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Envelopes created by circle families in the plane

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Abstract. In this paper, on envelopes created by circle families in the plane, all four basic problems (existence problem, representation problem, problem on the number of envelopes, problem on relationships of definitions) are solved.

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1. Introduction

Throughout this paper, I is an open interval and all functions, mappings are of class C^∞ unless otherwise stated.

Envelopes of planar regular curve families have fascinated many pioneers since the dawn of differential analysis (for instance, see [3]). In most typical cases, straight line families have been studied. However, even for envelopes created by straight line falimies, there were several unsolved problems until very recently. For instance, as the following Example 1 shows, the well-known method to represent the envelope for a given straight line family creating an envelope is useless in some cases. and the representation problem was open until very recently.

Example 1. Consider the elementary plane curve $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = \left(t, t^3\right)$. The regular curve f gives a parametrization of the non-singular cubic curve

$$C = \left\{(X, Y) \in \mathbb{R}^2 \,|\, Y = X^3\right\}.$$

The affine tangent line L_t to C at a point $\left(t, t^3\right)$ may be defined by

$$\left(X - t, Y - t^3\right) \cdot \left(-3t^2, 1\right) = 0,$$

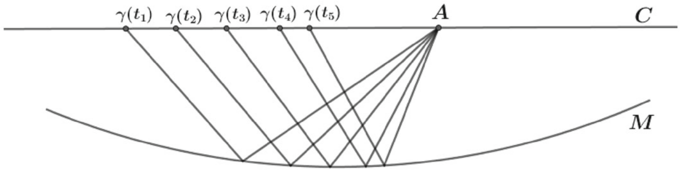

FIGURE 1 Reflection of sound waves

where the dot in the center stands for the standard scalar product of two vectors $\left(X-t, Y-t^3\right)$ and $\left(-3t^2, 1\right)$. Since the straight line family $\{L_t\}_{t\in\mathbb{R}}$ is the affine tangent line family to C, the non-singular cubic curve C must be an envelope of $\{L_t\}_{t\in\mathbb{R}}$. Set

$$F(X,Y,t) = \left(X-t, Y-t^3\right)\cdot\left(-3t^2,1\right) = -3t^2X + Y + 2t^3.$$

We have the following.

$$\begin{aligned}\mathcal{D} &= \left\{(X,Y)\in\mathbb{R}^2 \;\middle|\; \exists t\in\mathbb{R} \text{ s.t. } F(X,Y,t) = \frac{\partial F}{\partial t}(X,Y,t) = 0\right\}\\ &= \left\{(X,Y)\in\mathbb{R}^2 \;\middle|\; \exists t\in\mathbb{R} \text{ s.t. } -3t^2X + Y + 2t^3 = -6t(X-t) = 0\right\}\\ &= \left\{(X,Y)\in\mathbb{R}^2 \;\middle|\; Y=0 \text{ or } Y = X^3\right\}\\ &\supsetneqq \mathcal{C}.\end{aligned}$$

Therefore, unfortunately, the well-known method to represent the envelope does not work well in this case and it must be applied under appropriate assumptions.

In [6], by solving four basic problems on envelopes created by straight line families in the plane (existence problem, representation problem, uniqueness problem and equivalence problem of definitions), the second author constructs a general theory for envelopes created by straight line families in the plane. On the other hand, circle families in the plane are non-negligible families because the envelopes of them have already had an important application, namely, an application to Seismic Survey. Following 7.14(9) of [1], a brief explanation of Seismic Survey is given as follows. In the Euclidean plane $\mathbb{R}^2$, consider the "ground level curve" C parametrized by $\gamma : I \to \mathbb{R}^2$. Suppose that there is a stratum of granite below the top layer of sandstone and that the dividing curve, denoted by M, is parametrized by $\widetilde{f} : I \to \mathbb{R}^2$. Seismic Survey is the following method to obtain an approximation of $\widetilde{f}$ as precisely as possible. Take one fixed point A of C and consider an explosion at A. Assume that the sound waves travel in straight lines and are reflected from M, arriving back at points $\gamma(t)$ of C where their times of arrival are exactly recorded by sensors located along C (see Fig. 1).

It is known that there exists a curve W parametrized by $f : I \to \mathbb{R}^2$ with well-defined normals such that each broken line of a reflected ray starting at A and finishing on C can be replaced by a straight line which is normal to W and of the same total length. The curve W is called the *orthotomic* of M

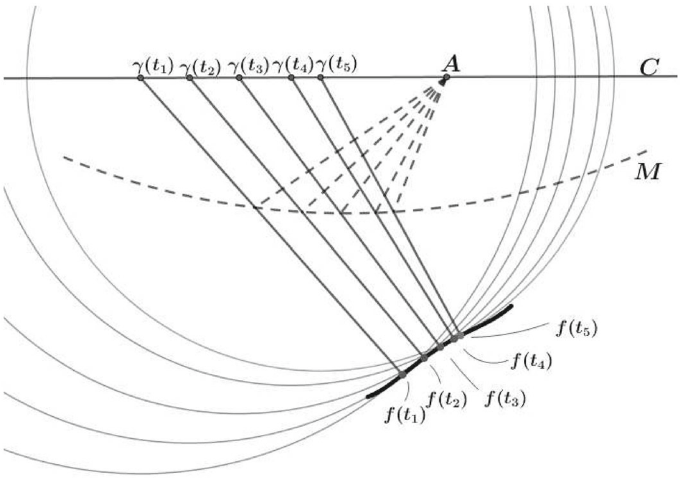

Figure 2 An envelope created by the circle family

relative to A and conversely the curve M is called the *anti-orthotomic* of W relative to A. Then, an envelope created by the circle family

$$\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; ||(x, y) - \gamma(t)|| = ||f(t) - \gamma(t)|| \right\}_{t \in I}$$

recovers W (see Fig. 2).

After obtaining the parametrization f of W, the parametrization $\widetilde{f}$ of M can be easily obtained by using the anti-orthotomic technique developed in [5]. Therefore, in order to investigate the parametrization of W as precisely as possible, it is very important to construct a general theory on envelopes created by circle families in the plane, which is the main purpose of this paper.

For a point P of $\mathbb{R}^2$ and a positive number λ, the circle $C_{(P,\lambda)}$ centered at P with radius λ is naturally defined as follows:

$$C_{(P,\lambda)} = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; ((x, y) - P) \cdot ((x, y) - P) = \lambda^2 \right\},$$

where the dot in the center stands for the standard scalar product. For a curve $\gamma : I \to \mathbb{R}^2$ and a positive function $\lambda : I \to \mathbb{R}_+$, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is naturally defined as follows. Here, $\mathbb{R}_+$ stands for the set consisting of positive real numbers.

$$\mathcal{C}_{(\gamma,\lambda)} = \left\{ C_{(\gamma(t),\lambda(t))} \right\}_{t \in I}.$$

It is reasonable to assume that at each point $\gamma(t)$ the normal vector to the curve γ is well-defined. Thus, we easily reach the following definition.

Definition 1. A curve $\gamma : I \to \mathbb{R}^2$ is called a *frontal* if there exists a mapping $\nu : I \to S^1$ such that the following identity holds for each $t \in I$, where S^1 is the unit circle in $\mathbb{R}^2$.

$$\frac{d\gamma}{dt}(t) \cdot \nu(t) = 0.$$

For a frontal γ, the mapping $\nu : I \to S^1$ given above is called the *Gauss mapping* of γ.

By definition, a frontal is a mapping giving a solution of the first order linear differential equation defined by Gauss mapping ν. Thus, for a fixed mapping $\nu : I \to S^1$ the set consisting of frontals with a given Gauss mapping $\nu : I \to S^1$ is a linear space. For frontals, [4] is recommended as an excellent reference. Hereafter in this paper, the curve $\gamma : I \to \mathbb{R}^2$ for a circle family $\mathcal{C}_{(\gamma,\lambda)}$ is assumed to be a frontal.

In this paper, the following is adopted as the definition of an envelope created by a circle family.

Definition 2. Let $\mathcal{C}_{(\gamma,\lambda)}$ be a circle family. A mapping $f : I \to \mathbb{R}^2$ is called an *envelope* created by $\mathcal{C}_{(\gamma,\lambda)}$ if the following two hold for any $t \in I$.

(1) $\frac{df}{dt}(t) \cdot (f(t) - \gamma(t)) = 0$.
(2) $f(t) \in C_{(\gamma(t),\lambda(t))}$.

By definition, as the same as an envelope created by a hyperplane family (see [6]), an envelope created by a circle family is a mapping giving a solution of a first order differential equation with one constraint condition. Moreover, again by definition, an envelope f created by a circle family $\mathcal{C}_{(\gamma,\lambda)}$ is a frontal with Gauss mapping $I \ni t \to \frac{f(t)-\gamma(t)}{||f(t)-\gamma(t)||} \in S^1$.

Problem 1. *Let $\gamma : I \to \mathbb{R}^2$ be a frontal with Gauss mapping $\nu : I \to S^1$ and let $\lambda : I \to \mathbb{R}_+$ be a positve function.*

(1) Find a necessary and sufficient condition for the circle family $\mathcal{C}_{(\gamma,\lambda)}$ to create an envelope in terms of γ, ν and λ.
(2) Suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope. Then, find a parametrization of the envelope created by $\mathcal{C}_{(\gamma,\lambda)}$ in terms of γ, ν and λ.
(3) Suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope. Then, find a criterion for the number of distinct envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$ in terms of γ, ν and λ.

Note 1. *(1) (1) of Problem 1 is a problem to seek the integrability conditions. There are various cases, for instance the concentric circle family $\{\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = t^2\}\}_{t\in\mathbb{R}_+}$ does not create an envelope while the parallel-translated circle family $\{\{(x,y) \in \mathbb{R}^2 \mid (x-t)^2 + y^2 = 1\}\}_{t\in\mathbb{R}}$ does create two envelopes. Thus, (1) of Problem 1 is significant.*
(2) The following Example 2 shows that the well-known method to represent the envelope is useless in this case. Thus, (2) of Problem 1 is important and the positive answer to it is much desired.
(3) The following Example 3 shows that there are at least three cases: the case having a unique envelope, the case having exactly two envelopes and the case having uncountably many envelopes. Thus, (3) of Problem 1 is meaningful and interesting.

Example 2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by $\gamma(t) = \left(t^3, t^6\right)$. Set $\nu(t) = \frac{1}{\sqrt{4t^6+1}}\left(-2t^3, 1\right)$. It is clear that the mapping γ is a frontal with Gauss mapping $\nu : \mathbb{R} \to S^1$. Let $\lambda : \mathbb{R} \to \mathbb{R}_+$ be the constant function defined by $\lambda(t) = 1$. Then, it seems that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates envelopes. Thus, we can expect that the created envelopes can be obtained by the well-known method. Set $F(x, y, t) = \left(x - t^3\right)^2 + \left(y - t^6\right)^2 - 1$. Then, we have the following.

$$\begin{aligned}
\mathcal{D} &= \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, \exists t \text{ such that } F(x, y, t) = \frac{\partial F}{\partial t}(x, y, t) = 0\right\} \\
&= \Big\{(x, y) \in \mathbb{R}^2 \,\Big|\, \exists t \text{ such that } \left(x - t^3\right)^2 + \left(y - t^6\right)^2 - 1 = 0,\ -6t^2\left(x - t^3\right) \\
&\quad -12t^5\left(y - t^6\right) = 0\Big\} \\
&= \Big\{(x, y) \in \mathbb{R}^2 \,\Big|\, \exists t \text{ such that } \left(x - t^3\right)^2 + \left(y - t^6\right)^2 - 1 = 0,\ t^2\left(\left(x - t^3\right)\right. \\
&\quad \left.+2t^3\left(y - t^6\right)\right) = 0\Big\} \\
&= \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1\right\} \bigcup \Big\{(x, y) \in \mathbb{R}^2 \,\Big|\, \left(x - t^3\right)^2 + \left(y - t^6\right)^2 - 1 = 0, \\
&\quad x = t^3 - 2t^3\left(y - t^6\right)\Big\} \\
&= \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1\right\} \bigcup \Big\{(x, y) \in \mathbb{R}^2 \,\Big|\, \left(-2t^3\left(y - t^6\right)\right)^2 + \left(y - t^6\right)^2 = 1, \\
&\quad x = t^3\left(1 - 2y + 2t^6\right)\Big\} \\
&= \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1\right\} \bigcup \left\{\left(t^3 \mp \frac{2t^3}{\sqrt{4t^6+1}}, t^6 \pm \frac{1}{\sqrt{4t^6+1}}\right) \in \mathbb{R}^2 \,\middle|\, t \in \mathbb{R}\right\}.
\end{aligned}$$

In Example 4 of Sect. 3, it turns out that the set $\mathcal{D}$ calculated here is actually larger than the set of envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$, namely the unit circle $\left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x^2 + y^2 = 1\right\}$ is redundant. Therefore, the well-known method to represent the envelopes does not work well in general and it must be applied under appropriate assumptions even for circle families. The circle family $\mathcal{C}_{(\gamma,\lambda)}$ and the candidates of its envelope are depicted in Fig. 3.

Example 3. (1) Let $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$ be the mapping defined by $\gamma(t) = (0, 1 + t)$. Then, it is clear that γ is a frontal. Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be the positive function defined by $\lambda(t) = 1 + t$. Then, it is easily seen that the origin $(0, 0)$ of the plane $\mathbb{R}^2$ itself is a created envelope by the circle family $\mathcal{C}_{(\gamma,\lambda)}$ and that there are no other envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$. Hence, the number of created envelopes is one in this case.

(2) The parallel-translated circle family $\{\{(x, y) \in \mathbb{R}^2 \mid (x-t)^2 + y^2 = 1\}\}_{t\in\mathbb{R}}$ creates exactly two envelopes.

(3) Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the constant mapping defined by $\gamma(t) = (0, 0)$. Then, it is clear that γ is a frontal. Let $\lambda : \mathbb{R} \to \mathbb{R}_+$ be the constant function defined by $\lambda(t) = 1$. Then, for any function $\theta : \mathbb{R} \to \mathbb{R}$, the mapping $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (\cos\theta(t), \sin\theta(t))$ is an envelope created by the circle family $\mathcal{C}_{(\gamma,\lambda)}$. Hence, there are uncountably many created envelopes in this case.

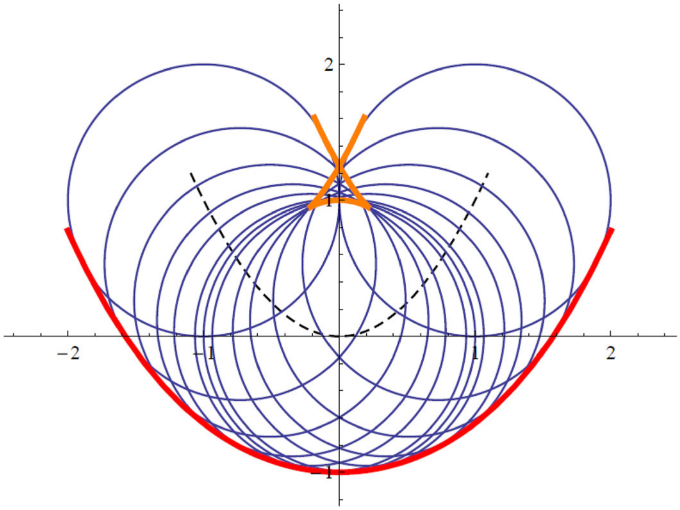

Figure 3 The circle family $\mathcal{C}_{(\gamma,\lambda)}$ and the candidates of its envelope

In order to solve Problem 1, we prepare several terminologies which can be derived from a frontal $\gamma : I \to \mathbb{R}^2$ with Gauss mapping $\nu : I \to S^1$ and a positive function $\lambda : I \to \mathbb{R}_+$. For a frontal $\gamma : I \to \mathbb{R}^2$ with Gauss mapping $\nu : I \to S^1$, following [2], we set $\mu(t) = J(\nu(t))$, where J is the anti-clockwise rotation by $\pi/2$. Then we have a moving frame $\{\mu(t), \nu(t)\}_{t\in I}$ along the frontal γ. Set

$$\ell(t) = \frac{d\nu}{dt}(t) \cdot \mu(t), \quad \beta(t) = \frac{d\gamma}{dt}(t) \cdot \mu(t).$$

The pair of functions (ℓ, β) is called the *curvature* of the frontal γ with Gauss mapping ν. We want to focus the ratio of $\frac{d\lambda}{dt}(t)$ and $\beta(t)$. The following definition is the key of this paper.

Definition 3. Let $\gamma : I \to \mathbb{R}^2$, $\lambda : I \to \mathbb{R}_+$ be a frontal with Gauss mapping $\nu : I \to S^1$ and a positive function respectively. Then, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is said to be *creative* if there exists a mapping $\widetilde{\nu} : I \to S^1$ such that the following identity holds for any $t \in I$.

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right).$$

Set $\cos\theta(t) = -\widetilde{\nu}(t) \cdot \mu(t)$. Then, the creative condition is equivalent to say that there exists a function $\theta : I \to \mathbb{R}$ satisfying the following identity for any $t \in I$.

$$\frac{d\lambda}{dt}(t) = \beta(t)\cos\theta(t).$$

By definition, any family of concentric circles with smoothly expanding radii is not creative, and it is clear that such the circle family does not create an envelope. Under the above preparation, Problem 1 is solved as follows.

Theorem 1. *Let $\gamma : I \to \mathbb{R}^2$ be a frontal with Gauss mapping $\nu : I \to S^1$ and let $\lambda : I \to \mathbb{R}_+$ be a positive function. Then, the following three hold.*

(1) The circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope if and only if $\mathcal{C}_{(\gamma,\lambda)}$ is creative.
(2) Suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope $f : I \to \mathbb{R}^2$. Then, the created envelope f is represented as follows.

$$f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t),$$

where $\widetilde{\nu} : I \to S^1$ is the mapping defined in Definition 3.
(3) Suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope. Then, the number of envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$ is characterized as follows.

(3-i) The circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates a unique envelope if and only if the set consisting of $t \in I$ satisfying $\beta(t) \neq 0$ and $\frac{d\lambda}{dt}(t) = \pm\beta(t)$ is dense in I.

(3-ii) There are exactly two distinct envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$ if and only if the set of $t \in I$ satisfying $\beta(t) \neq 0$ is dense in I and there exists at least one $t_0 \in I$ such that the strict inequality $|\frac{d\lambda}{dt}(t_0)| < |\beta(t_0)|$ holds.

(3-∞) There are uncountably many distinct envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$ if and only if the set of $t \in I$ satisfying $\beta(t) \neq 0$ is not dense in I.

By the assertion (2) of Theorem 1, it is reasonable to call $\widetilde{\nu}$ the *creator* for an envelope f created by $\mathcal{C}_{(\gamma,\lambda)}$.

This paper is organized as follows. Theorem 1 is proved in Sect. 2. In Sect. 3, several examples to which Theorem 1 is effectively applicable are given. Finally, in Sect. 4, relations of several definitions of an envelope created by a circle family are investigated.

2. Proof of Theorem 1

2.1. Proof of the assertion (1) of Theorem 1

Suppose that $\mathcal{C}_{(\gamma,\lambda)}$ is creative. By definition, there exists a mapping $\widetilde{\nu} : I \to S^1$ such that the equality $\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$ holds for any $t \in I$. Set

$$f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t).$$

Then, since $(f(t) - \gamma(t)) \cdot (f(t) - \gamma(t)) = \lambda^2(t)$, it follows $f(t) \in C_{(\gamma(t),\lambda(t))}$. Moreover, since

$$\frac{df}{dt}(t) = \frac{d\gamma}{dt}(t) + \frac{d\lambda}{dt}(t)\widetilde{\nu}(t) + \lambda(t)\frac{d\widetilde{\nu}}{dt}(t),$$

we have the following.

$$\begin{aligned}
&\frac{df}{dt}(t)\cdot(f(t)-\gamma(t))\\
&=\left(\frac{d\gamma}{dt}(t)+\frac{d\lambda}{dt}(t)\widetilde{\nu}(t)+\lambda(t)\frac{d\widetilde{\nu}}{dt}(t)\right)\cdot(\lambda(t)\widetilde{\nu}(t))\\
&=\frac{d\gamma}{dt}(t)\cdot(\lambda(t)\widetilde{\nu}(t))+\frac{d\lambda}{dt}(t)\lambda(t)\\
&=(\beta(t)\mu(t))\cdot(\lambda(t)\widetilde{\nu}(t))+(-\beta(t)\left(\widetilde{\nu}(t)\cdot\mu(t)\right))\lambda(t)\\
&=\beta(t)\lambda(t)\left(\mu(t)\cdot\widetilde{\nu}(t)\right)-\beta(t)\lambda(t)\left(\widetilde{\nu}(t)\cdot\mu(t)\right)\\
&=0.
\end{aligned}$$

Hence, f is an envelope created by the circle family $\mathcal{C}_{(\gamma,\lambda)}$.

Conversely, suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope $f: I\to\mathbb{R}$. Then, by definition, it follows that $f(t)\in C_{(\gamma(t),\lambda(t))}$ and $\frac{df}{dt}(t)\cdot(f(t)-\gamma(t))=0$. The condition $f(t)\in C_{(\gamma(t),\lambda(t))}$ implies that there exists a mapping $\widetilde{\nu}: I\to S^1$ such that the following equality holds for any $t\in I$.

$$f(t)=\gamma(t)+\lambda(t)\widetilde{\nu}(t).$$

Then, since

$$\frac{df}{dt}(t)=\frac{d\gamma}{dt}(t)+\frac{d\lambda}{dt}(t)\widetilde{\nu}(t)+\lambda(t)\frac{d\widetilde{\nu}}{dt}(t),$$

we have the following.

$$\begin{aligned}
0&=\frac{df}{dt}(t)\cdot(f(t)-\gamma(t))\\
&=\left(\frac{d\gamma}{dt}(t)+\frac{d\lambda}{dt}(t)\widetilde{\nu}(t)+\lambda(t)\frac{d\widetilde{\nu}}{dt}(t)\right)\cdot(\lambda(t)\widetilde{\nu}(t))\\
&=(\beta(t)\mu(t))\cdot(\lambda(t)\widetilde{\nu}(t))+\frac{d\lambda}{dt}(t)\lambda(t)\\
&=\lambda(t)\left(\beta(t)\left(\mu(t)\cdot\widetilde{\nu}(t)\right)+\frac{d\lambda}{dt}(t)\right).
\end{aligned}$$

Since $\lambda(t)$ is positive for any $t\in I$, it follows

$$\beta(t)\left(\mu(t)\cdot\widetilde{\nu}(t)\right)+\frac{d\lambda}{dt}(t)=0.$$

Therefore, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is creative. □

2.2. Proof of the assertion (2) of Theorem 1

The proof of the assertion (1) given in Sect. 2.1 proves the assertion (2) as well. □

2.3. Proof of the assertion (3) of Theorem 1

2.3.1. Proof of (3-i). Suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates a unique envelope. Then, by the assertion (2) of Theorem 1, the set $\{f : I \to \mathbb{R}^2 \mid f(t) = \gamma(t)+\lambda(t)\widetilde{\nu}(t) \quad (t \in I)\}$ consists of only one mapping. Therefore, for any $t \in I$ the unit vector $\widetilde{\nu}(t)$ satisfying

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$$

must be uniquely determined. Hence, under considering continuity of two functions $\frac{d\lambda}{dt}$ and β, it follows that the set consisting of $t \in I$ satisfying $\frac{d\lambda}{dt}(t) = \pm\beta(t) \neq 0$ must be dense in I.

Conversely, suppose that the set consisting of $t \in I$ satisfying $\frac{d\lambda}{dt}(t) = \pm\beta(t) \neq 0$ is dense in I. Then, under considering continuity of the function $t \mapsto \widetilde{\nu}(t) \cdot \mu(t)$, it follows that for any $t \in I$, $\widetilde{\nu}(t) \cdot \mu(t) = 1$ (resp., $\widetilde{\nu}(t) \cdot \mu(t) = -1$) if $\frac{d\lambda}{dt}(t) = -\beta(t)$ (resp., $\frac{d\lambda}{dt}(t) = \beta(t)$). Thus, the created envelope $f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t)$ must be unique. □

2.3.2. Proof of (3-ii). Suppose that there are exactly two distinct envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$. Then, by the equality $\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$, the set consisting of $t \in I$ satisfying $\beta(t) \neq 0$ must be dense in I. Suppose moreover that the equality $\frac{d\lambda}{dt}(t) = \pm\beta(t)$ holds for any $t \in I$. Then, it follows that the set consisting of $t \in I$ satisfying $\frac{d\lambda}{dt}(t) = \pm\beta(t) \neq 0$ is dense in I. Then, by the assertion (3-i), the given circle family must create a unique envelope. This contradicts the assumption that there are exactly two distinct envelopes. Hence, there must exist at least one $t_0 \in I$ such that the strict inequality $|\frac{d\lambda}{dt}(t_0)| < |\beta(t_0)|$ holds.

Conversely, suppose that the set of $t \in I$ satisfying $\beta(t) \neq 0$ is dense in I and there exists at least one $t_0 \in I$ such that the strict inequality $|\frac{d\lambda}{dt}(t_0)| < |\beta(t_0)|$ holds. Then, it follows that there must exist an open interval $\widetilde{I}$ in I such that the absolute value $|\widetilde{\nu}(t) \cdot \mu(t)| = |\cos\theta(t)|$ is less than 1 for any $t \in \widetilde{I}$. Thus, it follows $\theta(t) \neq -\theta(t)$ for any $t \in \widetilde{I}$. Hence, for any $t \in \widetilde{I}$, there exist exactly two distinct unit vectors $\widetilde{\nu}_+(t), \widetilde{\nu}_-(t)$ corresponding $\widetilde{\nu}_+(t) \cdot \mu(t) = -\cos\theta(t)$ and $\widetilde{\nu}_-(t) \cdot \mu(t) = -\cos\left(-\theta(t)\right)$ respectively. Therefore, by the assertion (2) of Theorem 1, the circle family must create exactly two distinct envelopes. □

2.3.3. Proof of (3-∞). Suppose that there are uncountably many distinct envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$. Suppose moreover that the set of $t \in I$ such that $\beta(t) \neq 0$ is dense in I. Then, from (3-i) and (3-ii), it follows that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ must create a unique envelope or two distinct envelopes. This contradicts the assumption that there are uncountably many distinct envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$. Hence, the set of $t \in I$ such that $\beta(t) \neq 0$ is never dense in I.

Conversely, suppose that the set of $t \in I$ such that $\beta(t) \neq 0$ is not dense in I. This assumption implies that there exists an open interval $\widetilde{I}$ in I such that

$\beta(t) = 0$ for any $t \in \widetilde{I}$. On the other hand, since $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope f_0, the equality

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$$

holds for any $t \in I$. Thus, there are no restrictions for the value $\widetilde{\nu}(t) \cdot \mu(t)$ for any $t \in \widetilde{I}$. Take one point t_0 of $\widetilde{I}$ and denote the $\widetilde{\nu}$ for the envelope f_0 by $\widetilde{\nu}_0$. Then, by using the standard technique on bump functions, we may construct uncountably many distinct creators $\widetilde{\nu}_a : I \to S^1$ $(a \in A)$ such that the following (a), (b), (c) and (d) hold, where A is a set consisting uncountably many elements such that $0 \notin A$.

(a) The equality $\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}_a(t) \cdot \mu(t)\right)$ holds for any $t \in I$ and any $a \in A$.
(b) For any $t \in I - \widetilde{I}$ and any $a \in A$, the equality $\widetilde{\nu}_a(t) = \widetilde{\nu}_0(t)$ holds.
(c) For any $a \in A$, the property $\widetilde{\nu}_a(t_0) \neq \widetilde{\nu}_0(t_0)$ holds.
(d) For any two distinct $a_1, a_2 \in A$, the property $\widetilde{\nu}_{a_1}(t_0) \neq \widetilde{\nu}_{a_2}(t_0)$ holds.

Therefore, by the assertion (2) of Theorem 1, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates uncountably many distinct envelopes. □

3. Examples

Example 4. We examine Example 2 by applying Theorem 1. In Example 2, $\gamma : \mathbb{R} \to \mathbb{R}^2$ is given by $\gamma(t) = \left(t^3, t^6\right)$. Thus, we can say that $\nu : \mathbb{R} \to S^1$ and $\mu : \mathbb{R} \to S^1$ are given by $\nu(t) = \frac{1}{\sqrt{4t^6+1}}\left(-2t^3, 1\right)$ and $\mu(t) = \frac{1}{\sqrt{4t^6+1}}\left(-1, -2t^3\right)$ respectively. Moreover, the radius function $\lambda : \mathbb{R} \to \mathbb{R}$ is the constant function defined by $\lambda(t) = 1$. Thus,

$$\frac{d\lambda}{dt}(t) = 0.$$

By calculation, we have

$$\beta(t) = \frac{d\gamma}{dt}(t) \cdot \mu(t) = \frac{-3t^2(1+4t^6)}{\sqrt{4t^6+1}}.$$

Therefore, the unit vector $\widetilde{\nu}(t) \in S^1$ satisfying

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$$

exists and it must have the form

$$\widetilde{\nu}(t) = \pm\nu(t) = \frac{\pm 1}{\sqrt{4t^6+1}}\left(-2t^3, 1\right).$$

Hence, by the assertion (1) of Theorem 1, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope $f : \mathbb{R} \to \mathbb{R}^2$. By the assertion (2) of Theorem 1, f is parametrized as follows.

$$\begin{aligned} f(t) &= \gamma(t) + \lambda(t)\widetilde{\nu}(t) \\ &= \left(t^3, t^6\right) \pm \frac{1}{\sqrt{4t^6+1}}\left(-2t^3, 1\right) \\ &= \left(t^3 \mp \frac{2t^3}{\sqrt{4t^6+1}}, t^6 \pm \frac{1}{\sqrt{4t^6+1}}\right). \end{aligned}$$

Finally, by the assertion (3-ii) of Theorem 1, the number of distinct envelopes created by the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is exactly two.

Therefore, Theorem 1 reveals that the set $\mathcal{D}$ calculated in Example 2 is certainly the union of the unit circle and the set of two envelopes of $\mathcal{C}_{(\gamma,\lambda)}$.

Example 5. We examine (1) of Example 3 by applying Theorem 1. In (1) of Example 3, $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$ is given by $\gamma(t) = (0, 1+t)$. Thus, if we define the unit vector $\nu(t) = (1, 0)$, $\nu : \mathbb{R}_+ \to S^1$ gives the Gauss mapping of γ. By definition, $\mu(t) = (0, 1)$ and thus we have $\beta(t) = \frac{d\gamma}{dt}(t) \cdot \mu(t) = 1$. On the other hand, the radius function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ has the form $\lambda(t) = 1 + t$ in this example. Thus, the creative condition

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$$

simply becomes

$$1 = -\left(\widetilde{\nu}(t) \cdot (0, 1)\right) \tag{*}$$

in this case. If we set $\widetilde{\nu}(t) = (0, -1)$, then the above equality holds for any $t \in \mathbb{R}_+$. Thus, by the assertion (1) of Theorem 1, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope. By the assertion (2) of Theorem 1, the parametrization of the created envelope is

$$f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t) = (0, 1+t) + (1+t)(0, -1) = (0, 0).$$

Finally, notice that for any $t \in \mathbb{R}_+$ the creative condition (*) in this case holds if and only if $\widetilde{\nu}(t) = (0, -1) = -\mu(t)$. Thus, by the assertion (3-i) of Theorem 1, the origin $(0, 0)$ is the unique envelope created by $\mathcal{C}_{(\gamma,\lambda)}$.

Example 6. Theorem 1 can be applied also to (2) of Example 3 as follows. In this example, $\gamma(t) = (t, 0)$ and $\lambda(t) = 1$. Thus, we may set $\nu(t) = (0, -1)$, $\mu(t) = (1, 0)$. It follows $\beta(t) = \frac{d\gamma}{dt}(t) \cdot \mu(t) = 1$. Since the radius function λ is a constant function, the creative condition

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$$

simply becomes

$$0 = -\left(\widetilde{\nu}(t) \cdot (0, 1)\right)$$

in this case. Thus, for any $t \in \mathbb{R}$, the creative condition is satisfied if and only if $\widetilde{\nu}(t) = \pm(1, 0)$. Hence, by the assertion (1) of Theorem 1, the circle family $\mathcal{C}_{(\gamma,\lambda)}$

creates an envelope. By the assertion (2) of Theorem 1, the parametrization of the created envelope is

$$f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t) = (t, 0) \pm (0, -1) = (t, \mp 1)\,.$$

Finally, by the assertion (3-ii) of Theorem 1, the number of envelope created by $\mathcal{C}_{(\gamma,\lambda)}$ is exactly two.

Example 7. Theorem 1 can be applied even to (3) of Example 3 as follows. In this example, $\gamma(t) = (0, 0)$ and $\lambda(t) = 1$. Thus, every mapping $\nu : \mathbb{R} \to S^1$ can be taken as Gauss mapping of γ. In particular, γ is a frontal. We have $\beta(t) = \frac{d\gamma}{dt}(t) \cdot \mu(t) = 0$. Since the radius function λ is a constant function $\lambda(t) = 1$, the creative condition

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$$

simply becomes

$$0 = 0$$

in this case. Thus, for any $\widetilde{\nu} : \mathbb{R} \to S^1$, the creative condition is satisfied. Hence, by the assertion (1) of Theorem 1, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope. By the assertion (2) of Theorem 1, the parametrization of the created envelope is

$$f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t) = (0, 0) + \widetilde{\nu}(t) = \widetilde{\nu}(t).$$

Finally, by the assertion (3-∞) of Theorem 1, there are uncountably many distinct envelope created by $\mathcal{C}_{(\gamma,\lambda)}$.

Example 8. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$ be the mapping defined by $\gamma(t) = (t, 0)$ and let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be the positive function defined by $\lambda(t) = t^2$. The circle family $\mathcal{C}_{(\gamma,\lambda)}$ and the candidate of its envelope is depicted in Fig. 4.

Defining the mapping $\nu : \mathbb{R}_+ \to S^1$ by $\nu(t) = (0, -1)$ clarifies that the mapping γ is a frontal. Then, $\mu(t) = J(\nu(t)) = (1, 0)$ and $\beta(t) = \frac{d\gamma}{dt}(t) \cdot \mu(t) = (1, 0) \cdot (1, 0) = 1$. We want to seek a mapping $\widetilde{\nu} : \mathbb{R}_+ \to S^1$ satisfying

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right),$$

namely, a mapping $\widetilde{\nu} : \mathbb{R}_+ \to S^1$ satisfying

$$2t = -((\widetilde{\nu}(t) \cdot (1, 0))).$$

Since $\widetilde{\nu}(t) \in S^1$, from the above expression, it follows that such $\widetilde{\nu}(t)$ does not exist if $\frac{1}{2} < t$. Thus, the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is not creative and it creates no envelopes by the assertion (1) of Theorem 1.

Example 9. This example is almost the same as Example 8. That is, $\gamma(t) = (t, 0)$ and $\lambda(t) = t^2$. The difference from Example 8 is only the parameter space. In Example 9, the parameter space I is $\left(0, \frac{1}{2}\right)$. That is to say, in this example,

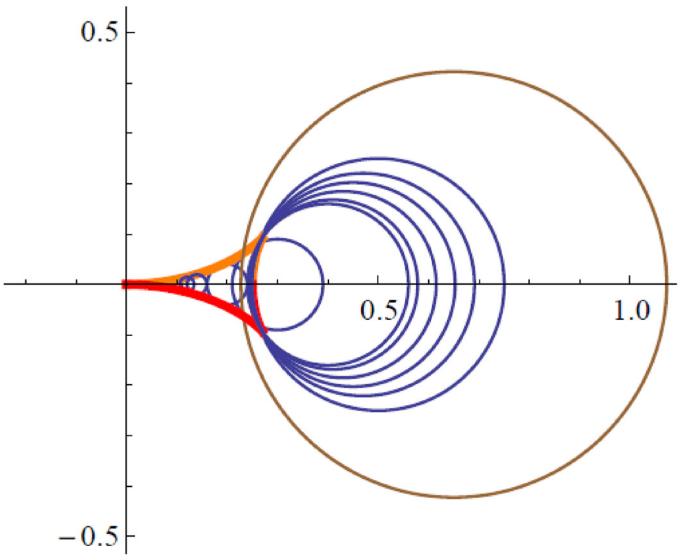

FIGURE 4 The circle family $\mathcal{C}_{(\gamma,\lambda)}$ and the candidate of its envelope

$\mathbb{R}_+$ in Example 8 is replaced by $\left(0, \frac{1}{2}\right)$ and all other settings in Example 8 remain without change.

Then, from calculations in Example 8, it follows that the given circle family $\mathcal{C}_{(\gamma,\lambda)}$ is creative. Thus, by the assertion (1) of Theorem 1, $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope. It is easily seen that the expression of $\widetilde{\nu}(t)$ must be as follows.

$$\widetilde{\nu}(t) = \left(-2t, \pm\sqrt{1-4t^2}\right).$$

Therefore, by the assertion (2) of Theorem 1, an envelope f created by $\mathcal{C}_{(\gamma,\lambda)}$ is parametrized as follows.

$$\begin{aligned} f(t) &= \gamma(t) + \lambda(t)\widetilde{\nu}(t) \\ &= (t, 0) + t^2\left(-2t,\ \pm\sqrt{1-4t^2}\right) \\ &= \left(t - 2t^3,\ \pm t^2\sqrt{1-4t^2}\right). \end{aligned}$$

Finally, by the assertion (3-ii) of Theorem 1, it follows that the number of distinct envelopes created by the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is exactly two.

Example 10. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by $\gamma(t) = (t^3, t^2)$ and let $\lambda : \mathbb{R} \to \mathbb{R}_+$ be the constant function defined by $\lambda(t) = 1$. The circle family $\mathcal{C}_{(\gamma,\lambda)}$ and the candidates of its envelope is depicted in Fig. 5.

It is easily seen that the mapping $\nu : \mathbb{R} \to S^1$ defined by $\nu(t) = \frac{1}{\sqrt{4+9t^2}}(2, -3t)$ gives the Gauss mapping for γ. Thus, γ is a frontal. By definition, the mapping

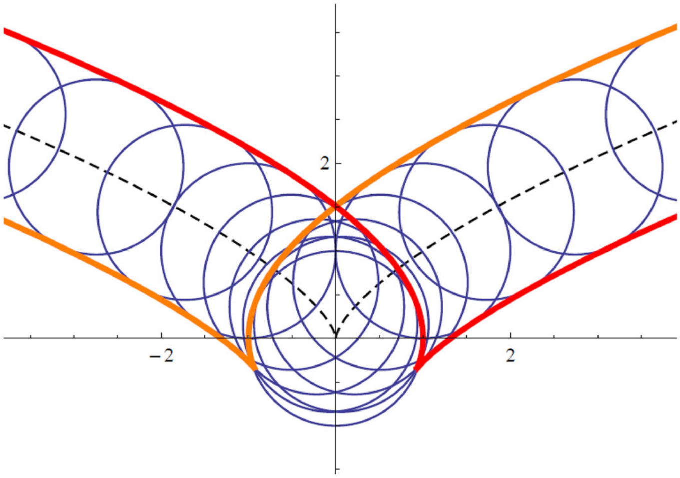

FIGURE 5 The circle family $\mathcal{C}_{(\gamma,\lambda)}$ and the candidate of its envelope

$\mu : \mathbb{R} \to S^1$ has the form $\mu(t) = \frac{1}{\sqrt{4+9t^2}}(3t, 2)$. By calculation, we have

$$\beta(t) = \frac{d\gamma}{dt}(t) \cdot \mu(t) = t\sqrt{4+9t^2}.$$

Since the radius function λ is constant, it follows $\frac{d\lambda}{dt}(t) = 0$. Thus, for any $t \in \mathbb{R}$, the unit vector $\widetilde{\nu}(t)$ satisfying

$$\frac{d\lambda}{dt}(t) = -\beta(t)\left(\widetilde{\nu}(t) \cdot \mu(t)\right)$$

always exists. Namely we have

$$\widetilde{\nu}(t) = \pm\nu(t) = \frac{\pm 1}{\sqrt{4+9t^2}}(2, -3t).$$

Thus, by the assertion (1) of Theorem 1, $\mathcal{C}_{(\gamma,\lambda)}$ creates an envelope, and the created envelope $f : \mathbb{R} \to \mathbb{R}^2$ has the following form by the assertion (2) of Theorem 1.

$$\begin{aligned} f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t) &= \left(t^3, t^2\right) \pm \frac{1}{\sqrt{4+9t^2}}(2, -3t) \\ &= \left(t^3 \pm \frac{2}{\sqrt{4+9t^2}}, t^2 \mp \frac{3t}{\sqrt{4+9t^2}}\right). \end{aligned}$$

Finally, by the assertion (3-ii) of Theorem 1, there are no other envelopes created by $\mathcal{C}_{(\gamma,\lambda)}$.

4. Alternative definitions

In Definition 2 of Sect. 1, the definition of envelope created by the circle family is given. In [1], the set consisting of the images of envelopes defined in Definition

2 is called E_2 *envelope* (denoted by E_2) and two alternative definitions (called E_1 *envelope* and $\mathcal{D}$ *envelope*) are given as follows.

Definition 4. (E_1 *envelope* [1]). Let $\gamma : I \to \mathbb{R}^2$, $\lambda : I \to \mathbb{R}_+$ be a frontal and a positive function respectively. Let t_0 be a parameter of I and fix it. Assume that

$$\lim_{\varepsilon \to 0} C_{(\gamma(t_0),\lambda(t_0))} \cap C_{(\gamma(t_0+\varepsilon),\lambda(t_0+\varepsilon))}$$

is not the empty set and denote the set by $I(t_0)$. Take one point $e_1(t_0) = (x(t_0), y(t_0))$ of $I(t_0)$. Then, the set consisting of the images of smooth mappings $e_1 : I \to \mathbb{R}^2$, if exists, is called an E_1 *envelope* created by the circle family $\mathcal{C}_{(\gamma,\lambda)}$ and is denoted by E_1.

Definition 5. ($\mathcal{D}$ *envelope* [1]). Let $\gamma : I \to \mathbb{R}^2$, $\lambda : I \to \mathbb{R}_+$ be a frontal and a positive function respectively. Set

$$F(x, y, t) = ||(x, y) - \gamma(t)||^2 - (\lambda(t))^2 .$$

Then, the following set is called the $\mathcal{D}$ *envelope* created by the circle family $\mathcal{C}_{(\gamma,\lambda)}$ and is denoted by $\mathcal{D}$.

$$\left\{(x, y) \in \mathbb{R}^2 \mid \exists t \in I \text{ such that } F(x, y, t) = \frac{\partial F}{\partial t}(x, y, t) = 0\right\}.$$

Concerning the relationships among E_1, E_2 and $\mathcal{D}$ for a given circle family $\mathcal{C}_{(\gamma,\lambda)}$, the following is known.

Fact 1. ([1]). *$E_1 \subset \mathcal{D}$ and $E_2 \subset \mathcal{D}$.*

In this section, we study more precise relationships among E_1, E_2 and $\mathcal{D}$.

4.1. The relationship between E_1 and E_2

We first establish the relationship between E_1 and E_2 as follows.

Theorem 2. *$E_1 = E_2$.*

Proof. We first show $E_1 \subset E_2$. Let t_0 be a parameter of I and let $\{t_i\}_{i=1,2,\ldots}$ be a sequence of I conversing to t_0. Take a point $(x(t_0), y(t_0))$ of E_1. Then, we may assume that a point $(x(t_i), y(t_i))$ is taken from the intersection of two circles $C(\gamma(t_i), \lambda(t_i)) \cap C(\gamma(t_0), \lambda(t_0))$ and satisfies

$$\lim_{t_i \to t_0} (x(t_i), y(t_i)) = (x(t_0), y(t_0)).$$

Then, we have the following.

$$||(x(t_i), y(t_i)) - \gamma(t_i)||^2 = (\lambda(t_i))^2 \tag{1}$$

$$||(x(t_i), y(t_i)) - \gamma(t_0)||^2 = (\lambda(t_0))^2 . \tag{2}$$

For $j = 0, 1, 2, \ldots$, set $\gamma(t_j) = (\gamma_x(t_j), \gamma_y(t_j))$. Subtracting (2) from (1) yields the following.

$$-2\left(x(t_i)\left(\gamma_x(t_i) - \gamma_x(t_0)\right) + y(t_i)\left(\gamma_y(t_i) - \gamma_y(t_0)\right)\right) + (\gamma_x(t_i))^2 - (\gamma_x(t_0))^2 + (\gamma_y(t_i))^2 - (\gamma_y(t_0))^2 (\lambda(t_i))^2 - (\lambda(t_0))^2 .$$

Since $\lim_{i\to\infty} t_i = t_0$ and $\lim_{t_i\to t_0}(x(t_i), y(t_i)) = (x(t_0), y(t_0))$, this equality implies

$$-2\left(x(t_0)\frac{d\gamma_x}{dt}(t_0) + y(t_0)\frac{d\gamma_y}{dt}(t_0)\right) + 2\left(\gamma_x(t_0)\frac{d\gamma_x}{dt}(t_0) + \gamma_y(t_0)\frac{d\gamma_y}{dt}(t_0)\right) = 2\lambda(t_0)\frac{d\lambda}{dt}(t_0).$$

Hence we have

$$-\frac{1}{\lambda(t_0)}\left(x(t_0) - \gamma_x(t_0), y(t_0) - \gamma_y(t_0)\right) \cdot \left(\frac{d\gamma_x}{dt}(t_0), \frac{d\gamma_y}{dt}(t_0)\right) = \frac{d\lambda}{dt}(t_0).$$

Notice that the vector

$$\frac{1}{\lambda(t_0)}\left(x(t_0) - \gamma_x(t_0), y(t_0) - \gamma_y(t_0)\right) = \frac{1}{\lambda(t_0)}\left((x(t_0), y(t_0)) - \gamma(t_0)\right)$$

is a unit vector and $\left(\frac{d\gamma_x}{dt}(t_0), \frac{d\gamma_y}{dt}(t_0)\right) = \beta(t_0)\mu(t_0)$. Thus the creative condtion is satisfied at $t = t_0$. Therefore, by the proof of the assertion (1) of Theorem 1, the point $(x(t_0), y(t_0))$ must belong to E_2.

Conversely, suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ creates an E_2 envelope $f : I \to \mathbb{R}^2$. By the assertion (2) of Theorem 1, f has the following representation.

$$f(t) = \gamma(t) + \lambda(t)\widetilde{\nu}(t).$$

For a point $P \in \mathbb{R}^2$ and a unit vector $\mathbf{v} \in S^1$, the straight line $L(P, \mathbf{v})$ is naturally defined as follows.

$$L_{(P,\mathbf{v})} = \left\{(x, y) \in \mathbb{R}^2 \mid ((x, y) - P) \cdot \mathbf{v} = 0\right\}.$$

Then, since

$$\frac{df}{dt}(t) \cdot \widetilde{\nu}(t) = \left(\frac{d\gamma}{dt}(t) + \frac{d\lambda}{dt}(t) \cdot \widetilde{\nu}(t) + \lambda(t)\frac{d\widetilde{\nu}}{dt}(t)\right) \cdot \widetilde{\nu}(t) = \frac{d\gamma}{dt}(t) \cdot \widetilde{\nu}(t) + \frac{d\lambda}{dt}(t) = \beta(t)\left(\mu(t) \cdot \widetilde{\nu}(t)\right) + \frac{d\lambda}{dt}(t) = 0,$$

f is an E_2 envelope created by the straight line family

$$\mathcal{L}_{(f,\widetilde{\nu})} = \left\{L_{(f(t),\widetilde{\nu}(t))}\right\}_{t\in\mathbb{R}}.$$

Take one parameter $t_0 \in I$ and let $\{t_i\}_{i=1,2,\ldots} \subset I$ be a sequence converging to t_0. Since for the straight line family $\mathcal{L}_{(f,\widetilde{\nu})}$ the image of E_2 envelope is the same as E_1 envelope (see the assertion (c) of Theorem 1 in [6]), for any sufficiently large $i \in \mathbb{N}$ there exists a point

$$(x(t_i), y(t_i)) \in L_{(f(t_0),\widetilde{\nu}(t_0))} \cap L_{(f(t_i),\widetilde{\nu}(t_i))}$$

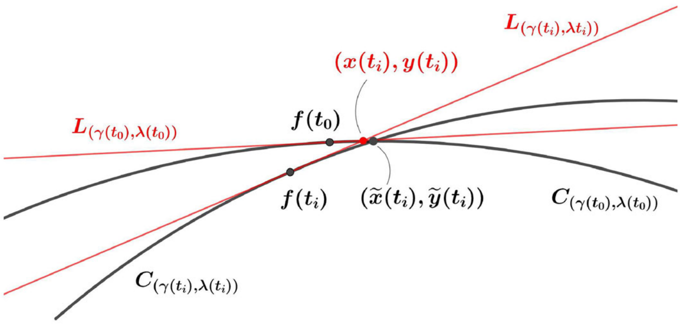

FIGURE 6 Existence of $(\widetilde{x}(t_i), \widetilde{y}(t_i)) \in C_{(\gamma(t_0),\lambda(t_0))} \cap C_{(\gamma(t_i),\lambda(t_i))}$ satisfying $\lim_{i\to\infty} (\widetilde{x}(t_i), \widetilde{y}(t_i)) = f(t_0)$

such that $\lim_{i\to\infty} (x(t_i), y(t_i)) = f(t_0)$. Hence for any sufficiently large $i \in \mathbb{N}$ there must exist a point

$$(\widetilde{x}(t_i), \widetilde{y}(t_i)) \in C_{(\gamma(t_0),\lambda(t_0))} \cap C_{(\gamma(t_i),\lambda(t_i))}$$

such that $\lim_{i\to\infty} (\widetilde{x}(t_i), \widetilde{y}(t_i)) = f(t_0)$ (see Fig. 6).

Therefore, the point $f(t_0) \in \mathbb{R}^2$ belongs to E_1. Since f is an arbitrary envelope created by $\mathcal{C}_{(\gamma,\lambda)}$ and t_0 is an arbitrary parameter in I, it follows that $E_2 \subset E_1$. □

4.2. A relationship between E_2 and $\mathcal{D}$

In this subsection, we prove the following theorem which asserts that $\mathcal{D} = E_2$ if and only if $\gamma : I \to \mathbb{R}^2$ is non-singular, and $\mathcal{D}$ contains not only E_2 but also the circle $C_{(\gamma(t),\lambda(t))}$ at a singular point t of γ when γ is singular.

Theorem 3. *Let $\gamma : I \to \mathbb{R}^2$, $\lambda : I \to \mathbb{R}_+$ be a frontal and a positive function respectively. Suppose that the circle family $\mathcal{C}_{(\gamma,\lambda)}$ is creative. Then, the following holds.*

$$\mathcal{D} = E_2 \cup \left(\bigcup_{t\in\Sigma(\gamma)} C_{(\gamma(t),\lambda(t))} \right).$$

Here, $\Sigma(\gamma)$ stands for the set consisting of singular points of $\gamma : I \to \mathbb{R}^2$.

Proof. Recall that

$$\mathcal{D} = \left\{ (x, y) \in \mathbb{R}^2 \mid \exists t \in I \text{ such that } F(x, y, t) = \frac{\partial F}{\partial t}(x, y, t) = 0 \right\}.$$

Let (x_0, y_0) be a point of $\mathcal{D}$. Since $F(x, y, t) = ||(x, y) - \gamma(t)||^2 - |\lambda(t)|^2$, it follows that there exists a $t_0 \in I$ such that the following (a) and (b) are satisfied.

(a) $((x_0, y_0) - \gamma(t_0)) \cdot ((x_0, y_0) - \gamma(t_0)) - (\lambda(t_0))^2 = 0.$

(b) $\frac{d\left(((x_0,y_0)-\gamma(t_0))\cdot((x_0,y_0)-\gamma(t_0))-(\lambda(t_0))^2\right)}{dt}=0.$

The condition (a) implies that there exists a unit vector $\nu_1(t_0)\in S^1$ such that

$$(x_0,y_0)=\gamma(t_0)-\lambda(t_0)\nu_1(t_0).$$

The condition (b) implies

$$\frac{d\gamma}{dt}(t_0)\cdot((x_0,y_0)-\gamma(t_0))-\frac{d\lambda}{dt}(t)\lambda(t_0)=0.$$

Since $\frac{d\gamma}{dt}(t_0)=\beta(t_0)\mu(t_0)$, just by substituting, we have the following.

$$\lambda(t_0)\left(\beta(t_0)\left(\mu(t_0)\cdot\nu_1(t_0)\right)+\frac{d\lambda}{dt}(t_0)\right)=0.$$

Since $\lambda(t)>0$ for any $t\in I$, it follows

$$\frac{d\lambda}{dt}(t_0)=-\beta(t_0)\left(\mu(t_0)\cdot\nu_1(t_0)\right).$$

On the other hand, since $\mathcal{C}_{(\gamma,\lambda)}$ is creative, there must exist a smooth unit vector field $\widetilde{\nu}:I\to S^1$ along $\gamma:I\to\mathbb{R}^2$ such that

$$\frac{d\lambda}{dt}(t)=-\beta(t)\left(\mu(t)\cdot\widetilde{\nu}(t)\right)$$

for any $t\in I$. Suppose that the parameter $t_0\in I$ is a regular point of γ. Then, $\beta(t_0)\neq 0$. Therefore, by the proof of the assertion (1) of Theorem 1, it follows

$$(x_0,y_0)\in E_2.$$

Suppose that the parameter $t_0\in I$ is a singular point of γ. Then, $\beta(t_0)=0$. Thus, for any unit vector $\mathbf{v}\in S^1$, the following holds.

$$\frac{d\lambda}{dt}(t_0)=-\beta(t_0)\left(\mu(t_0)\cdot\mathbf{v}\right).$$

Hence, at the singular point $t_0\in I$ of γ, we may choose any unit vector $\mathbf{v}\in S^1$ as the unit vector $\nu_1(t_0)$. Therefore, it follows

$$\mathcal{D}_0=C_{(\gamma(t),\lambda(t))},$$

where $\mathcal{D}_0=\left\{(x,y)\in\mathbb{R}^2\,|\,F(x,y,t_0)=\frac{\partial F}{\partial t}(x,y,t_0)=0\right\}$. □

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Declarations

Conflict of interest The authors declare that there is no conflict of interest in this work.

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