



Sphere theorems with and without smoothing

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Abstract. We show two sphere theorems for the Riemannian manifolds with scalar curvature bounded below and the non-collapsed $\text{RCD}(n-1, n)$ spaces with mean distance close to $\frac{\pi}{2}$.

1. Introduction

Beginning with the Gauss-Bonnet theorem, sphere theorems show how geometry can be used to decide the topology of a manifold. By asking the 1/4-pinching question, which is now a theorem, Hopf opened a door to the study of sphere theorems. More previous results and history can be found in the survey papers [1–3] etc.

Continuing the paradigm, we will add two kinds of sphere theorems to enrich the subject. Details about the conditions will be given later.

Theorem A. Let (M, g) be an orientable closed Riemannian n -manifold with scalar curvature $\geq n(n-1)$. Suppose that there exists a $(1, \wedge^1)$ -contracting map $f : M \rightarrow S^n$ of non-zero degree and that the map f is harmonic with condition $C \leq 0$, then f is an isometric map.

Remark 1. The idea of the proof is based on Chern-Goldberg's argument. Their argument originated in S.S. Chern's results that generalize Schwartz lemma to Hermitian manifolds in [6] and was used to form the Riemannian version of Schwarz lemma in [5]. In fact, the target of the map can be relaxed to general Einstein manifolds, which is shown by our proof.

Theorem B. Let (X, d, \mathcal{H}^n) be a compact non-collapsed $\text{RCD}(n-1, n)$ space with full support and $\text{md}(X)$ be the mean distance of (X, d, \mathcal{H}^n) , then

I. $\text{md}(X) \leq \frac{\pi}{2}$.

- II. $\text{md}(X) = \frac{\pi}{2}$ if and only if X is isometric to the standard round n -sphere S^n and $\mathcal{H}^n = a \text{dVol}$ for some $a > 0$, where dVol is the volume form of S^n .
- III. there is an $\epsilon(n) > 0$ such that $\text{md}(X) \geq \frac{\pi}{2} - \epsilon(n)$ implies that X is homeomorphic to S^n .

Remark 2. Mean distance is an old and well-studied metric invariant in graph theory, see [11]. Note that mean distance is also called distance covariance in probability, see [26].

Remark 3. We will use Ketterer’s maximal diameter theorem [21, Theorem 1.4] and Honda-Mondello’s topological sphere theorem [20, Theorem A] (for the metric measure space) to prove Theorem B. We also define a whole family of variants of the concept of mean distance for metric measure spaces and prove sphere theorems for all of them.

The paper is organized as follows. In Sect. 2, we introduce the notions of harmonic map with condition C and prove Theorem A. In Sect. 3, we collect facts about $\text{RCD}(n - 1, n)$ spaces and show Theorem B.

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2. Harmonic maps with condition C

Let S^n or (S^n, g_{st}) be the standard round n -sphere, then one has many rigidity theorems about S^n . We will focus on the rigidity theorem with scalar curvature bounded below.

Definition 4. Let M and N be two Riemannian manifolds of the same dimension n . A smooth map $f : M \rightarrow N$ is said to be ϵ -contracting if $\|f_*\nu\| \leq \epsilon\|\nu\|$ for all tangent vectors ν on M .

A smooth map $f : M \rightarrow N$ is said to be (ϵ, \wedge^k) -contracting if $\|f^*\varphi\| \leq \epsilon\|\varphi\|$, for all k -forms $\varphi \in \wedge^k(N)$ and $1 \leq k \leq n$.

Notice that 1-contracting and volume contracting with contraction constant 1 means $(1, \wedge^1)$ - and $(1, \wedge^n)$ -contracting, respectively. If f is $(1, \wedge^p)$ -contracting, then f is $(1, \wedge^{p+k})$ -contracting for $k \geq 1$.

Theorem 5. (Llarull [24]) *For all n, k, D with $n \geq k \geq 3$ and $D > 0$, there exists a Riemannian n -manifold M^n with scalar curvature $\geq D$ and a $(1, \wedge^k)$ -contracting map $f : M^n \rightarrow S^n$ of degree 1.*

In particular, a volume decreasing and non-zero degree map for M^n (with scalar curvature bounded below) into S^n generally will not be an isometric map.

By confirming and generalizing Gromov’s conjecture [17], Llarull showed the following rigidity theorem using index theoretical method.

Theorem 6. (Llarull [25]) *Let (M^n, g) be a closed, connected, Riemannian spin n -manifold with scalar curvature $\geq n(n - 1)$. Assume that there exists a 1-contracting map or $(1, \wedge^2)$ -contracting map $f : M^n \rightarrow S^n$ of non-zero degree, then f is an isometric map.*

Note that the spin condition on M^n can be relaxed to require that the map f is the spin map. S^n can be replaced by the Riemannian manifold, which satisfies a non-negative curvature operator and certain harmonic spinor conditions [19, Lemma 1.1]. Llarull’s rigidity theorem can also be generalized to the weighted rigidity theorem on weighted Riemannian manifold with positive weighted scalar curvature in [8, Proposition 4.21].

Remark 7. It is natural to appeal to use Schoen-Yau’s minimal surface method [31] to give another proof of Llarull’s rigidity theorem. Since Gromov-Lawson’s index theoretical approach [16] and Schoen-Yau’s minimal surface method are two of the fundamental methods of studying scalar curvature, one can try to apply one approach to give a proof of results that has been showed by the other.

Here we will use harmonic maps to approach the rigidity problem on scalar curvature. Furthermore, our argument can be applied to the rigidity question of the arbitrary Einstein manifolds.

We follow the setup and notations in [15]. Let M and N be Riemannian n -manifolds. Let ds_M^2 and ds_N^2 be the Riemannian metrics of M and N , respectively. Then we can write, locally,

$$ds_M^2 = \omega_1^2 + \dots + \omega_n^2, \quad ds_N^2 = \omega_1^{*2} + \dots + \omega_n^{*2},$$

where ω_i ($1 \leq i \leq n$) and ω_a^* ($1 \leq a \leq n$) are linear differential forms in M and N , respectively. The structure equations in M are

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l.$$

The Ricci tensor R_{ij} is defined as

$$R_{ij} = \sum_k R_{ikjk}$$

and the scalar curvature is defined as

$$R = \sum_i R_{ii}.$$

Similar equations are valid in N and the corresponding quantities are denoted in the same notation with asterisks.

Let f be a C^∞ -mapping of M into N and

$$f^*\omega_a^* = \sum_i A_i^a \omega_i.$$

Later, we will drop f^* from such formulas when its presence is clear in the context.

Let e_i (resp. e_a^*) be a frame that is dual to the coframe ω_i (resp. ω_a^*), then we have

$$f_*e_i = \sum_a A_i^a e_a^*.$$

The covariant differential of A_i^a is defined by

$$DA_i^a := dA_i^a + \sum_j A_j^a \omega_{ji} + \sum_b A_i^b \omega_{ba}^* := \sum_j A_{ij}^a \omega_j \tag{1}$$

with

$$A_{ij}^a = A_{ji}^a \tag{2}$$

The following geometrical interpretation of A_{ij}^a was given by Chern-Goldberg [5, P.136]: Let $x \in M$ and let T_x and $T_{f(x)}$ be the tangent spaces at x and $f(x)$, respectively. The mapping

$$f_{**} : T_x \rightarrow T_{f(x)}$$

defined by

$$f_{**}(\nu) = \sum_{a,i,j} A_{ij}^a \lambda_i \lambda_j e_a^*, \quad \nu = \sum_i \lambda_i e_i$$

is quadratic. The mapping has the property that if ν is a unit vector, $f_{**}(\nu)$ is the acceleration vector of $f(\gamma)$ at $f(x)$, where γ is the geodesic tangent to ν at x .

Definition 8. (*Harmonic maps*) The mapping f is called harmonic if

$$\sum_i A_{ii}^a = 0.$$

Notice that the tensor field with the components $\sum_i A_{ii}^a$ is the tension vector field of Eells-Sampson [13]. Eells and Sampson proved that the mapping f is harmonic (in Eells-Sampson's sense) if and only if $\sum_i A_{ii}^a = 0$.

Differentiating the Eq. (1) and using the structure equations in M and N , we get

$$\sum_j DA_{ij}^a \wedge \omega_j = -\frac{1}{2} \sum_{j,k,l} A_l^a R_{jikl} \omega_k \wedge \omega_l - \frac{1}{2} \sum_{b,c,d} A_i^b R_{bacd}^* \omega_c^* \wedge \omega_d^*, \tag{3}$$

where

$$DA_{ij}^a := dA_{ij}^a + \sum_b A_{ij}^b \omega_{ba}^* + \sum_k A_{kj}^a \omega_{ki} + \sum_k A_{ik}^a \omega_{kj} := \sum_k A_{ijk}^a \omega_k.$$

Thus, we have

$$A_{ijk}^a - A_{ikj}^a = - \sum_l A_l^a R_{likj} - \sum_{b,c,d} A_i^b A_k^c A_j^d R_{abcd}^* \tag{4}$$

According to (2) and (4), we can calculate the Laplacian

$$\Delta A_i^a := \sum_k A_{ikk}^a = \sum_k A_{kik}^a = \sum_k A_{kki}^a + \sum_l A_l^a R_{li} - \sum_{b,c,d,k} R_{abcd}^* A_k^b A_k^c A_i^d \tag{5}$$

The ration $A := \frac{f^* d\nu_N}{d\nu_M}$ of volume elements has the expression $A = \det(A_i^a)$. Let (B_a^i) be the adjoint of (A_j^a) , i.e., $\sum_a B_a^i A_j^a = \delta_j^i A$. Then, by (1),

$$dA = \sum_{i,a} B_a^i dA_i^a = \sum_{i,j,a} B_a^i dA_{ij}^a \omega_j =: \sum_j \mathbb{A}_j \omega_j$$

Let $V = A^2$, then we will compute the Laplacian of V and use it to prove Theorem A.

First, let

$$dV = \sum_k V_k \omega_k, \tag{6}$$

where

$$V_k = 2A \sum_{i,a} B_a^i A_{ik}^a \tag{7}$$

Second, exterior differentiation of (6) gives

$$\sum_k (dV_k - \sum_i V_i \omega_{ki}) \wedge \omega_k = 0.$$

Hence we can write

$$dV_k - \sum_i V_i \omega_{ki} = \sum_j V_{jk} \omega_j, \tag{8}$$

where

$$V_{jk} = V_{kj}.$$

Then the Laplacian of V is by definition equal to

$$\Delta V = \sum_k V_{kk}.$$

Third, by differentiating (7), using (8), and simplifying, we get

$$\frac{1}{2} V_{kj} = 2\mathbb{A}_j \mathbb{A}_k - \sum_{i,l,a,b} B_a^i B_b^l A_{lj}^a A_{ik}^b + A \sum_{i,a} B_a^i A_{ikj}^a.$$

Using (5), the Laplacian of V can be given by

$$\frac{1}{2} \Delta V = 2 \sum_j (\mathbb{A}_j)^2 + V(R - \sum_{b,c,j} R_{b,c}^* A_j^b A_j^c) - C + A \sum_{a,i,j} B_a^i A_{jji}^a, \tag{9}$$

where C is a scalar invariant of the mapping given by

$$C = \sum_{\substack{a,b \\ i,j,k}} B_a^i B_b^k A_{kj}^a A_{ij}^b. \tag{10}$$

If f is harmonic, then the last term of (9) vanishes. That is, see [5, P.141, formula (57)],

(for a harmonic map)

$$\frac{1}{2} \Delta V = 2 \sum_j (\mathbb{A}_j)^2 + V(R - \sum_{b,c,j} R_{b,c}^* A_j^b A_j^c) - C. \tag{11}$$

The geometric meaning of Condition C (10) was also given by Chern-Goldberg [5, P.142, Remark] as follows: the scalar C may be interpreted geometrically as a weighted measure of the deviation of the square length of the tensor C_{ijk} from the square length of its symmetric part, where C_{ijk} is the pullback of A_{ij}^a under f .

Proof of Theorem A. From the fact that the round sphere S^n is an Einstein manifold, we get

$$\sum_{b,c,j} R_{b,c}^* A_j^b A_j^c = \frac{R^*}{n} \sum_{a,i} (A_i^a)^2$$

Since M is compact and f is non-zero degree, V does attain its maximum at the point x in M . Then $V(x) > 0$ and $\Delta V(x) \leq 0$. Notice that $V(x)$ is independent of the choice of the frame and coframe. At the point x , we choose a local g -orthonormal frame e_1, \dots, e_n on $T_x M$ and a local g_{st} -orthonormal frame e_1^*, \dots, e_n^* on $T_{f(x)} S^n$, such that there exists $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$, with $f_* e_i = \lambda_i e_i^*$. This can be done by diagonalizing $f^* g_{st}$ with respect to the metric g . As f is $(1, \wedge^1)$ -contracting, we have $\lambda_i \leq 1$ for all $1 \leq i \leq n$. That means

$$\sum_{b,c,j} R_{b,c}^* A_j^b A_j^c = \frac{R^*}{n} \sum_{a,i} (A_i^a)^2 \leq R^*. \tag{12}$$

On account of $R^* = n(n - 1)$, $R \geq n(n - 1)$, and $C \leq 0$, we have

$$\frac{1}{2} \Delta V(x) \geq V(x)(R - n(n - 1)) \geq 0.$$

Then by combining $V(x) > 0$ and $\Delta V(x) \leq 0$, we get $R = n(n - 1)$ and $\lambda_i = 1$ for all $1 \leq i \leq n$. That means f is an isometric map. \square

In fact, the target of the map can be relaxed to an arbitrary Einstein manifold as (12) shows.

Corollary 9. *Let (N, g) be an orientable closed (Riemannian) Einstein n -manifold with scalar curvature R^* and (M, g) be an orientable closed Riemannian n -manifold with scalar curvature R .*

Suppose $R \geq R^*$, there exists a $(1, \wedge^1)$ -contracting map $f : M \rightarrow N$ of non-zero degree, and the map f is harmonic with condition $C \leq 0$, then f is a locally isometric map. If the fundamental group of N is trivial, then f is an isometric map.

Proof. As N is an Einstein manifold, then the inequality (12) and $R \geq R^*$ implies f is a locally isometric map. □

It is not clear whether the method of the poof in this section can be generalized to weighted Riemannian manifold with positive weighted scalar curvature in [8].

Conjecture 10. *Let (M, g) be an orientable closed Riemannian n -manifold with scalar curvature $\geq n(n-1)$. Suppose that there exists a $(1, \wedge^1)$ -contracting map $f : M \rightarrow S^n$ of non-zero degree, then f is an isometric map.*

Remark 11. If Conjecture 10 is confirmed, then Theorem B in [9] holds for all dimensions. And then we can get a new obstruction of the existence of a Riemannian metric with positive scalar curvature on an arbitrary closed smooth manifold.

3. Non-collapsed RCD spaces and mean distance

The framework of Riemannian manifolds with Ricci curvature bounded below is generalized to the metric measure space (X, d, μ) , which satisfies Riemannian curvature-dimension condition. And the metric measure spaces are called $RCD(K, N)$ spaces.

Definition 12. (*Non-collapsed $RCD(K, n)$ spaces*) Let $K \in \mathbb{R}$, $n \in \mathbb{N}$ and $n \geq 1$. The metric measure space (X, d, μ) is called a non-collapsed $RCD(K, n)$ space, if

1. (X, d, μ) is an $RCD(K, n)$ space.
2. [Non-collapsed condition] $\mu = \mathcal{H}^n$, where \mathcal{H}^n is the n -dimensional Hausdorff measure with respect to the metric d .

The concept of non-collapsed $RCD(K, N)$ space was defined by De Philippis and Gigli [12, Definition 1.1]. De Philippis-Gigli showed that N must be an integer. The proof of our theorem below is mainly based on De Philippis and Gigli’s results.

Definition 13. (*Mean distance*) Let (X, d, μ) be a compact metric measure space with full support, then the mean distance of X is defined as

$$\text{md}(X) := \int_{X \times X} \frac{d(\cdot, \cdot)}{\mu(X) \times \mu(X)} \mu \otimes \mu$$

Using Cheng’s maximal diameter theorem and Cheeger-Colding’s differentiable sphere theorem [4], Kokkendorff [22, Theorem 4] showed the following theorem.

Theorem 14. (Kokkendorff) *Let $(M, g, dVol_g)$ be a Riemannian n -manifold ($n \geq 2$) with Ricci curvature $\geq n - 1$, where $dVol_g$ is the Riemannian volume, then*

1. $md(M) \leq \frac{\pi}{2}$
2. $md(M) = \frac{\pi}{2}$ if and only if M is isometric to the standard round n -sphere (S^n, d_{S^n}) .
3. there is an $\epsilon'(n) > 0$ such that $md(M) \geq \frac{\pi}{2} - \epsilon'(n)$ implies that M is diffeomorphic to S^n .

We generalize Kokkendorff’s theorem to compact non-collapsed $RCD(n - 1, n)$ spaces.

The strategy of the proof of Theorem B is the same as Kokkendorff’s proof, but we need to replace Cheng’s maximal diameter theorem and Cheeger-Colding’s differentiable sphere theorem (for the Riemannian manifold) with Ketterer’s maximal diameter theorem [21, Theorem 1.4] and Honda-Mondello’s topological sphere theorem [20, Theorem A] (for metric measure spaces).

The radius $rad(X, d)$ of a metric space (X, d) is defined as

$$rad(X, d) := \inf_{x \in X} \sup_{y \in X} d(x, y).$$

Before giving the proof of Theorem B, let us collect the relevant properties of a non-collapsed $RCD(n - 1, n)$ space with full support so that we can use them in the proof.

- (1). The diameter $diam(X, d)$ is at most π (in particular $rad(X, d) \leq \pi$) [21, Theorem 4.8].
- (2). (X, d, \mathcal{H}^n) satisfies the generalized Bishop-Gromov inequality:

$$\frac{\mathcal{H}^n(B_r(x))}{\mathcal{H}^n(B_R(x))} \geq \frac{Vol_{S^n}(B_r)}{Vol_{S^n}(B_R)}$$

for any $x \in X$ and $0 \leq r \leq R \leq \pi$, where $B_r(x)$ is the closed r -ball with center x in X , B_s is a closed s -ball in the sphere S^n , and Vol is the volume on the S^n . Equivalently

$$\frac{\mathcal{H}^n(B_r(x))}{\mathcal{H}^n(B_R(x))} \geq \frac{\int_0^r [\sin(t)]^{n-1} dt}{\int_0^R [\sin(t)]^{n-1} dt}$$

for any $x \in X$ and $0 \leq r \leq R \leq \pi$ [12].

- (3). [Ketterer’s maximal diameter theorem] If $rad(X, d) = \pi$, then (X, d) is isometric to (S^n, d_{S^n}) [21, Theorem 1.4].
- (4). [Honda-Mondello’s topological sphere theorem] For all $n \in \mathbb{N}_{\geq 2}$, there exists a positive constant $\epsilon_1(n) > 0$ such that if a compact metric space (X, d) satisfies that $rad(X, d) \geq \pi - \epsilon_1(n)$, and that (X, d, μ) is an

RCD($n - 1, n$) space for some Borel measure μ on X with full support, then X is homeomorphic to the n -dimensional sphere [20, Theorem A].

Proof of Theorem B. Since the diameter of a non-collapsed RCD($n - 1, n$) space is at most π , we can apply the argument in the proof of Theorem 1.3 in [14] to prove I.

Let $D(x) := \sup\{d(x, y) : y \in X\} \leq \pi$ for a fixed $x \in X$, $h(y) := d(x, y)$, and $\mu_0 := \frac{\mathcal{H}^n}{\mathcal{H}^n(X)}$, then we have

$$\text{md}(x) := \int_X h(y)\mu_0(dy) = \int_0^\infty \mu_0(\{h \geq s\})ds \tag{13}$$

$$= \int_0^{D(x)} [1 - \mu_0(h^{-1}([0, s]))]ds = \int_0^{D(x)} [1 - \mu_0(B_s(x))]ds. \tag{14}$$

We have the generalized Bishop-Gromov inequality for RCD($n - 1, n$) space, i.e.,

$$\mu_0(B_s(x)) \geq \frac{\text{Vol}(B_s)}{\text{Vol}(S^n)} \tag{15}$$

for $x \in X$ and $0 \leq s \leq \pi$. Hence, we get

$$\text{md}(x) = \int_X h(y)\mu_0(dy) \leq \pi - \int_0^\pi \frac{\text{Vol}(B_s)}{\text{Vol}(S^n)}ds. \tag{16}$$

The right hand side of the inequality (16) coincides with $\text{md}(p)$ for any $p \in S^n$ by virtue of the formula in (14). As $\text{md}(S^n) = \frac{\pi}{2}$ was showed by Kokkendorff above, we have $\text{md}(x) \leq \frac{\pi}{2}$. That means $\text{md}(X) \leq \frac{\pi}{2}$.

If we have equality $\text{md}(X) = \frac{\pi}{2}$, then Bishop-Gromov inequality (15) must be equality for all $0 \leq s \leq \pi$. It implies that for μ_0 -a.e. point x , there must exist a point x' with $d(x, x') = \pi$. And this must hold even for every $x \in X$, because X is compact and μ_0 is full support. Therefore, $\text{rad}(X) = \pi$ and then (X, d) is isometric to (S^n, d_{S^n}) by Ketterer's maximal diameter theorem.

We will prove III. by showing that $\text{md}(X)$ close to $\frac{\pi}{2}$ implies that $\text{rad}(X, d)$ is closed to π and by applying Honda-Mondello's topological sphere theorem.

We will prove the claim by contradiction. Suppose that there exists $x_0 \in X$ such that $D(x_0) < \pi - \epsilon_1(n)$ and that the x_0 realizes $\text{rad}(X, d)$. Here we take $\epsilon_1(n)$ from Honda-Mondello's topological sphere theorem. Then $\mu_0(B_s(x_0))$ will achieve its maximum value 1 for $\pi - \epsilon_1(n)$. Hence, we get an estimate

$$\begin{aligned} \text{md}(x_0) &= \int_0^{D(x_0)} [1 - \mu_0(B_s(x_0))]ds \leq \int_0^{\pi - \epsilon_1(n)} \left[1 - \frac{\text{Vol}(B_s)}{\text{Vol}(S^n)}\right] ds \\ &= \int_0^\pi \left[1 - \frac{\text{Vol}(B_s)}{\text{Vol}(S^n)}\right] ds - \int_{\pi - \epsilon_1(n)}^\pi \left[1 - \frac{\text{Vol}(B_s)}{\text{Vol}(S^n)}\right] ds \\ &=: \frac{\pi}{2} - \delta(\epsilon_1(n), n). \end{aligned}$$

Since D is 1-Lipschitz, we have

$$D(y) < \pi - \frac{\epsilon_1(n)}{2}$$

for any $y \in B_{\frac{\epsilon_1(n)}{2}}(x_0)$. This implies that we have the estimate

$$\text{md}(y) < \frac{\pi}{2} - \delta\left(\frac{\epsilon_1(n)}{2}, n\right)$$

for any $y \in B_{\frac{\epsilon_1(n)}{2}}(x_0)$. Finally, we have

$$\begin{aligned} \text{md}(X) &= \int_X \text{md}(x)\mu_0(dx) = \int_{X \setminus B_{\frac{\epsilon_1(n)}{2}}(x_0)} \text{md}(x)\mu_0(dx) \\ &\quad + \int_{B_{\frac{\epsilon_1(n)}{2}}(x_0)} \text{md}(x)\mu_0(dx) \\ &< \frac{\pi}{2} - \mu_0(B_{\frac{\epsilon_1(n)}{2}}(x_0))\delta\left(\frac{\epsilon_1(n)}{2}, n\right) \leq \frac{\pi}{2} - \frac{\text{Vol}(B_{\frac{\epsilon_1(n)}{2}})}{\text{Vol}(S^n)}\delta\left(\frac{\epsilon_1(n)}{2}, n\right). \end{aligned}$$

Now we take

$$\epsilon(n) = \frac{\text{Vol}(B_{\frac{\epsilon_1(n)}{2}})}{\text{Vol}(S^n)}\delta\left(\frac{\epsilon_1(n)}{2}, n\right) > 0.$$

Then $\text{md}(X) \geq \frac{\pi}{2} - \epsilon(n)$ implies $\text{rad}(X, d) \geq \pi - \epsilon_1(n)$. Otherwise, it would be a contradiction. Then X is homeomorphic to the n -dimensional sphere by Honda-Mondello’s topological sphere theorem. \square

Remark 15. The definition of metric measure spaces with the measure contraction property $\text{MCP}(K, n)$ was given independently by Ohta [27] and Sturm [30]. On general metric measure spaces, the two definitions slightly differ, but on essentially non-branching spaces they coincide [7, Appendix A]. $\text{MCP}(K, n)$ spaces are another kind of generalization of Riemannian manifolds with Ricci curvature lower bound, but a measure contraction property is weaker than the usual curvature dimension conditions [30, Remark 5.6].

However, $\text{MCP}(K, n)$ spaces also satisfy generalized Bishop–Gromov inequality and the generalized Bonnet-Myers theorem [30, Remark 5.2] [27, Theorem 4.3; Theorem 5.1]. Therefore, Theorem B. I. also holds for a compact non-collapsed $\text{MCP}(n - 1, n)$ space with full support.

Ketterer and Rajala show that the non-branching assumption is essential in Ohta’s topological rigidity [28, Theorem 5.5] in [23, Theorem 2]. It is not clear (for the author) whether the maximal diameter (rigidity) theorem holds for a compact non-collapsed and non-branching $\text{MCP}(n - 1, n)$ space with full support.

Remark 16. A notion for lower bounds of Ricci curvature on Alexandrov spaces was introduced by Zhang-Zhu in [32, 33]. Bisho-Gromov inequality and maximal diameter theorem hold for Zhang-Zhu’s spaces [32, Corollary 5.1]. Thus, one can show that Theorem B. I. and II. hold for an n -dimensional Alexandrov

space without boundary, and with full support and Ricci curvature $\geq n - 1$ in Zhang-Zhu's sense.

Inspired by the concept of mean distance, one can define similar metric invariants. For a compact metric measure space (X, d, μ) with full support, given a continuous function $\alpha : [0, \infty) \rightarrow \mathbb{R}$, we can define the metric invariant $M_\alpha(X)$ as

$$M_\alpha(X) := \int_{X \times X} \frac{\alpha(d(\cdot, \cdot))}{\mu(X) \times \mu(X)} \mu \otimes \mu$$

And then one can try to generalize Theorem B to $M_\alpha(X)$ case.

In particular, Erbar and Sturm [14] defines

$$M_f(X) := \int_{X \times X} \frac{f(d(\cdot, \cdot))}{\mu(X) \times \mu(X)} \mu \otimes \mu,$$

$$M_{f,n}^* := \frac{\int_0^\pi f(r) [\sin(r)]^{n-1} dr}{\int_0^\pi [\sin(r)]^{n-1} dr},$$

where $f : [0, \pi] \rightarrow \mathbb{R}$ is a continuous and strictly increasing function.

Theorem 17. (Erbar-Sturm) *Let (X, d, μ) be a compact non-collapsed $\text{RCD}(n-1, n)$ space with $n \geq 1$, then $M_f(X) \leq M_{f,n}^*$ and $M_f(X) = M_{f,n}^*$ if and only if X is isometric to the standard round n -sphere S^n .*

Erbar-Sturm also noticed that an analogous statement (with $M_f(X) \geq M_{f,n}^*$ in place of $M_f(X) \leq M_{f,n}^*$) holds for strictly decreasing f . Without loss of generality, let us assume that f is continuous and strictly increasing. Then we have the following corollary by combining Erbar-Sturm's theorem and the proof of Theorem B.

Corollary 18. *Let (X, d, \mathcal{H}^n) be a compact non-collapsed $\text{RCD}(n-1, n)$ space with full support, then there is an $\epsilon_2(f, n) > 0$ such that $M_f(X) \geq M_{f,n}^* - \epsilon_2(f, n)$ implies that X is homeomorphic to S^n .*

Proof. We only need to show that $M_f(X)$ close to $M_{f,n}^*$ implies that $\text{rad}(X, d)$ is closed to π . We will prove the claim by contradiction.

Suppose that there exists $x_0 \in X$ such that $D(x_0) < \pi - \epsilon_1(n)$ and the x_0 realizes $\text{rad}(X, d)$. Here we take $\epsilon_1(n)$ from Honda-Mondello's topological sphere theorem. Let $g(y) := f(d(x_0, y))$, $\mu_0 := \frac{\mathcal{H}^n}{\mathcal{H}^n(X)}$, and $M_f(x_0) := \int_X g(y) \mu_0(dy)$, then $\mu_0(B_s(x_0))$ will achieve its maximum value 1 for $\pi - \epsilon_1(n)$.

Hence, we get an estimate

$$M_f(x_0) < M_{f,n}^* - \int_{f(\pi - \epsilon_1(n))}^{f(\pi)} \left[1 - \frac{\int_0^{f^{-1}(s)} [\sin(t)]^{n-1} dt}{\int_0^\pi [\sin(t)]^{n-1} dt} \right] ds$$

$$:= M_{f,n}^* - \delta_1(f, \epsilon_1(n), n).$$

Since D is 1-Lipschitz and f is continuous and strictly increasing, we have

$$f(D(y)) < f\left(\pi - \frac{\epsilon_1(n)}{2}\right)$$

for any $y \in B_{\frac{\epsilon_1(n)}{2}}(x_0)$. It implies that we have the estimate

$$M_f(y) < M_{f,n}^* - \delta_1\left(f, \frac{\epsilon_1(n)}{2}, n\right)$$

for any $y \in B_{\frac{\epsilon_1(n)}{2}}(x_0)$. Finally, we have

$$\begin{aligned} M_f(X) &= \int_X M_f(x)\mu_0(dx) = \int_{X \setminus B_{\frac{\epsilon_1(n)}{2}}(x_0)} M_f(x)\mu_0(dx) \\ &\quad + \int_{B_{\frac{\epsilon_1(n)}{2}}(x_0)} M_f(x)\mu_0(dx) \\ &< M_{f,n}^* - \mu_0(B_{\frac{\epsilon_1(n)}{2}}(x_0))\delta_1\left(f, \frac{\epsilon_1(n)}{2}, n\right) \\ &\leq M_{f,n}^* - \frac{\text{Vol}(B_{\frac{\epsilon_1(n)}{2}})}{\text{Vol}(S^n)}\delta_1\left(f, \frac{\epsilon_1(n)}{2}, n\right). \end{aligned}$$

Now we take

$$\epsilon_2(f, n) = \frac{\text{Vol}(B_{\frac{\epsilon_1(n)}{2}})}{\text{Vol}(S^n)}\delta_1\left(f, \frac{\epsilon_1(n)}{2}, n\right) > 0.$$

Then $M_f(X) \geq M_{f,n}^* - \epsilon(n)$ implies $\text{rad}(X, d) \geq \pi - \epsilon_1(n)$. Otherwise, it would be a contradiction. Then X is homeomorphic to the n -dimensional sphere by Honda-Mondello's topological sphere theorem. \square

Remark 19. The result of Corollary 18 is even new (for $f \neq \text{id}$) for a smooth Riemannian n -manifold with Ricci curvature bounded below by $(n - 1)$ and X is diffeomorphic to S^n by Cheeger-Colding's differentiable sphere theorem.

Gromov defines the observable diameter $\text{ObsDiam}(X; -\kappa)$ for metric measure spaces (X, d, μ) in [18] and shows the following theorem. The detailed proof can be found in [29, Theorem 2.29].

Theorem 20. (Gromov) *Let X be a closed n -dimensional Riemannian manifold with Ricci curvature $\geq (n - 1)$. Then, for any κ with $0 < \kappa \leq 1$, we have*

$$\text{ObsDiam}(X; -\kappa) \leq \text{ObsDiam}(S^n; -\kappa) = \pi - 2v^{-1}\left(\frac{\kappa}{2}\right)$$

where

$$v(r) := \frac{\int_0^r [\sin(t)]^{n-1} dt}{\int_0^\pi [\sin(t)]^{n-1} dt}.$$

It is not clear whether Gromov's theorem can be generalized to non-collapse $\text{RCD}(n - 1, n)$ spaces. If it can, can we show the sphere theorem about it?

The note has no associated data.

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References

- [1] Abresch, U., Meyer, W.T.: Injectivity radius estimates and sphere theorems. In: Comparison Geometry (Berkeley, CA, 1993–94), vol. 30 of *Math. Sci. Res. Inst. Publ.*, pp. 1–47. Cambridge Univ. Press, Cambridge (1997)
- [2] Brendle, S., Schoen, R.: Sphere theorems in geometry. In: Surveys in Differential Geometry. Vol. XIII. Geometry, Analysis, and Algebraic Geometry: Forty Years of the Journal of Differential Geometry, volume 13 of *Surv. Differ. Geom.*, pp. 49–84. Int. Press, Somerville, MA (2009)
- [3] Brendle, S., Schoen, R.: *Curvature, sphere theorems, and the Ricci flow*. Bull. Am. Math. Soc. **48**(1), 1–32 (2011)
- [4] Cheeger, J., Colding, T.H.: *On the structure of spaces with Ricci curvature bounded below*. I. J. Differ. Geom. **46**(3), 406–480 (1997)
- [5] Chern, S.S., Goldberg, S.I.: *On the volume decreasing property of a class of real harmonic mappings*. Am. J. Math. **97**, 133–147 (1975)
- [6] Chern, S.-s.: On holomorphic mappings of hermitian manifolds of the same dimension. In: Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966), pp. 157–170. Amer. Math. Soc., Providence, R.I. (1968)

- [7] Cavalletti, F., Mondino, A.: *Optimal maps in essentially non-branching spaces*. Commun. Contemp. Math. **19**(6), 1750007 (2017)
- [8] Deng, J.: *Curvature-dimension condition meets Gromov's n -volumic scalar curvature*. SIGMA Symm. Integr. Geom. Methods Appl. **17**, 013 (2021)
- [9] Deng, J.: *Enlargeable length-structure and scalar curvatures*. Ann. Glob. Anal. Geom. **60**(2), 217–230 (2021)
- [10] Deng, J.: *Foliated Positive Scalar Curvature*. PhD thesis, University of Goettingen (2021)
- [11] Doyle, J.K., Graver, J.E.: *Mean distance in a graph*. Discrete Math. **7**(2), 147–154 (1977)
- [12] De Philippis, G., Gigli, N.: *Non-collapsed spaces with Ricci curvature bounded from below*. J. Éc. Polytech. Math. **5**, 613–650 (2018)
- [13] Eells, J., Jr., Sampson, J.H.: *Harmonic mappings of Riemannian manifolds*. Am. J. Math. **86**, 109–160 (1964)
- [14] Erbar, M., Sturm, K.T.: *Rigidity of cones with bounded Ricci curvature*. J. Eur. Math. Soc. (JEMS) **23**(1), 219–235 (2021)
- [15] Goldberg, S.I., Ishihara, T.: *Harmonic quasiconformal mappings of Riemannian manifolds*. Am. J. Math. **98**(1), 225–240 (1976)
- [16] Gromov, M., Lawson, H.B., Jr.: *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*. Inst. Hautes Études Sci. Publ. Math. **58**, 83–196 (1984)
- [17] Gromov, M.: *Large Riemannian manifolds*. In: *Curvature and Topology of Riemannian Manifolds* (Katata, 1985), volume 1201 of *Lecture Notes in Math.*, pp. 108–121. Springer, Berlin (1986)
- [18] Gromov, M.: *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, (1999). Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
- [19] Goette, S., Semmelmann, U.: *Scalar curvature estimates for compact symmetric spaces*. Differ. Geom. Appl. **16**(1), 65–78 (2002)
- [20] Honda, S., Mondello, I.: *Sphere theorems for RCD and stratified spaces*. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **22**(2), 903–923 (2021)
- [21] Ketterer, C.: *Cones over metric measure spaces and the maximal diameter theorem*. J. Math. Pures Appl. (9) **103**(5), 1228–1275 (2015)
- [22] Kokkendorff, S.L.: *Characterizing the round sphere by mean distance*. Differ. Geom. Appl. **26**(6), 638–644 (2008)
- [23] Ketterer, C., Rajala, T.: *Failure of topological rigidity results for the measure contraction property*. Potential Anal. **42**(3), 645–655 (2015)
- [24] Llarull, M.: *Scalar curvature and volume of a Riemannian manifold*. Geom. Dedicata **56**(1), 1–4 (1995)
- [25] Llarull, M.: *Sharp estimates and the Dirac operator*. Math. Ann. **310**(1), 55–71 (1998)
- [26] Lyons, R.: *Distance covariance in metric spaces*. Ann. Probab. **41**(5), 3284–3305 (2013)

- [27] Ohta, S.: *On the measure contraction property of metric measure spaces*. Comment. Math. Helv. **82**(4), 805–828 (2007)
- [28] Ohta, S.I.: *Products, cones, and suspensions of spaces with the measure contraction property*. J. Lond. Math. Soc. (2) **76**(1), 225–236 (2007)
- [29] Shioya, T.: *Metric measure geometry*, volume 25 of *IRMA Lectures in Mathematics and Theoretical Physics*. EMS Publishing House, Zürich, 2016. Gromov’s theory of convergence and concentration of metrics and measures
- [30] Sturm, K.T.: *On the geometry of metric measure spaces*. II. Acta Math. **196**(1), 133–177 (2006)
- [31] Schoen, R., Yau, S.T.: *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*. Ann. Math. (2) **110**(1), 127–142 (1979)
- [32] Zhang, H.C., Zhu, X.P.: *Ricci curvature on Alexandrov spaces and rigidity theorems*. Commun. Anal. Geom. **18**(3), 503–553 (2010)
- [33] Zhang, H., Zhu, X.: *On a new definition of Ricci curvature on Alexandrov spaces*. Acta Math. Sci. Ser. B (Engl. Ed.) **30**(6), 1949–1974 (2010)

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