Notes on flat fronts in hyperbolic space

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Abstract. We give a short introduction to discrete flat fronts in hyperbolic space and prove that any discrete flat front in the mixed area sense admits a Weierstrass representation.

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1. Introduction

Flat fronts in hyperbolic space—that is, intrinsically flat surfaces in hyperbolic space that may have certain types of singularities—yield an interesting class of surfaces/fronts for their relations to various areas in differential geometry and, in particular, also for the study of singularities of special surfaces: they admit a Weierstrass type representation [10] that is of a rather geometric nature as it is directly linked to the geometry of a pair of hyperbolic Gauss maps that form a curved flat in the integrable systems sense [4]. For example, this observation establishes a relation with Darboux pairs of minimal surfaces in Euclidean space [12], c.f. [15]. In a sphere geometric context, these hyperbolic Gauss maps are related to a pair of isothermic sphere congruences that put flat fronts in the context of Demoulin’s Ω surfaces and, in particular, of linear Weingarten surfaces [5]; on the other hand, flat fronts occur in parallel families as orthogonal surfaces to a cyclic system spanned by the pair of hyperbolic Gauss maps, c.f. [4].

Two approaches have been taken to discretize flat fronts in hyperbolic space, in an integrable systems sense: firstly, by discretizing their Weierstrass representation in [14], secondly, as particular linear Weingarten surfaces in a hyperbolic space form in [6]. A relation has been established by showing that the nets obtained by the discrete Weierstrass-type representation satisfy the

FIGURE 1 Two parallel discrete flat front patches with their common hyperbolic Gauss maps

second definition [18], though the converse seems not to be established yet. However, in [13] we recently related discrete flat fronts in the former sense to Darboux pairs (curved flats of [9]) of holomorphic maps into the 2-sphere, c.f. Fig. 1, which is a building block towards proving the converse.

These notes grew out of the first author's MSc thesis, supervised by the other authors: we aim to give a relatively gentle and low-tech introduction to the geometry of (discrete) flat fronts in hyperbolic space, following various threads and approaches and to knit them together. As a new result, we shall prove that Darboux pairs (curved flats) of discrete holomorphic maps admit a Weierstrass-type representation. As a consequence, any discrete linear Weingarten net in hyperbolic space with mixed area Gauss curvature $K \equiv 1$ then admits a Weierstrass-type representation.

2. Hyperbolic geometry

From a differential geometric point of view it is convenient to embed hyperbolic space into a (flat) vector space geometry, so that covariant differentiation becomes just ordinary differentiation followed by a projection: thus we consider hyperbolic space as one of the sheets of a two-sheeted hyperboloid in a Minkowski space $\mathbb{R}^{3,1} = (\mathbb{R}^4, (.,.))$:

$$H^3 = \{x \in \mathbb{R}^{3,1} \mid (x, x) = -1, (x, o) < 0\},$$

where $o \in H^3$ denotes some unit timelike vector, that distinguishes one sheet of the hyperboloid. To visualize objects in this 3-dimensional hyperbolic geometry we then project to a (non-flat) Poincaré model, where H^3 is conformally

embedded into Euclidean 3-space, for example, by means of the stereographic projection into the unit ball of $\mathbb{R}^3 = \{o\}^\perp \subset \mathbb{R}^{3,1}$:

$$\sigma : H^3 \rightarrow \mathbb{R}^3, \quad x \mapsto \sigma(x) := \frac{x + o(o, x)}{1 - (o, x)}.$$

In what follows we discuss two enhancements of this setup that provide for a better geometric handling of flat surfaces and fronts in hyperbolic space, and for an algebraic setting that caters for their Weierstrass type representation, respectively.

2.1. Hyperbolic geometry as a subgeometry of Lie sphere geometry

Hyperbolic geometry, as any other space form geometry, may be considered as a subgeometry of Lie sphere geometry: the classical incarnation of contact geometry. This approach is, in particular, well suited to investigate fronts, that is, surfaces $x : \Sigma^2 \rightarrow H^3$ that may have certain types of singularities but still admit a smooth unit normal field (Gauss map) $n : \Sigma^2 \rightarrow S^{2,1}$ such that the pair (x, n) is an immersion. Our short description here will be tailored to the specific case of hyperbolic geometry, for more general and comprehensive introductions see [1] or [8].

Thus consider $\mathbb{R}^{3,1} \subset \mathbb{R}^{4,2} = \mathbb{R}^{1,1} \oplus_\perp \mathbb{R}^{3,1}$ and let (p, q) denote an orthonormal basis of $\mathbb{R}^{1,1}$, that is, $-(p, p) = (q, q) = 1$ and $p \perp q$; hence $\mathbb{R}^{3,1} = \langle p, q \rangle^\perp$, where $\langle \dots \rangle$ denotes the linear span of vectors. We now consider the affine quadrics

$$\begin{aligned} Q^3 &:= \{y \in \mathcal{L} \mid (y, p) = 0, (y, q) = -1\}, \\ P^3 &:= \{y \in \mathcal{L} \mid (y, p) = -1, (y, q) = 0\}, \end{aligned} \quad \text{where } \mathcal{L} := \{y \in \mathbb{R}^{4,2} \mid (y, y) = 0\}$$

denotes the light cone of $\mathbb{R}^{4,2}$. Clearly, P^3 and Q^3 are, as offsets of the unit 1- and 2-sheeted hyperboloids in $\mathbb{R}^{3,1}$ by p and $-q$, respectively, isometric to these hyperboloids. In particular, each component of Q^3 yields a model of hyperbolic 3-space in the light cone $\mathcal{L} \subset \mathbb{R}^{4,2}$. The common infinity boundary of these two hyperbolic spaces is formed by the null directions $\langle x \rangle$ in $\mathbb{R}^{3,1} \subset \mathbb{R}^{4,2}$, where $x \in \mathcal{L} \cap \langle p, q \rangle^\perp$, and inherits a 2-dimensional Möbius geometry from the hyperbolic geometry of Q^3 and accordingly the group of Lie sphere transformations that fix p and q .

More generally, points $\langle s \rangle$ of the Lie quadric (the projective light cone of $\mathbb{R}^{4,2}$, i.e., $s \in \mathcal{L}$) represent oriented spheres: in hyperbolic geometry these come in various flavours; the “point spheres”, $s \perp p$, of either sheet of Q^3 or of their common infinity boundary have been discussed above. Accordingly, we call p the “point sphere complex”.

A hyperbolic “distance sphere” with centre $c \in H^3 \subset \mathbb{R}^{4,2}$ and radius $r \in \mathbb{R}$ is represented by $\langle s \rangle$ with $s = p \sinh r - q \cosh r + c$: the intersection $Q^3 \cap \langle s \rangle^\perp$ consists of points in distance r from $c - q$, as evaluating $0 = (x(t), s)$ on an arc-length parametrization $t \mapsto x(t) = c \cosh t + c' \sinh t - q$ of a geodesic in Q^3 emanating from $c - q$ in a unit direction $c' \in T_{c-q}Q^3 = T_cH^3$ shows. Note that a sign swap of the radius r corresponds to reflection of s in $\langle p \rangle^\perp$: this

reflection yields the orientation reversal; point spheres ($r = 0$) are invariant under this reflection. A reflection in $\langle q \rangle^\perp$, on the other hand, swaps the sheets of Q^3 , hence moving a distance sphere from one hyperbolic space into the other.

Spheres represented by $\langle p \cosh r + q \sinh r + n \rangle$ with $n \in S^{2,1}$ and $r \in \mathbb{R}$ intersect the infinity sphere of the two sheets of Q^3 : $\langle n, p, q \rangle^\perp \cap \mathcal{L} \neq \{0\}$. In particular, if $r = 0$ then $p + n \in P^3$ is invariant under a reflection in $\langle q \rangle^\perp$, that is, under an inversion in the infinity boundary, hence represents a plane in both hyperbolic spaces. Accordingly, we call q the “plane (sphere) complex”.

Given a point $x \in H^3$ and a unit vector $n \in T_x H^3$, any sphere through x with normal n at x is represented by $s \in \langle x - q, n + p \rangle$, for example, all distance spheres $(x - q) \cosh r + (n + p) \sinh r$ touch at the point x . Note that $\langle x - q, n + p \rangle \subset \mathbb{R}^{4,2}$ is a null 2-plane, hence defines a line in the Lie quadric. Such lines will be called “contact elements”.

Two spheres $\langle s \rangle$ and $\langle s' \rangle$ “touch” (are in oriented contact) if they span a contact element, that is, if $(s, s') = 0$. In particular, a sphere is a horosphere if it touches either of the oriented infinity spheres $\langle q \pm p \rangle$ of the two hyperbolic sheets of Q^3 : thus a horosphere is represented by $h = q \pm p + x$, where $x \in \mathcal{L} \cap \langle p, q \rangle^\perp$ yields (homogeneous coordinates of) the point of contact with the infinity boundary.

Lie sphere transformations are given by the orthogonal transformations of $\mathbb{R}^{4,2}$, hence act on the space of oriented spheres (the Lie quadric) and preserve contact. Hyperbolic isometries are those Lie sphere transformations that preserve the point and plane sphere complexes, $\langle p \rangle$ and $\langle q \rangle$, and that do not swap the sheets of Q^3 . Of interest will also be the “parallel transformations”, given by Lorentz boosts in q -direction, $(p, q) \mapsto (p, q) \begin{pmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{pmatrix}$, that shift the radius of all distance spheres:

$$p \sinh r - q \cosh r + c \mapsto p \sinh(r + s) - q \cosh(r + s) + c.$$

2.2. Hyperbolic geometry using Pauli matrices

Weierstrass-type representations of surfaces in hyperbolic space typically employ a model of hyperbolic geometry that is based on Hermitian matrices (or forms), using an explicit incarnation of the (universal) covering of the group of hyperbolic motions (the restricted Lorentz group) by the special linear group $\mathrm{Sl}(2, \mathbb{C})$, cf [3, 10] and [16, Sect. 4.2]. Here we will briefly recall this model of hyperbolic geometry, for completeness.

Thus consider the (real) 4-dimensional space of Hermitian 2×2 -matrices $\mathrm{Herm}(2)$: equipped with the quadratic form $-\det : \mathrm{Herm}(2) \rightarrow \mathbb{R}$ this space becomes a 4-dimensional real Minkowski space, $(\mathrm{Herm}(2), -\det) \cong \mathbb{R}^{3,1}$, that $\mathrm{Sl}(2, \mathbb{C})$ acts transitively on as its restricted Lorentz group via

$$\mathrm{Sl}(2, \mathbb{C}) \times \mathrm{Herm}(2) \rightarrow \mathrm{Herm}(2), \quad (G, X) \mapsto G \cdot X = GXG^*.$$

Together with the identity matrix the Pauli matrices form an orthonormal basis (E_0, E_1, E_2, E_3) of $\text{Herm}(2)$,

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hyperbolic geometry is then modelled on $H^3 \cong \{X \in \text{Herm}(2) \mid \det X = 1, \text{tr } X > 0\}$ with the group $\text{Sl}(2, \mathbb{C})$ acting on H^3 by hyperbolic motions; note that orientation reversing hyperbolic isometries (e.g., hyperbolic reflections) are not obtained from the action of $\text{Sl}(2, \mathbb{C})$ on $\text{Herm}(2)$.

In this setting of hyperbolic geometry, its boundary Möbius geometry comes in two flavours: as the Möbius geometry of restricted Lorentz transformations on the projective light cone of $\mathbb{R}^{3,1}$; and as the geometry of the complex projective line $\mathbb{CP}^1$, acted upon by complex Möbius transformations. The relation between the two models is established by the Hermitian dyadic square,

$$\mathbb{CP}^1 \ni \langle x \rangle \mapsto \langle xx^* \rangle \in \mathbb{P}(\mathcal{L}^3), \quad \text{where } x \in \mathbb{C}^2 \text{ and } \mathcal{L}^3 = \{X \in \text{Herm}(2) \mid \det X = 0\},$$

which is obviously compatible with the actions of $\text{Sl}(2, \mathbb{C})$ on $\mathbb{CP}^1$ by linear fractional transformations and on the projective light cone $\mathbb{P}(\mathcal{L}^3)$ of $\text{Herm}(2)$ by the restricted Lorentz group. Also note that this map has an inverse as every singular Hermitian matrix is a Hermitian dyadic square.

To obtain the hyperbolic Gauss maps of a surface or front in H^3 we will later be interested to determine those points, where a geodesic emanating from a point $C \in H^3$ with a unit initial velocity $C' \in T_C H^3$ hits the infinity sphere. Using, as before, an arc-length parametrization we take the limits

$$\lim_{t \rightarrow \pm\infty} \langle C \cosh t + C' \sinh t \rangle = \langle C \pm C' \rangle$$

and observe that $-\det(C \pm C') = (C, C) \pm 2(C, C') + (C', C') = 0$, that is, $C \pm C' \in \mathcal{L}^3$ indeed. If, further, $C = G \cdot E_0$ and $C' = G \cdot E_3$ are given as images of the basis vectors E_0 and E_3 under a hyperbolic rotation $G = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then

$$\begin{aligned} C + C' &= G(E_0 + E_3)G^* = 2 \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}^* \quad \text{and} \\ C - C' &= G(E_0 - E_3)G^* = 2 \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}^*, \end{aligned}$$

that is, the columns of the matrix $G \in \text{Sl}(2, \mathbb{C})$ yield (homogeneous coordinates for) the points at infinity of the geodesic, as points in $\mathbb{CP}^1$.

3. Discrete flat fronts in hyperbolic space

Discrete flat fronts in hyperbolic space have been approached from two rather different angles:

- in [14, Sect. 4.3] they were defined in terms of a Weierstrass representation and shown to be principal (circular) nets (Thm 4.6 and Rem 4.7 in [14]);

- in [6, Expl 4.3] they were investigated in terms of their mixed area (extrinsic) Gauss curvature, as a special instance of discrete linear Weingarten surfaces.

A relation was established in [18, Sect. 5], where the discrete flat fronts of [14] were shown to be discrete linear Weingarten surfaces with a constant mixed area Gauss curvature $K \equiv 1$, see [18, Prop 5.1 and Lemma 5.3]. Further, in [17], the Weierstrass-type representation of [14] is given a sphere geometric interpretation, cf [17, Thm 4.9 and Expl 4.8]. Another sphere geometric approach to flat fronts, in terms of (discrete) cyclic systems, has been discussed in [13, Sect. 4.2], where a focus is on the geometry of the two hyperbolic Gauss maps $h^\pm$ of a flat front — which draws a connection to the discrete curved flats of [9, Chap 8 (Thm 8.4.1)].

Here we aim to knit the various threads together, to obtain further insight into the geometry of (discrete) flat fronts and, in particular, to prove that every flat front in the sense of [6] admits a Weierstrass-type representation of [14]. In the process we will present some short and simple proofs for some of the results described above.

3.1. A Weierstrass-type representation

To set the scene we fix some notations, cf [6] or [7]: we shall consider maps that “live” on cells of some dimension of a quadrilateral cell complex, $\Sigma^2 = (\Sigma_0^2, \Sigma_1^2, \Sigma_2^2)$, where Σ_0^2 denotes the set of vertices, Σ_1^2 the set of (oriented) edges, and Σ_2^2 the set of (oriented) quadrilateral faces; the elements of Σ_1^2 and Σ_2^2 will usually be denoted by their vertices, i.e., $(ij) \in \Sigma_1^2$ denotes an edge from $i \in \Sigma_0^2$ to $j \in \Sigma_0^2$, and $(ijkl) \in \Sigma_2^2$ denotes an oriented face with vertices $i, j, k, l \in \Sigma_0^2$ and edges $(ij), (jk), (kl), (li) \in \Sigma_1^2$.

We will generally assume Σ^2 to be *simply connected*, that is, any two points $i, j \in \Sigma_0^2$ can be connected by an (edge-)path and any closed (edge-)path is null-homotopic, via “face-flips” and via dropping single edge “return trips”, cf [7, Sect. 2.3].

Given a vector-valued function $i \mapsto g_i$ on vertices, its *derived function* and discrete *derivative* will be denoted by

$$g_{ij} := \frac{g_i + g_j}{2} \quad \text{and} \quad dg_{ij} := g_j - g_i,$$

so that a Leibniz rule $d(fg)_{ij} = df_{ij}g_{ij} + f_{ij}dg_{ij}$ (for any product of vectors) holds; note that the derived edge function is a function on unoriented edges while the derivative is a discrete 1-form,

$$g_{ij} = g_{ji} \quad \text{while} \quad dg_{ij} = -dg_{ji}.$$

If $g : \Sigma_0^2 \rightarrow \mathbb{C}$ is a complex-valued vertex function, then we may compute the cross ratio on (quadrilateral) faces,

$$\text{cr}(g_i, g_j, g_k, g_l) := \frac{dg_{ij}}{dg_{jk}} \frac{dg_{kl}}{dg_{li}};$$

note that this does not define a map on faces, but it does after factoring out the anharmonic group. Then g will be called *holomorphic* if it is a (totally umbilic) discrete isothermic net:

$$\forall(ijkl) \in \Sigma_2^2 : \text{cr}(g_i, g_j, g_k, g_l) = \frac{a_{ij}}{a_{jk}}, \quad (3.1)$$

where $a : \Sigma_1^2 \rightarrow \mathbb{R}$ denotes a (real) *edge-labelling*, that is, a function on edges so that

$$\forall(ij) \in \Sigma_1^2 : a_{ij} = a_{ji} \text{ and } \forall(ijkl) \in \Sigma_2^2 : a_{ij} = a_{kl}.$$

For regularity, we further assume that $\forall(ijkl) \in \Sigma_2^2 : \text{cr}(g_i, g_j, g_k, g_l) \neq 0, 1, \infty$.

Figure 1 shows (the inverse stereographic projections of) two holomorphic maps (blue and green) with (constant) cross ratios -1 , thus with edge labellings with values ± 1 , up to scaling.

We are now in a position to recall the representation of discrete flat fronts from [14, Sect. 4.3]

Weierstrass representation Let $g : \Sigma_0^2 \rightarrow \mathbb{C}$ be a discrete holomorphic function, with edge-labelling $a : \Sigma_1^2 \rightarrow \mathbb{R}$, and fix $t \in \mathbb{R} \setminus \{0, \frac{1}{a_{ij}} \mid (ij) \in \Sigma_1^2\}$; then there is a frame $F : \Sigma_0^2 \rightarrow \text{Sl}(2, \mathbb{C})$, satisfying

$$F_j = F_i W_{ij} \text{ with } W_{ij} := \begin{pmatrix} 1 & dg_{ij} \\ \frac{ta_{ij}}{dg_{ij}} & 1 \end{pmatrix} \frac{1}{\sqrt{1 - ta_{ij}}}, \quad (3.2)$$

that yields a 1-parameter family of parallel discrete flat fronts $(X, N) : \Sigma_0^2 \rightarrow H^3 \times S^{2,1}$, where

$$X := F(E_0 \cosh s + E_3 \sinh s) F^* \text{ and } N := F(E_0 \sinh s + E_3 \cosh s) F^* \quad (3.3)$$

for $s \in \mathbb{R}$ and $E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \text{Herm}(2)$, as above.

Remark By Christoffel's formula $dg_{ij}^* = \frac{a_{ij}}{d\bar{g}_{ij}}$ with the (isothermic) dual g^* of g , cf [11, §5.7.7]; hence

$$W = \begin{pmatrix} 1 & dg \\ t d\bar{g}^* & 1 \end{pmatrix} \frac{1}{\sqrt{1 - t\bar{a}}}.$$

Note that $W : \Sigma_1^2 \rightarrow \text{Sl}(2, \mathbb{C})$ defines a *discrete connection* on the vector bundle $\Sigma_0^2 \times \text{Herm}(2)$, since

$$W_{ij} W_{ji} = \begin{pmatrix} 1 & dg_{ij} \\ t d\bar{g}_{ij}^* & 1 \end{pmatrix} \begin{pmatrix} 1 & -dg_{ij} \\ -t d\bar{g}_{ij}^* & 1 \end{pmatrix} \frac{1}{\sqrt{1 - ta_{ij}}^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Integrability. To substantiate the claims in relation to the above Weierstrass-type representation, first note that the frame F indeed exists: as Σ^2 is assumed to be simply connected, the integrability of (3.2) is equivalent to the (metric) connection W being flat (cf [7, Prop 2.17]), that is,

$$\forall(ijkl) \in \Sigma_2^2 : W_{ij} W_{jk} = W_{il} W_{lk}. \quad (3.4)$$

Using that $a : \Sigma_1^2 \rightarrow \mathbb{R}$ is an edge-labelling, the flatness (3.4) of the connection W reduces to the integrability of the Christoffel equation for g^* , and to the cross ratio condition (3.1) for g to be holomorphic, cf [14, Thm 4.6 (proof)] and [18, Sect. 4.3].

Remark. The Weingarten equation (3.2) yields a propagation for the pair (X, N) via reflections. Namely, let

$$Y_{ij} := \begin{pmatrix} |dg_{ij}|^2 & dg_{ij} \\ d\bar{g}_{ij} & ta_{ij} \end{pmatrix} \frac{1}{|dg_{ij}| \sqrt{1 - ta_{ij}}} \in \text{Herm}(2)$$

and observe that $F_i Y_{ij} F_i^* + F_j Y_{ji} F_j^* = F_j (W_{ji} Y_{ij} + Y_{ji} W_{ij}^*) F_i^* = 0$; hence $R_{ij} := F_i Y_{ij} F_i^*$ yields a well defined reflection on $\mathbb{R}^{3,1} \cong (\text{Herm}(2), -\det)$,

$$X \mapsto \varrho_{ji}(X) := X - 2R_{ij}(R_{ij}, X) = X + R_{ij} \frac{\det(X + R_{ij}) - \det(X - R_{ij})}{2}.$$

As $E_k - 2Y_{ij}(Y_{ij}, E_k) = W_{ij} E_k W_{ij}^*$ for $k = 0, 3$ we find that these reflections transport the points between adjacent geodesics from X in direction N ,

$$X_j = \varrho_{ji}(X_i) \text{ and } N_j = \varrho_{ji}(N_i) \text{ for } s \in \mathbb{R}.$$

Note that, in accordance with [13, Sect. 4.2], $\langle R_{ij} \rangle = \langle d(X \pm N)_{ij} \rangle$.

Circularity. In fact, the map $(X, N) : \Sigma_0^2 \rightarrow H^3 \times S^{2,1}$ satisfies a discrete *Rodrigues equation* on edges,

$$0 = dN_{ij} + k_{ij} dX_{ij}, \text{ where } k_{ij} = -\frac{(dN_{ij}, R_{ij})}{(dX_{ij}, R_{ij})}$$

is the *principal curvature* of the edge; note that $k_{ij} = k_{ji}$. To obtain a Lie geometric interpretation we embed $\mathbb{R}^{3,1} \subset \mathbb{R}^{4,2} = \mathbb{R}^{3,1} \oplus_{\perp} \mathbb{R}^{1,1}$ with $\mathbb{R}^{1,1} = \langle p, q \rangle$, as before, then consider the discrete *sphere congruences* $\langle x \rangle$ and $\langle n \rangle$ given by the light cone maps

$$X - q =: x : \Sigma_0^2 \rightarrow Q^3 \text{ and } N + p =: n : \Sigma_0^2 \rightarrow P^3.$$

Clearly, $dn = dN$ and $dx = dX$ satisfy the same Rodrigues equation, hence adjacent contact elements intersect in a common (edge-) *curvature sphere* $\langle \kappa_{ij} \rangle = \langle x, n \rangle_i \cap \langle x, n \rangle_j$ —which characterizes the contact element map $\langle x, n \rangle$ as a discrete *Legendre map*, cf [2, Sect. 3.5] or [6, Def 2.1]. As a consequence, each map $X : \Sigma_0^2 \rightarrow H^3$ resp $x : \Sigma_0^2 \rightarrow Q^3$ of the parallel family is a *principal* (or, *circular*) *net*, cf [14, Thm 4.6] and [18, Lemma 5.1].

Remark. As $x : \Sigma_0^2 \rightarrow Q^3$ and $n : \Sigma_0^2 \rightarrow P^3$, these are the point sphere and tangent plane congruences of the Legendre map $\langle x, n \rangle$, respectively. Note how the aforementioned hyperbolic parallel transformations of Sect. 2.1 move through the parallel family in the Weierstrass representation:

$$\begin{aligned} & \langle X - (p \sinh s + q \cosh s), N + (p \cosh s + q \sinh s) \rangle \\ &= \langle (X \cosh s + N \sinh s) - q, (X \sinh s + N \cosh s) + p \rangle. \end{aligned}$$

Curvature. Finally, we argue that $(X, N) : \Sigma_0^2 \rightarrow H^3 \times S^{2,1}$ defines a flat front; to this end we compute face diagonals $\delta H_{ik}^{\pm} := H_k^{\pm} - H_i^{\pm}$ for its hyperbolic

Gauss maps, $H^\pm = X \pm N$, then determine its (extrinsic) mixed area *Gauss curvature* K on faces from the equation (cf [6, Def 2.3])

$$A(H^+, H^-) = A(X, X) - A(N, N) = A(X, X)(1 - K),$$

where $A : \Sigma_2^2 \rightarrow \Lambda^2 \mathbb{R}^{3,1}$ denotes the (vector-valued) mixed area, a symmetric bilinear form given by its quadratic form

$$A(H, H)_{ijkl} := \frac{1}{2} \delta H_{ik} \wedge \delta H_{jl}.$$

Thus let $E^\pm := E_0 \pm E_3$, so that $H^\pm = FE^\pm F^*$, and let $r : \Sigma_0^2 \rightarrow \mathbb{R}$ be a function that factorizes $\frac{|dg|^2}{a} : \Sigma_1^2 \rightarrow \mathbb{R}$, so that $dg_{ij}^* = \frac{a_{ij}}{d\bar{g}_{ij}} = \frac{dg_{ij}}{r_i r_j}$ (cf [2, Thm 2.31] or [6, (3.1)]); hence compute

$$\begin{aligned} F_i^{-1} \delta H_{ik}^+(F_k^*)^{-1} &= W_{ij} W_{jk} E^+ - (W_{kl} W_{li} E^+)^* \\ &= \begin{pmatrix} \left(\frac{dg_{ij}}{d\bar{g}_{jk}} \frac{a_{jk}}{r_k} - \frac{dg_{il}}{d\bar{g}_{lk}} \frac{a_{ij}}{r_i} \right) r_k & \delta g_{ik}^* \\ \delta \bar{g}_{ik}^* & 0 \end{pmatrix} \frac{2t}{\sqrt{1-ta_{ij}} \sqrt{1-ta_{jk}}}, \\ F_i^{-1} \delta H_{jl}^-(F_k^*)^{-1} &= W_{il} E^- W_{kl}^* - W_{ij} E^- W_{kj}^* \\ &= \begin{pmatrix} \left(\frac{dg_{ij}}{d\bar{g}_{jk}} a_{jk} r_j - \frac{dg_{il}}{d\bar{g}_{lk}} a_{ij} r_l \right) r_k & \delta g_{jl} \\ \delta \bar{g}_{jl} & 0 \end{pmatrix} \frac{2}{\sqrt{1-ta_{ij}} \sqrt{1-ta_{jk}}}. \end{aligned}$$

Since $0 = \frac{\delta g_{ik}^*}{\delta(1/r)_{ik}} + \frac{\delta g_{jl}}{\delta r_{jl}}$, from the Christoffel equation (cf [2, (2.37)] or [6, (3.3)]), we conclude that

$$\forall (ijkl) \in \Sigma_2^2 : \delta H_{ik}^+ \parallel \delta H_{jl}^- \Rightarrow A(H^+, H^-) \equiv 0 \Rightarrow K \equiv 1,$$

that is, (X, N) has the extrinsic Gauss curvature of a (smooth) flat surface in hyperbolic space, cf [18, Lemma 5.3]. Furthermore, as $\delta(x \pm n) \parallel \delta H^\pm$, we learn that the lifts $x \pm n : \Sigma_0^2 \rightarrow \mathcal{L}$ of the enveloped horosphere congruences $\langle x \pm n \rangle$ are Königs dual lifts of isothermic sphere congruences, showing that the Legendre map $\langle x, n \rangle = \langle x+n, x-n \rangle$ is an Ω -net in the sense of [6, Def 3.1] or [7, Def 6.1], cf [18, Prop 5.1].

Summary. Any discrete flat front $(X, N) : \Sigma_0^2 \rightarrow H^3 \times S^{2,1}$ obtained from the Weierstrass representation (3.3) is a discrete linear Weingarten net with constant Gauss curvature $K \equiv 1$.

3.2. Darboux pairs of holomorphic maps

This last observation, about the enveloped horosphere congruences $\langle x \pm n \rangle$ of a flat front (x, n) in hyperbolic space being isothermic with Königs dual lifts, leads to a relation between flat fronts and Darboux pairs of discrete holomorphic maps, see [13, Thm 31]: *the hyperbolic Gauss maps of a discrete flat front (in the mixed area sense) form a Darboux pair of holomorphic maps in the 2-sphere; conversely, any Darboux pair of discrete holomorphic maps in a 2-sphere yields a hyperbolic line congruence so that its orthogonal nets have mixed area Gauss curvature $K \equiv 1$.* The configuration is illustrated in Fig. 1.

Clearly the first assertion of [13, Thm 31], for discrete flat fronts in the sense of [6], now follows directly also for discrete flat fronts in the sense of [14]:

Lemma. The two hyperbolic Gauss maps $\langle H^\pm \rangle$ of a discrete flat front $(X, N) : \Sigma_0^2 \rightarrow H^3 \times S^{2,1}$, obtained from the Weierstrass representation (3.3), form a Darboux pair in $S^2 \cong \mathbb{P}(\mathcal{L}^3)$.

Alternatively, this Lemma is more directly verified by straightforward cross ratio computations: using the $\mathbb{CP}^1$ -versions $\langle h^\pm \rangle : \Sigma_0^2 \rightarrow \mathbb{CP}^1$, $h^\pm = Fe^\pm$ with $e^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, of the hyperbolic Gauss maps, the Weierstrass representation (3.2) yields

$$\begin{aligned} \text{cr}(\langle h_i^\pm \rangle, \langle h_j^\pm \rangle, \langle h_k^\pm \rangle, \langle h_l^\pm \rangle) &= \frac{\det(e^\pm, W_{ij}e^\pm)}{\det(e^\pm, W_{jk}e^\pm)} \frac{\det(W_{lk}e^\pm, e^\pm)}{\det(W_{il}e^\pm, e^\pm)} = \frac{a_{ij}}{1-ta_{ij}} \frac{1-ta_{jk}}{a_{jk}}, \\ \text{cr}(\langle h_i^\pm \rangle, \langle h_j^\pm \rangle, \langle h_j^\mp \rangle, \langle h_i^\mp \rangle) &= \frac{\det(e^+, W_{ij}e^+)}{\det(e^+, e^-)} \frac{\det(W_{ij}e^-, e^-)}{\det(e^-, e^+)} = -\frac{ta_{ij}}{1-ta_{ij}}. \end{aligned} \quad (3.5)$$

Thus $\langle h^\pm \rangle : \Sigma_0^2 \rightarrow \mathbb{CP}^1 \cong S^2$ form a Darboux pair of isothermic nets, with cross ratio factorizing edge-labelling

$$b := -\frac{a}{1-ta} \Leftrightarrow 1 = (1-ta)(1-tb). \quad (3.6)$$

Remark. By [9, Thm 8.4.1] we just confirmed that the point pair map $(\langle h^+ \rangle, \langle h^- \rangle) : \Sigma_0^2 \rightarrow S^2 \times S^2$ qualifies as a discrete curved flat. Observe how this resonates with the change of the cross ratio factorizing edge-labelling (3.6), cf [11, §5.7.34 and §5.7.16], and the fact that (3.2) is a discretization of the smooth curved flat system $dF = F\Phi$ with $\Phi = \begin{pmatrix} 0 & dg \\ td\bar{g}^* & 0 \end{pmatrix}$, cf [11, §5.5.20] and [4, Sect. 3].

Mission. We would like to obtain the converse: given a Darboux pair (curved flat) of discrete holomorphic maps $\langle h^\pm \rangle : \Sigma_0^2 \rightarrow \mathbb{CP}^1$ into a 2-sphere $S^2 \cong \mathbb{CP}^1$, we would like to show the existence of—and derive a defining equation for—lifts $h^\pm$ so that $F = (h^+, h^-)$ satisfies (3.2).

Setting. Thus let $h^\pm : \Sigma_0^2 \rightarrow \mathbb{C}^2$ denote (any lifts of) the maps of a Darboux pair of discrete holomorphic maps into the 2-sphere $S^2 \cong \mathbb{CP}^1$, with parameter $t \in \mathbb{R} \setminus \{0\}$ and an edge-labelling $b : \Sigma_1^2 \rightarrow \mathbb{R}$, that is (cf [11, §5.7.12]),

$$\begin{aligned} \forall (ijkl) \in \Sigma_2^2 : \text{cr}(\langle h_i^\pm \rangle, \langle h_j^\pm \rangle, \langle h_k^\pm \rangle, \langle h_l^\pm \rangle) &= \frac{b_{ij}}{b_{jk}} \text{ and} \\ \forall (ij) \in \Sigma_1^2 : \text{cr}(\langle h_i^+ \rangle, \langle h_j^+ \rangle, \langle h_j^- \rangle, \langle h_i^- \rangle) &= tb_{ij}. \end{aligned}$$

Further, let $F' := (h^+, h^-) : \Sigma_0^2 \rightarrow \text{Gl}(2, \mathbb{C})$ and $W' : \Sigma_1^2 \rightarrow \text{Gl}(2, \mathbb{C})$, $(ij) \mapsto W'_{ij} := (F'_i)^{-1}F'_j$.

Cross ratios. Writing $W' = \begin{pmatrix} u & v \\ x & y \end{pmatrix}$, the assumption $F' : \Sigma_0^2 \rightarrow \text{Sl}(2, \mathbb{C})$ and the condition for the cross ratio of corresponding edges yield, analogously to (3.5),

$$\left. \begin{aligned} uy - vx &= 1 \\ -vx &= tb \end{aligned} \right\} \Rightarrow \begin{cases} x = -\frac{tb}{v}, \\ y = \frac{1-tb}{u}; \end{cases} \quad (3.7)$$

hence the factorizing cross ratios of the isothermic nets $h^\pm$ on $(ijkl) \in \Sigma_2^2$ reduce to a single equation,

$$\frac{b_{ij}}{b_{jk}} = \text{cr}(\langle h_i^\pm \rangle, \langle h_j^\pm \rangle, \langle h_k^\pm \rangle, \langle h_l^\pm \rangle) = \frac{v_{ij}}{v_{jk}} \frac{v_{lk}}{v_{il}} \Leftrightarrow b_{jk} \frac{v_{ij}}{v_{jk}} = b_{ij} \frac{v_{il}}{v_{lk}}. \quad (3.8)$$

Maurer–Cartan equation. Next we wheel out the conditions on W' to define a flat connection: firstly, using (3.7), the condition $W'_{ji} = (W'_{ij})^{-1}$ on W' to define a connection produces two equations

$$\left. \begin{aligned} 1 &= u_{ij}u_{ji} - tb_{ij}\frac{v_{ij}}{v_{ji}} \\ 0 &= u_{ij}u_{ji} + (1 - tb_{ij})\frac{v_{ij}}{v_{ji}} \end{aligned} \right\} \Leftrightarrow \begin{cases} v_{ji} = -v_{ij}, \\ u_{ij}u_{ji} = 1 - tb_{ij}, \end{cases} \quad (3.9)$$

in particular, we conclude that v is a discrete 1-form; secondly, we use (3.7) and (3.8) to extract two equations from the flatness $W'_{ij}W'_{jk} = W'_{il}W'_{lk}$ of W' ,

$$u_{ij}u_{jk} - u_{il}u_{lk} = t(b_{jk}\frac{v_{ij}}{v_{jk}} - b_{ij}\frac{v_{il}}{v_{lk}}) = 0 \quad (3.10)$$

and

$$v_{ij}\frac{1 - tb_{jk}}{u_{jk}} + v_{jk}u_{ij} = v_{il}\frac{1 - tb_{ij}}{u_{lk}} + v_{lk}u_{il}. \quad (3.11)$$

Gauge transformation. Finally consider the remaining freedom $F = F'G = (h^+w, h^-w^{-1})$ for the choice of lifts (h^+, h^-) of the Darboux pair and the corresponding gauge transformation for the induced connection,

$$W_{ij} = G_i^{-1}W'_{ij}G_j = \begin{pmatrix} \frac{u_{ij}w_j}{w_i} & \frac{v_{ij}}{w_iw_j} \\ -tb_{ij}\frac{w_iw_j}{v_{ij}} & (1 - tb_{ij})\frac{w_i}{u_{ij}w_j} \end{pmatrix},$$

seeking lifts so that W has the appropriate shape of (3.2): using (3.6) this leads to the difference equation

$$\left. (1 - tb_{ij})\frac{\frac{u_{ij}w_j}{w_i}}{u_{ij}w_j} \right\} = \frac{1}{\sqrt{1 - ta_{ij}}} = \sqrt{1 - tb_{ij}} \Leftrightarrow w_j = \frac{\sqrt{1 - tb_{ij}}}{u_{ij}} w_i,$$

the integrability of which is granted by (3.10) and the fact that b is an edge-labelling. Further, using the difference equation for w , (3.9) and (3.11) yield the existence of a potential $g : \Sigma_0^2 \rightarrow \mathbb{C}$ of the 1-form

$$\Sigma_1^2 \ni (ij) \mapsto \frac{v_{ij}}{w_iw_j\sqrt{1 - tb_{ij}}} = dg_{ij} \in \mathbb{C},$$

which is holomorphic by (3.8), with cross ratio factorizing edge-labelling $a = -\frac{b}{1 - tb}$ by (3.6).

Thus we have proved:

Lemma. Any Darboux pair of discrete holomorphic maps in the 2-sphere $S^2 \cong \mathbb{CP}^1$ admits a Weierstrass representation (3.2), that is, there exist lifts $h^\pm : \Sigma_0^2 \rightarrow \mathbb{C}^2$ of the two holomorphic maps $\langle h^\pm \rangle : \Sigma_0^2 \rightarrow \mathbb{CP}^1$ so that $F := (h^+, h^-) : \Sigma_0^2 \rightarrow \text{Sl}(2, \mathbb{C})$ satisfies (3.2).

And, as the hyperbolic Gauss maps of a discrete flat front in the mixed area sense of [6] form a Darboux pair of holomorphic maps by [13, Thm 31], we also learn that every flat front in hyperbolic space admits a Weierstrass representation:

Corollary. Any discrete flat front $(X, N) : \Sigma_0 \rightarrow H^3 \times S^{2,1}$, defined as a linear Weingarten net with mixed area Gauss curvature $K \equiv 1$, admits a Weierstrass-type representation (3.2).

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