



Finsler metrics and semi-symmetric compatible linear connections

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Abstract. Finsler metrics are direct generalizations of Riemannian metrics such that the quadratic Riemannian indicatrices in the tangent spaces of a manifold are replaced by more general convex bodies as unit spheres. A linear connection on the base manifold is called compatible with the Finsler metric if the induced parallel transports preserve the Finslerian length of tangent vectors. Finsler manifolds admitting compatible linear connections are called generalized Berwald manifolds Wagner (Dokl Acad Sci USSR (N.S.) 39:3–5, 1943). Compatible linear connections are the solutions of the so-called compatibility equations containing the components of the torsion tensor as unknown quantities. Although there are some theoretical results for the solvability of the compatibility equations (monochromatic Finsler metrics Bartelmeß and Matveev (J Diff Geom Appl 58:264–271, 2018), extremal compatible linear connections and algorithmic solutions Vincze (Aequat Math 96:53–70, 2022)), it is very hard to solve them in general because compatible linear connections may or may not exist on a Finsler manifold and may or may not be unique. Therefore special cases are of special interest. One of them is the case of the so-called semi-symmetric compatible linear connection with decomposable torsion tensor. It is proved Vincze (Publ Math Debrecen 83(4):741–755, 2013 (see also Vincze (Euro J Math 3:1098–1171, 2017))) that such a compatible linear connection must be uniquely determined. The original proof is based on averaging in the sense that the 1-form in the decomposition of the torsion tensor can be expressed by integrating differential forms on the tangent manifold over the Finslerian indicatrices. The integral formulas are very difficult to compute in practice. In what follows we present a new proof for the uniqueness by using linear algebra and some basic facts about convex bodies. We present an explicit formula for the solution without integration. The method has a new contribution to the problem as well: necessary conditions of the solvability are formulated in terms of intrinsic equations without unknown quantities.

Mathematics Subject Classification. 53C60, 58B20.

Keywords. Convex bodies, Tangent hyperplanes, Minkowski norm, Finsler spaces, Generalized Berwald spaces, Semi-symmetric linear connections, Intrinsic Geometry.

1. Compatibility equations in Finsler geometry

Let M be a smooth connected manifold with a local coordinate system u^1, \dots, u^n . The induced local coordinate system on the tangent manifold TM consists of the functions x^1, \dots, x^n and y^1, \dots, y^n given by $x^i(v) := u^i \circ \pi(v) = u^i(p) =: p^i$, where $\pi: TM \rightarrow M$ is the canonical projection and $y^i(v) = v(u^i)$, $i = 1, \dots, n$. Throughout the paper, we will use the shorthand notations

$$\partial_i := \frac{\partial}{\partial x^i} \quad \text{and} \quad \dot{\partial}_i := \frac{\partial}{\partial y^i}. \quad (1.1)$$

A Finsler metric [1] on a manifold is a smoothly varying family of Minkowski norms in the tangent spaces. It is a direct generalization of Riemannian metrics such that the inner products (quadratic indicatrices) in the tangent spaces are replaced by Minkowski norms (smooth strictly convex bodies).

Definition 1.1. A **Finsler metric** is a non-negative continuous function $F: TM \rightarrow \mathbb{R}$ satisfying the following conditions: F is smooth on the complement of the zero section (*regularity*), $F(tv) = tF(v)$ for all $t > 0$ (*positive homogeneity*), $F(v) = 0$ if and only if $v = \mathbf{0}$ (*definiteness*) and the Hessian $g_{ij} = \partial_i \dot{\partial}_j E$ of the energy function $E = F^2/2$ is positive definite at all non-zero elements $v \in T_p M$ (*strong convexity*). The pair (M, F) is called a **Finsler manifold**.

On a Riemannian manifold we obviously have compatible linear connections in the sense that the induced parallel transports preserve the Riemannian length of tangent vectors (metric linear connections). Following the classical Christoffel process it is clear that such a linear connection is uniquely determined by the torsion tensor. In contrast to the Riemannian case, non-Riemannian Finsler manifolds admitting compatible linear connections form a special class of spaces in Finsler geometry. They are called generalized Berwald manifolds [2]. It is known that some Finsler manifolds do not admit any compatible linear connections because of topological constraints, some have infinitely many compatible linear connections and it can also happen that the compatible linear connection is uniquely determined [3], see also [4].

Definition 1.2. A linear connection is **compatible** with the Finsler metric if the induced parallel transports preserve the Finslerian length of tangent vectors. Finsler manifolds admitting compatible linear connections are called **generalized Berwald manifolds**.

In terms of local coordinates, equations

$$X_i^{h^*} F := \partial_i F - y^j (\Gamma_{ij}^k \circ \pi) \dot{\partial}_k F = 0 \quad (i = 1, \dots, n) \tag{1.2}$$

form necessary and sufficient conditions for a linear connection ∇ to be compatible with the Finsler metric F over a local neighbourhood. Equations (1.2) are called **compatibility equations** or CEQ for short¹. The fundamental result of generalized Berwald manifold theory states that a compatible linear connection ∇ is always Riemann metrizable [6], i.e. ∇ must be a metric linear connection with respect to a Riemannian metric γ . Such a Riemannian metric can be given by integration of g_{ij} on the indicatrix hypersurfaces [6], see also [7] and [8]. It is the so-called **averaged Riemannian metric**. Therefore CEQ can be reformulated by replacing the Christoffel symbols Γ_{ij}^k by the torsion tensor components [9], see also [4]. Using the horizontal vector fields

$$X_i^{h^*} := \partial_i - y^j (\Gamma_{ij}^{k^*} \circ \pi) \dot{\partial}_k \quad (i = 1, \dots, n),$$

where $\Gamma_{ij}^{k^*}$ are the Christoffel symbols of the Levi-Civita connection of the averaged Riemannian metric γ , CEQ takes the form

$$y^j (T_{jk}^l \gamma^{kr} \gamma_{il} + T_{ik}^l \gamma^{kr} \gamma_{jl} - T_{ij}^r) \dot{\partial}_r F = -2X_i^{h^*} F \quad (i = 1, \dots, n). \tag{1.3}$$

The unknown quantities T_{ab}^c are the torsion tensor components of the compatible linear connection. **In the rest of the paper we use normal coordinates** with respect to the Riemannian metric γ around a given point $p \in M$. The coordinate vector fields $\partial/\partial u^1, \dots, \partial/\partial u^n$ form an orthonormal basis in $T_p M$, i.e. $\gamma_{ij}(p) = \delta_{ij}$ and $\Gamma_{ij}^{k^*}(p) = 0$. Therefore $X_i^{h^*}(v) = \partial_i(v)$ for any $v \in T_p M$ and CEQ takes the form

$$\sum_{a < b, c} \sigma_{ab;i}^c T_{ab}^c = -2\partial_i F \quad (i = 1, \dots, n), \tag{1.4}$$

where the coefficients are

$$\sigma_{ab;i}^c := \delta_i^a f_{cb} + \delta_i^b f_{ac} + \delta_i^c f_{ab}, \quad f_{ij} := y^i \dot{\partial}_j F - y^j \dot{\partial}_i F. \tag{1.5}$$

If none of the indices a, b, c is equal to i then $\sigma_{ab;i}^c = 0$. Otherwise the table shows the possible cases, where indices are separated according to their values (equal indices are put into the same cell and different cells contain different values). To complete the list we also collect the remaining three trivial cases: $\sigma_{aa;i}^i = 0$ ($i = c, a = b$), $\sigma_{ii;i}^c = 0$ ($i = a = b, c$) and $\sigma_{ii;i}^i = 0$ ($i = a = b = c$). Therefore the i -th compatibility equation at the point p is

$$\sum_a' 2f_{ia} T_{ia}^i + \sum_{a < b}' f_{ab} (T_{ib}^a + T_{ab}^i + T_{ai}^b) = -2\partial_i F, \tag{1.6}$$

¹The local solutions only very implicitly indicate the existence of a global solution. Roughly speaking, if we have at least continuous local solutions (connection parameters) then the local parallel vector fields provide linear isometries between the tangent spaces neighbourhood to neighbourhood. Therefore the manifold is locally monochromatic. It becomes a global property in case of a connected manifold and the main result in [5] implies that we have a generalized Berwald manifold. However, **it is enough to consider the local solutions as the consequences of the global solution in our present paper.**

	Indices			The coefficients
1.	$i = a$	b	c	$\sigma_{ib;i}^c = f_{cb}$
2.	$i = a$	$b = c$		$\sigma_{ib;i}^b = 0$
3.	$i = b$	a	c	$\sigma_{ai;i}^c = f_{ac}$
4.	$i = b$	$a = c$		$\sigma_{ai;i}^a = 0$
5.	$i = c$	a	b	$\sigma_{ab;i}^i = f_{ab}$
6.	$i = a = c$	b		$\sigma_{ib;i}^i = 2f_{ib}$
7.	$i = b = c$	a		$\sigma_{ai;i}^i = 2f_{ai}$

where the primed summation means summing for $a \neq i$ in the first one, and in the second one, a and b where $i \neq a, b$.

2. The geometry of the tangent spaces

The following table shows a panoramic view about the geometric structures of the tangent space T_pM due to the simultaneously existing Finsler and Riemannian metrics.

Finsler structure		Riemannian structure
Minkowski norm	Metric on T_pM	Euclidean norm and inner product
Finslerian spheres $F_p(\lambda)$	Level sets of $\lambda \in \mathbb{R}_+$	Euclidean spheres $R_p(\lambda)$
\mathcal{F}_v	Tangent hyperplanes of level sets at $v \in T_pM$	\mathcal{R}_v
$\mathcal{LF}_v := \mathcal{F}_v - v$	Linear tangent hyperplanes at $v \in T_pM$	$\mathcal{LR}_v := \mathcal{R}_v - v$
$G := \text{grad } F = [\dot{\partial}_1 F, \dots, \dot{\partial}_n F]$	Normal vector fields (w.r.t. γ)	$C := [y^1, \dots, y^n]$

Both gradient vector fields G and C are non-zero everywhere on $T_p^\circ M := T_pM \setminus \{0\}$. For every element $v \in T_p^\circ M$ the tangent hyperplanes of the Finslerian and the Riemannian (Euclidean) spheres passing through v can be related as follows.

- If $\mathcal{F}_v = \mathcal{R}_v$, i.e. $G_v \parallel C_v$, we call the point v a **vertical contact point** of the metrics.

- If $\mathcal{F}_v \neq \mathcal{R}_v$, i.e. G_v and C_v are linearly independent, then the intersection $\mathcal{L}\mathcal{F}_v \cap \mathcal{L}\mathcal{R}_v$ is the orthogonal complement of $\text{span}(C_v, G_v)$, and thus $\mathcal{F}_v \cap \mathcal{R}_v$ is an affine subspace of dimension $n - 2$.

Using normal coordinates with respect to the Riemannian metric γ around a given point $p \in M$, let us define the vector field

$$H_v := [\partial_1 F(v), \dots, \partial_n F(v)], \tag{2.1}$$

where $v \in T_p^\circ M$. An element $v \in T_p^\circ M$ is a **horizontal contact point** of the metrics if H_v is the zero vector. It can be easily seen that if $T_p M$ is a vertical contact tangent space, i. e. all of its non-zero elements are vertical contact, then the Finsler metric is a scalar multiple of γ at the point $p \in M$. Moreover, if the indicatrix of a generalized Berwald metric is quadratic at a single point, then it is quadratic at every point of the (connected) base manifold because the tangent spaces are related by linear isometries due to the parallel transports with respect to the compatible linear connection with the Finsler metric. Therefore such a generalized Berwald manifold reduces to a Riemannian manifold. At a horizontal contact point, equations of (1.4) are homogeneous. If $T_p M$ is a horizontal contact tangent space, i. e. all of its non-zero elements are horizontal contact, then $T = 0$ is a solution of CEQ at $p \in M$.

2.1. A useful family of vector fields

Let us define the vector fields

$$f_i(v) := [f_{i1}(v), f_{i2}(v), \dots, f_{in}(v)]^T \quad (i = 1, \dots, n) \tag{2.2}$$

on $T_p^\circ M$ to help in proving some elementary properties and solving CEQ.

Lemma 2.1. $\text{span}(f_1(v), \dots, f_n(v)) \subseteq \text{span}(G_v, C_v)$ for any $v \in T_p^\circ M$.

Proof. Observe that f_i can be written as

$$f_i = \begin{bmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{in} \end{bmatrix} = \begin{bmatrix} y^i \dot{\partial}_1 F - y^1 \dot{\partial}_i F \\ y^i \dot{\partial}_2 F - y^2 \dot{\partial}_i F \\ \vdots \\ y^i \dot{\partial}_n F - y^n \dot{\partial}_i F \end{bmatrix} = y^i \begin{bmatrix} \dot{\partial}_1 F \\ \dot{\partial}_2 F \\ \vdots \\ \dot{\partial}_n F \end{bmatrix} - \dot{\partial}_i F \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix}, \tag{2.3}$$

i.e. $f_i = y^i \cdot G - \dot{\partial}_i F \cdot C$. □

Lemma 2.2. At a vertical contact point $v \in T_p^\circ M$, $f_i(v) = 0$ and CEQ takes the form $0 = \partial_i F(v)$ ($i = 1, \dots, n$).

Proof. If v is vertically contact, then $G_v = \lambda C_v$ for some non-zero $\lambda \in \mathbb{R}$. Substituting into (2.3), we get

$$f_i(v) = v^i \cdot \lambda C_v - \lambda v^i \cdot C_v = 0 \quad (i = 1, \dots, n),$$

i.e. the coordinates of f_i are all zero at v and the reformulation (1.6) of CEQ implies the statement. □

Corollary 2.3. *In order for CEQ to have a solution, all vertically contact points must be horizontal contact.*

Lemma 2.4. *At a not vertically contact $v \in T_p^\circ M$ there are indices such that $f_{ij}(v) \neq 0$ and the vectors f_i and f_j are linearly independent over some neighborhood U of v in $T_p M$.*

Proof. Suppose that all the $f_{ij}(v)$, and consequently, all the vectors $f_i(v)$ are zero. Since v is not vertically contact, G_v and C_v are linearly independent. In particular, neither of them is the zero vector, so one of their coordinates is non-zero, meaning that for some index i , (2.3) gives the zero vector as a linear combination of the independent vectors C_v and G_v with non-zero coefficients. This is a contradiction, so there must be an $f_{ij}(v)$, and thus two vectors $f_i(v)$ and $f_j(v)$ different from zero. Since the matrix

$$\begin{bmatrix} f_i \\ f_j \end{bmatrix} = \begin{bmatrix} f_{i1} \dots 0 \dots f_{ij} \dots f_{in} \\ f_{j1} \dots f_{ji} \dots 0 \dots f_{jn} \end{bmatrix}$$

has rank 2 at v (choose the i -th and j -th columns), they are linearly independent at v and the same is true at the points of some adequately small neighborhood of v in $T_p M$ by a continuity argument. \square

Corollary 2.5. *At a point $v \in T_p^\circ M$,*

$$\text{span}(f_1(v), \dots, f_n(v)) = \begin{cases} \{\mathbf{0}\} & \text{if } v \text{ is vertically contact,} \\ \text{span}(G_v, C_v) & \text{if } v \text{ is not vertically contact.} \end{cases}$$

Proof. For any $v \in T_p^\circ M$, we have $\text{span}(f_1(v), \dots, f_n(v)) \subseteq \text{span}(G_v, C_v)$ by Lemma 2.1. If v is vertically contact, all the vectors $f_i(v)$ are zero according to Lemma 2.2. If not, there are 2 independent vectors among them according to Lemma 2.4, thus generating the whole $\text{span}(G_v, C_v)$. \square

Lemma 2.6. *At a not vertically contact $v \in T_p^\circ M$, let us choose indices $i \neq j$ such that $f_{ij}(v) \neq 0$. Then (f_i, f_j) is a basis of $\text{span}(G, C)$ over some neighborhood U of v in $T_p M$ and*

$$f_k = \frac{f_{kj}}{f_{ij}} \cdot f_i + \frac{f_{ik}}{f_{ij}} \cdot f_j \quad (k = 1, \dots, n) \tag{2.4}$$

at any point of U .

Proof. By Lemma 2.4, we know that (f_i, f_j) is a basis of $\text{span}(G, C)$ at the points of U . Let us choose an index $k \in \{1, \dots, n\}$ and write $f_k = \lambda_1 f_i + \lambda_2 f_j$. By (2.3), we can write that

$$f_k = y^k \cdot G - \dot{\partial}_k F \cdot C = \lambda_1 \left(y^i \cdot G - \dot{\partial}_i F \cdot C \right) + \lambda_2 \left(y^j \cdot G - \dot{\partial}_j F \cdot C \right).$$

By comparing the coefficients in the basis (G, C) ,

$$\begin{bmatrix} y^k \\ \dot{\partial}_k F \end{bmatrix} = \begin{bmatrix} y^i & y^j \\ \dot{\partial}_i F & \dot{\partial}_j F \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

and, consequently,

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{f_{ij}} \begin{bmatrix} \dot{\partial}_j F & -y^j \\ -\dot{\partial}_i F & y^i \end{bmatrix} \cdot \begin{bmatrix} y^k \\ \dot{\partial}_k F \end{bmatrix} = \frac{1}{f_{ij}} \begin{bmatrix} y^k \dot{\partial}_j F - y^j \dot{\partial}_k F \\ y^i \dot{\partial}_k F - y^k \dot{\partial}_i F \end{bmatrix}.$$

□

3. The semi-symmetric compatible linear connection and its uniqueness

Although there are some theoretical results for the solvability of the compatibility equations (monochromatic Finsler metrics [5], extremal compatible linear connections and algorithmic solutions [9]), it is very hard to solve them in general because compatible linear connections may or may not exist on a Finsler manifold and may or may not be unique. Therefore special cases are of special interest. One of them is the case of the so-called semi-symmetric compatible linear connection with decomposable torsion tensor.

Definition 3.1. A linear connection is called **semi-symmetric** if its torsion tensor can be written as

$$T(X, Y) = \rho(Y)X - \rho(X)Y \tag{3.1}$$

for some differential 1-form ρ on the base manifold.

It is proved [10] that a semi-symmetric compatible linear connection must be uniquely determined.

Theorem 3.2. [10] *A non-Riemannian Finsler manifold admits at most one semi-symmetric compatible linear connection.*

The original proof is based on averaging in the sense that the 1-form ρ can be expressed by integrating differential forms on the tangent manifold over the Finslerian indicatrices. The integral formulas are very difficult to compute in practice. In what follows we present a new proof for the uniqueness by using linear algebra and some basic facts about convex bodies. We present an explicit formula for the solution without integration. The method has a new contribution to the problem as well: necessary conditions of the solvability are formulated in terms of intrinsic equations without unknown quantities.

3.1. The proof of Theorem 3.2

Since

$$\begin{aligned} T \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) &= \rho \left(\frac{\partial}{\partial u^j} \right) \frac{\partial}{\partial u^i} - \rho \left(\frac{\partial}{\partial u^i} \right) \frac{\partial}{\partial u^j} = \rho_j \frac{\partial}{\partial u^i} - \rho_i \frac{\partial}{\partial u^j} \\ &= (\delta_i^k \rho_j - \delta_j^k \rho_i) \frac{\partial}{\partial u^k}, \end{aligned}$$

the torsion components are

$$T_{ij}^k = \delta_i^k \rho_j - \delta_j^k \rho_i. \tag{3.2}$$

In particular, all torsion components with 3 different indices are zero, and

$$T_{ij}^i = \delta_i^i \rho_j - \delta_j^i \rho_i = \rho_j \quad (j \neq i).$$

Substituting the torsion components into the general form (1.6) of CEQ at a point p , it takes the (matrix) form

ρ_1	ρ_2	ρ_3	\cdots	ρ_n	RHS
0	f_{12}	f_{13}	\cdots	f_{1n}	$-\partial_1 F$
f_{21}	0	f_{23}	\cdots	f_{2n}	$-\partial_2 F$
f_{31}	f_{32}	0	\cdots	f_{3n}	$-\partial_3 F$
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
f_{n1}	f_{n2}	f_{n3}	\cdots	0	$-\partial_n F$

(3.3)

The problem is to solve (3.3) for ρ_1, \dots, ρ_n , considered as the coordinates of a vector $\rho \in T_p M$, as v ranges over $T_p^\circ M$. To prove Theorem 3.2 it is enough to consider the homogeneous version H-CEQ with vanishing right hand side of (3.3). We are going to verify that the only solution of H-CEQ is $\rho_1 = \dots = \rho_n = 0$. Since the rows of the matrix on the left hand side are exactly the vectors f_1, \dots, f_n defined in (2.2), solving H-CEQ at a fixed element v means finding the orthogonal complement of $\text{span}(f_1(v), \dots, f_n(v))$. By Corollary 2.5,

- it is $T_p M$ for any vertically contact element v ,
- it is the orthogonal complement of $\text{span}(G_v, C_v)$ for any not vertically contact element v , i.e. the intersection $\mathcal{L}_{\mathcal{F}_v} \cap \mathcal{L}_{\mathcal{R}_v}$ of the linear tangent hyperplanes of the Finslerian and Riemannian spheres.

The solution of H-CEQ at the point p is the intersection of all the solution spaces as the element v ranges over $T_p^\circ M$. Note that the homogeneity of the coefficients implies that it is enough to consider the intersection of all the solution spaces as the element v ranges over the Finslerian (or the Riemannian) unit sphere.

- If all the elements of $T_p M$ are vertically contact and the Finsler manifold admits a compatible (semi-symmetric) linear connection ∇ , then it is a Riemannian manifold because the linear isometries via the parallel transports with respect to ∇ extend the quadratic Finslerian (esp. Riemannian) indicatrix at the point p to the entire (connected) manifold.
- If there is a not vertically contact element v , then, by a continuity argument, we can consider a neighborhood $U \subseteq T_p M$ containing only not vertically contact elements. For the solution vector ρ we have

$$\rho \in \bigcap_{v \in U} (\mathcal{L}_{\mathcal{F}_v} \cap \mathcal{L}_{\mathcal{R}_v}) \subseteq \left(\bigcap_{v \in U} \mathcal{L}_{\mathcal{F}_v} \right) \cap \left(\bigcap_{v \in U} \mathcal{L}_{\mathcal{R}_v} \right).$$

It is clear that the right hand side contains only the zero vector because the normal vectors at the points of any open set on the boundary of a

Euclidean sphere (or any smooth strictly convex body) span the entire space. Therefore $\rho = 0$ is the only solution of H-CEQ at p and, consequently, CEQ admits at most one solution for the components of the torsion tensor of a semi-symmetric linear connection point by point.

□

3.2. Intrinsic equations and v -solvability of CEQ

In this section we investigate (3.3) evaluated at non-zero tangent vectors in T_pM :

ρ_1	ρ_2	ρ_3	\cdots	ρ_n	RHS
0	$f_{12}(v)$	$f_{13}(v)$	\cdots	$f_{1n}(v)$	$-\partial_1 F(v)$
$f_{21}(v)$	0	$f_{23}(v)$	\cdots	$f_{2n}(v)$	$-\partial_2 F(v)$
$f_{31}(v)$	$f_{32}(v)$	0	\cdots	$f_{3n}(v)$	$-\partial_3 F(v)$
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
$f_{n1}(v)$	$f_{n2}(v)$	$f_{n3}(v)$	\cdots	0	$-\partial_n F(v)$

(3.4)

Definition 3.3. The system of the compatibility equations is called v -solvable at the point $p \in M$ if (1.4) is solvable for any non-zero element $v \in T_pM$. The system of the compatibility equations is called v -solvable for ρ at the point $p \in M$ if (3.4) is solvable for any non-zero element $v \in T_pM$.

Remark 3.4. The system of the compatibility equations is v -solvable if and only if all vertical contact vectors are horizontal contact. It is an obvious necessary condition because the coefficient matrix of CEQ is zero at a vertically contact point and the system must be homogeneous with vanishing horizontal derivatives of the Finsler metric with respect to the Levi-Civita connection of the averaged Riemannian metric. The sufficiency is based on the idea of the extremal compatible linear connection [9]. The extremal solution typically depends on the reference element $v \in T_pM$ but does not take a decomposable form in general. Therefore v -solvability for ρ needs additional conditions. Using Corollary 2.5 and basic linear algebra we can formulate the following characterizations of v -solvability in case of semi-symmetric compatible linear connections.

Lemma 3.5. *The system of the compatibility equations is v -solvable for ρ at the point $p \in M$ if and only if the following conditions are satisfied:*

- all vertically contact elements are also horizontal contact in T_pM and
- the rank of the augmented matrix of system (3.4) is 2 at all not vertically contact elements $v \in T_pM$. In other words $H_v \in \text{span}(f_1(v), \dots, f_n(v))$, where the vector H_v is defined by formula (2.1).

Proposition 3.6. *The system of the compatibility equations is v -solvable for ρ at the point $p \in M$ if and only if the following conditions are satisfied:*

- all vertically contact elements are also horizontal contact in T_pM and
- for any triplets of distinct indices i, j, k we have

$$f_{ij}(v) \partial_k F(v) + f_{jk}(v) \partial_i F(v) + f_{ki}(v) \partial_j F(v) = 0 \tag{3.5}$$

provided that $f_{ij}(v) \neq 0$.

Proof. If v is vertically contact, (3.5) stands trivially. Otherwise we are going to show that it is equivalent to the augmented matrix having rank 2. Suppose that v is not vertically contact and choose indices $i \neq j$ such that $f_{ij}(v) \neq 0$ and $(f_i(v), f_j(v))$ is a basis of $\text{span}(G_v, C_v)$. By Lemma 2.6, if $k \neq i, j$ then

$$f_k = \frac{f_{kj}}{f_{ij}} \cdot f_i + \frac{f_{ik}}{f_{ij}} \cdot f_j \quad (k = 1, \dots, n).$$

In other words we can eliminate the k -th row for any $k \neq i, j$. The elimination must also yield zeroes on the right-hand side of (3.4) to have a solution, i.e. for $k \neq i, j$ we must have

$$\begin{aligned} -\partial_k F - \frac{f_{kj}}{f_{ij}} \cdot (-\partial_i F) - \frac{f_{ik}}{f_{ij}} \cdot (-\partial_j F) &= 0 \\ f_{ij} \partial_k F - f_{kj} \partial_i F - f_{ik} \partial_j F &= 0. \end{aligned}$$

Equation (3.5) follows by interchanging the indices in f_{kj} and f_{ik} . □

Remark 3.7. Equations (3.5) do not contain unknown quantities. They are intrinsic conditions of the solvability. Taking $f_{ij}(v) \neq 0$ for some fixed indices i and j at a not vertically contact element, they provide $n - 2$ equations to be automatically satisfied because $k = 1, \dots, n$, but $k \neq i, j$. The missing equations are

$$\left. \begin{aligned} \langle f_i, \rho \rangle &= -\partial_i F \\ \langle f_j, \rho \rangle &= -\partial_j F \end{aligned} \right\}, \tag{3.6}$$

where $\langle f_i, \rho \rangle$ and $\langle f_j, \rho \rangle$ stand for the inner product at $p \in M$ coming from the Riemannian metric γ . They provide the only possible solution ρ in an explicit form.

3.3. The only possible solution of CEQ at the point $p \in M$

Recall that if the tangent space at $p \in M$ contains only vertically contact non-zero elements (vertically contact tangent space) and the Finsler manifold admits a compatible (semi-symmetric) linear connection ∇ , then it is a Riemannian manifold because the linear isometries via the parallel transports with respect to ∇ extend the quadratic Finslerian (esp. Riemannian) indicatrix at the point p to the entire (connected) manifold. Therefore we present the solution of CEQ in the generic case of non-Riemannian Finsler manifolds. Let us choose a not vertically contact element $v \in T_p^\circ M$ and indices $i \neq j$ with $f_{ij}(v) \neq 0$, i.e. f_i, f_j are linearly independent over some neighborhood U of v in T_pM . Using (2.3), the eliminated form (3.6) of CEQ gives that

$$\left. \begin{aligned} y^i \langle G, \rho \rangle - \dot{\partial}_i F \langle C, \rho \rangle &= -\partial_i F \\ y^j \langle G, \rho \rangle - \dot{\partial}_j F \langle C, \rho \rangle &= -\partial_j F \end{aligned} \right\} \iff \begin{bmatrix} y^i & -\dot{\partial}_i F \\ y^j & -\dot{\partial}_j F \end{bmatrix} \cdot \begin{bmatrix} \langle G, \rho \rangle \\ \langle C, \rho \rangle \end{bmatrix} = \begin{bmatrix} -\partial_i F \\ -\partial_j F \end{bmatrix},$$

and, consequently,

$$\begin{bmatrix} \langle G, \rho \rangle \\ \langle C, \rho \rangle \end{bmatrix} = \frac{1}{f_{ji}} \begin{bmatrix} -\dot{\partial}_j F & \dot{\partial}_i F \\ -y^j & y^i \end{bmatrix} \cdot \begin{bmatrix} -\partial_i F \\ -\partial_j F \end{bmatrix}.$$

We are going to concentrate on the second row

$$\langle C, \rho \rangle = \frac{1}{f_{ji}} (y^j \partial_i F - y^i \partial_j F) =: \frac{f_{ji}^h}{f_{ji}} \tag{3.7}$$

at the points of the open neighborhood U around v . Let us choose a value $\varepsilon > 0$ such that all the elements

$$\begin{aligned} w_1 &:= v - \varepsilon \cdot \partial / \partial u^1(p) = [v^1 - \varepsilon, v^2, v^3, \dots, v^n] \\ w_2 &:= v - \varepsilon \cdot \partial / \partial u^2(p) = [v^1, v^2 - \varepsilon, v^3, \dots, v^n] \\ &\vdots \\ w_n &:= v - \varepsilon \cdot \partial / \partial u^n(p) = [v^1, v^2, v^3, \dots, v^n - \varepsilon] \end{aligned}$$

are contained in U . Then (3.7) implies the system

$$\begin{bmatrix} v^1 - \varepsilon & v^2 & v^3 & \dots & v^n \\ v^1 & v^2 - \varepsilon & v^3 & \dots & v^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v^1 & v^2 & v^3 & \dots & v^n - \varepsilon \end{bmatrix} \cdot \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} = \begin{bmatrix} f_{ji}^h / f_{ji}(w_1) \\ f_{ji}^h / f_{ji}(w_2) \\ \vdots \\ f_{ji}^h / f_{ji}(w_n) \end{bmatrix} \tag{3.8}$$

of linear equations. Using the notation

$$V := \begin{bmatrix} v^1 & v^2 & \dots & v^n \\ \vdots & \vdots & \ddots & \vdots \\ v^1 & v^2 & \dots & v^n \end{bmatrix},$$

we have to investigate the regularity of the matrix $V - \varepsilon I$, where I denotes the identity matrix of the same type as V .

Lemma 3.8. *The matrix $V - \varepsilon I$ is regular if and only if $\varepsilon \notin \{0, \tilde{v} := v^1 + \dots + v^n\}$.*

Proof. The determinant $\det(V - \varepsilon I)$ is the characteristic polynomial of V with ε as the variable. It is zero if and only if ε is an eigenvalue of V . Consider the transpose of V as the matrix of a linear transformation φ (the eigenvalues are the same as those of V). Since the image of φ is the line generated by v , it follows that φ has a kernel of dimension $n - 1$ and v is an eigenvector corresponding to the eigenvalue $\tilde{v} := v^1 + \dots + v^n$ because of

$$\varphi(v) = \begin{bmatrix} v^1 & \dots & v^1 \\ \vdots & \ddots & \vdots \\ v^n & \dots & v^n \end{bmatrix} \cdot \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \begin{bmatrix} v^1(v^1 + \dots + v^n) \\ \vdots \\ v^n(v^1 + \dots + v^n) \end{bmatrix} = \tilde{v} \cdot v.$$

□

Lemma 3.9. *Choosing $\varepsilon \notin \{0, \tilde{v} := v^1 + \dots + v^n\}$, we have*

$$[V - \varepsilon I]^{-1} = \frac{1}{(\tilde{v} - \varepsilon)\varepsilon} [V - (\tilde{v} - \varepsilon)I].$$

Proof. We shall prove the formula

$$[V - \varepsilon I] \cdot [V - (\tilde{v} - \varepsilon)I] = (\tilde{v} - \varepsilon)\varepsilon I.$$

Rearranging the left-hand side,

$$V \cdot V - (\tilde{v} - \varepsilon)V - \varepsilon V + \varepsilon(\tilde{v} - \varepsilon)I = (\tilde{v} - \varepsilon)\varepsilon I$$

because of $V \cdot V = \tilde{v} \cdot V$. □

Returning to (3.8),

$$\begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_n \end{bmatrix} = \frac{1}{(\tilde{v} - \varepsilon)\varepsilon} \begin{bmatrix} v^1 + \varepsilon - \tilde{v} & v^2 & \dots & v^n \\ v^1 & v^2 + \varepsilon - \tilde{v} & \dots & v^n \\ \vdots & \vdots & \ddots & \vdots \\ v^1 & v^2 & \dots & v^n + \varepsilon - \tilde{v} \end{bmatrix} \cdot \begin{bmatrix} f_{ji}^h/f_{ji}(w_1) \\ f_{ji}^h/f_{ji}(w_2) \\ \vdots \\ f_{ji}^h/f_{ji}(w_n) \end{bmatrix}.$$

Theorem 3.10. *If a non-Riemannian Finsler manifold admits a semi-symmetric compatible linear connection, then the values of the components ρ_k in formula (3.2) for its torsion at the point $p \in M$ are*

$$\rho_k = \frac{1}{\varepsilon} \left(\frac{1}{\tilde{v} - \varepsilon} \sum_{l=1}^n v^l \frac{f_{ji}^h}{f_{ji}}(w_l) - \frac{f_{ji}^h}{f_{ji}}(w_k) \right), \tag{3.9}$$

where

- $v = [v^1, \dots, v^n]$ is a not vertically contact vector in $T_p^\circ M$,
- $\varepsilon \notin \{0, \tilde{v} := v^1 + \dots + v^n\}$,
- $w_i = [v^1, \dots, v^{i-1}, v^i - \varepsilon, v^{i+1}, \dots, v^n]$, $i = 1, \dots, n$,
- $f_{ji}^h = y^j \partial_i F - y^i \partial_j F = y^j \frac{\partial F}{\partial x^i} - y^i \frac{\partial F}{\partial x^j}$,
- $f_{ji} = y^j \dot{\partial}_i F - y^i \dot{\partial}_j F = y^j \frac{\partial F}{\partial y^i} - y^i \frac{\partial F}{\partial y^j}$ and the coordinates on the base manifold form a normal coordinate system with respect to the averaged Riemannian metric γ around the point $p \in M$.

Acknowledgements

Márk Oláh is supported by the UNKP-21-3 New National Excellence Program of the Ministry for Innovation and Technology from the source of the National Research, Development and Innovation Fund of Hungary. The authors would like to thank the referee for his/her careful work and valuable suggestions to improve the quality of the paper.

Funding Information Open access funding provided by University of Debrecen.

Data Availability Statement No datasets were generated or analyzed during the current study.

Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Received: May 13, 2022.

Revised: August 10, 2022.

Accepted: August 11, 2022.