - \tau_n)} \left(1 + O\left(\frac{\log n}{n}\right)\right).$$

Using (C.7) and the estimate $c_n = 2n\pi\left(1 + O\left(\frac{1}{n}\right)\right)$ (cf Proposition 2) one obtains the claimed asymptotic. □

Combining the two Lemmas above, one obtains

Corollary 44 *For $k \neq n$ and $q \in L_0^2$ with $\gamma_k(q) = 0$,*

$$\frac{\partial \eta_{n,k}}{\partial q(x)} = c_{n,k}\left(y_1(x,\lambda_{2k},q)y_2(x,\lambda_{2k},q) + d_k y_2^2(x,\lambda_{2k},q)\right).$$

Proof of Proposition 41 Formula (C.1) follows from Lemma 42 and Corollary 44. It remains to prove that the series in (C.1) converges in H^2.

For $k \geq K$, $y_1(x, \lambda_{2k}, q)\ y_2(x, \lambda_{2k}, q)$ and $y_2^2(x, \lambda_{2k}, q)$ are in H^2. Using that $c_{n,k}$ and $c_{n,k}d_k$ are $O\left(\frac{1}{k^2}\right)$ (Lemma 43) and the following estimates of $y_1 \equiv y_1(x, \lambda_{2k}, q)$ and $y_2 \equiv y_2(x, \lambda_{2k}, q)$ (cf [PT])

$$y_1 = \cos \pi kx + O_\infty\left(\frac{1}{k}\right); \qquad y_2 = \frac{\sin \pi kx}{\pi k} + O_\infty\left(\frac{1}{k^2}\right); \tag{C.8}$$

$$y_1' = -\pi k \sin \pi kx + O_\infty(1); \qquad y_2' = \cos \pi kx + O_\infty\left(\frac{1}{k}\right) \tag{C.9}$$

one obtains, by a straightforward computation, the convergence of the series in H^2. □

To state the next result, recall that $\tilde\theta_n := \sum_{k \neq n} \eta_{n,k}$. For $q \in L_0^2$ with $\lambda_{2n-1}(q) = \lambda_{2n}(q)$ introduce an orthonormal basis $\tilde f_{2n}, \tilde f_{2n-1}$ of *span* $\langle y_1(\cdot, \lambda_{2n}), y_2(\cdot, \lambda_{2n})\rangle$ with $\tilde f_{2n} := \frac{y_2}{||y_2||}$ and $\tilde f_{2n-1}(0) > 0$. Then $\tilde f_{2n-1}$ is of the form ($y_j \equiv y_j(\cdot, \lambda_{2n})$, $j = 1, 2$)

$$\tilde f_{2n-1} = \frac{y_1 + b_n y_2}{||y_1 + b_n y_2||}; \qquad b_n := -\frac{\langle y_1, y_2\rangle_{L^2}}{\langle y_2, y_2\rangle_{L^2}}.$$

Lemma 45 *Let $q \in L_0^2$ with $\lambda_{2n-1}(q) = \lambda_{2n}(q)$. Then*

$$\frac{\partial x_n}{\partial q(x)} = \xi_n \left(\cos \tilde\theta_n \frac{\tilde f_{2n}^2 - \tilde f_{2n-1}^2}{2} - \kappa_n \sin \tilde\theta_n\ \tilde f_{2n} \tilde f_{2n-1}\right) \tag{C.10}$$

$$\frac{\partial y_n}{\partial q(x)} = \xi_n \left(\sin \tilde\theta_n \frac{\tilde f_{2n}^2 - \tilde f_{2n-1}^2}{2} + \kappa_n \cos \tilde\theta_n\ \tilde f_{2n} \tilde f_{2n-1}\right) \tag{C.11}$$

where $\kappa_n \equiv \kappa_n(q)$ satisfies $\kappa_n \neq 0$. If q is a finite gap potential, one has for $n \to \infty$

$$\kappa_n = -1 + O\left(\frac{\log n}{n}\right).$$

Proof. Formulas (C.10) and (C.11) are derived in a similar fashion, so we prove only (C.10). Let $(q_m)_{m\geq1}$ be a sequence in L_0^2, convergent to q, such that $\mu_n(q_m) = \lambda_{2n}(q_m) > \lambda_{2n-1}(q_m)\ \forall m \geq 1$. For $p \in L_0^2 \setminus D_n$, $x_n = \sqrt{2I_n}\cos\theta_n = \frac{1}{2}\xi_n\gamma_n\cos\theta_n$. Therefore,

$$\frac{\partial x_n}{\partial q(x)} = \frac{1}{2}\lim_{m\to\infty}\left[\frac{\partial \xi_n}{\partial q(x)}\gamma_n\cos\theta_n + \xi_n\frac{\partial\gamma_n}{\partial q(x)}\cos\theta_n - \xi_n\gamma_n\sin\theta_n\frac{\partial\theta_n}{\partial q(x)}\right]\Bigg|_{q_m}.$$

By definition, $\eta_{n,n}(p) = 0$ for p with $\lambda_{2n-1}(p) < \mu_n(p) = \lambda_{2n}(p)$. Hence $\theta_n(q_m) = \sum_{k\neq n}\eta_{n,k}(q_m)$. As $\sum_{k\neq n}\eta_{n,k}$ is analytic, the following limit exists,

$$\tilde\theta_n := \lim_{m\to\infty}\theta_n(q_m) = \sum_{k\neq n}\eta_{n,k}(q).$$

As $\xi_n(\cdot)$ is analytic and $\lim_{m\to\infty}\gamma_n(q_m)=0$, we obtain

$$\lim_{m\to\infty}\frac{\partial\xi_n}{\partial q(x)}\gamma_n\cos\theta_n\bigg|_{q_m}=0.$$

Thus

$$\frac{\partial x_n}{\partial q(x)}=\frac{1}{2}\xi_n(q)\left[\cos\tilde\theta_n\lim_{m\to\infty}\frac{\partial\gamma_n}{\partial q(x)}\bigg|_{q_m}-\sin\tilde\theta_n\lim_{m\to\infty}\gamma_n\frac{\partial\theta_n}{\partial q(x)}\bigg|_{q_m}\right]. \tag{C.12}$$

Step 1 : Computation of the first limit on the right side of (C.12). For $p\in L_0^2\setminus D_n$, $\frac{\partial\gamma_n}{\partial q(x)}\Big|_p=f_{2n}^2(p)-f_{2n-1}^2(p)$, where f_{2n-1} and f_{2n} are L^2-normalized eigenfunctions corresponding to λ_{2n-1} and λ_{2n}. As $\lambda_{2n}(q_m)=\mu_n(q_m)$, the eigenfunction $f_{2n}(q_m)$ can be chosen to be $f_{2n}(q_m)=\frac{y_2}{||y_2||}$. Then

$$\lim_{m\to\infty}f_{2n}^2(q_m)=\tilde f_{2n}^2.$$

Notice that, as $\lambda_{2n-1}(q_m)<\lambda_{2n}(q_m)$, the eigenfunction $f_{2n-1}(q_m)$ is orthogonal to the eigenfunction $f_{2n}(q_m)$. Choose

$$f_{2n-1}=a_n(y_1(x,\lambda_{2n-1},q_m)+b_ny_2(x,\lambda_{2n-1},q_m))$$

with $a_n\equiv a_n(q_m)=||y_1+b_ny_2||^{-1}$ and $b_n\equiv b_n(q_m)$ (m sufficiently large). From

$$\langle f_{2n-1}(q_m),f_{2n}(q_m)\rangle_{L^2}=0 \tag{C.13}$$

it follows that

$$\langle y_2,f_{2n}\rangle_{L^2}\,b_n=-\langle y_1,f_{2n}\rangle_{L^2}$$

where $f_{2n}=f_{2n}(x,q_m)$ and $y_j=y_j(x,\lambda_{2n-1}(q_m),q_m)$ $(j=1,2)$. Notice that

$$\langle y_2,f_{2n}\rangle_{L^2}\to||y_2(\cdot,\lambda_{2n}(q),q)||\neq 0\quad(m\to\infty).$$

Hence for m sufficiently large $\langle y_2,f_{2n}\rangle_{L^2}\neq 0$ and

$$b_n=-\frac{\langle y_1,f_{2n}\rangle_{L^2}}{\langle y_2,f_{2n}\rangle_{L^2}}.$$

Define $Q(q_m)=||y_1+b_ny_2||$ (m sufficiently large) and notice that $Q(q_m)\to Q(q)$ with $Q(q)\neq 0$ as $y_1(x,\lambda_{2n}(q),q)$ and $y_2(x,\lambda_{2n}(q),q)$ are linearly independent. Hence $a_n(q_m):=1/Q(q_m)$ is well defined for m large and

$$a_n(q_m)\to a_n(q)>0\quad(m\to\infty).$$

We conclude that $\lim_{m\to\infty}f_{2n-1}(q_m)=\tilde f_{2n-1}(q)$ where

$$\tilde f_{2n-1}(q)=\frac{y_1+b_n\tilde f_{2n}}{||y_1+b_n\tilde f_{2n}||}$$

with

$$b_n(q) := -\frac{\langle y_1, y_2\rangle_{L^2}}{\langle y_2, y_2\rangle_{L^2}}.$$

It follows that

$$||\tilde{f}_{2n-1}|| = 1; \quad \left\langle \tilde{f}_{2n-1}, \tilde{f}_{2n}\right\rangle_{L^2} = 0 \tag{C.14}$$

and $\lim_{m\to\infty} f_{2n-1}^2(q_m) = \tilde{f}_{2n-1}^2$. Thus we have proved that

$$\lim_{m\to\infty} \frac{\partial\gamma_n}{\partial q(x)}(q_m) = \tilde{f}_{2n}^2 - \tilde{f}_{2n-1}^2.$$

Step 2 : Computation of the second limit on the right side of (C.12). We have to compute $\lim_{m\to\infty} \frac{\gamma_n}{2}\frac{\partial\theta_n}{\partial q(x)}\Big|_{q_m}$. As $\sum_{k\neq n}\eta_{n,k}$ is analytic, its gradient $\frac{\partial}{\partial q(x)}\Big|_p \sum_{k\neq n}\eta_{n,k}$ depends continuously on p. Therefore, as $\lim_{m\to\infty}\gamma_n(q_m) = 0$, we obtain

$$\lim_{m\to\infty} \gamma_n \left.\frac{\partial\theta_n}{\partial q(x)}\right|_{q_m} = \lim_{m\to\infty} \gamma_n \left.\left(\frac{\partial\eta_{n,n}}{\partial q(x)} + \frac{\partial\sum_{k\neq n}\eta_{n,k}}{\partial q(x)}\right)\right|_{q_m} = \lim_{m\to\infty} \gamma_n \left.\frac{\partial\eta_{n,n}}{\partial q(x)}\right|_{q_m}.$$

By Lemma 42

$$\begin{aligned}\lim_{m\to\infty} \gamma_n \left.\frac{\partial\eta_{n,n}}{\partial q(x)}\right|_{q_m} &= \left(\lim_{m\to\infty}\gamma_n(q_m)c_{n,n}(q_m)\right) y_1(x,\lambda_{2n},q)y_2(x,\lambda_{2n},q)\\ &\quad + \left(\lim_{m\to\infty}\gamma_n(q_m)c_{n,n}(q_m)d_n(q_m)\right) y_2^2(x,\lambda_{2n},q).\end{aligned}$$

By Lemma 43,

$$\tilde{c}_{n,n} := \lim_{m\to\infty}\gamma_n(q_m)c_{n,n}(q_m) = -4\pi n\left(1 + O\left(\frac{\log n}{n}\right)\right) \neq 0$$

and $\lim_{m\to\infty} d_n(q_m) = d_n(q) = O(1)$. Hence

$$\lim_{m\to\infty} \gamma_n \left.\frac{\partial\theta_n}{\partial q(x)}\right|_{q_m} = \tilde{c}_{n,n}\left(y_1(x,\lambda_{2n},q)y_2(x,\lambda_{2n},q) + d_n(q)y_2^2(x,\lambda_{2n},q)\right).$$

To obtain the claimed statement it remains to interprete the right side of the equation above. As $\theta_n(q+c) = \theta(q)$ for any c, we have $\int_0^1 \gamma_n \left.\frac{\partial\theta_n}{\partial q(x)}\right|_{q_m} dx = 0$ for any m. Therefore $0 = \int_0^1 \left(y_1(x)y_2(x) + d_n y_2^2(x)\right)dx$. Hence $y_1 + d_n y_2$ and y_2 are orthogonal and thus $d_n = b_n$. It follows that

$$\frac{1}{2}\tilde{c}_{n,n}(y_1y_2 + d_n y_2^2) = \kappa_n \tilde{f}_{2n}\tilde{f}_{2n-1}$$

with $\kappa_n := \frac{1}{2}\tilde{c}_{n,n}||y_2||\,||y_1 + b_n y_2|| \neq 0$ and

$$\begin{aligned}\kappa_n &= \frac{1}{2}(-4\pi n)\left(1+O\left(\frac{\log n}{n}\right)\right)\left(\frac{1}{n\pi}\frac{1}{\sqrt{2}}+O\left(\frac{1}{n^2}\right)\right)\left(\frac{1}{\sqrt{2}}+O\left(\frac{1}{n}\right)\right)\\ &= -1+O\left(\frac{\log n}{n}\right).\end{aligned}$$

In view of (C.12), formula (C.10) and the claimed asymptotics for κ_n are thus proved. □

D Appendix

In this appendix, for the convenience of the reader, we review the sampling formula (cf [MT1]) in the form used in this paper. Recall that for $q \in L_0^2$, $j \geq 1$, $\psi_j(\lambda, q) = \frac{c_j}{j^2\pi^2}\prod_{n\neq j}\frac{\mu_n^{(j)}-\lambda}{n^2\pi^2}$ denote the functions introduced in section 2. The following interpolation formula is an incidence of the sampling formula (cf [MT1]).

Proposition 46 *For $q \in L_0^2$, $j \geq 1$,*

$$\sum_{k=1}^{\infty}\frac{\psi_j(\mu_k(q), q)}{\dot{m}_{12}(\mu_k(q), q)}\frac{m_{12}(\lambda, q)}{\lambda - \mu_k(q)} = \psi_j(\lambda, q) \quad (\lambda \in \mathbb{C}) \tag{D.1}$$

where $\dot{}$ *denotes the derivative with respect to λ and $m_{12}(\lambda, q) = y_2(1, \lambda, q)$.*

Proposition 46 follows by a limiting argument from the corresponding one for finite gap potentials. Denote by $Gap^0_{\leq K}$ the set of K-gap potentials $Gap^0_{\leq K} := \{q \in L_0^2 \mid \gamma_k = 0 \text{ iff } k > K\}$ ($1 \leq K < \infty$ arbitrary).

Lemma 47 *For $q \in Gap^0_{\leq K}$, $1 \leq j \leq K$, and $\lambda \in \mathbb{C}$*

$$\sum_{k=1}^{K}\frac{\psi_j(\mu_k(q), q)}{\dot{m}_{12}(\mu_k(q), q)}\frac{m_{12}(\lambda, q)}{\lambda - \mu_k(q)} = \psi_j(\lambda, q) \tag{D.2}$$

Proof. Denote the left and right hand side of (D.2) by $LHS_j(q, \lambda)$ and $RHS_j(q, \lambda)$ respectively. Using the product representation for ψ_j and for m_{12} (cf. [PT]), we conclude that

$$\begin{aligned}\frac{m_{12}(\lambda, q)}{\lambda - \mu_k(q)} &= \frac{-1}{k^2\pi^2}\left(\prod_{\substack{1\leq l\leq K\\ l\neq k}}\frac{\mu_l(q)-\lambda}{l^2\pi^2}\right)G_1(\lambda, q);\\ \psi_j(\lambda, q) &= \frac{c_j(q)}{j^2\pi^2}\left(\prod_{\substack{1\leq l\leq K\\ l\neq j}}\frac{\mu_l^{(j)}(q)-\lambda}{l^2\pi^2}\right)G_{2,j}(\lambda, q);\end{aligned}$$

where

$$G_1(\lambda, q) := \prod_{k>K} \frac{\mu_k(q) - \lambda}{k^2\pi^2}; \quad G_{2,j}(\lambda, q) := \prod_{k>K} \frac{\mu_k^{(j)}(q) - \lambda}{k^2\pi^2}.$$

As $q \in Gap_{\leq K}$, for $k > K$, $\mu_k(q) = \mu_k^{(j)}(q) = \lambda_{2k-1}(q) = \lambda_{2k}(q)$ and $G_1(\lambda, q) = G_{2,j}(\lambda, q) =: G(\lambda, q)$. Thus $LHS_j(\lambda, q) = P_{1,j}(\lambda, q)G(\lambda, q)$ and $RHS_j(\lambda, q) = P_{2,j}(\lambda, q)G(\lambda, q)$ where $P_{1,j}(\lambda, q)$ and $P_{2,j}(\lambda, q)$ are polynomials in λ of degree at most $K - 1$. As $m_{12}(\mu_k(q), q) = 0$ for $k \geq 1$, we obtain, by L'Hopital's rule, that $LHS_j(\mu_k(q), q) = RHS_j(\mu_k(q), q)$. Clearly, $G(\mu_k(q), q) \neq 0$ for $1 \leq k \leq K$, thus $P_{1,j}(\mu_k(q), q) = P_{2,j}(\mu_k(q), q)$ for $1 \leq k \leq N$ which means that P_1 and P_2, both being polynomials of degree at most $K - 1$, coincide. □

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